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A Note on a Result of J. Ax

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James Ax has proved [1, Sect. 2, Proposition 2′] that if (K, V) is a henselian rank one valued field which is perfect of characteristic $p > 0$ and if α is an element of an algebraic closure $\bar{K}$ of K, then there exists $a \in K$ such that $V(\alpha - a) \geqslant \Delta(\alpha)$, where

$$\Delta(\alpha) = \min\{\bar{V}(\alpha' - \alpha)\colon \alpha' \text{ runs over } K\text{-conjugates of } \alpha,\ \bar{V} \text{ is an extension of } V \text{ to } \bar{K}\}.$$

We wish to point out in this note that this result is false by giving a simple counterexample.

Let k_0 be the algebraic closure of the finite field F_p of p elements and $K_0 = k_0((T))$ be the field of Laurent series in T with valuation v_0 given by $v_0(T) = 1$. Fix an algebraic closure L of K_0 with a valuation v such that v extends v_0 and the value group of v is contained in the set of rational numbers. Let K be the inseparable closure of K_0 in L. It is readily verified (cf. [2, Chap. II, Sect. 4]) that K is the union of the fields $k_0((T^{p^{-n}}))$, n running over all natural numbers. K satisfies the hypothesis of Ax's proposition: it is perfect and being an algebraic extension of a complete rank 1 valued field, is henselian. Let α be an element of L, satisfying the relation $\alpha^p - \alpha - T^{-1} = 0$. Then $v(\alpha) = -1/p$ and α is not in K in view of a lemma proved below. So the conjugates of α are $\alpha, \alpha + 1, \ldots, \alpha + p - 1$ and $\Delta(\alpha) = 0$. We claim that there does not exist an element a in K for which $v(\alpha - a) \geqslant 0$; for if there exists such an element a in K, then $v(a) = v(\alpha) = -1/p$. Also,

$$\begin{aligned} v(a^p - a - T^{-1}) &= v(a^p - a - T^{-1} - (\alpha^p - \alpha - T^{-1})) \\ &\geqslant v(\alpha - a) \geqslant 0, \end{aligned}$$

which contradicts the following lemma.

LEMMA. *If z is any element of K (K as constructed above) with $v(z) = -1/p$, then $v(z^p - z - T^{-1}) < 0$.*

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Proof. Suppose $z \in k_0((T^{p^{-n}}))$; then we can write

$$z = \sum_{i \in Z,\, i \geqslant -p^{n-1}} a_i T^{ip^{-n}}, \qquad a_i \neq 0 \text{ if } i = -p^{n-1}.$$

Let $i_0 p^{-n}$ be the largest among the negative exponents of T occurring in z. Now looking at the finitely many negative exponents of T in $z^p - z - T^{-1}$, one has

$$v(z^p - z - T^{-1}) \leqslant i_0 p^{-n} < 0.$$

References

1. J. Ax, Zeroes of polynomials over local fields—The Galois action, *J. Algebra* **15** (1970), 417–428.
2. J. P. Serre, "Local Fields," Springer-Verlag, New York, 1979.