Finite 2-Groups with Small Centralizer of an Involution

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1. INTRODUCTION

The starting point is the following theorem of Berkovich [2]. For a finite $p$-group $G$, one of the following holds:

(a) $G$ has no maximal elementary abelian subgroup of order $p^2$.

(b) $|\Omega_2(G)| \leq p^3$.

(c) There exists in $G$ an element $x$ of order $p$ such that $C_G(x) = \langle x \rangle \times Q$, where $Q$ is a cyclic or generalized quaternion. Furthermore, $G$ has no normal subgroup of order $p^{p+1}$ and exponent $p$.

The classification of $p$-groups containing an element $t$ of order $p$ such that $C_G(t) = \langle t \rangle \times Z$, where $Z$ is cyclic of order $p^m$, is very difficult. This question is solved for $m = 1$ by Suzuki (see [5, Satz 14.23]). In this case we have a well-known characterization of $p$-groups of maximal class. For $p = 2$ and $m = 2$, the problem is solved in Gorenstein [4, Proposition 10.27]. For $p > 2$ there are some results of Blackburn [3] which only show that the problem is a difficult one indeed. For $p = 2$, the problem is solved in a special case (with two other assumptions) in Berkovich [2, Theorem 9.2].

In this paper we consider the case $p = 2$ in general. In fact we shall prove the following classification result and in the case where $G$ has elementary abelian subgroups of order 8, we get exactly four infinite classes of 2-groups which we give in terms of generators and relations.

**Theorem 1.1.** Let $G$ be a finite nonabelian 2-group containing an involution $t$ such that the centralizer $C_G(t) = \langle t \rangle \times C$, where $C$ is a cyclic group...
of order $2^m$, $m \geq 1$. Then $G$ has no elementary abelian subgroup of order 16 and $G$ is generated by at most three elements.

(A1) If $G$ has no elementary abelian subgroup of order 8, then one of the following assertions holds:

(a) $G$ is a dihedral group $D_{2^m}$ of order $2^n$ ($n \geq 3$) or $G$ is a semidihedral group $SD_{2^n}$ of order $2^n$ ($n \geq 4$). Here we have $m = 1$.

(b) $G$ is isomorphic to the group $M_{2^n}$ of order $2^n$ ($n \geq 4$) and this is the unique 2-group of class 2 and order $2^n$ which has a cyclic subgroup of index 2. Here we have $m = n - 2$.

(c) $|G : C_G(t)| = 2$, $t \in \Phi(G)$, $Z(G)$ is a cyclic subgroup of order $\geq 4$ not contained in $\Phi(C_G(t))$, $G/Z(G)$ is dihedral with the cyclic subgroup $T/Z(G)$ of index 2, $T$ is abelian of type $(2,2^n)$, $m \geq 3$ and $t \in T$. If $x$ is an element of maximal order in $G \setminus T$, then $(x^2) = Z(G)$. Here we have $C_G(t) = T$ and $|G| = 2^{m+2}$.

(d) $G$ has a subgroup $S$ of index $\leq 2$, where $S = AL$, the subgroup $L$ is normal in $G$, $L = \langle b, t \mid b^{2^n-1} = 1, t^2 = 1, b^i = b^{-1}, n \geq 3 \rangle \cong D_{2^n}$, $A = \langle a \rangle$ is cyclic order $2^m$, $m \geq 2$, $A \cap L = Z(L)$, $[a,t] = 1$, $\Omega_1(G) = \Omega_2(S) = \Omega_2(A) \ast L$ which is the central product of $\Omega_2(A)$ and $L$, where $\Omega_2(A) \cap L = Z(L)$. If $|G : S| = 2$, then there is an element $x \in G \setminus S$ so that $t^x = tb$ and $C_G(t) = (t) \times \langle a \rangle$.

(A2) If $G$ has an elementary abelian subgroup of order 8, then $Z(G)$ is of order 2, $C_G(t) = (t) \times \langle a \rangle$, where $A = \langle a \rangle$ has order $2^m$ ($m \geq 2$), $G$ has a normal subgroup $L = \langle b, t \mid b^{2^n-1} = 1, t^2 = 1, b^i = b^{-1}, n \geq 3 \rangle \cong D_{2^n}$, $A \cap L = Z(L) = \langle z \rangle$, $S = AL$ is a normal subgroup of index $\leq 2$ in $G$, and $G$ is isomorphic to one of the following groups:

(e) $G = \langle a,b,t \mid a^{2^n} = 1, m \geq 3, b^{2^n-1} = 1, n \geq 4, t^2 = 1, b^i = b^{-1}, [a,t] = 1, a^{2^n-1} = b^{2^n-2} = z, b^a = b^{i+2}, i = n-m \geq 2 \rangle$. We have $G = AL = S$ and the cyclic group $\langle a \rangle / \langle z \rangle$ of order $2^{m-1}$ acts faithfully on $L$.

(f) $G = \langle a,b,t,s \mid a^{2^n} = 1, a^{2^n-1} = z, m \geq 4, b^{2^n-1} = 1, n \geq 5, b^{2^n-2} = z, t^2 = 1, b^i = b^{-1}, [t,a] = 1, b^a = b^{i+2}, i = n-m+1, s^2 = 1, b^i = b^{-1}, t^i = tb, s^a = a^{-1} a^{-1} b^{2^n-1} s, i \geq 2 \rangle$. Here $S = AL$ is a subgroup of index 2 in $G$ and $\Omega_1(S) = \Omega_2(S) \ast L$. Also $G = S(s)$, where $M = L(s) \cong D_{2^{m+1}}$, $N_G(M) = M \langle a^2 \rangle$ and $s$ acts invertingly on $\Omega_2(A)$. We have $|G : S| = 2$ and so the order of $G$ is $2^{m+n}$.

(g) Berkovich groups $G = F(m, n)$ from [2, Theorem 9.2]. Here we have $G = \langle a,b,t,s \mid a^{2^n} = 1, a^{2^n-1} = z, m \geq 2, b^{2^n-1} = 1, n \geq 3, b^{2^n-2} = z, t^2 = 1, b^i = b^{-1}, [t,a] = 1, [b,a] = 1, s^2 = 1, b^s = b^{-1}, t^s = tb, a^i = a^{-1} z^i, v = 0, 1, and if $v = 1$, then $m \geq 3$]. Here $S = A \ast L$ is the central product of $A$ and $L$ and $G = S(s)$, where $|G : S| = 2$. We have $M = L(s) \cong D_{2^{m+1}}$ and $s$ acts invertingly on $\langle a \rangle$ or in case $m \geq 3$ we can have also $a^i = a^{-1} z$.
(h) \( G = \langle a, b, t, s \mid a^{2m} = 1, m \geq 4, b^{2n-1} = 1, n \geq 4, a^{2m-1} = b^{2n-2} = z, t^2 = 1, b^t = b^{-1}, s^t = s, b^s = b^{-1}, t^b = tb, [a, t] = 1, b^s = bz, a^t = a^{-1+2n-2}b^{2n-2} \rangle \). Here we have again \( G = S\langle s \rangle \) so that \( |G : S| = 2 \). Finally, \( M = L\langle s \rangle \) is isomorphic to \( D_{2n+1} \).

2. NOTATION AND KNOWN RESULTS

Let \( C_m \) be the cyclic group of order \( m \), \( E_{pm} \) the elementary abelian group of order \( p^m \) \((p \text{ prime})\), \( D_{2n} \) the dihedral group of order \( 2^m \) \((m \geq 3)\), \( SD_{2n} \) the semidihedral group of order \( 2^m \) \((m \geq 4)\), \( Q_{2n} \) the generalized quaternion group of order \( 2^m \) \((m \geq 3)\),

\[
M_{2n} = \langle x, y \mid x^2 = y^{2m-1} = 1, m \geq 4, [y, x] = y^{2n-1} \rangle,
\]

\( C_G(M) \) the centralizer of a subset \( M \) in \( G \), \( N_G(H) \) the normalizer of a subgroup \( H \) in \( G \), \( G' \) the derived group of \( G \), \( Z(G) \) the center of \( G \), \( \Phi(G) \) the Frattini subgroup of \( G \), and for a \( p \)-group \( G \) \((p \text{ is a prime})\) we set \( \Omega_n(G) = \langle x \in G \mid x^{p^n} = 1 \rangle \) and \( \Omega_n(G) = \langle x^{p^n} \mid x \in G \rangle \).

A \( p \)-group \( G \) of order \( p^m \) is said to be of maximal class if \( m \geq 3 \) and the class of \( G \) is \( m - 1 \). For \( x \in G \), we denote by \( ccl_G(x) \) the conjugacy class of \( x \) in \( G \). Two elements \( x, y \) in \( G \) are said to be fused in \( G \) if there is \( g \in G \) such that \( x^g = g^{-1}xg = y \). Finally, \( [x, y] = x^{-1}y^{-1}xy \) is the commutator of \( x \) and \( y \).

**Proposition 2.1** (Berkovich [1, Proposition 19(b)]). Let \( B \) be a subgroup of a nonabelian \( p \)-group \( G \) such that \( C_G(B) \subseteq B \). If \( |B| = p^2 \), then \( G \) is of maximal class.

**Proposition 2.2** (Huppert [5, p. 90]). Let \( G \) be a nonabelian 2-group which possesses a cyclic subgroup of index 2. Then \( G \) is isomorphic to one of the following groups: \( D_2(n \geq 3), SD_2(n \geq 4), Q_{2n}(n \geq 3), \) or \( M_{2n}(n \geq 4) \).

**Proposition 2.3** (Berkovich [2, Theorem 10.3]). Let \( G \) be a nonabelian 2-group. If \( |G : G'| = 4 \), then \( G \) has a cyclic subgroup of index 2. In particular, if \( G \) is a 2-group of maximal class, then \( G \) is isomorphic to \( D_{2^r}, SD_{2^r}, \) or \( Q_{2^r} \).

**Proposition 2.4** (Huppert [5, p. 304]). Let \( G \) be a \( p \)-group in which every normal abelian subgroup is cyclic. Then \( G \) is either cyclic or a 2-group of maximal class.

**Proposition 2.5** (Huppert [5, p. 84]). Let \( G = \langle b \rangle \) be a cyclic group of order \( 2^n \) \((n \geq 3)\). Then the automorphism group \( \text{Aut}(G) \) is abelian of type \((1, n - 2)\) and \( G \equiv \langle \alpha \rangle \times \langle \beta \rangle \), where \( \alpha \) is induced by \( b^n = b^{-1} \) and \( \beta \) is induced by \( b^\beta = b^5 = b^{1+2} \). Here \( \alpha \) is of order 2 and \( \beta \) is of order \( 2^{n-2} \). We have \( C_G(\alpha) = \langle b^{2^{n-1}} \rangle \), which is the fixed subgroup of \( \alpha \) in \( G \) \((\text{of order } 2)\). The
The fixed subgroup of \( \beta \) in \( G \) is \( \langle b^{2^{i-2}} \rangle \) which is of order 4. The fixed subgroup of \( \beta^2 \) in \( G \) has order \( 2^{2j+1} \) \((0 \leq j \leq n - 2)\). On the other hand, the automorphism \( \gamma \) induced by \( b^\gamma = b^{1+2^i} \) \((2 \leq i \leq n)\) has the fixed subgroup of order \( 2^i \). Hence there is an odd number \( r \) so that

\[
b^{\beta^2 r - 2} = b^{1+2^i} \quad (2 \leq i \leq n).
\]

**Proposition 2.6.** The automorphism group \( \text{Aut}(G) \) of \( G = \langle b, t \mid b^{2n} = 1, t^2 = 1, b^t = b^{-1} = D_{2^{n+1}} \) \((n \geq 3)\) is generated by the inner automorphism group \( \text{Inn}(G) \) (which is isomorphic to \( D_{2^n} \)) and “outer” automorphisms \( \alpha \) and \( \beta \), where \( \alpha \) is of order 2 and is induced with \( t^\alpha = tb \), \( b^\alpha = b^{-1} \), and \( \beta \) is of order \( 2^{n-2} \) and is induced with \( t^\beta = t \), \( b^\beta = b^5 \). We have \( [\alpha, \beta] = i_{2^n} \), which is the inner automorphism of \( G \) induced (by conjugation) with the element \( b^5 \). Hence the outer automorphism group \( \text{Aut}(G)/\text{Inn}(G) \) is abelian of type \((1, n - 2)\). Furthermore \( \text{CG}(\beta^2) \cong D_{2^{j+3}} \) \((0 \leq j \leq n - 2)\).

**Proof.** The inner automorphisms of \( G \) induced with elements contained in \( \langle b \rangle \) fuse the \( 2^n \) involutions in \( G \setminus Z(G) \) in two orbits \( O_1 = t\langle b^5 \rangle \) and \( O_2 = tb\langle b^5 \rangle \) of lengths \( 2^{n-1} \) each. The inner automorphism \( i_t \) (induced by \( t \)) fixes \( t \) and acts invertingly on \( \langle b \rangle \). Consider the outer automorphism \( \alpha \) of order 2 induced by \( t^\alpha = tb \), \( b^\alpha = b^{-1} \) so that \( \alpha \) fuses \( O_1 \) and \( O_2 \). Let \( B \supseteq \text{Inn}(G) \) be the subgroup of \( \text{Aut}(G) \) which fixes \( O_1 \) (and \( O_2 \)) so that \( |\text{Aut}(G) : B| = 2 \). \( B \) is normal in \( \text{Aut}(G) \), and \( \text{Aut}(G) = \langle \alpha \rangle B \) which is a semidirect product of \( \langle \alpha \rangle \) and \( B \). Let \( \beta'' \) be any outer automorphism from \( B \setminus \text{Inn}(G) \). Then multiplying \( \beta'' \) with an inner automorphism \( i \) induced with an element contained in \( \langle b \rangle \), we may assume that \( \beta'' = \beta''i \) fixes the involution \( t \). But \( (\beta')' \) must act faithfully on the (characteristic) cyclic subgroup \( \langle b \rangle \) of \( G \). Multiplying \( \beta' \) with \( i_t \) (if necessary), we may assume that \( \beta_0 = \beta'i_{2^e} \) \((e = 0, 1)\) fixes \( t \) and centralizes a subgroup of order \( \geq 4 \) in \( \langle b \rangle \). By Proposition 2.5, we see that \( \beta_0 \) acts on \( \langle b \rangle \) as a power of the automorphism \( \beta \) from Proposition 2.5. Hence if we consider the outer automorphism \( \beta \) of order \( 2^{n-2} \) induced by \( t^\beta = t \), \( b^\beta = b^5 \), we see that \( B \) is a semidirect product of \( \langle \beta \rangle \) and \( \text{Inn}(G) \) and so \( \text{Aut}(G) = \text{Inn}(G)(\alpha, \beta) \). Finally, we compute that \( [\alpha, \beta] = i_{2^n} \) and so \( \text{Aut}(G)/\text{Inn}(G) \) is abelian of type \((1, n - 2)\). It follows from Proposition 2.5 that the fixed subgroup of \( \beta^2 \) in \( G \) is isomorphic to \( D_{2^{j+3}} \) \((0 \leq j \leq n - 2)\).

**3. PROOF OF THEOREM 1.1**

Let \( G \) be a nonabelian 2-group containing an involution \( t \) such that \( C_G(t) = \langle t \rangle \times C \), where \( C \cong C_{2^m} \), \( m \geq 1 \). If \( m = 1 \), then by Propositions 2.1 and 2.3, we have that \( G \) is isomorphic to \( D_{2^n} \), \( n \geq 3 \), or \( SD_{2^n} \), \( n \geq 4 \). In what follows we assume that \( m \geq 2 \). Since \( G \) is not of maximal class, it follows by
Proposition 2.4 that $G$ possesses a normal four-subgroup $U$. Set $T = C_G(U)$ so that we have $[G : T] \leq 2$. Suppose in addition that $t \in T$. This forces $t \in U$ since $C_G(t)$ does not have an elementary abelian subgroup of order 8. Also we have $[G : T] = 2$ and $T = \langle t \rangle \times C$, where $C \cong C_m$, $m \geq 2$. We may also assume that $G$ does not possess any other normal four-subgroup which possibly does not contain $t$ because we shall consider that case later. Set $\Omega_1(C) = \langle a \rangle = \Omega_{m-1}(T)$ so that $U = \langle t, u \rangle = \Omega_1(T)$ and $\langle u \rangle \subseteq Z(G)$. Since $Z(G) \subseteq T$, so $\langle u \rangle = \Omega_1(Z(G))$. We have $\Phi(G) = \Omega_1(G) \subseteq T$ and $\Phi(G)$ contains $\Omega_1(C)$ which is of order $2^{m-1}$. If $t$ is not in $\Phi(G)$, then there exists a maximal subgroup $M$ of $G$ which does not contain $t$. Set $M_0 = T \cap M$ so that $M_0 \cong C_m$ is a cyclic subgroup of index 2 in $M$ and $C_G(t) = \langle t \rangle \times M_0$. If $M$ is cyclic, then $t$ acts faithfully on $M$ and $t$ centralizes the maximal subgroup $M_0$ of $M$ which gives that $G$ is isomorphic to $M_{2m+2}$, $m \geq 2$, and this is case (b) of Theorem 1.1. So assume now that $M$ is not cyclic but $M$ and the cyclic subgroup $M_0$ of index 2. If $M$ were abelian, then $\Omega_1(M)$ is a normal four-subgroup of $G$ which does not contain $t$, contrary to our assumption. Thus $M$ is nonabelian. If $M$ is not of maximal class, then again $\Omega_1(M)$ is a normal four-subgroup of $G$ which does not contain $t$. Hence $M$ is a group of maximal class and we have $Z(M) = \langle u \rangle$, where $\langle t, u \rangle = U$. Let $\langle v \rangle$ be the cyclic subgroup of order 4 contained in $M_0$ and let $y \in M \setminus M_0$ so that we have $v^y = v^{-1} = vu$ and $v' = tu$. Hence we have $(tv)^y = tv$, where $(tv)$ is a cyclic subgroup of order 4 in $T \setminus M_0$ with $(tv)^2 = u$ and $C_G(tv) \supseteq \langle T, y \rangle = G$. It follows that $G$ is the central product of a cyclic group of order 4 and a 2-group of maximal class. But then in any case $G$ contains a dihedral subgroup of order $2^{m-1}$, $m \geq 2$, and the center of $G$ is cyclic of order 4. This group is then a special case of groups in (d) of Theorem 1.1 (with $S = G$ and $A$ is of order 4 centralising $L$). It remains to consider the case $t \in \Phi(G)$. In this case we have $\Phi(G) = \langle t \rangle \times \Omega_1(C)$ and so the group $G$ generated by two elements. If $m = 2$, then $\Phi(G) = \langle t, u \rangle$. Since $\Omega_1(T) = \langle u \rangle$, there is an element $x$ in $G \setminus T$ such that $x^2 \in U \setminus \langle u \rangle$. But then $x$ centralizes $U$, which is a contradiction. Hence we must have $m \geq 3$. There is an element $x \in G \setminus T$ such that $x^2 \in \Phi(G) \setminus \Omega_1(C) \setminus U$. Set $C = \langle a \rangle$ which is of order $2^m$, $m \geq 3$, so that $a^{2^{m-1}} = u$, $T = C_G(t) = \langle t \rangle \times \langle a \rangle$, and $U = \Omega_1(T) = \langle t, u \rangle$. Also, $\Omega_1(C) = \Omega_1(C) = \langle a^2 \rangle$ and $\Phi(G) = \langle t, a^2 \rangle$, where $a^2$ is of order $\geq 4$. It follows from the choice of $x$ that $\Phi(G) = \langle a^2, x^2 \rangle$. We have $t^2 = tu$ since $T = C_G(t)$. If $x$ would centralize $a^2$, then $x$ centralizes $\langle a^2, x^2 \rangle = \Phi(G)$ and $\Phi(G)$ contains $t$ so $x \in C_G(t) = T$, which is not the case. Also $C_G(x^2) \supseteq \langle T, x \rangle = G$, so $x^2 \in C_G(t)$. Since $t \notin Z(G)$, so $Z(G) \leq T$ is cyclic. We claim that $\langle x^2 \rangle = Z(G)$. If not, then there is an element $y \in Z(G)$ with $y^2 = x^2$. But $y^2 \in \Omega_1(C) = \Phi(T)$ and so $x^2 \in \Omega_1(C)$ which is not the case. Moreover, if $y \in G \setminus T$, then $y^2 \in Z(G)$ since $T$ is abelian and generates $G$ together with $y$. As we saw, $a^2 \notin Z(G)$. All ele-
ments in \((G/Z(G))\setminus(T/Z(G))\) are involutions. Since \(Z(G) \not\subset \Phi(T)\), it follows that \(T/Z(G)\) is cyclic. In that case, \(G/Z(G)\) is dihedral. Indeed, since \(\Phi(G) \not\subset Z(G)\), we conclude that \(G/Z(G)\) is not abelian of type \((2,2)\). We have obtained a group in part (c) of the theorem.

In the rest of the proof we assume that \(G\) possesses a normal four-subgroup \(U\) which does not contain our involution \(t\). Set again \(T = C_G(U)\) and so \(t\) is not in \(T\) so that \(G = \langle t \rangle T\). By Dedekind law, \(G_0 = C_G(t) = \langle t \rangle \times A\), where \(A = C_T(t)\) is cyclic of order \(2^m\), \(m \geq 2\), and so \(Z = \Omega_1(A) = C_T(t) = \Omega_1(Z(G))\). Since \(Z(G) \trianglelefteq A\), so \(Z(G)\) is cyclic. Set \(G_1 = N_G(G_0)\). Since \(G\) is nonabelian we have \(G_1 \neq G_0\). Since \(\Omega_1(G_0) = \langle t, z \rangle\) where \(z = Z\), \(t\) has only two conjugates, \(t\) and \(tz\), in \(G_1\). It follows that \(|G_1 : G_0| = 2\) and so \(G_1 \cong G_0 U\) because \(|G_0 U : G_0| = 2\). Set \(D_0 = U\langle t \rangle\) so that \(D_0\) is a dihedral group of order 8 and \(G_1\) is the central product \(G_1 = A \ast D_0\) with \(A \cap D_0 = \langle z \rangle\). We have \(U = \langle u, z \rangle \subset D_0\). Let \(v\) be an element of order 4 in \(D_0\) and let \(y\) be an element of order 4 in \(A\) so that we have \(y^2 = v^2 = z\) and so \(x = yv\) is an involution in \(G_1 \setminus D_0\). Since \(x^t = xz\), we see that \(D_1 = \langle x, t \rangle\) is a dihedral group of order 8. Because \(A = Z(G_1)\), we also have \(G_1 = A \ast D_1\), \(D_1 \cong D_8\), \(t \in D_1\), and \(D_1 \cap U = Z(D_1) = \langle z \rangle\).

In the rest of the proof we consider a subgroup \(S\) of \(G\) which is maximal subject to the following conditions:

1. \(S\) contains \(G_1 = A \ast D_1\).
2. \(S = A L\), where \(L\) is a normal subgroup of \(S\) and \(L \cong D_{2^n}, n \geq 3\).
3. \(A \cap L = \langle z \rangle = Z(L)\) and \(A \cap S = \langle z \rangle\).
4. \(L \cap G_1 = D_1 \cong D_8\).

We note that \(A = \langle a \rangle\) is cyclic of order \(2^m\), \(m \geq 2\), and \(\Omega_1(A) = \langle z \rangle = Z(D_1) = Z(L) = \Omega_1(Z(G))\). We have \(t \in D_1\), \(C_G(t) = \langle t \rangle \times A\) and \(A = Z(G_1)\). Hence \(Z(G) \subseteq A\) and so \(Z(G)\) is cyclic of order \(\leq 2^m\). Also, \(G_1\) contains \(U = \langle z, u \rangle\) which is a normal four-subgroup of \(G\) and \(u^t = uz\). Now we act with \(A = \langle a \rangle\) on the dihedral group \(L\), where we set \(L = \langle b, t \mid b^{2^{n-1}} = 1, t^2 = 1, b^t = b^{-1}\rangle\). Here \(\langle b \rangle\) is the unique cyclic subgroup of index 2 in \(L\) and so \(\langle b \rangle\) is \(A\)-admissible. We have \(\langle b \rangle \cap D_1 \geq 4\), \(A\) centralizes \(D_1\) and \(C_L(a) = \langle t \rangle (C(a) \cap \langle b \rangle)\) and so \(C_L(a)\) is a dihedral subgroup of \(L\) of order \(\geq 8\) containing \(D_1\). Looking at \(\text{Aut}(L)\) (Proposition 2.6) we see that \(A\) induces on \(L\) a cyclic group of automorphisms of order at most \(2^{n-3}\) and so \(|A : C_A(L)| \leq 2^{n-3}\). Since \(S/L\) is cyclic, we have \(\Omega_1(S) = UL\). Set \(\Omega_1(S) \cap A = \langle y \rangle\), where \(y\) is an element of order 4 with \(y^2 = z\) and we have \(\langle y \rangle = \Omega_2(A)\). Let us consider at first the case that \(y\) acts faithfully on \(L\) so that \(C_A(L) = \langle z \rangle\) and \(\langle a \rangle / \langle z \rangle\) acts faithfully on \(L\). Then \(y\) induces an automorphism of order 2 on \(L\) and since \(C_L(a) \supseteq D_1\) we must have \(n \geq 4\). \([y, t] = 1\), and \(b^y = bz\) so that \(C_L(y) = \langle t, b^y \rangle \cong D_{2^{n-1}}\) and \(C_L(y)\) is a maximal subgroup of \(L\). Let \(v\) be an element of order 4 in \(D_1\). Since
$U = \langle z, u \rangle \subseteq A D_1$, we may set $u = yv$. We have $y^b = yz = y^{-1}$ and so $u^{t^b} = (uz)^b = (yvz)^b = y^{-1}vz = yv = u$ so that $\langle z, u, tb \rangle$ is an elementary abelian subgroup of order 8. Obviously, $C(u) \cap \Omega_1(S) = \langle u \rangle \times \langle tb, b^2 \rangle$ and since $C_G(t) = A \times \langle t \rangle$ does not contain an $E_8$, $t$ cannot be fused in $G$ to any involution contained in $C(u) \cap \Omega_1(S)$. Naturally, $t$ cannot be fused in $G$ to any involution in $U$ because $U$ is normal in $G$. On the other hand, it is easy to compute that any involution in $\Omega_1(S)$ lies either in $L$ or in $C(u) \cap \Omega_1(S)$. This forces $N_G(S) = S$ and so $S = G$. If $m = 2$, we have the Berkovich groups for $m = 2$ stated in (g) of Theorem 1.1 since $\langle t, b^2 \rangle \cong D_{2^{m-1}}$ centralizes $A$ and the involution $tb$ acts invertingly on $A$. So assume that $m \geq 3$.

Then we may set (see Proposition 2.6) $b^a = b^{1+2^i}$, where $i \geq 2$ because $C_L(a) \cong L$. Since $C_L(a) \cong D_{2^{m-1}}$ and $C_L(a^{2^{m-1}}) \cong D_{2^{m-1}}$ and $a^{2^{m-1}} = z$, $C_L(a^{2^{m-1}}) = L$. It follows that $i = n - m$. We have obtained exactly the groups stated in part (e) of Theorem 1.1.

In what follows we shall assume always that $\langle y \rangle = \Omega_2(A)$ centralizes $L$ so that $\Omega_1(S)$ is the central product of $\Omega_2(A)$ and $L$. In this case each involution in $\Omega_1(S)$ is contained either in $L$ or in $U$ and $L$ is the subgroup $L$ being generated by its own involutions is a characteristic subgroup of $\Omega_1(S)$ and so of $S$. If $S = G$, then we get some groups stated in part (d) of Theorem 1.1.

From now on we shall assume in addition that $S \neq G$. Let $W = N_G(\Omega_1(S))$. Since $W$ fuses $ccl(t) \cap \Omega_1(S)$ with $ccl(tb) \cap \Omega_1(S)$, where both classes are of size $2^{n-2}$ and both are contained in $L$, we get $|W : S| = 2$, $S$ is normal in $W$, so $L$ is normal in $W$. If $\Omega_1(S) = \Omega_1(W)$, then we have $W = G$. Suppose that $W \neq G$ and let $g \in N_G(W) \setminus W$ so that $g^2 \in W$. We have $\langle ccl(t) \cap W \rangle = L$, $U$ is normal in $G$, and all involutions in $\Omega_1(S)$ lie in $U$ or in $L$. Therefore (replacing $g$ with $gw$, $w \in W$, if necessary) we may assume that $s = t^s \in W \setminus S$. Since $s$ normalizes $L$ and $C(t) \cap (L(s)) = \langle t, z \rangle$, we have $L(s) \cong D_{2^{m-1}}$. Now $L$ is normal in $W$, $L^g$ is normal in $W$, $L^g \cap S = L^g \cap L$, $L^gS = W$, $|L^gS : S| = |L^g : (L^g \cap S)| = |L^g : (L^g \cap L)| = 2$, and $|LL^g| = 2^{n+1}$ so $LL^g = L(s)$. Hence $(LL^g)^{2^i} = LL^g$ and so $L(s)$ is a dihedral group of order $2^{n+1}$ which is normal in $W$. We have $W = A(L(s))$ and this contradicts the maximality of $S = AL$. Hence we must have $W = G$ in any case.

In what follows we assume in addition that $\Omega_1(S) \neq \Omega_1(G)$. Then for each involution $s \in G \setminus S$, we have that $s$ normalizes $L$ and so $M = L(s) \cong D_{2^{m-1}}$ (since $C(t) \cap M = \langle t, z \rangle$). Because of the maximality of $S$, we have that $M$ is not normal in $G$. But $L$ is normal in $G$ and so $A_0 = C_G(L) = C_A(L)$ (containing $\Omega_1(A)$) is also normal in $G$ and we have $A_0 = Z(S)$, which is of order $\geq 4$. We look at the structure of $\tilde{G} = G/L$ which is a group with a cyclic subgroup $\tilde{S} = \tilde{A} = (AL)/L \cong A/(z)$ (bar convention) of order $2^{m-1}$ and index 2 and with an involution $\tilde{M} = M/L$ outside of
the cyclic group. Since $M$ is not normal in $G$, $\tilde{G}$ is nonabelian. We have $G = \tilde{A}M$, $\tilde{A} \cap M = 1$, and $|\tilde{A}| \geq 4$, so that $m \geq 3$. Now $|A_0| \geq 4$ and so $|A_0| \geq 2$ and $\tilde{A}_0$ is normal in $\tilde{G}$ and $A_0 \subseteq \tilde{A}$. Since $\tilde{G}/A_0$ is a subgroup of the outer automorphism group of the dihedral group $L$, it follows that $\tilde{G}/A_0$ is abelian.

Suppose at first that $\tilde{G}$ is not of maximal class so that $\Omega_1(\tilde{G}) = (UM)/L$ and $UM = \Omega_1(G)$. Since $M$ is not normal in $G$, it follows that $s$ must act faithfully on $Z(\Omega_1(S)) = \Omega_2(A)$ and so $\langle z, u, s \rangle \cong E_8$. Namely, if $s$ centralizes $\Omega_2(A)$, then $\Omega_1(G) = \Omega_2(A) * M$ and each involution in $\Omega_1(G)$ lies in $U$ or in $M$ and so $M$ would be normal in $G$, which contradicts the maximality of $S$. If we set $A = \langle a \rangle$, then $N_G(M) = M/\langle a^2 \rangle$, which follows from the structure of $\tilde{G}$ (see Propositions 2.2 and 2.3). Since $\Omega_2(A)$ centralizes $L$ but $\Omega_2(A)/\langle z \rangle$ acts faithfully on $M$, it follows that $\langle a^2 \rangle/\langle z \rangle$ acts faithfully on $M$ and so $\langle a^2 \rangle/\Omega_2(A)$ acts faithfully on $L$ and $C_G(L) = \Omega_2(A) = A_0$. It follows that $Z(\tilde{G})$ is of order 2 and $A/\Omega_2(A)$ acts faithfully on $L$. Because $s$ acts faithfully on $A$, $|A| \geq 8$ and $m \geq 4$. We may assume that the involution $s$ acts on $L = \langle b, t \rangle$ as follows: $b^s = b^{-1}$ and $t^s = tb$ and we set $b^{2z} = z$, where $\langle z \rangle = Z(G)$. Replacing $a$ with $a'$ (r odd) we may assume $b^s = b_i b_i^{2z}$, $i \geq 2$ because $C_i(a) \subseteq D_1 \subseteq D_n$. Hence we have $C_i(a) \cong D_{2i+1}$ and so $C_i(\langle a^{2n-2} \rangle) \cong D_{2i+1} \cong 4 \cong D_{2i+4}$, where we have taken into account that $\langle a \rangle/\langle a^{2n-2} \rangle = A/\Omega_2(A)$ acts faithfully on $L$ and $\Omega_2(A)$ centralizes $L$. Thus $i + m - 1 = n$ and so $i = n - m + 1$. Since $i \geq 2$ we must have $n \geq m + 1$. Because $m \geq 4$, we have here $n \geq 5$.

It remains to determine the commutator $[a, s]$. We know that $A = \langle a \rangle$ does not normalize $M$ and so $\langle [a, s] \rangle = A_0$ which gives $[a, s] = a_0l$, where $\langle a_0 \rangle = A_0$ has order 4, $a_0 = z$ and $l \in L$. We can also write $[a, s] = (a_0z)(bl) = a_0^{-1}(bl)$ and so replacing $l$ with $bl$, if necessary, we may assume that $a_0 = a_0^{2n-2}$ and $s$ inverts $a_0$. Hence we get $a^{-1}sa = a_0l$ and so $a^{-1}sa = a_0l$. Since $a_0l$ is an involution, we must have $l = b^i$ for a suitable integer $j$. Therefore $i^{a_0^{-1}a} = t^{a_0^{-1}b^i}$ which gives $b^a = b_{1-2i}$. On the other hand $b^a = b_{1+2i}$ and so $j \equiv -2^{-1} \pmod{2n-2}$. This gives (4) $a^{-1}sa = a_0^{2n-2} b_{1-2i}$. This gives (5) $a^{-1}sa = a_0^{2n-2} b_{1-2i}$. However, replacing $b$ with $b' = b_1^{1-2i+1}$ and $t$ with $t' = tb_{2n-2i}$, we see that all the relations obtained so far in this case go into the same relations with $b'$ instead of $b$ and $t'$ instead of $t$ but the relation (4) goes into relation (5). Therefore we may choose the relation (4). We have determined the structure of $G$ uniquely in this case and this is the group given in (f) of Theorem 1.1.

Suppose now that $\tilde{G} = G/L$ is a group of maximal class. Since $\tilde{G}/\tilde{G}'$ is of order 4 and $\tilde{G}/A_0$ is abelian, we have $|\tilde{A} : A_0| \leq 2$.

We consider at first the case $\tilde{A} = A_0$ which means that $A = C_G(L)$ and so $S$ is the central product of $A$ and $L$ with $A \cap L = \langle z \rangle = Z(L)$. Hence $A = Z(S)$ and so $A$ is normal in $G$. Since $(A(s))/\langle z \rangle$ is of maximal class and the case $[A, s] \subseteq \langle z \rangle$ is not possible (because $L/\langle s \rangle$ is not normal in $G$),
we have either $a^t = a^{-1}$ or $a^t = a^{-1}z$, which is possible only when $m \geq 3$. Our group $G$ is isomorphic to a Berkovich group $F(m, n)$ as stated in part (g) of Theorem 1.1. Also we have here $\langle z, u, s \rangle \cong E_8$.

It remains to consider the case where $|A : A_0| = 2$ so that $C_G(L) = A_0$ is of index 2 in $A$. We have $A_0 = \langle a^2 \rangle = Z(S)$ and so $a$ induces an automorphism of order 2 on $L$ with $C_L(a) \supseteq D_1$. This gives at once that $b^2 = bzb$ and so $n \geq 4$. Hence we also have $a^b = az$. Since $[A, L] = \langle z \rangle$ and $L/\langle z \rangle$ is a (nonabelian) dihedral group of order $2^{n-1}$, it follows that $C_L(L/\langle z \rangle) = A \ast \langle b' \rangle$, where $\langle b' \rangle$ is the cyclic subgroup of order 4 in $L$. Hence $A \ast \langle b' \rangle$ is normal in $G$. If $s$ normalizes $A$, then $a^t = a^{-1}z^\alpha$, $\alpha = 0, 1$ since $A(s)/\langle z \rangle$ is of maximal class. But from $sts = tb$ we get acting on $a : a^t = az$ which gives $az = a$ and this is a contradiction. Hence $s$ does not normalize $A$. Since $A^s \subseteq A \ast \langle b' \rangle$ and $(A \langle b' \rangle)/\langle b' \rangle)(s)$ is of maximal class, we may set (6) $a^t = a^{-1}b'$ or (7) $a^t = a^{-1+2n^{-2}}b'$. We must have $a^t = a$ since $s^2 = 1$. But in case (6) this gives $(a^{-1}b')^t = a$ and so $az = a$ which is a contradiction. Hence we must have the relation (7). Replacing $a$ with $a^{-1}$ in (7), we get $(a^{-1})^t = (a^{-1})^{-1+2n^{-2}}b^{-1}$ and other relations in this case are not changed. Thus we may put $b' = b^{2n^{-2}}$ in (7). This determines the group $G$ as stated in part (h) of Theorem 1.1. Here we have also $\langle z, u, s \rangle \cong E_8$.

An inspection of obtained groups shows that in any case $G$ has no elementary abelian subgroups of order 16 and $G$ is generated by at most 3 elements. Also in the case where $G$ possesses an elementary abelian subgroup of order 8, we see that $Z(G)$ has order 2. Theorem 1.1 is proved.

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