Summing over all topologies in CDT string field theory

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ABSTRACT

By explicitly allowing for topology to change as a function of time, two-dimensional quantum gravity defined through causal dynamical triangulations gives rise to a new continuum string field theory. Within a matrix-model formulation we show that – rather remarkably – the associated sum over all genera can be performed in closed form, leading to *a* nonperturbative definition of CDT string field theory. We also obtain explicit formulas for the *n*-loop correlation functions. Our construction exhibits interesting parallels with previous, purely Euclidean treatments.

1. Introduction

By general acknowledgement, string theory is in need of a genuinely nonperturbative definition if it is to serve as a fundamental theory of all interactions. String theory's perturbative expansion does not serve this purpose, since it is only asymptotic and becomes useless when the string coupling approaches order one. It is thus of great interest to study toy models like non-critical string theory for which one can define the sum over all genera, a possibility first outlined in the seminal papers [1–3]. Unfortunately, the results of such summations are plagued by ambiguities: even in the simplest cases, like pure two-dimensional Euclidean quantum gravity ($c = 0$ non-critical string theory) and all unitary non-critical string theories, the "nonperturbative" partition function is not Borel-summable [4–7] (see also [8] for a recent lucid discussion). Attempts to perform the summation in these theories typically result in complex partition functions or, perhaps more accurately, constructions which *do* obtain a real partition function are often guided by the desire for a real outcome, rather than by applying stringent physical criteria. (For other approaches see, for example, [9,10].)

Given this somewhat unsatisfactory state of affairs, any new physical model for which one has analytic control over the sum over topologies is potentially valuable, to understand possible ambiguities in the summation and their genericity, to explore their

physical meaning, and to see whether new insights can be gained into the nature of surfaces of infinite genus, a topic which has maybe not received the attention it deserves.

The present Letter deals with just such a new system in two spacetime dimensions. Its origin lies in the attempt to formulate a nonperturbative theory of four-dimensional quantum gravity based on conventional quantum field theory, more precisely, the path integral approach, in the framework of "Causal Dynamical Triangulations" (or "CDT" for short), which has led to a variety of intriguing and unprecedented results [11,12]. However, the construction can in principle be applied in any dimension, leading to nonperturbative models of dynamical quantum geometry with causal, Lorentzian properties. The case of two spacetime dimensions is a particularly attractive testing ground because of the availability of a whole range of analytical tools. The original, strictly causal and purely gravitational model was solved exactly in 2d [13], and has since been generalized by the inclusion of matter [14,15] and isolated (cap or branching) points where causality is violated. The latter has culminated in the recent definition of a fully fledged CDT string field theory (in zero-dimensional target space), which is the subject of the remainder of the present work.

In the mid-nineties, Kawai, Ishibashi and collaborators developed a string field theory for non-critical strings [16–21]. While its original starting point was the explicit realization of non-critical string theory in terms of dynamical triangulations (of the Euclidean variety), it eventually was formulated entirely in the language of continuum field theory. In [22,23] we repeated this continuum analysis for the two-dimensional CDT model of quantum gravity. We demonstrated how it is connected to the conventional matrix model of 2d Euclidean quantum gravity [24,25] and how the *continuum* Dyson–Schwinger equations of the CDT string field

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theory are related to a particular matrix model.[1] In what follows, we will use this matrix-model formulation to perform the sum over all topologies in the continuum CDT string field theory and compute a number of associated geometric observables, so-called loop–loop correlation functions. It turns out that this can be done in a surprisingly simple manner.

The remainder of this Letter is organized as follows: in Section 2 we introduce the CDT matrix model and related observables, in Section 3 we show how to calculate these quantities summed over all genera, in Section 4 we put our results into the context of previous Euclidean models, and finally, in Section 5 we discuss the nature of our nonperturbative results.

2. The CDT matrix model

The approach of causal dynamical triangulations aims to define quantum gravity nonperturbatively via a path integral over geometries $[g_{\mu\nu}]$,

$$Z(G_N, \lambda) = \int \mathcal{D}[g_{\mu\nu}] e^{-S[g_{\mu\nu}]}, \tag{1}$$

where S denotes the (Euclidean) Einstein–Hilbert action of gravity, G_N is Newton's constant and λ the cosmological constant. We are presently interested in the case of two spacetime dimensions for which the action is given by

$$S[g_{\mu\nu}] = -\frac{1}{2\pi G_N} \int d^2\xi \sqrt{\det g_{\mu\nu}}\, R + \lambda \int d^2\xi \sqrt{\det g_{\mu\nu}}. \tag{2}$$

One proceeds in two steps, first setting up the CDT lattice regularization of the path integral in *Lorentzian* signature, and then rotating each individual triangulation in the Lorentzian ensemble to Euclidean signature. The resulting Euclidean path integral (1) differs from a standard Euclidean path integral since by construction each geometric "path" has a time-foliation and thus a "memory" of the causal properties it possessed before the Wick rotation. We refer the reader to the original article [13] for technical details.

The observables we will discuss in this Letter are generalized Hartle–Hawking amplitudes involving 2d geometries with a fixed number of boundaries. The simplest one is that for a single boundary, with all boundary points sharing a common time (in the time-foliation referred to above). This so-called Hartle–Hawking or disc amplitude $W(l)$ is the amplitude that the spatial boundary (consisting of a closed, one-dimensional loop) has length l. Depending on whether one thinks of the boundary as being initial or final, it is the amplitude of the universe of length l vanishing into or coming from "nothing". A generalized such amplitude is one where the spatial boundary (still at equal time) consists of n disconnected loop components with specified lengths $l_1, \ldots, l_n$. Spacetime may remain disconnected or reconnect in various ways at other times. We denote the corresponding amplitudes $W_d(l_1, \ldots, l_n)$, where the subscript "d" indicates that for $n > 1$ it includes the possibility of having disconnected universes. The connected component of $W_d(l_1, \ldots, l_n)$ will be called $W_c(l_1, \ldots, l_n)$. To lowest order $n = 1$ we have $W_d(l) = W_c(l) = W(l)$, while for $n = 2$

$$W_c(l_1, l_2) = W_d(l_1, l_2) - W(l_1)W(l_2), \tag{3}$$

which generalizes to higher-loop correlators in standard fashion.

Instead of considering a situation where the boundary lengths l_i are kept fixed, one can introduce by hand boundary cosmologi-

cal constants x_i and include corresponding terms $x_i \cdot l_i$ in the action. Now the lengths l_i are allowed to fluctuate, but their average values are determined by the values of the x_i. This leads to another kind of generalized Hartle–Hawking wave functions, denoted by $\tilde{W}_d(x_1, \ldots, x_n)$ and $\tilde{W}_c(x_1, \ldots, x_n)$, and related to the previous $W_{c,d}(l_1, \ldots, l_n)$ by Laplace transformation according to

$$\tilde{W}_{c,d}(x_1, \ldots, x_n)$$
$$= \int_0^\infty dl_1 \cdots dl_n\, e^{-(x_1 l_1 + \cdots + x_n l_n)} W_{c,d}(l_1, \ldots, l_n). \tag{4}$$

The Dyson–Schwinger equations, an infinite set of coupled equations relating $\tilde{W}_c(x_1, \ldots, x_n)$ for different n, follow from the continuum CDT string field theory as developed in [22]. They are obtained from the cubic matrix model

$$Z_N(g, \lambda) = \int dM \exp\left[-\frac{N}{g} \mathrm{tr}\left(\lambda M - \frac{1}{3}M^3\right)\right], \tag{5}$$

where M is an $N \times N$ Hermitian matrix. In this matrix model the observables introduced above take the form

$$\tilde{W}_d(x_1, \ldots, x_n) = \frac{1}{N^n}\left\langle \mathrm{tr}\left(\frac{1}{x_1 - M}\right) \cdots \mathrm{tr}\left(\frac{1}{x_n - M}\right)\right\rangle, \tag{6}$$

$$\tilde{W}_c(x_1, \ldots, x_n) = \frac{1}{N^n}\left\langle \mathrm{tr}\left(\frac{1}{x_1 - M}\right) \cdots \mathrm{tr}\left(\frac{1}{x_n - M}\right)\right\rangle_{\text{connected}}, \tag{7}$$

where the expectation value of an operator $\mathcal{O}$ is defined by

$$\langle \mathcal{O}(M) \rangle = \frac{1}{Z_N(g, \lambda)} \int dM\, \mathcal{O}(M)$$
$$\times \exp\left[-\frac{N}{g} \mathrm{tr}\left(\lambda M - \frac{1}{3}M^3\right)\right]. \tag{8}$$

Of course an expression like (5) is formal and should always be understood as a suitable expansion in powers of M. This is most clearly exhibited by making the variable change

$$M = \sqrt{t\lambda}\, Y - \sqrt{\lambda}, \quad t = \frac{g}{\lambda^{3/2}} \equiv e^{-1/G_N}, \tag{9}$$

where t is a dimensionless coupling constant, which is related to the coupling g of Eq. (5) and the gravitational coupling G_N as specified. In terms of the matrix Y and the coupling t the matrix integral reads

$$Z_N(t(g, \lambda), \lambda) = \exp\left(\frac{2N}{3t}\right)(t\lambda)^{N^2/2}$$
$$\times \int dY \exp\left[-N \mathrm{tr}\left(Y^2 - \frac{\sqrt{t}}{3}Y^3\right)\right], \tag{10}$$

which after expanding $\exp(\sqrt{t}Y^3/3)$ and performing the Gaussian integrals becomes a formal power expansion in t. The coefficient of t^n in this power series is positive, in agreement with the interpretation that it represents the number of 2d surfaces with a time foliation with n degenerate points (see [22] for details). Let us emphasize that the pre-factor, which does not have a power expansion in t, will cancel in any expectation values of observables and should not really be considered part of the partition function. In fact we could have defined the partition function as

$$\tilde{Z}_N(t) = \frac{\int dY \exp[-N \mathrm{tr}(Y^2 - \frac{\sqrt{t}}{3}Y^3)]}{\int dY \exp[-N \mathrm{tr}\, Y^2]}, \tag{11}$$

which would then only contain positive powers of t. However, we find it convenient to use (10) in the rest of this Letter, with the understanding that the exponential growth of $Z_N(t)$ for $t \to 0$ should not be considered as a reflection of unphysical behaviour, but is merely a consequence of a particular choice of normalization.

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3. Summing over all genera

In Ref. [24] the matrix model (5) was related to the CDT Dyson–Schwinger equations by (i) introducing into the latter an expansion parameter α, which kept track of the genus of the two-dimensional spacetime, and (ii) identifying this parameter with $1/N^2$, where N is the size of the matrix in the matrix integral. The $1/N$-expansion of our matrix model therefore plays a role similar to the $1/N$-expansion originally introduced by 't Hooft [30]: it reorganizes an asymptotic expansion in a coupling constant (t in our case) into convergent sub-summations in which the kth summand appears with a coefficient N^{-2k}. In QCD applications, the physically relevant value is $N = 3$, to which the leading-order terms in the large N-expansion can under favourable circumstances give a reasonable approximation.

As we will see, for the purposes of solving our string field-theoretic model nonperturbatively, an additional expansion in inverse powers of N (and thus an identification of the contributions at each particular genus) is neither essential nor does it provide any new insights. This means that we will consider the entire sum over topologies "in one go", which simply amounts to setting $N = 1$, upon which the matrix integral (5) reduces to the ordinary integral[2]

$$Z(g, \lambda) = \int dm \exp\left[-\frac{1}{g}\left(\lambda m - \frac{1}{3}m^3\right)\right], \qquad (14)$$

while the observables (6) can be written as

$$\tilde{W}_d(x_1, \ldots, x_n) = \frac{1}{Z(g, \lambda)} \int dm \frac{\exp[-\frac{1}{g}(\lambda m - \frac{1}{3}m^3)]}{(x_1 - m) \cdots (x_n - m)}. \qquad (15)$$

Again, these integrals should be understood as formal power series in the dimensionless variable t as mentioned below Eq. (10). Any choice of an integration contour which makes the integral well defined and reproduces the formal power series is a potential nonperturbative definition of these observables. However, different contours might produce different nonperturbative contributions (i.e. which cannot be expanded in powers of t), and there may even be nonperturbative contributions which are not captured by any choice of integration contour. As usual in such situations, additional physics input is needed to fix these contributions.

To illustrate the point, let us start by evaluating the partition function given in (14). We have to decide on an integration path

in the complex plane in order to define the integral. One possibility is to take a path along the negative axis and then along either the positive or the negative imaginary axis. The corresponding integrals are

$$Z(g, \lambda) = \sqrt{\lambda} t^{1/3} F_{\pm}\left(t^{-2/3}\right),$$
$$F_{\pm}\left(t^{-2/3}\right) = 2\pi e^{\pm i\pi/6} \mathrm{Ai}\left(t^{-2/3} e^{\pm 2\pi i/3}\right), \qquad (16)$$

where Ai denotes the Airy function. Both $F_{\pm}$ have the same asymptotic expansion in t, with positive coefficients. Had we chosen the integration path entirely along the imaginary axis we would have obtained ($2\pi i$ times) $\mathrm{Ai}(t^{-2/3})$, but this has an asymptotic expansion in t with coefficients of oscillating sign, which is at odds with its interpretation as a probability amplitude. In the notation of [31] we have

$$F_{\pm}(z) = \pi \left(\mathrm{Bi}(z) \pm i\mathrm{Ai}(z)\right), \qquad (17)$$

from which one deduces immediately that the functions $F_{\pm}(t^{-2/3})$ are not real. However, since $\mathrm{Bi}(t^{-2/3})$ grows like $e^{\frac{2}{3t}}$ for small t while $\mathrm{Ai}(t^{-2/3})$ falls off like $e^{-\frac{2}{3t}}$, their imaginary parts are exponentially small in $1/t$ compared to the real part, and therefore do not contribute to the asymptotic expansion in t. An obvious way to *define* a partition function which is real and shares the same asymptotic expansion is by symmetrization,

$$\frac{1}{2}(F_+ + F_-) \equiv \pi \mathrm{Bi}. \qquad (18)$$

The situation parallels the one encountered in the double scaling limit of the "old" matrix model [6], and discussed in detail in [8], but is less complicated. We will return to a discussion of this in the next section.

Presently, let us collectively denote by $F(z)$ any of the functions $F_{\pm}(z)$ or $\pi \mathrm{Bi}(z)$, leading to the tentative identification

$$Z(g, \lambda) = \sqrt{\lambda} t^{1/3} F\left(t^{-2/3}\right), \qquad F''(z) = zF(z), \qquad (19)$$

where we have included the differential equation satisfied by the Airy functions for later reference. In preparation for the computation of the observables $\tilde{W}_d(x_1, \ldots, x_n)$ we introduce the dimensionless variables

$$x = y\sqrt{\lambda}, \qquad m = g^{1/3}\beta,$$
$$\tilde{W}_d(x_1, \ldots, x_n) = \lambda^{-n/2} \tilde{w}_d(y_1, \ldots, y_n). \qquad (20)$$

Assuming $y_k > 0$, we can write

$$\frac{1}{y - t^{1/3}\beta} = \int_0^\infty d\alpha \exp[-(y - t^{1/3}\beta)\alpha]. \qquad (21)$$

We can use this identity to re-express the pole terms in Eq. (15) to obtain the integral representation

$$\tilde{w}_d(y_1, \ldots, y_n) = \int_0^\infty \prod_{i=1}^n d\alpha_i \, e^{-(y_1\alpha_1 + \cdots + y_n\alpha_n)}$$
$$\times \frac{F(t^{-2/3} - t^{1/3} \sum_{i=1}^n \alpha_i)}{F(t^{-2/3})} \qquad (22)$$

for the amplitude with dimensionless arguments. From the explicit expression of the Laplace transform, Eq. (4), we can now read off the generalized Hartle–Hawking amplitude as function of the boundary lengths,

$$W_d(l_1, \ldots, l_n) = \frac{F(t^{-2/3} - t^{1/3}\sqrt{\lambda}(l_1 + \cdots + l_n))}{F(t^{-2/3})}. \qquad (23)$$

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[2] Starting from a matrix integral for $N \times N$-matrices like (5), performing a *formal* expansion in (matrix) powers commutes with setting $N = 1$, as follows from the following property of expectation values of products of traces, which holds for any $n = 1, 2, 3, \ldots$ and any set of non-negative integers $\{n_k\}$, $k = 1, \ldots, 2n$, such that $\sum_{k=1}^{2n} n_k = 2n$. For any particular choice of such numbers, consider

$$\left\langle \prod_{k=1}^{2n}\left(\frac{1}{N}\mathrm{tr}\, M^{n_k}\right)\right\rangle \equiv \frac{\int dM \, e^{-\frac{1}{2}\mathrm{tr}\, M^2} \prod_{k=1}^{2n}(\mathrm{tr}\, M^{n_k}/N)}{\int dM \, e^{-\frac{1}{2}\mathrm{tr}\, M^2}} = \sum_{m=-n}^{n} \omega_m N^m, \qquad (12)$$

where the last equation *defines* the numbers ω_m as coefficients in the power expansion in N of the expectation value. Now, we have that

$$\sum_{m=-n}^{n} \omega_m = (2n-1)!! \qquad (13)$$

independent of the choice of partition $\{n_k\}$. The number $(2n-1)!!$ simply counts the "Wick contractions" of x^{2n} which we could have obtained directly as the expectation value $\langle x^{2n}\rangle$, evaluated with a one-dimensional Gaussian measure. In the model at hand, we will calculate sums of the form $\sum_{m=-n}^{n} \omega_m$ directly, since we are summing over all genera *without* introducing an additional coupling constant for the genus expansion. In other words, the dimensionless coupling constant t in this case already contains the information about the splitting and joining of the surfaces, and the coefficient of t^k contains contributions from 2d geometries whose genus ranges between 0 and $[k/2]$. We cannot disentangle these contributions further unless we introduce N as an extra parameter.

For the special case $n = 1$ we find

$$W(l) = \frac{F(t^{-2/3} - t^{1/3}\sqrt{\lambda}l)}{F(t^{-2/3})} \tag{24}$$

for the disc amplitude, together with the remarkable relation[3]

$$W_d(l_1, \ldots, l_n) = W(l_1 + \cdots + l_n). \tag{25}$$

By Laplace transformation this formula implies the relation

$$\tilde{W}_d(x_1, \ldots, x_n) = \sum_{i=1}^{n} \frac{\tilde{W}(x_i)}{\prod_{j \neq i}^{n}(x_j - x_i)}. \tag{26}$$

Before turning to a discussion of the nonperturbative expression for $W(l)$ we have just derived, let us remark that the asymptotic expansion in t of course agrees with that obtained by recursively solving the CDT Dyson–Schwinger equations. Using the standard asymptotic expansion of the Airy function [31] one obtains

$$W(l) = e^{-\sqrt{\lambda}l}e^{th(t,\sqrt{\lambda}l)} \frac{\sum_{k=0}^{\infty} c_k t^k (1 - t\sqrt{\lambda}l)^{-\frac{3}{2}k - \frac{1}{4}}}{\sum_{k=0}^{\infty} c_k t^k}, \tag{27}$$

where the coefficients c_k are given by $c_0 = 1$, $c_k = \frac{1}{k!}(\frac{3}{4})^k(\frac{1}{6})_k(\frac{5}{6})_k$, $k > 0$. In (27), we have rearranged the exponential factors to exhibit the exponential fall-off in the length variable l, multiplied by a term containing the function

$$h(t, \sqrt{\lambda}l) = \frac{2}{3t^2}\left[(1 - t\sqrt{\lambda}l)^{3/2} - 1 + \frac{3}{2}t\sqrt{\lambda}l\right], \tag{28}$$

which has an expansion in positive powers of t.

Finally, let us derive an expression for the amplitude $\tilde{W}(x)$. Since $F(z)$ satisfies the Airy differential equation, we have the identity

$$F(t^{-2/3} - t^{1/3}\alpha) = \left(\frac{d^2}{d\alpha^2} + t\alpha\right)F(t^{-2/3} - t^{1/3}\alpha). \tag{29}$$

Inserting this into Eq. (22) and performing the partial integrations, we obtain a first-order differential equation for $\tilde{w}(y)$, namely,

$$\tilde{w}(y) = \left(y^2 - t\frac{d}{dy}\right)w(y) + \frac{t^{1/3}F'(t^{-2/3})}{F(t^{-2/3})} - y. \tag{30}$$

Its solution is given by

$$\tilde{w}(y) = t^{-1}e^{-\frac{1}{t}(y - \frac{1}{3}y^3)}\int_y^{\infty} dv\, e^{\frac{1}{t}(v - \frac{1}{3}v^3)}(v - 1 - f(t)), \tag{31}$$

where the function $f(t)$ is defined by

$$f(t) = \frac{t^{1/3}F'(t^{-2/3})}{F(t^{-2/3})} - 1. \tag{32}$$

Performing a variable shift $v = y + t\xi$ in the integral in (31), $\tilde{w}(y)$ is conveniently written as

$$\tilde{w}(y) = \int_0^{\infty} d\xi\, e^{-(y^2-1)\xi}e^{-ty\xi^2 - t^2\xi^3/3}\big[(y - 1) - f(t) + t\xi\big]. \tag{33}$$

From (33) it follows by expanding the exponential containing t that $\tilde{w}(y)$ has an asymptotic expansion in t (for $y > 1$). The same

expansion represents $\tilde{w}(y)$ by an expansion in inverse powers of $(y + 1)$, corresponding to the expansion (27). Explicitly, one finds

$$\tilde{w}(y) = \frac{1}{y + 1} + t\frac{y + 3}{4(y + 1)^3} + O(t^2), \tag{34}$$

which, as already stated, of course agrees with the perturbative expansion of $\tilde{w}(y)$ derived previously in CDT string field theory [22].

4. Relation with other models

We should perhaps not be too surprised to meet the Airy function as part of our nonperturbative analysis, since it has already appeared previously in non-critical string theory, more specifically, in the so-called 2d topological quantum gravity. This theory can be described in two ways. On the one hand, a set of observables of the theory, the intersection indices of Riemann surfaces, can be calculated by the Kontsevich matrix integral, which *is* the matrix generalization of the Airy function. This is a cubic integral like (5), but with λ a matrix, which gives it more structure than (5) and allows for the calculation of the intersection indices. Analogous to our case (5), it is a matrix representation of a *continuum* theory, namely, 2d topological quantum gravity.

On the other hand, 2d topological quantum gravity also has a "conventional" one-matrix representation, which in fact is the simplest one possible. Recall that the (p, q) minimal conformal field theories coupled to 2d Euclidean quantum gravity in the cases $(p, q) = (2, 2m - 1)$, $m = 2, 3, \ldots$, can be described as double-scaling limits of one-matrix models with certain fine-tuned matrix potentials of order at least $m + 1$. Formally, the case $m = 1$, which corresponds to a somewhat degenerate $(2, 1)$ conformal field theory with central charge $c = -2$, is then described by a special double-scaling limit of the purely Gaussian matrix model (see the review [34]). In this double-scaling limit one obtains for the so-called FZZT brane precisely the Airy function, see [8,29,35] for recent discussions.

Our model is *not* equivalent to 2d Euclidean topological quantum gravity, but is dual to it in a specific way. The arguments presented below suggest that our CDT string field theory can also be identified as a continuum theory associated with $c = -2$, but corresponding to the unconventional, "wrong" branch of the KPZ equation. Recall that for a given conformal field theory coupled to Euclidean 2d quantum gravity, we have the KPZ formula

$$\gamma_- = \frac{-(1 - c) - \sqrt{(1 - c)(25 - c)}}{12} \tag{35}$$

for the susceptibility (for spherical topology). The parameter γ_- corresponds to the "right" choice of branch of the quadratic KPZ equation, i.e. the branch which leads to a weakly coupled Liouville theory as $c \to -\infty$. Choosing instead the other branch, one obtains

$$\gamma_+ = \frac{-(1 - c) + \sqrt{(1 - c)(25 - c)}}{12} = -\frac{\gamma_-}{1 - \gamma_-}. \tag{36}$$

The interpretation of this γ_+ in terms of matrix models and geometry can be found in [38–40], and for earlier related work see [41,42]. The simplest example is again given by $c = -2$: topological quantum gravity has $\gamma_- = -1$ whose dual is the "wrong" $\gamma_+ = 1/2$, which happens to be the value occurring generically in the theory of branched polymers (see, e.g., [43,44] for a discussion of why branched polymers and baby universes are generic and even dominant in many situations). While it is possible to describe 2d topological quantum gravity by a double-scaling limit of the Gaussian matrix model, the most natural geometric interpretation

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[3] Formula (25) has a structure quite similar to the one encountered in the "old" matrix model [32,33], where again the multi-loop correlator is "almost" a function of $l_1 + \cdots + l_n$ only.

of the Gaussian matrix model is in terms of branched polymers, in the sense that the integral

$$\frac{\int dM \, \mathrm{tr}\, M^{2n} e^{-\frac{1}{2} \mathrm{tr}\, M^2}}{\int dM \, e^{-\frac{1}{2} \mathrm{tr}\, M^2}} \qquad (37)$$

can be thought of as the gluing of a boundary of length $2n$ into a double-line branched polymer of length n. Since the branched polymers are also allowed to form closed loops, their partition function contains a sum over topologies 'en miniature', and one can indeed define a double-scaling limit of the model. When solving for the partition function in this limit, one obtains precisely our $Z(g,\lambda)$ of Eq. (19)![4] One should not jump to the conclusion that the two models are identical, since the CDT string field theory has a much richer structure of observables, with no obvious analogues in the branched-polymer set-up. Nevertheless, it is obvious that the partition function captures the essentials of the counting of branchings and joinings, which for the case of the CDT model is insensitive to the fact that the geometries are genuinely extended. (The latter is obvious from the nontrivial dependence on the boundary cosmological constants or, equivalently, the boundary lengths.)

The fact that the CDT string field theory shares some properties of branched polymers is maybe less surprising in view of the fact that the original two-dimensional CDT model without branching can be mapped to a one-dimensional random-walk model [37]. The generalization implemented by the CDT string field theory corresponds to adding "branches" to the random walks, resulting again in branched polymers.

5. Discussion

The central result of this Letter is the derivation of the explicit, nonperturbative expressions (19) and (23) for the partition function $Z(g,\lambda)$ and the Hartle–Hawking amplitude $W_d(l_1,\ldots,l_n)$ of the CDT string field theory, both incorporating the infamous "sum over topologies" (2d spacetimes of all possible genera). It is rather remarkable that these sums can be performed and – with hindsight – in a manner which is technically not very involved. As was already the case for the two-dimensional CDT quantum gravity theory with fixed spacetime topology, the results are genuinely different from those of the corresponding purely Euclidean models. Nevertheless, as we have tried to argue above, from the point of view of conformal field theory, the CDT string field theory can probably be understood as the continuum theory "in the wrong branch" with conformal charge $c = -2$ and susceptibility $\gamma_+ = 1/2$, and thus as "dual" to topological quantum gravity in two dimensions.

The nonperturbative aspects of our theory suffer from the same fundamental ambiguity as string theory, and which are rooted in the non-Borel summability of the perturbation series. Beyond the perturbative expansion, which unfortunately is only asymptotic, there is no real definition of the theory. We have managed to sum the asymptotic series and produce a closed formula for $W(l)$, but like in non-critical string theory with $c \geqslant 0$, i.e. unitary field theories coupled to 2d Euclidean quantum gravity, we lack a clear physical principle which would allow us to decide which nonperturbative completion of the perturbative expansion to choose.

If one insists on a *real* partition function (in the Euclidean sector), it is natural to take $F(z) = \pi \mathrm{Bi}(z)$ in formulas (19) and (23). However, this choice is only unique within the matrix-model realization. While it is true that matrix models in non-critical string theory have been able to incorporate physics they were not originally designed to incorporate, like the physics of ZZ-branes, we are not aware of any argument that would identify matrix models as *the* correct, nonperturbative definitions of continuum theories including a sum of genera, should they indeed exist.

Our string-field theoretic model highlights in a particularly simple and transparent manner the limited amount of information contained in the perturbative expansion. When comparing explicitly the closed-form nonperturbative results (19) and (24) with their asymptotic expansions, one finds that the latter are only good approximations in a small range of their arguments t and l. This can simply be traced to the fact that the asymptotic expansion of for instance $\mathrm{Bi}(t^{-2/3})$, terminated after k terms, is only valid for $t \ll 4/k$. In a similar vein, many aspects of the nonperturbative solution could not possibly have been guessed from the perturbative series. Consider, for instance, the behaviour of the nonperturbative disk amplitude $W(l)$ as a function of the boundary length l for fixed g and λ. Each term in the perturbative expansion of $W(l)$ falls off exponentially as $e^{-\sqrt{\lambda}l}$ for $\sqrt{\lambda}l \gg 1$ *and* is positive. However, while the full nonperturbative function $W(l)$ will initially decrease with increasing l, as expected from each of the (positive) terms in its asymptotic expansion, it becomes oscillatory when $l > 1/(t\sqrt{\lambda}) = \lambda/g$. The same oscillatory behaviour occurs when t is increased while l and λ are kept fixed, i.e. when the coupling g is increased. This oscillation is a genuinely nonperturbative effect, which is opposite to the behaviour of *each* term in the perturbative expansion of $W(l)$ in powers of t and may be indicative of a phase transition to spacetimes completely dominated by topology changes.

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