JOURNAL OF ALGEBRA 105, 116–148 (1987)

Crossed-Products Orders over Discrete Valuation Rings

DARRELL E. HAILE*

Department of Mathematics, Indiana University, Bloomington, Indiana 47405

Communicated bu I. N. Herstein

Received February 1, 1985

INTRODUCTION

Let R be a discrete valuation ring with field of fractions F and let Σ be a central simple F-algebra. There exists a well-developed theory of the R-orders in Σ, that is those R-subalgebras A of Σ that are finitely generated as R-modules and for which $AF = \Sigma$. In this paper we describe an alternate approach to part of this theory, employing the generalized cohomology theory first developed in Haile, Larson, and Sweedler [5]. In the present setting the two-cocycles of that theory can be used to form "crossed-product orders," analogous to the crossed-product algebras in the theory of central simple algebras.

The collection of crossed-product orders contains, up to a suitable notion of equivalence, all the maximal orders over R (assuming the residue field of R is perfect). Moreover, the concrete nature of the construction allows a different perspective on the structure of the crossed-product orders, and so in particular on maximal orders. On the other hand, this class of orders is to some extent complementary to that determined by standard homological considerations: if a crossed-product order is hereditary, then it is in fact maximal. In this sense the crossed-product construction provides a collection of orders that occur naturally, yet different from those studied classically. In this paper, however, the main emphasis is on those aspects of the theory related to maximal orders.

We want to be more precise. Let K/F be a finite Galois extension of fields with group G and let S be the integral closure of R in K. Assume S/R is unramified (so that S/R is itself a Galois extension). Let $S^{\#} = S - \{0\}$. Consider normalized two-cocycles $f\colon G \times G \to S^{\#}$, that is, functions satisfying $f^{\sigma}(\tau, \gamma)\, f(\sigma, \tau\gamma) = f(\sigma, \tau)\, f(\sigma\tau, \gamma)$ for all $\sigma, \tau, \gamma \in G$ and

* This work was partially supported by NSF Grant MCS-82-0118.

0021-8693/87 $3.00

$f(1,\sigma)=f(\sigma,1)=1$ for all $\sigma \in G$. From such a cocycle we can form a crossed-product order A_f, given by $A_f = \coprod_{\sigma \in G} Sx_\sigma$ with the usual rules of multiplication ($x_\sigma s = \sigma(s)x_\sigma$ for all $s \in S$, $\sigma \in G$, $x_\sigma x_\tau = f(\sigma,\tau)\, x_{\sigma\tau}$). This R-algebra A_f is an order in the classical crossed product F-algebra $\Sigma_f = \coprod_{\sigma \in G} Kx_\sigma$.

This then is the class of orders we wish to consider. In the first section we derive some basic properties of the cocycles and the orders. We show how to associate a finite graph to each cocycle. One of the themes of the paper is the relationship between properties of this graph and the structure of the order. Also in this section we show that if the residue field is perfect, then every maximal order is equivalent to a crossed-product order.

In the second and third sections the orders are considered in more detail. In Section 2 we assume that S is a discrete valuation ring (DVR). In this special case the structure of the order is quite rigid and quite explicit results are obtained. For example, in this case the crossed-product orders are primary, that is, have a unique maximal ideal, and there is a simple characterization of those cocycles (in terms of the associated graph) which give rise to maximal orders. Also in this section we show how to determine the ideals in the order and give necessary and sufficient conditions for two orders A_{f_1} and A_{f_2} to be isomorphic as R-algebras. In particular we prove that if A_{f_1} and A_{f_2} are maximal, then A_{f_1} is isomorphic to A_{f_2} as an R-algebra if and only if f_1 and f_2 are cohomologous over S (in the usual sense). Again this is all in the case where S is a discrete valuation ring.

In Section 3 we take up the general case (S/R unramified but S not necessarily local). Here things are much more complicated. In particular, A_f is no longer necessarily primary and the first important result is a condition on the cocycle f equivalent to A_f being primary. Let G and S be as above and let M be a maximal ideal of S with decomposition group D_M. If $f\colon G \times G \to S$ is a cocycle, then A_f is primary if and only if there are coset representatives $g_1,..., g_r$ of D_M in G, that is $G = \bigcup_i D_M g_i$, with $f(g_i, g_i^{-1}) \notin M$ for all i. (As it turns out, the existence of such a set of representatives for one maximal ideal S implies the existence of suitable sets of representatives for all the maximal ideals of S.) Using this result, we show that the primary crossed-product orders are very well-behaved: If f_M: $D_M \times D_M \to S \subseteq S_M$ denotes the restriction of f (and S_M is the localization of S at M), then we can form the new crossed-product order A_{f_M}. Since S_M is local, we know the structure of A_{f_M} from Section 2. If A_f is primary, then there is a one-to-one product preserving correspondence between the ideals of A_f and the ideals of A_{f_M}. This is proved in the same way as an analogous result of Harada (Lemma 1 of [6]), the crucial point being the existence of the "good" set of coset representatives of D_M in G. In particular, in the case where A_f is primary, we show that A_f is maximal if and only if A_{f_M} is maximal and in this sense we are back in the nice situation of Section 2.

Along the way we obtain results on the relationship between the graph of f and the graphs of the cocycles f_M, as M varies through the maximal ideal of S. We also show how to compute the ideals of the primary orders and end the section with a determination of when two maximal crossed-product orders are R-isomorphic.

In the final section, examples are given of the various definitions and results of the preceding sections.

I want to thank Moss Sweedler and Richard Larson for useful conversations about various aspects of this paper.

1. Generalities

Let R be a discrete valuation ring (DVR) with field of fractions F, maximal ideal $m=(\pi)$ and residue field k. Let K/F be a finite Galois extension with group G. Let S be the integral closure of R in K and let $S^{\#}$ denote $S-\{0\}$. Let $U(S)$ denote the group of units of S. Let $Z^2(G, S^{\#})$ denote the set of normalized cocycles f: $G \times G \to S^{\#}$, that is such functions f satisfying $f^{\sigma}(\tau, \gamma)\, f(\sigma, \tau\gamma) = f(\sigma, \tau)\, f(\sigma\tau, \gamma)$ for all $\sigma, \tau, \gamma \in G$ and $f(\sigma, 1) = f(1, \sigma) = 1$ for all $\sigma \in G$. Call two such cocycles f and g cohomologous over S, and write $f \sim_S g$, if there is a one-cochain α: $G \to U(S)$ such that $f(\sigma, \tau) = (\alpha(\sigma)\, \alpha^{\sigma}(\tau)/\alpha(\sigma\tau))\, g(\sigma, \tau)$ for all $\sigma, \tau \in G$. The set of equivalence classes, denoted $N^2(G, S)$, is a monoid under pointwise multiplication.

There is a canonical map $N^2(G, S) \to H^2(G, K)$ which is a homomorphism of monoids. This map is easily seen to be surjective. There is also a canonical map $N^2(G, S) \to M^2(G, \bar{S})$ where $\bar{S} = S/mS$ and $M^2(G, \bar{S})$ denotes the cohomology theory of Haile *et al.* (HLS) [5]. The map is given by reducing the values of the cocycle modulo m. (Note that the reduced cocycle may take on noninvertible values, for example zero, so the image lies in $M^2(G, \bar{S})$ rather than $H^2(G, \bar{S})$.)

If f: $G \times G \to S^{\#}$ is a (normalized) cocycle, we let A_f denote the corresponding crossed-product R-algebra, that is, $A_f = \amalg_{\sigma \in G} Sx_{\sigma}$, where each x_{σ} is an indeterminate and we multiply by the rules $x_{\sigma}s = \sigma(s)x_{\sigma}$ for all $\sigma \in G$, $s \in S$ and $x_{\sigma}x_{\tau} = f(\sigma, \tau)x_{\sigma\tau}$ for all σ, $\tau \in G$. The resulting R-algebra is associative with identity $1 = x_1$ and center $R = Rx_1$. In fact, A_f is clearly an R-order in the central simple crossed-product F-algebra $\Sigma_f = \amalg_{\sigma \in G} Kx_{\sigma}$.

Our first aim is to give a partial characterization of the orders that appear this way, in the case where S/R is unramified. The following lemma is useful for this and other purposes.

Lemma 1. *Assume S/R is unramified. Let f: $G \times G \to K^{\times}$ be a cocycle and let $\Sigma_f = \amalg_{\sigma} Kx_{\sigma}$ be the corresponding crossed-product algebra. Let T be a* *finitely generated $S \otimes_R S$-submodule of Σ_f (where S acts on the left and right via the inclusion $S \subseteq \Sigma_f$). Then $T = \amalg_\sigma (T \cap Kx_\sigma)$.*

Proof. For each $\sigma \in G$ let $K_\sigma = \{k \in K \mid kx_\sigma + \sum_{\tau \neq \sigma} k_\tau x_\tau \in T$ for some $k_\tau \in K\}$. Then K_σ is the image of $T \cap Kx_\sigma$ in K under the canonical homomorphism (K is viewed as an $S \otimes S$-module via the action $(s_1 \otimes s_2) \cdot k = s_1 k \sigma(s_2)$). Since T is finitely generated over $S \otimes S$, so is K_σ. It follows that K_σ is an S-fractional ideal and hence $K_\sigma = Sk_\sigma$ for some $k_\sigma \in K$. Let $y_\sigma = k_\sigma x_\sigma$. Then $T \subseteq \sum_\sigma Sy_\sigma$.

We need to show $T \subseteq \sum_\sigma T \cap Kx_\sigma$. Since $T \cap Kx_\sigma \subseteq Sy_\sigma$, it suffices to show that if $\sum_\sigma s_\sigma y_\sigma \in T$, where $s_\sigma \in S$ for all σ, then $s_\sigma y_\sigma \in T$ for all σ. Suppose this is not true and let $t = \sum_{i=1}^r s_i y_{\sigma_i}$ be a counterexample with r as small as possible. We have $r \geqslant 2$. Let $I = \{s \in S \mid ss_1 y_{\sigma_1} \in T\}$, an ideal of S. We want to show $I = S$. If $I \neq S$, then there is a maximal ideal M of S such that $I \subseteq M$. Since S/R is unramified it is Galois, and so there is an element $s \in S$ such that $\sigma_1(s) - \sigma_2(s) \notin M$. Consider $\sigma_2(s)t - ts = (\sigma_2(s) - \sigma_1(s))\, s_1 y_{\sigma_1} + (\sigma_2(s) - \sigma_3(s))\, s_3 y_{\sigma_3} + \cdots + (\sigma_2(s) - \sigma_r(s))\, s_r y_{\sigma_r}$. This is an element of T and so by minimality, $(\sigma_2(s) - \sigma_1(s))\, s_1 y_{\sigma_1} \in T$. Hence $\sigma_2(s) - \sigma_1(s) \in I \subseteq M$, a contradiction. ▌

COROLLARY 1.2. *If $f: G \times G \to S^\#$ is a cocycle and $A_f = \amalg_{\sigma \in G} Sx_\sigma$ is the corresponding order, then every $S \otimes S$-submodule T of A_f (in particular every ideal of A_f) satisfies $T = \amalg_\sigma (T \cap Sx_\sigma)$.*

Proof. Given the lemma we need only observe that $T \cap Kx_\sigma \subseteq A_f \cap Kx_\sigma = Sx_\sigma$. ▌

PROPOSITION 1.3. *Assume S/R is unramified and let $f: G \times G \to K^\times$ be a cocycle. Let Σ_f be the corresponding crossed-product algebra and let $A \subseteq \Sigma_f$ be an R-order. There is a cocycle $g: G \times G \to S^\#$, $g \sim f$ over K, such that $A = A_g$ (viewed as a subalgebra of Σ_f in the natural way) if and only if $A \supseteq S$.*

Proof. If $A = A_g$, g as in the statement, then certainly $A \supseteq S$. Conversely, suppose $A \supseteq S$. Then A is a finitely generated $S \otimes S$-submodule of Σ_f, so $A = \amalg_\sigma A \cap Kx_\sigma$ by Lemma 1.1. Moreover as in the proof of that lemma, for each $\sigma \in G$, $A \cap Kx_\sigma = Sy_\sigma$ for some $y_\sigma \in Kx_\sigma$. Since A is an order in Σ_f, $y_\sigma \neq 0$ for all $\sigma \in G$. Hence if $g: G \times G \to S^\#$ is defined by $g(\sigma, \tau)\, y_{\sigma\tau} = y_\sigma y_\tau$, then g is a cocycle and $A = A_g$. ▌

COROLLARY 1.4. *If $A \subseteq \Sigma_f$ is a maximal order, then A is conjugate to a crossed-product order in Σ_f.*

Proof. The ring $S \subseteq \Sigma_f$ can be embedded in a maximal order B. By the proposition B is a crossed-product order. Since all maximal orders in a fixed central simple F-algebra are conjugate, we are done. ∎

Let A, B be R-orders (in some, possibly different, F-central simple algebras). Following Auslander and Goldman [3], we will call A and B *equivalent* if there are positive integers m and n such that $A \otimes_R M_m(R) \cong B \otimes_R M_n(R)$ as R-algebras. They show that if A is a maximal order and B is equivalent to A, then B is also maximal. (See Proposition 8.6 of [3].)

PROPOSITION 1.5. *Assume k is perfect. Let A be a maximal R-order. Then there is Galois extension K of F such that S, the integral closure of R in K, is unramified over R and a cocycle f: $G \times G \to S^\#$ such that A is equivalent to A_f.*

Proof. Let $\Sigma = A \otimes_R F$. By [1, Theorem 3.3], there is a K and an S as in the statement such that K splits Σ. It follows that Σ is Brauer equivalent to a crossed product algebra Σ_g, for some cocycle g: $G \times G \to K^*$. By Corollary 1.4 we may assume $g(G \times G) \subseteq S^\#$ and A_g is a maximal order in Σ_g. Let m and n be chosen so that $\Sigma \otimes_F M_n(F) \cong \Sigma_g \oplus_F M_m(F)$. Then $A \otimes_R M_n(R)$ is a maximal order in $\Sigma \otimes_F M_n(F)$ and $A_g \otimes M_m(R)$ is maximal in $\Sigma_g \otimes M_m(F)$. Hence $A \otimes M_n(R) \cong A_g \otimes M_m(R)$ and we are done. ∎

Thus we see that even in the rather special situation where S/R is unramified, we are able to capture, up to equivalence, all the maximal R-orders (assuming k is perfect). With this excuse *we are going to assume from this point forward that S/R is an unramified extension.*

Let f: $G \times G \to S^\#$ be a cocycle and let $A_f = \amalg_\sigma S x_\sigma$ be the corresponding order. Let $H = \{\sigma \in G \mid f(\sigma, \sigma^{-1})$ is a unit in $S\}$. Then H is a subgroup of G and $H = \{\sigma \in G \mid x_\sigma$ is invertible in $A_f\}$. As in HLS, we can associate to f a partial ordering on G/H by the rule $\sigma H \leqslant \tau H$ if $f(\sigma, \sigma^{-1}\tau)$ is a unit. It is easily checked that this is well defined and a partial ordering, and depends only on the cohomology class of f on S. Moreover, this ordering has the coset H as its unique least element and is *lower subtractive*: Given $\sigma H \leqslant \tau H$, we have $\sigma H \leqslant \gamma H \leqslant \tau H$ if and only if $\sigma^{-1}\gamma H \leqslant \sigma^{-1}\tau H$. For each subgroup T of G and each lower subtractive partial ordering θ on G/T with unique least element T, we let $N^2_\theta(G, S) = \{[f] \in N^2(G, S) \mid T$ is the subgroup associated to f and θ is the partial ordering on G/T determined by $f\}$. Then $N^2_\theta(G, S)$ is a submonoid of $N^2(G, S)$, possibly empty. Putting these pieces together, we obtain a decomposition $N^2(G, S) = \bigcup_\theta N^2_\theta(G, S)$, where the union is disjoint and taken over all partial orderings as described above.

Under the map $N^2(G, S) \to M^2(G, \bar{S})$ described earlier the image of $N^2_\theta(G, S)$ lies in the group $M^2_{e_\theta}(G, \bar{S})$, where e_θ is the idempotent cosickle corresponding to the partial ordering θ. (See HLS, Sect. 7.) On the other hand, the image of $N_\theta^2(G, S)$ in $H^2(G, K)$ is easily seen to be a subgroup (because $H^2(G, K)$ is torsion) and so we have the diagram

$$\begin{array}{ccc} N_\theta^2(G, S) & \longrightarrow & H^2(G, K) \\ \downarrow & & \\ M_{e\theta}^2(G, \bar{S}). & & \end{array}$$

Moreover, the Brauer group of S/R acts on each of these objects in canonical ways and the maps are $B(S/R)$-set maps.

Let $M_1, M_2, \ldots, M_r$ be the maximal ideals of S and let $M_i = (\pi_i)$, $\pi_i \in S$, $1 \leqslant i \leqslant r$. Let P be the submonoid of $S^\#$ generated by the π_i's, so $P = \{\pi_1^{k_1} \cdots \pi_r^{k_r} \mid k_i \geqslant 0 \text{ for all } i\}$. If $f\colon G \times G \to S^\#$ is a cocycle, we can decompose f uniquely into $f = f_u f_p$, where $f_p(G \times G) \subseteq P$ and $f_u(G \times G) \subseteq U(S)$. It is easy to see that f_u and f_p are again cocycles. We will make use of this decomposition in a later section.

Finally, again for later use, we record the following result on automorphisms.

Proposition 1.6. *Let $f\colon G \times G \to S^\#$ be a cocycle. Let ϕ be an automorphism of A_f such that $\phi|_S =$ identity. Then there is a unit u in S such that $\phi(a) = uau^{-1}$ for all $a \in A_f$. In particular, ϕ is inner.*

Proof. Let $A_f = \amalg_\sigma S x_\sigma$ as usual. The automorphism ϕ extends to an automorphism $\tilde{\phi}$ of Σ_f such that $\tilde{\phi}|_K =$ identity. By the Skolem–Noether theorem, there is an invertible element a in Σ_f such that $\tilde{\phi} = I_a$, the inner automorphism determined by a. Moreover, since $\tilde{\phi}$ is the identity on K, we conclude that a centralizes K, so $a \in K$. Returning to A_f, since ϕ is the identity on S, it follows easily that for all $\sigma \in G$, $\phi(x_\sigma) = u_\sigma x_\sigma$ for some unit u_σ in S. Hence $a/\sigma(a) = u_\sigma$ for all $\sigma \in G$. As in the discussion preceding this proposition, let $M_i = (\pi_i)$, $i = 1, 2, \ldots, r$ be the maximal ideals of S. Let $a = v\pi_1^{k_1} \cdots \pi_r^{k_r}$, where each k_i is an integer and v is a unit of S. From the condition that $a/\sigma(a)$ is a unit and the fact that G acts transitively on the maximal ideals of S, it follows that $k_i = k_j$ for all i and j. Hence $a = u\pi^k$ for some integer k and some unit u of S. Since $\pi \in R$, we have $\tilde{\phi} = I_a = I_u$ and so $\phi = I_u$ as desired. ∎

2. The Case Where S Is a DVR

In this section and the next we undertake an investigation of the structure of the crossed-product orders. We will look at the relationship between that structure and properties of the graphs associated to the cocycle, and determine what about the graph makes the order maximal.

Let R, F, S, K, G, k, and $m=(\pi)$ be as in Section 1. Recall that we are assuming throughout that S/R is unramified. *In this section we will assume that S is itself a* DVR.

Let $f\colon G\times G\to S^{\#}$ be a normalized cocycle and let A_f denote the corresponding crossed-product order. Let H be the subgroup associated to f. We have $A_f=B_f\oplus J$, where $B_f=\amalg_{\sigma\in H}Sx_\sigma$ and $J=\amalg_{\sigma\notin H}Sx_\sigma$ and the sum is direct as R-modules.

PROPOSITION 2.1. (a) *The set B_f is an R-subalgebra of A_f. Moreover, B_f is Azumaya with center S^H.*

(b) *The ideal $mB_f\oplus J$ is the radical of A_f and is the unique maximal (2-sided) ideal of A_f.*

Proof. (a) Since H is a subgroup of G, it follows easily that B_f is a subalgebra of A_f. Now $f(H\times H)\subseteq U(S)$: if $h_1, h_2\in H$, then $f^{h_1}(h_2, h_2^{-1})=f(h_1,h_2)\,f(h_1h_2,h_2^{-1})$, so $f(h_1,h_2)\in U(S)$. Clearly then B_f is the crossed product algebra determined by $f|_{H\times H}$. It follows that B_f is Azumaya over its center S^H (see DeMeyer and Ingraham [4]). Since S^H is unramified over R, B_f is in fact separable over R.

(b) The k-algebra $\bar{A}_f=A_f/mA_f$ is the crossed product algebra for the cosickle $\bar{f}\colon G\times G\to S/mS$ in the sense of HLS. In particular, $\bar{A}_f$ has radical $\bar{J}$ and $\bar{A}_f/\bar{J}$ is simple. The desired result follows. ∎

Remark. A ring A is called *primary* if $A/\mathrm{rad}(A)$ is simple Artinian. By Proposition 2.1 each A_f is primary.

In the last section we showed that every maximal R-order is equivalent to a crossed-product order. We are now heading for a characterization of those cocycles, and hence those partial orderings, which give rise to maximal orders.

PROPOSITION 3.2. *Assume S is a* DVR. *Let f be a cocycle with associated subgroup H. Suppose A_f is maximal. Then there is an element $\sigma\in G$, $\sigma\notin H$, such that $\sigma H\leqslant\tau H$ for all $\tau\in G-H$.*

Proof. Let $r=|H|$. Number the elements of G, say $\sigma_1, \sigma_2,\ldots, \sigma_n$ in such a way that the following two conditions hold.

(1) $\sigma_i H=\sigma_j H$ if $kr+1\leqslant i, j\leqslant (k+1)r$ for some k, $0\leqslant k<n/r$ and

(2) if $\sigma_i H\lneqq\sigma_j H$, then $i\lneqq j$.

It is easy to see that such a numbering exists. Note that since H is the unique minimal element in the ordering on G/H, we have $H=\{\sigma_1, \sigma_2,\ldots, \sigma_r\}$.

Now suppose A_f is maximal. Then there is an element $y\in A_f$ such that $\mathrm{rad}(A_f)=yA_f=A_fy$ (see Reiner [7, Theorem 18.7]). Let $y=\sum_{i=1}^n b_{\sigma_i}x_{\sigma_i}$.

Since $y \in \operatorname{rad}(A_f) = \pi(\coprod_{\sigma \in H} Sx_\sigma) + \coprod_{\sigma \notin H} Sx_\sigma$, we see that $b_\sigma \in mS$, for $\sigma \in H$. Given i, $1 \leqslant i \leqslant n$, there are elements c_{ij} in S, $1 \leqslant j \leqslant n$, such that

$$\left(\sum_j c_{ij} x_{\sigma_j}\right) y = \begin{cases} \pi x_{\sigma_i}, & \sigma_i \in H \\ x_{\sigma_i}, & \sigma_i \notin H. \end{cases}$$

Expanding the left-hand side and computing the coefficient of a given x_{σ_k}, we obtain for each i,

$$\sum_j c_{ij} b^{\sigma_j}_{\sigma_j^{-1}\sigma_k} f(\sigma_j, \sigma_j^{-1}\sigma_k) = \begin{cases} \pi, & k = i \text{ and } \sigma_i \in H \\ 1, & k = i \text{ and } \sigma_i \notin H \\ 0, & k \neq i. \end{cases}$$

Let C be the $n \times n$ matrix whose (i, j) component is c_{ij} and let B be the $n \times n$ matrix whose (j, k) component $b^{\sigma_j}_{\sigma_j^{-1}\sigma_k} f(\sigma_j, \sigma_j^{-1}\sigma_k)$. The relations given above can be expressed as the matrix equation

$$CB = \left(\begin{array}{c|c} \pi I_r & 0 \\ \hline 0 & I \end{array}\right),$$

where I_r is the $r \times r$ identity. By our assumption on the ordering $f(\sigma_j, \sigma_j^{-1}\sigma_k) \in mS$ if $\sigma_j H \neq \sigma_k H$ and $j > k$. Moreover, if $\sigma_j H = \sigma_k H$, then $\sigma_j^{-1}\sigma_k \in H$ and $b_{\sigma_j^{-1}\sigma_k} \in mS$. Hence letting a "bar" denote reduction modulo m, we see that $\bar{B}$ is block strictly upper triangular, that is,

$$\bar{B} = \begin{pmatrix} 0 & * & * & \cdots & * \\ & 0 & * & & * \\ & & 0 & & \vdots \\ \bigcirc & & & \ddots & * \\ & & & & 0 \end{pmatrix},$$

where each asterisk denotes an $r \times r$ block. In addition, we have

$$\bar{B}\bar{C} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & I \end{array}\right),$$

where I is the $(n-r) \times (n-r)$ identity. It follows that the matrix obtained from $\bar{B}$ by eliminating the first column and last row of blocks is invertible (over S/mS). Hence each of its diagonal blocks is invertible. Let k be an integer, $1 \leqslant k \leqslant (n/r) - 1$. A typical such diagonal $r \times r$ block has (i, j) entry equal to

$$\bar{b}^{\sigma_{kr+i}}_{\sigma_{kr+i}^{-1}\sigma_{(k+1)r+j}} \bar{f}(\sigma_{kt+i}, \sigma_{kr+i}^{-1}\sigma_{(k+1)r+j}), \qquad 1 \leqslant i, j \leqslant r.$$

But either $\bar{f}(\sigma_{kr+i}, \sigma_{kr+i}^{-1}\sigma_{(k+1)r+j})$ is zero for all i, j, or is nonzero for all i, j, since all the σ_{kr+i} belong to the same coset modulo H as i varies (and similarly for the $\sigma_{(k+1)r+j}$). By the invertibility of the block we conclude that $f(\sigma_{kr+i}, \sigma_{kr+i}^{-1}\sigma_{(k+1)r+j})$ is a unit in S for all k, i, j, where $1 \leqslant k < n/r$, $1 \leqslant i, j \leqslant r$. Hence $\sigma_{r+1}H \leqslant \sigma_{2r+1}H \leqslant \cdots \leqslant \sigma_{n-r+1}H$. Letting $\sigma = \sigma_{r+1}$, we are done. ∎

THEOREM 2.3. *Assume S is a* DVR. *Let f: $G \times G \to S^{\#}$ be a cocycle and let H be its associated subgroup. The crossed-product order A_f is maximal if and only if the following conditions are satisfied:*

(1) *The subgroup H is normal in G and G/H is cyclic.*

(2) *There is an element $\sigma \in G$ such that $G/H = \langle \sigma H \rangle$ and $f(\sigma, \sigma^{-1}) \in mS - m^2S$, and*

(3) The graph of f is the simple chain $H \leqslant \sigma H \leqslant \sigma^2 H \leqslant \cdots \leqslant \sigma^{m-1}H$, *where $m = |G/H|$.*

Moreover, under these conditions, $\mathrm{rad}(A_f) = A_f x_\sigma = x_\sigma A_f$.

Proof. First assume A_f is maximal. From the previous proposition we know there is an element $\sigma \in G - H$ such that $\sigma H \leqslant \tau H$ for all $\tau \in G - H$. Let t be minimal with $\sigma^t \in H$. We first *claim* $\sigma H \leqslant \sigma^2 H \leqslant \cdots \leqslant \sigma^{t-1}H$. In fact, if $1 \leqslant i < t-1$, then $\sigma H \leqslant \sigma^{i+1}H$ and $\sigma H \leqslant \sigma^i H$. Thus, by lower subtractivity, $\sigma H \leqslant \sigma^i H \leqslant \sigma^{i+1}H$. Next let $g \in G - H$. Choose i maximal such that $1 \leqslant i \leqslant t-1$ and $\sigma^i H \leqslant gH$. We *claim* $gH = \sigma^i H$. If not, then from $\sigma^i H \leqslant gH$ and $\sigma H \leqslant \sigma^{-i}gH$ we conclude by lower subtractivity that $\sigma^i H \leqslant \sigma^{i+1}H \leqslant gH$, a contradiction. Hence we see that $t = m$ and the graph of f is the chain $H \leqslant \sigma H \leqslant \sigma^2 H \leqslant \cdots \leqslant \sigma^{m-1}H$.

We now show H is normal. From lower subtractivity it follows that the action of H on G/H by left multiplication preserves the order. Since σH is the unique element of height one, we have $h\sigma H = \sigma H$ for all $h \in H$. Thus $\sigma H \sigma^{-1} = H$, so H is normal.

Next we show that $f(\sigma, \sigma^{-1}) \in mS - m^2S$. Suppose $f(\sigma, \sigma^{-1}) \in m^2S$. An easy cocycle computation gives $f(\sigma^i, \sigma^{-i}\sigma^k) \in m^2S$ for all k, i with $0 \leqslant k < i \leqslant m-1$. Let $A_f = \coprod_\gamma Sx_\gamma \subseteq \coprod_\gamma Kx_\gamma = \Sigma_f$ as usual. Define elements $y_\gamma \in \Sigma_f$, $\gamma \in G$, by the formula

$$y_\gamma = \begin{cases} \dfrac{1}{\pi} x_\gamma, & \gamma \in \sigma^{m-1}H \\ x_\gamma, & \gamma \notin \sigma^{m-1}H. \end{cases}$$

It is straightforward to show that for all δ, $\gamma \in G$, we have $y_\delta y_\gamma \in Sy_{\delta\gamma}$.

Thus $\tilde{A}_f = \coprod_\gamma Sy_\gamma$ is an R-order in Σ_f properly containing A_f. This contradicts the maximality of A_f.

The converse statement will follow if we show that given f satisfying (1), (2), and (3), then $\operatorname{rad}(A_f) = A_f x_\sigma = x_\sigma A_f$. Recall that $\operatorname{rad}(A_f) = \pi \coprod_{h \in H} Sx_h + \coprod_{\tau \notin H} Sx_\tau$. Since $\sigma H \leqslant \tau H$ for all $\tau \notin H$ and $(x_\sigma x_{\sigma^{-1}})S = \pi S$, it follows that $x_\sigma A_f = \operatorname{rad}(A_f)$. To show $A_f x_\sigma = \operatorname{rad}(A_f)$, it suffices to show $f(\tau\sigma^{-1}, \sigma)$ is a unit for all $\tau \notin H$. But $\tau H = \sigma^k H$ for some k, $1 \leqslant k \leqslant m-1$. By the normality of H, we see $\tau\sigma^{-1}H = \sigma^{k-1}H \leqslant \sigma^k H = \tau H$, as desired. ∎

It is instructive to compare this result with the exact sequence of Auslander and Brumer [1]. Under the conditions of the theorem, they derive the sequence

$$0 \to B(S/R) \to B(K/F) \to \chi(G) \to 0,$$

where $B(S/R)$ is the subgroup of elements of $B(R)$ split by S, $B(K/F)$ is the analogous subgroup for K/F and $\chi(G)$ is the character group of G. If we begin with a crossed-product algebra Σ_f, $[f] \in B(K/F)$, then the sequence associates to $[f]$ a character of G, that is a normal subgroup H of G with cyclic quotient and a distinguished generator σH of G/H (where the character sends σH to $(1/|H|\,\mathbb{Z}$ in $Q/\mathbb{Z})$.

If we assume, as we may, that $f(G \times G) \subseteq S$ and A_f is maximal, then the previous theorem in particular associates a normal subgroup with cyclic quotient and a distinguished generator for that quotient. It is not difficult to show that this leads to the same character as determined by the sequence. In conjunction with the decomposition of Proposition 3.1, this gives a different perspective on the role of that character.

We next want to describe the ideals of the crossed-product orders. Let f: $G \times G \to S^{\#}$ be a cocycle (recall that we are assuming S is a DVR). Let I be an ideal of $A_f = \coprod_\sigma Sx_\sigma$. Let $A = A_f$. Since I is in particular an $S \otimes S$-submodule of A we may apply Lemma 1.1 and obtain $I = \coprod_{\sigma \in G} (I \cap Sx_\sigma) = \coprod_\sigma I_\sigma x_\sigma$, where $I_\sigma = \{s \in S \mid sx_\sigma \in I\}$. In particular, $I = \sum_\sigma AI_\sigma x_\sigma A$. Since S is a DVR, all the ideals of S are G-stable, so we see that $I = \sum_\sigma I_\sigma(Ax_\sigma A)$. We have therefore proved the following result.

PROPOSITION 2.4. *If I is an ideal of $A = A_f = \coprod_{\sigma \in G} Sx_\sigma$, then $I = \sum_\sigma I_\sigma(Ax_\sigma A)$, where $I_\sigma = \{s \in S \mid sx_\sigma \in I\}$.* ∎

Since each I_σ is an ideal in S, we see that to determine the ideals of A_f it suffices to describe the ideals generated by the elements x_σ, $\sigma \in G$. We will see that this can be done by using the graphs associated to f. Let v: $K \to Z$ be the valuation associated to S (so $v(\pi) = 1$).

PROPOSITION 2.5. *If $\sigma \in G$, then $Ax_\sigma A = \coprod_\sigma T_\sigma x_\sigma$, where $T_\gamma = \pi^{k_\gamma} S$ and $k_\gamma = \min_{\tau \in G}\{v(f(\sigma, \sigma^{-1}\tau)) + v(f(\gamma\tau^{-1}, \tau))\}$.*

Proof. Clearly $Ax_\sigma A = \sum_{\alpha,\beta\in G} Sx_\alpha x_\sigma x_\beta$. Hence $T_\gamma x_\gamma = \sum_{\alpha\in G} Sx_\alpha x_\sigma x_{\sigma^{-1}\alpha^{-1}\gamma}$. This can be written in a more useful way by letting $\alpha=\gamma\tau^{-1}$. We obtain $T_\gamma x_\gamma = \sum_\tau Sx_{\gamma\tau^{-1}} x_\sigma x_{\sigma^{-1}\tau} = [\sum_\tau Sf^{\gamma\tau^{-1}}(\sigma, \sigma^{-1}\tau)$ $f(\gamma\tau^{-1}, \tau)] x_\gamma$. The proposition follows easily. ∎

The integers k_γ, $\gamma\in G$, can be determined by considering "weighted" graphs. For each coset σH, we construct a copy of the left graph of f and weight it by attaching to the coset τH the integer $v(f(\sigma, \sigma^{-1}\tau))$, which in some sense measures how far σH is from being less than τH. Similarly, for each coset $H\sigma$, we construct a copy of the right graph of f and weight it by attaching to the coset $H\tau$ the integer $v(f(\tau\sigma^{-1}, \sigma))$. Clearly the integers k_γ and hence the ideals $Ax_\sigma A$ can be determined from these $2[G : H]$ graphs. An example will be given in the last section.

The last thing we want to do in this section is to determine when two crossed-products orders are R-algebra isomorphic. Let f: $G\times G \to S^\#$ be a cocycle and let H be its associated subgroup. Let $\sigma_1, \sigma_2, \ldots, \sigma_m$ be a set of left coset representatives of H in G (i.e., $G=\bigcup \sigma_i H$).

PROPOSITION 2.6. *Let ϕ: $S\to A_f$ be an R-algebra imbedding. There is an integer i, $1\leqslant i\leqslant m$ and an invertible element $a\in A_f$ such that $a\phi(s)a^{-1}=\sigma_i(a)$ for all $s\in S$. In particular, $\phi(S)$ is conjugate to S.*

Proof. Let $A=A_f$. The map ϕ allows us to put an $A\otimes_R S$-module structure on A via the formula $(a\otimes s)x = ax\phi(s)$ for all a, $x\in A$, $s\in S$. Let A_ϕ denote A with this module structure. Similarly, for each i the map σ_i endows A with the $A\otimes S$-module structure given by $(a\otimes s)x = ax\sigma_i(s)$. Let A_i denote A with this module structure. We *claim* that for some i, A_i is isomorphic to A_ϕ as an $A\otimes S$-module. We will assume the claim for the moment and show how to complete the proof. Let ψ: $A_\phi \to A_i$ be an $A\otimes S$-module isomorphism. Then $\psi((a\otimes s)x) = (a\otimes s)\psi(x)$ for all a, $x\in A$, $s\in S$. Hence $\psi(ax\phi(s)) = a\psi(s)\sigma_i(s)$. It follows easily that $\psi(a) = a\psi(1)$ for all $a\in A$ and $\phi(s)\psi(1) = \psi(1)\sigma_i(s)$ for all $s\in S$. Since ψ is an isomorphism, the element $\psi(1)$ is invertible. But then $\psi(1)^{-1}\phi(s)\psi(1) = \sigma_i(s)$ for all $s\in S$, as desired.

We now proceed to prove the claim. First observe that each of the modules A_ϕ, A_i, $1\leqslant i\leqslant m$, is projective over $A\otimes S$ and isomorphic to A as a left A-module (where A is viewed as a subring of $A\otimes S$). To see the projectivity consider, for example, the module A_ϕ. Since S/R is Galois, there is a unique minimal idempotent e in $\phi(S)\otimes S$ such that $(1\otimes s)e = (\phi(s)\otimes 1)e$ for all $s\in S$. The ideal $(\phi(S)\otimes S)(1-e)$ is the kernel of the homomorphism $\phi(S)\otimes S\to\phi(S)$ given by $\phi(s)\otimes t\mapsto\phi(st)$. There is a left $A\otimes S$-module homomorphism $A\otimes S\to A_\phi$ given by $a\otimes s\mapsto a\phi(s)$. This map is surjective and sends $e\in\phi(S)\otimes S\subseteq A\otimes S$ to 1. It is then easy to see that the map $A_\phi \to A\otimes S$ given by $a\mapsto (a\otimes 1)e$ is an $A\otimes S$-module homomorphism and a splitting. Hence A_ϕ is projective. The modules A_i are handled in the same way.

Let "$\sim$" denote reduction modulo the radical of $A \otimes S$ and let "$-$" denote reduction modulo m. If N_1, N_2 are projective $A \otimes S$-modules, then they are isomorphic if and only if $\tilde{N}_1$, $\tilde{N}_2$ are isomorphic as $A \otimes S$-modules. Now let $A_f = B_f \oplus J$ as usual. As noted before, $\bar{A}_f = \bar{B}_f \oplus \bar{J}$ is the crossed product algebra for the cosickle $\bar{f}$ in the sense of Section 10 of HLS. In particular, $\bar{B}_f$ is simple with center $L = \bar{S}^H$. (Recall that $\bar{S}/k$ is Galois with group G.) Then $\bar{A} \otimes_k \bar{S} = (\bar{B} \otimes \bar{S}) \oplus (\bar{J} \otimes \bar{S})$ has radical $\bar{J} \otimes \bar{S}$ and $\widetilde{A \otimes S} \cong \bar{B} \otimes_k \bar{S}$. The algebra $\bar{B} \otimes \bar{S}$ is semisimple with center $L \otimes_k \bar{S} \cong \bar{S}_1 \oplus \cdots \oplus \bar{S}_m$, where $\bar{S}_i$ is k-isomorphic to $\bar{S}$ and isomorphism is given by $l \otimes s \mapsto \sum_i \sigma_i(l)s$. Moreover, $\bar{B} \otimes \bar{S} = \coprod_i \bar{B} \otimes_L \bar{S}_i$, where $\bar{S}_i$ is viewed as a left L-module via σ_i. Each of these components is simple and has dimension $[\bar{S} : L]^2$ over its center (which is $\bar{S}_i$ for the ith component). For each i we can make $\bar{B}$ into a left $\bar{B} \otimes \bar{S}$-module by setting $(b \otimes s)c = bc\sigma_i(s)$ for b, $c \in \bar{B}$, $s \in \bar{S}$. Call the resulting module $\bar{B}_i$. Then clearly $\tilde{A}_i \cong \bar{B}_i$ over $\widetilde{A \otimes S} \cong \bar{B} \otimes_k \bar{S}$. Since $[\bar{B}_i : \bar{S}_i] = [\bar{S} : L]$, it follows that $\bar{B}_i$ is an irreducible module over $\bar{B} \otimes_L \bar{S}_i$ and that $\bar{B} \otimes_L \bar{S}_i$ is split. Moreover, any module for $\bar{B} \otimes_k \bar{S}$ of dimension $[\bar{S} : L]$ over $\bar{S}$ must be irreducible and isomorphic to some $\bar{B}_i$. But A_ϕ is isomorphic to A as a left A-module, and so $\tilde{A}_\phi$ is isomorphic to $\bar{B}$ as a left $\bar{B} \otimes \bar{S}$-module. In particular, $[\tilde{A}_\phi : \bar{S}] = [\bar{S} : L]$ and so $\tilde{A}_\phi \cong \tilde{B}_i$ for some i. Hence $A_\phi \cong A_i$ over $A \otimes S$. ∎

The group G acts on $N^2(G, S)$ by the rule $(\sigma \cdot f)(\alpha, \beta) = f^\sigma(\sigma^{-1}\alpha\sigma, \sigma^{-1}\beta\sigma)$ for σ, α, $\beta \in G$. The next theorem says that the R-algebra isomorphism classes of crossed product orders are in one-to-one correspondence with the orbits of this action (when S is a DVR).

THEOREM 2.7. *Assume S is a* DVR. *Let $[f_1]$, $[f_2] \in N^2(G, S)$. Let H be the subgroup associated to f_1 and let $\sigma_1, \sigma_2, \ldots, \sigma_m$ be a set of left coset representatives of H in G (i.e., $G = \cup \sigma_i H$). Then $A_{f_1} \cong A_{f_2}$ as R-algebras if and only if $f_2 \sim \sigma_i^{-1} \cdot f_1$ for some i.*

Proof. We first show that for all $\tau \in G$, $A_{f_1} \cong A_{\tau \cdot f_1}$. In fact, if $A_{f_1} = \coprod_\sigma Sx_\sigma$ and $A_{\tau \cdot f_1} = \coprod_\sigma Sy_\sigma$, then one can easily check that the map from A_{f_1} to $A_{\tau \cdot f_1}$ given by $\sum_\sigma s_\sigma x_\sigma \mapsto \sum_\sigma \tau(s_\sigma) y_{\tau\sigma\tau^{-1}}$ is an R-algebra isomorphism.

Conversely, suppose $A_{f_1} \cong A_{f_2}$. It follows from Proposition 2.6 that there is an integer i and an R-algebra isomorphism ψ: $A_{f_2} \to A_{f_1}$ such that $\psi(s) = \sigma_i(s)$ for all $s \in S$. By the first part of this proof there is an isomorphism ϕ: $A_{f_1} \to A_{\sigma_i^{-1} f_1}$ such that $\phi(s) = \sigma_i^{-1}(s)$. The composite $\phi\psi$: $A_{f_2} \to A_{\sigma_i^{-1} \cdot f_1}$ is then the identity on S. It follows by standard arguments that $f_2 \sim \sigma_i^{-1} \cdot f_1$ over S. ∎

COROLLARY 2.8. *Assume S is a* DVR. *Suppose A_{f_1} and A_{f_2} are maximal orders. Then $A_{f_1} \cong A_{f_2}$ as R-algebras if and only if $f_1 \sim f_2$ over S.*

Proof. If $f_1 \sim f_2$ over S, then A_{f_1} and A_{f_2} are clearly isomorphic. Conversely, suppose $A_{f_1} \cong A_{f_2}$. Since A_{f_1} is maximal, we know by Theorem 2.3 that H is normal in G with G/H cyclic and the graph of f is of the form $H \leqslant \sigma H \leqslant \sigma^2 H \leqslant \cdots \leqslant \sigma^{m-1} H$, where $G/H = \langle \sigma H \rangle$, $|G/H| = m$, and $f(\sigma, \sigma^{-1}) \in mS - m^2 S$. It is easy to see that it follows that for all $\alpha, \beta \in G$, $f(\alpha, \beta) \notin m^2 S$ (and, of course, $f(\alpha, \beta) \in U(S)$ if and only if $\alpha \in H$ or $\beta \in H$). Now by the theorem we know that $f_2 \sim \sigma^i \cdot f_1$ for some i, $0 \leqslant i \leqslant m-1$. It suffices then to show that $\sigma^i \cdot f_1 \sim f_1$ for all i. For that it suffices to show there is an R-algebra automorphism ψ_i of A_{f_1} such that $\psi_i = \sigma^i$ on S. We will show that if $A_{f_1} = \coprod_\tau S x_\tau$, then $x_\sigma A_{f_1} x_\sigma^{-1} = A_{f_1}$, where the inverse x_σ^{-1} is taken in Σ_{f_1}. This automorphism equals σ on S, so it and its powers will then settle the issue. Now to see that $x_\sigma A_{f_1} x_\sigma^{-1} = A_{f_1}$, it is enough to show that for all $\tau \in G$, $x_\sigma x_\tau x_\sigma^{-1} = u_{\sigma\tau\sigma^{-1}} x_{\sigma\tau\sigma^{-1}}$ for some unit $u_{\sigma\tau\sigma^{-1}}$ in S. But $x_\sigma^{-1} = f(\sigma^{-1}, \sigma)^{-1} x_{\sigma^{-1}}$. Hence $x_\sigma x_\tau x_\sigma^{-1} = f^{\sigma\tau}(\sigma^{-1}, \sigma)^{-1} f(\sigma, \tau) f(\sigma\tau, \sigma^{-1}) x_{\sigma\tau\sigma^{-1}} = f(\sigma, \tau) f(\sigma\tau\sigma^{-1}, \sigma) x_{\sigma\tau\sigma^{-1}}$. By the remarks above either $v(f(\sigma, \tau)) = v(f(\sigma\tau\sigma^{-1}, \sigma)) = 1$ or $v(f(\sigma, \tau)) = v(f(\sigma\tau\sigma^{-1}, \sigma)) = 0$. In either case the result is a unit multiple of $x_{\sigma\tau\sigma^{-1}}$, as desired. ∎

Remark. It should be observed that we can now determine the outer automorphism group of a maximal order A_f quite explicitly: By Proposition 2.6 any automorphism $\tilde{\phi}$ is congruent modulo an inner automorphism to an automorphism $\tilde{\phi}$ which preserves S (and so is equal to σ^i on S, where $H \leqslant \sigma H \leqslant \cdots \leqslant \sigma^{m-1} H$ is the graph of f and i is some integer, $0 \leqslant i \leqslant m-1$). By Lemma 1.6 and the proof of Corollary 2.8, the automorphism $\tilde{\phi}$ is congruent modulo an inner to conjugation by x_{σ^i}. But conjugation by x_{σ^i} is not inner because x_{σ^i} is not invertible in A_f. Hence $\mathrm{Out}(A_f) = \langle \phi_\sigma \rangle$, where ϕ_σ is the image of the automorphism given by conjugation by x_σ. In particular, $\mathrm{Out}(A_f)$ is cyclic of order $[G : H]$. This should be compared with Corollary 37.32 of [7]. In particular, we see by that corollary that $[G : H]$ is the index of ramification of the division algebra part of the completion of A_f.

3. THE GENERAL CASE

In this section we investigate the structure of the crossed product orders when S is not necessarily a DVR. Let R, F, S, K, G be as usual (S/R unramified). The basic idea is to reduce to the case of a DVR by replacing A_f by the algebra $C_{A_f}(S^D) \oplus S^D_M$, where M is a maximal ideal of S, D is the decomposition group of M, and $C_{A_f}(S^D)$ is the centralizer of S^D, the fixed ring under D, in A_f (the tensor product is over S^D). This moves the setting from S/R to S_M/S_M^D and S_M is a DVR.

To begin let f: $G \times G \to S$ be a cocycle with associated subgroup H. As before we have the decomposition $A_f = B_f \oplus J$, where $B_f = \coprod_{h \in H} Sx_h$ and $J = \coprod_{\sigma \notin H} Sx_\sigma$. We want to determine the radical of A_f. If $\sigma \in G$, we let $I_\sigma = \prod M$, where the product is taken over those maximal ideals M of S such that $f(\sigma, \sigma^{-1}) \notin M$. In other words, $I_\sigma = (mS; f(\sigma, \sigma^{-1})) = \{x \in S \mid xf(\sigma, \sigma^{-1}) \in mS\}$.

PROPOSITION 3.1. (a) *The set B_f is a subalgebra of A_f. Moreover, B_f is Azumaya with center S^H.*

(b) *The radical of A_f is given by* $\operatorname{rad}(A_f) = \coprod_{\sigma \in G} I_\sigma x_\sigma$.

Proof. (a) The argument of part (a) of Proposition 2.1 applies.

(b) We first show $I = \coprod_{\sigma \in G} I_\sigma x_\sigma$ is an ideal in A_f. To see that I is a right ideal, it suffices to show that $(I_\sigma x_\sigma) x_{\sigma^{-1}\tau} \subseteq I_\tau x_\tau$ for all $\sigma, \tau \in G$. That is, we need to show $I_\sigma f(\sigma, \sigma^{-1}\tau) \subseteq I_\tau$. From the identity $f^\sigma(\sigma^{-1}\tau, \tau^{-1})\ f(\sigma, \sigma^{-1}) = f(\sigma, \sigma^{-1}\tau)\ f(\tau, \tau^{-1})$, we obtain $I_\sigma f(\sigma, \sigma^{-1}\tau)$ $f(\tau, \tau^{-1}) \subseteq I_\sigma f(\sigma, \sigma^{-1}) \subseteq mS$. Hence $I_\sigma f(\sigma, \sigma^{-1}\tau) \subseteq I_\tau$ as desired. Similarly, to show I is a left ideal, we need $I_\sigma^{\tau\sigma^{-1}} f(\tau\sigma^{-1}, \sigma) \subseteq I_\tau$ for all σ, $\tau \in G$. From the identity $f^{\tau\sigma^{-1}}(\sigma, \tau^{-1})\ f(\tau\sigma^{-1}, \sigma\tau^{-1}) = f(\tau\sigma^{-1}, \sigma)\ f(\tau, \tau^{-1})$ we see that it suffices to show $I_\sigma^{\tau\sigma^{-1}} f^{\tau\sigma^{-1}}(\sigma, \tau^{-1}) f(\tau\sigma^{-1}, \sigma\tau^{-1}) \subseteq mS$. But $I_\sigma f(\sigma, \tau^{-1}) f^{\sigma\tau^{-1}}(\tau\sigma^{-1}, \sigma\tau^{-1}) = I_\sigma f(\sigma, \tau^{-1}) f(\sigma\tau^{-1}, \tau\sigma^{-1}) = I_\sigma f^\sigma(\tau^{-1}, \tau\sigma^{-1})\ f(\sigma, \sigma^{-1}) \subseteq I_\sigma f(\sigma, \sigma^{-1}) \subseteq mS$ as desired.

To see that I is in fact the radical of A_f, first note that $I \supseteq mA_f$, so we may work modulo mA_f. Since $\bar{A}_f = A_f/mA_f$ is a finite-dimensional $k = R/m$ algebra, it suffices to show $\bar{I}$ is the maximal nilpotent ideal of $\bar{A}_f$. To show $\bar{I}$ is nilpotent, it is enough to show that $\bar{I}$ has a k-basis of nilpotent elements, that is, it suffices to show $\overline{I_\sigma x_\sigma}$ is nilpotent for all σ. But if r is the order of σ in G, then $(I_\sigma x_\sigma)^r = I_\sigma I_\sigma^\sigma \cdots I_\sigma^{\sigma^{r-1}} f(\sigma, \sigma)\ f(\sigma^2, \sigma) \cdots f(\sigma^{r-1}, \sigma) \subseteq I_\sigma^{\sigma^{-1}} f(\sigma^{-1}, \sigma) = (I_\sigma f(\sigma, \sigma^{-1}))^{\sigma^{-1}} \subseteq mS$, Thus $\overline{I_\sigma x_\sigma}^r = 0$. We now have $I \subseteq \operatorname{rad}(A_f)$. Suppose the inclusion is strict. We know $\operatorname{rad}(A_f) = \coprod_\sigma(\operatorname{rad}(A_f) \cap Sx_\sigma)$ by Lemma 2.1. Hence there are elements $\sigma \in G$ and $a_\sigma \in S - I_\sigma$ such that $a_\sigma x_\sigma \in \operatorname{rad}(A_f)$. But then $\operatorname{rad}(A_f) \ni (a_\sigma x_\sigma) x_{\sigma^{-1}} = a_\sigma f(\sigma, \sigma^{-1})$. By the remarks above $(a_\sigma f(\sigma, \sigma^{-1}))^r \in mS$ for some r. But then $a_\sigma f(\sigma, \sigma^{-1}) \in mS$, so $a_\sigma \in I_\sigma$. ∎

If S is not a DVR, it is not necessarily true that $\operatorname{rad}(A_f)$ is a maximal ideal, i.e., that A_f is primary. Maximal orders are primary so we first want to characterize the condition of being primary in terms of the cocycle f. For each maximal ideal M of S we let D_M denote the decomposition group of M, that is, $D_M = \{\sigma \in G \mid M^\sigma = M\}$. Since S/R is unramified, the group D_M may be identified with the Galois group of S/M over k.

THEOREM 3.2. *Let f: $G \times G \to S^{\#}$ be a cocycle. The crossed product order A_f is primary if and only if for every maximal ideal M of S there is a set of right coset representatives $g_1, g_2, \ldots, g_r$ of D_M in G (i.e., G is the disjoint union $\bigcup_i D_M g_i$) such that for all i, $f(g_i, g_i^{-1}) \notin M$.*

Proof. Let $A = A_f$. If I is an ideal of A, then $I = \coprod_\sigma (I \cap Sx_\sigma)$. It follows that A is primary if and only if the following condition holds: If $\sigma \in G$ and T is an ideal of S such that $T \not\subset I_\sigma$, then $ATx_\sigma A = A$.

Now suppose A is primary. Let M be a maximal ideal of S and let $\hat{M} = \prod_{N \max, N \neq M} N$. Since $I_1 = mS$, the criterion above gives $A = A\hat{M}x_1 A = A\hat{M}A$. It follows that $S = \sum_{\sigma \in G} x_\sigma \hat{M} x_{\sigma^{-1}} = \sum_\sigma \hat{M}^\sigma f(\sigma, \sigma^{-1})$. Now let $G = \bigcup_{i=1}^r h_i D_M$ be a left coset decomposition. Then

$$S = \sum_i \sum_{d \in D_M} \hat{M}^{h_i} f(h_i d, d^{-1} h_i^{-1}) = \sum_i \hat{M}^{h_i} \left(\sum_d f(h_i d, d^{-1} h_i^{-1}) \right).$$

As i varies from 1 to r, the ideals M^{h_i} range over the r maximal ideals of S. It must then be the case that for all i, $\sum_d f(h_i d, d^{-1} h_i^{-1}) \notin M^{h_i}$. Hence for each i there is an element $d_i \in D_M$ such that $f(h_i d_i, d_i^{-1} h_i^{-1}) \notin M^{h_i}$. Replacing h_i by $\tilde{h}_i = h_i d_i$ we have a set of left coset representatives $\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_r$ of D_M in G such that $f(\tilde{h}_i, \tilde{h}_i^{-1}) \notin M^{\tilde{h}_i}$. Letting $g_i = \tilde{h}_i^{-1}$ we obtain a set of right coset representatives of D_M in G and $f(g_i, g_i^{-1}) = f^{g_i}(g_i^{-1}, g_i) \notin M$.

We proceed to the converse. Suppose $\sigma \in G$ and T is an ideal of S such that $T \not\subseteq I_\sigma$. We need to show $ATx_\sigma A = A$. Since $T \not\subseteq I_\sigma$, there is a maximal ideal M of S such that $f(\sigma, \sigma^{-1}) \notin M$ and $T \not\subseteq M$. Since it does no harm to replace T by a possible smaller ideal of S, we may assume that $T \subseteq \hat{M}$ and $T \not\subseteq M$.

By hypothesis we have a coset decomposition $G = \bigcup_i D_M g_i$ with $f(g_i, g_i^{-1}) \notin M$. Thus $ATx_\sigma A \supseteq \sum_i x_{g_i^{-1}} T x_\sigma x_{\sigma^{-1} g_i} = \sum_i T^{g_i^{-1}} f^{g_i^{-1}}(\sigma, \sigma^{-1} g_i)$ $f(g_i^{-1}, g_i) = \sum_i S_i$ (say). But $f^\sigma(\sigma^{-1}, g_i)\, f(\sigma, \sigma^{-1} g_i) = f(\sigma, \sigma^{-1}) \notin M$, so $f^{g_i^{-1}}(\sigma, \sigma^{-1} g_i) \notin M^{g_i^{-1}}$. Also $f(g_i^{-1}, g_i) = f^{g_i^{-1}}(g_i, g_i^{-1}) \notin M^{g_i^{-1}}$. Hence for each i we have $S_i \not\subseteq M^{g_i^{-1}}$ and $S_i \subseteq \hat{M}^{g_i^{-1}}$. It follows that $\sum_i S_i = S$ and so $ATx_\sigma A = A$. ∎

Let M be a maximal ideal of S. The cocycle f: $G \times G \to S^{\#}$ determines a cocycle f_M: $D_M \times D_M \to S_M^{\#}$ by restriction (and the inclusion of S in the localization S_M). Let $T = S^{D_M}$, the fixed ring of D_M. The centralizer $C_{A_f}(T)$ of T in A_f can be expressed as $\coprod_{d \in D_M} Sx_d$ and is a T-order. The algebra A_{f_M} is the localization of $C_{A_f}(T)$ at the maximal ideal $M \cap T$ of T. Let H_M be the subgroup of D_M associated to f_M, that is, $H_M = \{d \in D_M \mid f_M(d, d^{-1}) \text{ is a unit}\} = \{d \in D_M \mid f(d, d^{-1}) \notin M\}$. We want to compare the orderings on G/H and D_M/H_M. To do this, we introduce an intermediate relation: For $\sigma, \tau \in G$, define $\sigma H \leqslant_M \tau H$ if $f(\sigma, \sigma^{-1}\tau) \notin M$. It is easy to see that this is well defined. The following proposition shows that the relation is transitive and a form of lower subtractivity holds.

PROPOSITION 3.3. (a) *Suppose* $\sigma H \leqslant_M \tau H$ *and* $\tau H \leqslant_M \gamma H$. *Then* $\sigma H \leqslant_M \gamma H$.

(b) *Suppose* $\sigma H \leqslant_M \gamma H$. *We have* $\sigma H \leqslant_M \tau H \leqslant_M \gamma H$ *if and only if* $\sigma^{-1}\tau H \leqslant_{M^{\sigma^{-1}\tau}} \sigma^{-1}\gamma H$.

Proof. Both statements follow easily from the identity $f(\sigma, \sigma^{-1}\tau)$ $f(\tau, \tau^{-1}\gamma) = f^{\sigma}(\sigma^{-1}\tau, \tau^{-1}\gamma)\, f(\sigma, \sigma^{-1}\gamma)$. ∎

It should be noted that this relation is not necessarily a partial ordering: The inequalities $\sigma H \leqslant_M \tau H$ and $\tau H \leqslant_M \gamma H$ do not imply $\sigma H = \tau H$ but only that $f(\sigma^{-1}\tau, \tau^{-1}\sigma) \notin M^{\sigma^{-1}\tau}$. Also it is clear that if σ, $\tau \in D_M$, then $\sigma H \leqslant_M \tau H$ if and only if $\sigma H_M \leqslant \tau H_M$.

Now assume A_f is primary and let $\sigma \in G$. By Theorem 3.2, there is an element $d \in D_M$ such that $f(d^{-1}\sigma, \sigma^{-1}d) \notin M$. From the identity $f(d, d^{-1}\sigma)$ $f(\sigma, \sigma^{-1}d) = f^{d}(d^{-1}\sigma, \sigma^{-1}d)$, we see that $dH \leqslant_M \sigma H$ and $\sigma H \leqslant_M dH$. Suppose r is another element of D_M with $f(r^{-1}\sigma, \sigma^{-1}r) \notin M$. Then $rH \leqslant_M \sigma H$ and $\sigma H \leqslant_M rH$, so $dH \leqslant_M rH$ and $rH \leqslant_M dH$. By the remarks following Proposition 3.3, we conclude that $dH_M = rH_M$. Hence d is uniquely determined by σ up to H_M. Moreover, it is easy to see that if $h \in H$, then $f(d^{-1}\sigma h,\ h^{-1}\sigma^{-1}d) \notin M$. Thus we have a well-defined function ϕ_M: $G/H \to D_M/H_M$ given by $\sigma H \mapsto dH_M$, where $f(d^{-1}\sigma, \sigma^{-1}d) \notin M$.

PROPOSITION 3.4. *Assume* A_f *is primary. Let* M *be a maximal ideal of* S.

(a) *The map* ϕ_M *described above is a* D_M*-set map and is surjective.*

(b) *For all* σ, $\tau \in G$, $\sigma H \leqslant_M \tau H$ *if and only if* $\phi_M(\sigma H) \leqslant \phi_M(\tau H)$. *In particular,* ϕ_M *is a map of partially ordered sets.*

(c) *The canonical map* ϕ: $G/H \to \prod_{M \max} D_M/H_M$ *is injective.*

Proof. (a) Since $\phi_M(dH) = dH_M$ for all $d \in D_M$, the map is surjective. Let $d \in D_M$, $\sigma \in G$. We want to show $\phi_M(d\sigma H) = d\phi_M(\sigma H)$. Let $\phi_M(\sigma H) = rH_M$, $r \in D_M$. Its suffices to show $f((dr)^{-1}\, d\sigma,\ (d\sigma)^{-1}\, dr) \notin M$. But this is clear.

(b) Let $\phi_M(\sigma H) = dH_M$ and $\phi_M(\tau H) = rH_M$, where d, $r \in D_M$. We have seen that $\sigma H \leqslant_M dH \leqslant_M \sigma H$ and $\tau H \leqslant_M rH \leqslant_M \tau H$. It follows that $\sigma H \leqslant_M \tau H$ if and only if $dH \leqslant_M rH$. But by the remarks following Proposition 3.3, this latter inequality is equivalent to $dH_M \leqslant rH_M$.

(c) If $\phi_M(\sigma H) = \phi_M(\tau H)$ for all maximal ideals M of S, then $\sigma H \leqslant_M \tau H$ and $\tau H \leqslant_M \sigma H$ for all M. Hence $f(\sigma, \sigma^{-1}\tau)$ and $f(\tau, \tau^{-1}\sigma)$ are units, so $\sigma H = \tau H$. ∎

Remark. The map ϕ_M: $G/H \to D_M H_M$ is defined independent of any particular choice of coset representatives satisfying the hypotheses of Theorem 3.2. However, for computational purposes, it should be noted that if $G = \bigcup D_M g$ is a coset decomposition with $f(g, g^{-1}) \notin M$ for all representatives g, then for $\sigma \in G$ with $\sigma = dg$, $d \in D_M$ and g a coset representative, we have $\phi_M(\sigma H) = dH_M$ (because $f(d^{-1}\sigma, \sigma^{-1}d) = f(g, g^{-1})$).

We want to obtain information about the relations among the cocycles f_M as M ranges through the maximal ideals of S_1 (in the case where A_f is primary).

The following lemma is very useful.

LEMMA 3.5. *Let $g \in G$ with $f(g, g^{-1}) \notin M$. Then we have:*

(a) *$f(g, x) \notin M$ for all $x \in G$,*

(b) *$f(x, g) \notin M^x$ for all $x \in G$,*

(c) *$f(g^{-1}, x) \notin M^{g^{-1}}$ for all $x \in G$.*

Proof. They are all straightforward. To see the first, use the identity $f^g(g^{-1}, gx)\, f(g, x) = f(g, g^{-1})$. The others are similar. ∎

We now introduce a function which is somewhat more natural than the cocycle, at least with respect to the graph of the cocycle. If $f\colon G \times G \to S^{\#}$ is a cocycle, we define $F\colon G \times G \to S^{\#}$ by $F(\alpha, \beta) = f(\alpha, \alpha^{-1}\beta)$ for $\alpha, \beta \in G$. Of course F is not a cocycle. Note that $F(\alpha, \beta)$ is a unit if and only if $\alpha H \leqslant \beta H$. For us this function is useful mostly because it simplifies notation. If M is a maximal ideal of S, let $v_M\colon K \to \mathbb{Z}$ be the corresponding valuation.

LEMMA 3.6. *Let f, F be as above.*

(a) *If M is a maximal ideal of S and $h_1, h_2 \in H_M$, then $v_M(F(\alpha h_1, \beta h_2)) = v_M(F(\alpha, \beta))$ for all $\alpha, \beta \in D_M$.*

(b) *If $h_1, h_2 \in H$, then $v_M(F(\alpha h_1, \beta h_2)) = v_M(F(\alpha, \beta))$ for all $\alpha, \beta \in G$ and all maximal ideals M of S (i.e., $F(\alpha h_1, \beta h_2)\, F(\alpha, \beta)^{-1}$ is a unit).*

Proof. We have the identities

$$f^{\alpha}(h_1, h_1^{-1}\alpha^{-1}\beta h_2)\, f(\alpha, \alpha^{-1}\beta h_2) = f(\alpha, h_1)\, f(\alpha h_1, h_1^{-1}\alpha^{-1}\beta h_2)$$

and

$$f^{\alpha}(\alpha^{-1}\beta, h_2)\, f(\alpha, \alpha^{-1}\beta h_2) = f(\alpha, \alpha^{-1}\beta)\, f(\beta, h_2).$$

Both parts of the lemma follow from these identities, in conjunction with Lemma 3.5. ∎

Because of this lemma we will abuse notation and write expressions of the form $v_M(F(\alpha H_M, \beta H_M))$ for $\alpha, \beta \in D_M$, meaning $v_M(F(\alpha h_1, \beta h_2))$ for any choice of $h_1, h_2 \in H_M$. We will also let ϕ_M denote both the map $G/H \to D_M/H_M$ and the induced map $G \to D_M/H_M$.

Before stating the next proposition a remark on notation is appropriate. If $g \in G$, then $f(g, g^{-1}) \notin M$ if and only if $f(g^{-1}, g) \notin M^{g^{-1}}$. Hence the existence of a right coset decomposition $G = \bigcup_i D_M g_i$ with $f(g_i, g_i^{-1}) \notin M$ is equivalent to the existence of a left coset decomposition $G = \bigcup_i r_i D_M$ with $f(r_i, r_i^{-1}) \notin M^{r_i}$. It is often more convenient to use the left decomposition

PROPOSITION 3.7. *Let $f\colon G \times G \to S^{\#}$ be a cocycle such that A_f is primary and let M be a maximal ideal of S.*

(1) *For all $\alpha, \beta \in G$, $v_M(F(\alpha, \beta)) = v_M(F(\phi_M(\alpha), \phi_M(\beta)))$.*

(2) *Let $g \in G$ with $f(g, g^{-1}) \notin M^g$. Then:*

(a) *$v_M(F(\alpha, \beta)) = v_M(F(\phi_M(g^{-1}\alpha), \phi_M(g^{-1}\beta)))$ for all $\alpha, \beta \in G$.*

(b) *If $d, r \in D_M$, then $v_{M^g}(F(gdg^{-1}, grg^{-1})) = v_M(F(d, r))$.*

(c) *We have $H_{M^g} = gH_M g^{-1}$ and the map $D_M/H_M \to D_{M^g}/H_{M^g}$ given by conjugation by g is an isomorphism of partially ordered sets.*

(d) *For all $\sigma \in G$, $\phi_{M^g}(\sigma) = g\phi_M(g^{-1}\sigma) g^{-1}$.*

Proof. (1) Let $G = \bigcup_i D_M g_i$ be a coset decomposition with $f(g_i, g_i^{-1}) \notin M$ (which exists by Theorem 3.2). Let $\alpha = dg_i$, $\beta = rg_j$, where $d, r \in D_M$. As was noted in the discussion following Proposition 3.4, $\phi_M(\alpha) = dH_M$ and $\phi_M(\beta) = rH_M$. The result now follows from part (a) of Lemma 3.6.

(2)(a) We have $f^g(g^{-1}\alpha, \alpha^{-1}\beta) f(g, g^{-1}\beta) = f(g, g^{-1}\alpha) f(\alpha, \alpha^{-1}\beta)$. Hence by Lemma 3.5, $v_{M^g}(F(\alpha, \beta)) = v_{M^g}(F^g(g^{-1}\alpha, g^{-1}\beta)) = v_M(F(g^{-1}\alpha, g^{-1}\beta))$, which equals $v_M(F(\phi_M(g^{-1}\alpha), \phi_M(g^{-1}\beta))$ by part (1).

(2)(b) By part (2)(a), $v_{M^g}(F(gdg^{-1}, grg^{-1}) = v_M(F(\phi_M(dg^{-1}), \phi_M(rg^{-1}))) = v_M(F(d, r))$ by part (a) of Proposition 3.4.

(c) This is an easy consequence of part (b).

(d) Let x be an element of D_{M^g} such that $g\phi_M(g^{-1}\sigma) g^{-1} = xH_{M^g}$. Then $v_{M^g}(F(x^{-1}\sigma, 1)) = v_M(F(\phi_M(g^{-1}x^{-1}\sigma), \phi_M(g^{-1}))) = v_M(F(g^{-1}x^{-1}g(\phi_M(g^{-1}\sigma), 1)) = v_M(F(g^{-1}H_{M^g} g, 1)) = v_M(F(H_M, 1)) = 0$, where we have used (2)(a), (2)(b), and part (a) of Proposition 3.4. By the definition of ϕ_{M^g}, we conclude that $\phi_{M^g}(\sigma) = xH_{M^g}$, as desired. ∎

Parts (2a) and (2b) in conjunction with part (d) of Proposition 3.4 give a fairly complete picture of the ordering on G/H in terms of the orderings on D_M/H_M as M ranges through the maximal ideals of S (in the case where A_f is primary). An example will be given in the last section.

We are now heading for Theorem 3.10, which says that if A_f is primary and M is a maximal ideal of S, then there is a one-to-one correspondence between the ideals of A_f and the ideals of A_{f_M}. The proof is based on an argument of Harada [6]. He proved the result in the case where the values of the cocycle are all units, but with a weaker assumption on S (tamely ramified).

Recall that if T is an ideal in the crossed product order A_f, then T is an $S \otimes S$-submodule of A_f and so by Lemma 1.1 we have $T = \coprod_{\sigma \in G}(T \cap Sx_\sigma) = \coprod_\sigma T_\sigma x_\sigma$, where $T_\sigma = \{s \in S \mid sx_\sigma \in T\}$.

LEMMA 3.8. *Let $T = \coprod_\sigma T_\sigma x_\sigma$ be an ideal of A_f.*

(1) *If $\sigma \in G$ and $h \in H$, then $T_{\sigma h} = T_\sigma$ and $T_{h\sigma} = T_\sigma^h$.*

(2) *If M is a maximal ideal of S and $\sigma \in D_M$, $h \in H_M$, then $v_M(T_{h\sigma}) = v_M(T_{\sigma h}) = v_M(T_\sigma)$.*

Proof. (1) If $h \in H$, then x_h is invertible in A_f. We have $T_{\sigma h}x_{\sigma h} \supseteq (T_\sigma x_\sigma)x_h = T_\sigma f(\sigma, h)x_{\sigma h} = T_\sigma x_{\sigma h}$ because $f(\sigma, h)$ is a unit. By the invertibility of x_h, it follows that $T_{\sigma h} = T_\sigma$. That $T_{h\sigma} = T_\sigma^h$ follows by considering $x_h T_\sigma x_\sigma$.

(2) This is proved in the same way as part (1), with the observation that if $h \in D_M$, then $v_M(T_\sigma^h) = v_M(T_\sigma)$. ∎

Because of this lemma, we will abuse notation and write expressions of the form $v_M(T_{\phi_M(\sigma)})$, meaning $v_M(T_d)$ for any choice of d for which $\phi_M(\sigma) = dH_M$.

PROPOSITION 3.9. *Assume A_f is primary. Let $T = \coprod_\sigma T_\sigma x_\sigma$ be an ideal of A_f.*

(1) *For all maximal ideals M of S, $v_M(T_\sigma) = v_M(T_{\phi_M(\sigma)})$.*

(2) *If M is a maximal ideal of S and $g \in G$ with $f(g, g^{-1}) \notin M^g$,*

then:

(a) *$v_{M^g}(T_{g\,dg^{-1}}) = v_M(T_d)$ for all $d \in D_M$.*

(b) *For all $\sigma \in G$, $v_{M^g}(T_\sigma) = v_M(T_{\phi_M(g^{-1}\sigma)})$.*

(c) *For all $\sigma, \tau \in G$, $v_{M^g}(T^\sigma_{\sigma^{-1}\tau}) = v_M(T_{\phi_M(g^{-1}\sigma)^{-1}\phi_M(g^{-1}\tau)})$.*

(*Note that this last expression makes sense by part* (2) *of Lemma* 3.8.)

Proof. (1) Let $\sigma = dg$, where $d \in D_M$ and $f(g, g^{-1}) \notin M$ (which is possible because A_f is primary). By the remark following Proposition 3.4, $\phi_M(\sigma) = dH_M$. Hence $v_M(T_\sigma) = v_M(T_{dg}) = v_M(T_d)$ by the preceding lemma.

(2a) Clearly $T_{g\,dg^{-1}}x_{g\,dg^{-1}} \supseteq x_g(T_d x_d)x_{g^{-1}} = T_d^g f(g, d) f(gd, g^{-1})x_{g\,dg^{-1}}$. By Lemma 3.5, $f(g, d) f(gd, g^{-1}) \notin M^g$. Thus $v_{M^g}(T_{g\,dg^{-1}}) \leqslant v_{M^g}(T_d^g) = v_M(T_d)$. The other direction is similar.

(2b) By part 2(d) of Proposition 3.7, $\phi_{M^g}(\sigma) = g\phi_M(g^{-1}\sigma)g^{-1}$. The result now follows from part (a).

(2c) Let $\sigma^{-1}g=\gamma d$, where $d\in D_M$ and $f(\gamma,\gamma^{-1})\notin M^{\gamma}$. Then $v_{M^g}(T^{\sigma}_{\sigma^{-1}\tau}) = v_{M^{\sigma^{-1}g}}(T_{\sigma^{-1}\tau}) = v_{M^{\gamma}}(T_{\sigma^{-1}\tau}) = v_M(T_{\phi_M(\gamma^{-1}\sigma^{-1}\tau)}) = v_M(T_{\phi_M(dg^{-1}\tau)})=v_M(T_{d\phi_M(g^{-1}\tau)})$. But $g^{-1}\sigma=d^{-1}\gamma^{-1}$ and $f(\gamma^{-1},\gamma)\notin M$, so $d^{-1}H_M=\phi_M(g^{-1}\sigma)$, as desired. ∎

If I is an ideal of S and M is a maximal ideal of S, let I_M denote the localization of I at M.

THEOREM 3.10. *If A_f is primary and M is a maximal ideal of S, then the map $T=\coprod_\sigma T_\sigma x_\sigma \mapsto \coprod_{d\in D_M}(T_d)_M x_d$ is a one-to-one, product-preserving correspondence between the ideals of A_f and the ideals of A_{f_M}.*

Proof. First note that if T is an ideal of A_f, then $\coprod_{d\in D_M}(T_d)_M x_d$ is just the localization at $M\cap S^{D_M}$ of $T\cap C_{A_f}(S^{D_M})$ and so is an ideal of A_{f_M}. (See the discussion following Proposition 3.2.)

Let $G=\bigcup gD_M$ be a coset decomposition with $f(g,g^{-1})\notin M^g$. We first show that the map from ideals of A_f to ideals of A_{f_M} is one-to-one. Suppose $T=\coprod_\sigma T_\sigma x_\sigma$ and $U=\coprod_\sigma U_\sigma x_\sigma$ are ideals of A_f such that $v_M(T_d)=v_M(U_d)$ for all $d\in D_M$. We need to show $T=U$. It suffices to show $v_{M^g}(T_\sigma)=v_{M^g}(U_\sigma)$ for all $\sigma\in G$ and all of our special coset representatives g. But by Proposition 3.9, $v_{M^g}(T_\sigma)=v_M(T_{\phi_M(g^{-1}\sigma)})=v_M(U_{\phi_M(g^{-1}\sigma)})=v_{M^g}(U_\sigma)$, where as usual we are using Lemma 3.6 to abuse notation.

To see that the map is surjective, let $\coprod_{d\in D_M} T_d x_d$ be an ideal of A_{f_M}. For each $\sigma\in G$, let U_σ be the ideal of S determined by the conditions $v_{M_g}(U_\sigma)=v_M(T_{\phi_M(g^{-1}\sigma)})$ for all coset representatives g. We *claim* that $U=\coprod_{\sigma\in G}U_\sigma x_\sigma$ is an ideal (if so, then $U\mapsto T_d x_d$ is clear). We need to show $Ux_\tau\subseteq U$ and $x_\tau U\subseteq U$ for all $\tau\in G$. This reduces to showing $U_\sigma f(\sigma,\sigma^{-1}\tau)\subseteq U_\tau$ and $U^{\sigma}_{\sigma^{-1}\tau}f(\sigma,\sigma^{-1}\tau)\subseteq U_\tau$ for all σ,τ. Let $F\colon G\times G\to S^{\#}$ be as usual.

To prove the first of these inclusions note that for every coset representative g,

$$\begin{aligned} v_{M^g}(U_\sigma f(\sigma,\sigma^{-1}\tau)) &= v_{M^g}(U_\sigma)+v_{M^g}(F(\sigma,\tau))\\ &= v_M(T_{\phi_M(g^{-1}\sigma)})+v_M(F(\phi_M(g^{-1}\sigma),\phi_M(g^{-1}\tau)))\\ &\qquad\qquad\text{by Proposition 3.9,}\\ &= v_M(T_{\phi_M(g^{-1}\sigma)}F(\phi_M(g^{-1}\sigma),\phi_M(g^{-1}\tau)))\\ &\geqslant v_M(T_{\phi_M(g^{-1}\tau)})=v_{M^g}(U_\tau), \end{aligned}$$

where the inequality follows from the fact that $\coprod T_d x_d$ is an ideal of A_{f_M}. Hence $U_\sigma F(\sigma,\tau)\subseteq U_\tau$.

To prove $U^{\sigma}_{\sigma^{-1}\tau}f(\sigma,\sigma^{-1}\tau)\subseteq U_\tau$ we first *claim* that $v_{M^g}(U^{\sigma}_{\sigma^{-1}\tau})=v_M(T_{\phi_M(g^{-1}\sigma)^{-1}\phi_M(g^{-1}\tau)})$. (This is the same argument as that given for part 2(c) of Proposition 3.9, except we do not know yet that U is an ideal.)

Let $\sigma^{-1}g=\gamma d$, where $d\in D_M$ and γ is one of our coset representatives. Then $v_{M^g}(U_{\sigma^{-1}\tau}) = v_{M^{\sigma^{-1}g}}(U_{\sigma^{-1}\tau}) = v_{M^\gamma}(U_{\sigma^{-1}\tau}) = v_M(T_{\phi_M(\gamma^{-1}\sigma^{-1}\tau)}) = v_M(T_{d\phi_M(g^{-1}\tau)})=v_M(T_{\phi_M(g^{-1}\sigma)^{-1}\phi_M(g^{-1}\tau)})$. This proves the claim. To prove the second inclusion we observe that by the claim and Proposition 3.9, $v_{M^g}(U_{\sigma^{-1}\tau}F(\sigma,\tau)) = v_M(T_{\phi_M(g^{-1}\sigma)^{-1}\phi_M(g^{-1}\tau)}F(\phi_M(g^{-1}\sigma), \phi_M(g^{-1}\tau))) \geqslant v_M(T_{\phi_M(g^{-1}\tau)})$ because $\coprod T_d x_d$ is an ideal of A_{f_M}.

Finally we need to show the correspondence preserves products. Let $U=\coprod_\sigma U_\sigma x_\sigma$ and $T=\coprod_\sigma T_\sigma x_\sigma$ be ideals of A_f. We want

$$\coprod_{d\in D_M} ((UT)_d)_M x_d=\left(\coprod_d (U_d)_M x_d\right)\left(\coprod_d (T_d)_M x_d\right).$$

Then inclusion "$\supseteq$" is clear. It suffices then to show that if $\sigma\in G$ and $d\in D_M$, then $(U_\sigma T^\sigma_{\sigma^{-1}d}F(\sigma,d))_M x_d\subseteq$ r.h.s. But $v_M(U_\sigma T^\sigma_{\sigma^{-1}d}F(\sigma,d))= v_M(U_{\phi_M(\sigma)}T_{\phi_M(\sigma)^{-1}\phi_M(d)}F(\phi_M(\sigma),\phi_M(d)))$ by Propositions 3.9 and 3.7. It follows that $(U_\sigma T^\sigma_{\sigma^{-1}d}F(\sigma,d))_M x_d = (U_{\phi_M(\sigma)})_M\ x_{\phi_M(\sigma)}(T_{\phi_M(\sigma)^{-1}\phi_M(d)})_M\ x_{\phi_M(\sigma)^{-1}\phi_M(d)}$, so we are done. ∎

COROLLARY 3.11. *Assume A_f is primary and M is a maximal ideal of S. The cross-product order A_f is maximal if and only if A_{f_M} is maximal.*

Proof. If C is a primary order over a discrete valuation T, then C is maximal if and only if some power of $\operatorname{rad}(C)$ is equal to mC where m is the maximal ideal of T. In our case, under the one-to-one correspondence of the theorem, rad (A_f) corresponds to $\operatorname{rad}(A_{f_M})$ and mA_f corresponds to $mA_{f_M}=(M\cap S^{D_M})\,A_{f_M}$. Since the correspondence preserves products, the result follows. ∎

The theorem also allows one to determine the ideals of a primary order A_f. In Section 2 we discussed a method for determining the ideals of A_{f_M}, where M is a maximal ideal of S. The one-to-one correspondence of Theorem 3.10, together with part (2b) of Proposition 3.9, allow that determination for A_f. We will give an example in last section.

We now obtain another result concerning the relationship between A_f and A_{f_M} for A_f primary.

THEOREM 3.12. *If A_f is primary and M is a maximal ideal of S, then $A_f/\operatorname{rad}(A_f)$ is isomorphic as a k-algebra to $M_r(A_{f_M}/\operatorname{rad}(A_{f_M}))$ (the ring of $r\times r$ matrices over $A_{f_M}/\operatorname{rad}(A_{f_M})$), where $r=[G:D_M]$.*

Proof. Let $\tilde{A}_f$ and $\tilde{A}_{f_M}$ denote the residue class algebras. It suffices to display a set of matrix units e_{ij} in $\tilde{A}_f$, $1\leqslant i, j\leqslant r$ such that $e_{11}\tilde{A}_f e_{11}\cong\tilde{A}_{f_M}$. Let $S/mS=K_1\oplus\cdots\oplus K_r$, where $K_i\cong S/M_i$, and $M=M_1, M_2,\ldots, M_r$ are the maximal ideals of S. Let e_{ii} be the minimal idempotent of S generating K_i. Let $G=\bigcup_{i=1}^r g_i D_M$ be a coset decomposition with $f(g_i, g_i^{-1})\notin M^{g_i}$ and let $A_f=\coprod Sx_\sigma$. Since $f^{g_i}(g_j^{-1}, g_j)=f(g_i, g_j^{-1})\,f(g_i g_j^{-1}, g_j)$ and

$f(g_j^{-1}, g_j) \notin M$, we see that $f(g_i g_j^{-1}, g_j) \notin M^{g_i}$ and so that $e_{ii} f(g_i g_j^{-1}, g_j) \neq 0$. Let $c_{ij} \in K_i$ be the inverse of $e_{ii} f(g_i g_j^{-1}, g_j)$, that is, $e_{ii} f(g_i g_j^{-1}, g_j) c_{ij} = e_{ii}$. Let $e_{ij} = c_{ij} \tilde{x}_{g_i g_j^{-1}}$, $1 \leq i, j \leq r$. We *claim* that the e_{ij} form a set of matrix units. That $\sum_i e_{ii} = 1$ is clear. We need to verify that $e_{ij} e_{kl} = \delta_{jk} e_{il}$. First note that because $G = \bigcup g_i D_M$, we have $g_i(K_1) = K_i$ for all i. Hence $g_i g_j^{-1}(K_j) = K_i$. If $j \neq k$, then $e_{ij} e_{kl} = c_{ij} \tilde{x}_{g_i g_j^{-1}} c_{kl} \tilde{x}_{g_k g_l^{-1}} = c_{ij} g_i g_l^{-1}(c_{kl}) \tilde{x}_{g_i g_j^{-1}} \tilde{x}_{g_k g_l^{-1}} = 0$ because $g_i g_j^{-1}(c_{kl}) \notin K_i$. Moreover $e_{ij} e_{jl} = c_{ij} g_i g_j^{-1}(c_{jl}) f(g_i g_j^{-1}, g_j g_l^{-1}) \tilde{x}_{g_i g_l^{-1}} = c_{il} \tilde{x}_{g_i g_l^{-1}}$, where to get the last equality we use the identity

$$
\begin{aligned}
&f^{g_i g_j^{-1}}(g_j g_l^{-1}, g_l) f(g_i g_j^{-1}, g_j) \\
&\quad = f(g_i g_j^{-1}, g_j g_l^{-1}) f(g_i g_j^{-1}, g_j g_l^{-1}) f(g_i g_l^{-1}, g_l).
\end{aligned}
$$

Finally, we need to compute $e_{11} \tilde{A}_f e_{11}$. Recall that $\mathrm{rad}(A_f) = \coprod_{\sigma \in G} I_\sigma x_\sigma$, where $I_\sigma = (mS; f(\sigma, \sigma^{-1}))$. Hence $e_{11} \tilde{A}_f e_{11} = \coprod_\sigma e_{11}(S/I_\sigma) \sigma(e_{11}) \tilde{x}_\sigma$. But $e_{11}(S/I_\sigma) \sigma(e_{11}) \neq 0$ if and only if $\sigma(e_{11}) = e_{11}$ and $e_{11} \notin I_\sigma$, that is, if and only if $\sigma \in \{\tau \in G \mid \tau \in D_M$ and $f(\tau, \tau^{-1}) \notin M\} = H_M$. Hence $e_{11} \tilde{A}_f e_{11} = \coprod_{d \in H_M} K_1 \tilde{x}_d = \tilde{A}_{f_M}$. ∎

Remark. Let A_f be maximal. An argument similar to that given for Theorem 3.12 shows that if $\hat{R}$ is the completion of R, then $A_f \otimes_R \hat{R} \cong M_r(A_{\hat{f}_M})$, where $\hat{f}_M$: $D_M \times D_M \to \hat{S}_M$ is the obvious "completion" of f_M (and has the same associated subgroup and graph). In fact, the formulas of the theorem again determine a complete set of matrix units (recall that $\hat{K} = K \otimes_F \hat{F}$ is isomorphic to $\coprod_i \hat{K}_i$ where $\hat{K}_i$ is the completion of K at M_i). In particular, the division algebra part of $\Sigma_f \otimes \hat{F}$ is the same as the division algebra part of $\Sigma_{\hat{f}_M}$ and hence that division algebra part has ramification index $[D_M : H_M]$ (see the remarks following Corollary 2.8). By Corollary 37.32 of [7], we conclude that the outer automorphism group of A_f has order $[D_M : H_M]$. We will use this observation later.

For any cocycle f: $G \times G \to S^{\#}$ it is easy to see that the center of $\tilde{A}_f = A_f/\mathrm{rad}(A_f)$ is $L = \{\bar{s} \in \bar{S} \mid \sigma(s) - s \in I_\sigma$ for all $\sigma \in G\}$. If A_f is primary, then L is a field and by the preceding proposition $L_i = LK_i$ is the center of $\tilde{A}_{f_{M_i}}$, $i = 1, 2, \ldots, r$. Hence $L_i = K_i^{H_{M_i}}$ and $[\tilde{A}_f : L] = [G : D_{M_i}]^2 [\tilde{A}_{f_{M_i}} : L_i] = [G : D_{M_i}]^2 [K_i : L_i]^2 = [G : H_{M_i}]^2$. We will use this computation in a subsequent result.

We are now heading for a determination of conditions on two cocycles f_1 and f_2 equivalent to A_{f_1} and A_{f_2} being R-algebra isomorphic, in the case where A_{f_1} and A_{f_2} are maximal. To begin suppose M is a maximal ideal of S and f: $D_M \times D_M \to S_M^{\#}$ is a cocycle. Let F: $D_M \times D_M \to S_M^{\#}$ be given by $F(\sigma, \tau) = f(\sigma, \sigma^{-1}\tau)$ for all $\sigma, \tau \in D_M$. Let $M = M_1, M_2, \ldots, M_r$ be the full set of maximal ideals of S and choose $\pi_i \in S$ such that $M_i = (\pi_i)$, $i = 1, 2, \ldots, r$. As in Section 1, let P be the submonoid of $S^{\#}$ generated by $\pi_1, \pi_2, \ldots, \pi_r$; that is, $P = \{\pi_1^{k_1} \cdots \pi_r^{k_r} \mid 0 \leq k_i$ for all $i\}$. We want to lift f to G. Let

$G=\bigcup_i g_i D_M=\bigcup_i D_M g_i^{-1}$ be a coset decomposition. Let $\phi: G \to D_M$ be given by $\phi(dg_i^{-1})=d$ if $d \in D_M$ and g_i^{-1} is one of the right coset representatives. Define $\tilde{F}$: $G \times G \to P \subseteq S^{\#}$ by the rules $v_{M^{g_i}}(\tilde{F}(\sigma, \tau))= v_M(F(\phi(g_i^{-1}\sigma), \phi(g_i^{-1}\tau))$ for all i. Let $\tilde{f}$: $G \times G \to P \subseteq S^{\#}$ be given by $\tilde{f}(\sigma, \tau)=\tilde{F}(\sigma, \sigma\tau)$.

PROPOSITION 3.13. *The function $\tilde{f}$ described above is a cocycle and $A_{\tilde{f}}$ is primary.*

Proof. We need to show that for all σ, τ, $\gamma \in G$ we have $\tilde{f}(\sigma, \sigma^{-1}\tau)\, \tilde{f}(\tau, \tau^{-1}\gamma)=\tilde{f}^{\sigma}(\sigma^{-1}\tau, \tau^{-1}\gamma)\, \tilde{f}(\sigma, \sigma^{-1}\gamma)$, that is, $\tilde{F}(\sigma, \tau)\, \tilde{F}(\tau, \gamma)= \tilde{F}^{\sigma}(\sigma^{-1}\tau, \sigma^{-1}\gamma)\, \tilde{F}(\sigma, \gamma)$. Since $\tilde{F}(G \times G) \subseteq P$, it suffices to show that the $v_{M^{g_i}}$ valuation of both sides is the same for all i. Let $g=g_i$. Then $v_{M^g}(\tilde{F}(\sigma, \tau))+ v_{M^g}(\tilde{F}(\tau, \gamma)) = v_M(F(\phi(g^{-1}\sigma), \phi(g^{-1}\tau))+v_M(F(\phi(g^{-1}\tau), \phi(g^{-1}\gamma)) = v_M(F^{\phi(g^{-1}\sigma)}(\phi(g^{-1}\sigma)^{-1}\ \phi(g^{-1}\tau), \phi(g^{-1}\sigma)^{-1}\ \phi(g^{-1}\gamma)))+v_M(F(\phi(g^{-1}\sigma), \phi(g^{-1}\gamma)))$ because F comes from the cocycle f. To compute the right-hand side, let $\sigma^{-1}g=hd$, where h is one of the coset representatives and $d \in D_M$. Note that $d=\phi(\sigma^{-1}g)^{-1}$. Then $v_{M^g}(\tilde{F}^{\sigma}(\sigma^{-1}\tau, \sigma^{-1}\gamma)) = v_{M^{\sigma^{-1}g}}(\tilde{F}(\sigma^{-1}\tau, \sigma^{-1}\gamma)) = v_{M^h}(\tilde{F}(\sigma^{-1}\tau, \sigma^{-1}\gamma)) = v_M(F(\phi(h^{-1}\sigma^{-1}\tau), \phi(h^{-1}\sigma^{-1}\gamma))) = v_M(F(\phi(dg^{-1}\tau), \phi(dg^{-1}\gamma))) = v_M(F(d\phi(g^{-1}\tau), d\phi(g^{-1}\gamma))) = v_M(F(\phi(g^{-1}\gamma)^{-1}\ \phi(g^{-1}\tau), \phi(g^{-1}\sigma)^{-1}\ \phi(g^{-1}\gamma)))$, as desired.

To show $A_{\tilde{f}}$ is primary it suffices, by Theorem 3.2, to find for each left coset representative g_i a full set of right coset representative h_{ij}, $1 \leqslant j \leqslant r$, such that $\tilde{f}(h_{ij}, h_{ij}^{-1}) \notin M^{g_i}$ for all j. Given i, let $h_{ij}=g_i g_j^{-1}$, $1 \leqslant j \leqslant r$. Then $G=\bigcup_i D_M g_j^{-1} = \bigcup_j g_i D_M g_i^{-1} g_j^{-1} = \bigcup_j D_{M^{g_i}} g_i g_j^{-1} g_i^{-1} = \bigcup_j D_{M^{g_i}} g_i g_j^{-1} = \bigcup_j D_{M^{g_i}} h_{ij}$. Moreover, $v_{M^{g_i}}(\tilde{f}(h_{ij}, h_{ij}^{-1})) = v_{M^{g_i}}(\tilde{F}(h_{ij}, 1)) = v_M(F(\phi(g_j^{-1}), \phi(g_i^{-1})))=v_M(F(1, 1))=0$. ∎

If f: $G \times G \to P \subseteq S^{\#}$ is a cocycle, M a maximal ideal of S and $G=\bigcup gD_M$ a coset decomposition, we can form a new cocycle $\tilde{f}$: $G \times G \to P$ by lifting f_M: $D_M \times D_M \to S_M^{\#}$ as described above. For an arbitrary cocycle f: $G \times G \to S^{\#}$ we can decompose $f=f_u f_p$ as described in Section 1, and given a coset decomposition $G=\bigcup gD_M$ we can form $\tilde{f}=f_u \tilde{f}_p$. We will call such a cocycle a *twist* of f.

PROPOSITION 3.14. *If f: $G \times G \to S^{\#}$ is a cocycle such that A_f is maximal and $\tilde{f}$ is a twist of f, then $A_{\tilde{f}} \cong A_f$ as* R-algebras.

Proof. First note that if d, $r \in D_M$, then $v_M(\tilde{F}(d,r))=v_M(F(d, r))$. In particular, f_M and $\tilde{f}_M$ determine the same graph. By the preceding proposition $A_{\tilde{f}}$ is primary, so we infer from Corollary 3.3 that $A_{\tilde{f}}$ is maximal. Since maximal orders in a fixed central simple algebra are isomorphic, it suffices to show that $f \sim \tilde{f}$ over K. Decompose $f=f_u f_p$ and $\tilde{f}=f_u \tilde{f}_p$. It suffices to show $f_p \sim \tilde{f}_p$ over K. We apply the exact sequence of Auslander and Brumer $0 \to B(S/R) \to B(K/F) \to \chi(D_M) \to 0$. Since f_p and $\tilde{f}_p$

determine the same character of D_M, it follows that $f_p \sim u\bar{f}_p$, where u: $G \times G \to U(S)$ is an invertible cocycle. Hence $f_p = (\partial a)\, u\bar{f}_1$, where ∂a is the coboundary of some cochain a: $G \to K^x$. If a is decomposed in the obvious way as the product $a = a_u a_p$, then because $f_p(G \times G) \subseteq P$ and $\bar{f}_p(G \times G) \subseteq P$, it follows that $f_p = (\partial a_p) \bar{f}_p$. ▮

Let f and $\bar{f}$ be as in Proposition 3.14. Since A_f and $A_{\bar{f}}$ are maximal, there are coset decompositions $G = \bigcup_i g_i D_M = \bigcup_i h_i D_M$ (where $g_i D_M = h_i D_M$ for all i) such that $f(g_i, g_i^{-1}) \notin M^{g_i}$ and $\bar{f}(h_i, h_i^{-1}) \notin M^{h_i}$. Suppose $f \sim \bar{f}$ over S. Then $f(h_i, h_i^{-1}) \notin M^{h_i} = M^{g_i}$. Since $f^{g_i^{-1}}(h_i, h_i^{-1} g_i)\ f(g_i^{-1}, g_i) = f(g_i^{-1}, h_i)\ f(g_i^{-1} h_i, h_i^{-1} g_i)$, it follows from Lemma 3.5 that $f(g_i^{-1} h_i, h_i^{-1} g_i) \notin M$. But $g_i^{-1} h_i \in D_M$, so $g_i^{-1} h_i \in H_M$ and $g_i H_M = h_i H_M$. Since f and $\bar{f}$ are cohomologous over S if and only if there is an R-algebra isomorphism ψ: $A_f \to A_{\bar{f}}$ such that $\psi(s) = s$ for all $s \in S$, such an isomorphism exists only if $g_i H_M = h_i H_M$ for all i.

Keeping the analysis above in mind, begin again and let f: $G \times G \to S^{\#}$ be a cocycle with A_f maximal. For each i, $i = 1, 2, \ldots, r$, let $\{d_{ij}\}$ be a set of left coset representatives of H_{M_i} in D_{M_i}. Let $G = \bigcup g_i D_M$ be a coset decomposition with $f(g_i, g_i^{-1}) \notin M^{g_i}$. Using the d_{ij} and this decomposition of G we obtain, in the obvious way, a total of $n = [D_M : H_M]^{[G:D_M]-1}$ different sets of coset representatives of D_M in G. Let $f = f_1, f_2, \ldots, f_n$ denote the twists determined by f and these sets of coset representatives. Proposition 3.14 and the analysis above show that $A_{f_i} \cong A_{f_j}$ as R-algebras for all i, j but that if f_i and f_j are cohomologous over S, then $i = j$. For each i let ψ_i: $A_{f_i} \to A_f$ be a fixed R-algebra isomorphism, $\psi_1 = id$. Let $A = A_f$. Each ψ_i endows A with the structure of a left $A \otimes_R S$-module via $(a \otimes s)x = ax\psi_i(s)$ for a, $x \in A$, $s \in S$. Let A_i denote A equipped with this module structure. As in the proof of Proposition 2.6, we see that each A_i is a projective, cyclic $A \otimes S$-module. We *claim* that if $A_i \cong A_j$ as $A \otimes S$-modules, then $i = j$. In fact, if ψ: $A_i \to A_j$ is an $A \otimes S$-module isomorphism, then the standard argument shows that $\psi(1)$ is invertible in A_j $(= A)$ and $\psi_j(s) = \psi(1)^{-1}\, \psi_i(s)\, \psi(1)$ for all $s \in S$. Let γ: $A \to A$ be the inner automorphism $a \mapsto \psi(1)^{-1}\, a\psi(1)$. The composite map $\psi_j^{-1} \gamma \psi_i$ from A_{f_i} to A_{f_j} is an R-algebra isomorphism and is the identity on S. Hence f_i and f_j are cohomologous over S, so $i = j$. In this way we have produced $n = [D_M : H_M]^{[G:D_M]-1}$ different $A \otimes S$-module structures on A. We can go further. For each i we have seen that A_{f_i} is maximal. By the remarks following Theorem 3.12, the order of the outer automorphism group of A_{f_i} is $t = [D_M : H_M]$. Let ϕ_{ij}, $1 \leqslant j \leqslant t$, be a full set of representatives of the inner automorphism group in the full automorphism group. For each i let ψ_i: $A_{f_i} \to A_f$ be a fixed R-algebra isomorphism. Let $\psi_{ij} = \psi_i \circ \phi_{ij}$ for $1 \leqslant j \leqslant t$. As usual each ψ_{ij} puts an $A \otimes_R S$-module structure on A. Let A_{ij} denote A equipped with this structure. We *claim* that if $A_{ij} \cong A_{iq}$ as $A \otimes S$-modules, $1 \leqslant j$, $q \leqslant t$, then $j = q$. To see this, suppose ψ: $A_{ij} \to A_{iq}$ is an $A \otimes S$-module isomorphism. Then as we saw above, it follows that $\psi(1)$ is invertible in A and $\psi(1)^{-1}\psi_{ij}(s)\psi(1) = \psi_{iq}(s)$ for all $s \in S$. Hence if γ denotes conjugation by $\psi(1)$, then $\psi_{iq}^{-1}\gamma\psi_{ij}$: $A_{f_i} \to A_{f_i}$ is an R-algebra automorphism which is the identity on S. But we have seen (Proposition 1.6) that such an automorphism is inner, given by conjugation by a unit of S. It then follows that $\psi_{iq}^{-1}\psi_{ij}$ is inner, so $j = q$.

By the argument above, we have produced $[D_M : H_M]$ module structures on A for each twist f_i. Since there are $[D_M : H_M]^{[G : D_M] - 1}$ such twists, we have accounted for $[D_M : H_M]^{[G : D_M]}$ different $A \otimes S$-module structures on A, namely the A_{ij}, $1 \leqslant i \leqslant [D_M : H_M]^{[G : D_M] - 1}$ and $1 \leqslant j \leqslant [D_M : H_M]$.

The next proposition shows that these are all the module structures of a certain type. Let $\widetilde{A \otimes S}$ denote the quotient of $A \otimes S$ by its radical. It is easy to see that $\widetilde{A \otimes S} \cong \tilde{A} \otimes_k \bar{S}$, where $\tilde{A} \cong A/\mathrm{rad}(A)$. Let $L \subseteq \bar{S}$ denote the center of $\widetilde{A \otimes S}$ and let $\tilde{A}_{ij}$ be the quotient module of A_{ij}. Let $n = [D_M : H_M]^{[G : D_M] - 1}$.

Proposition 3.15. *If N is a left $A \otimes S$ module that is faithful over $1 \otimes \bar{S}$ and has L-dimension equal to $[G : H_M]^2$, then $N \cong \tilde{A}_{ij}$ for some i, j, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant [D_M : H_M]$.*

Proof. Let $M = M_1, M_2, \ldots, M_r$ be the maximal ideals of S and let $\bar{S} = \coprod_i K_i$ as usual. Let $L_i = LK_i$, a subfield of K_i isomorphic to L. Then $\widetilde{A \otimes S} \cong \tilde{A} \otimes_k \bar{S} \cong \coprod_i \tilde{A} \otimes_k K_i$. Let $N_i = (\tilde{A} \otimes K_i)N$. Then $N = \coprod_i N_i$ is a direct sum decomposition into $\widetilde{A \otimes S}$ submodules. Since N is faithful over $\bar{S}$ (identified with $1 \otimes \bar{S} \subseteq \widetilde{A \otimes S}$), each N_i is nonzero. For each i, $\tilde{A} \otimes K_i \cong \coprod_j \tilde{A} \otimes_{L_i \smile \sigma_{ij}} K_i$, where $\{\sigma_{ij} \mid 1 \leqslant j \leqslant [L_i : k]\}$ is the set of distinct embeddings of L_i into K_i. By Theorem 3.2, $\tilde{A} \cong M_r(\tilde{A}_{f_{M_i}})$ for all i. In particular, $\tilde{A}$ is split by K_i for all i. Hence an irreducible $\tilde{A} \otimes_{L_i \smile \sigma_{ij}} K_i$ module has L_i-dimension equal to $[\tilde{A} : L]^{1/2}[K_i : L_i]$. Using the computation following Theorem 3.2, this dimension then equals $[G : H_{M_i}][K_i : L_i] = [G : H_{M_i}][D_{M_i} : H_{M_i}] = [G : H_M][D_M : H_M]$ since these numbers are the same for all i. Hence $[N_i : L_i] \geqslant [G : H_M][D_M : H_M]$, so $[N : L] = \sum_i [N_i : L_i] \geqslant [G : D_M][G : H_M][D_M : H_M] = [G : H_M]^2$. But by assumption $[N : L] = [G : H_M]^2$. It follows that each N_i is an irreducible $\tilde{A} \otimes K_i$-module. Hence there are exactly $[L : F]^{[G : D_M]} = [D_M : H_M]^{[G : D_M]}$ possibilities for N. But each of the modules $\tilde{A}_{ij}$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant [G : D_M]$ satisfies the hypotheses of the proposition too. Moreover since each A_{ij} is projective, if $\tilde{A}_{ij} \cong \tilde{A}_{pq}$, then $A_{ij} \cong A_{pq}$ and so as we have seen $i = p$ and $j = q$. Hence $N \cong \tilde{A}_{ij}$ for some i, j. ∎

Theorem 3.16. *Let t: $G \times G \to S^{\#}$ be a cocycle with A_t maximal. Then $A_t \cong A_f$ as R-algebras if and only if $t \sim f_i$* (over S) *for some i, $1 \leqslant i \leqslant n$.*

Proof. By Proposition 3.14, $A_f \cong A_{f_i}$ for all i. Suppose then that ϕ: $A_t \to A_f$ is an R-algebra isomorphism. As before we endow A_f with a left $A_f \otimes S$ module structure via $(a \otimes s)x = ax\phi(s)$ for a, $x \in A_f$, $s \in S$. If A_ϕ denotes A_f with this new module structure, then $\tilde{A}_\phi$ satisfies the hypotheses of the proposition. Hence $\tilde{A}_\phi \cong \tilde{A}_{ij}$ for some i, j and since A_ϕ, A_i are projective, we get $A_\phi \cong A_{ij}$. Now the argument preceding the proposition shows that $t \sim f_i$ over S and we are done. ▌

4. Examples

In this section we present some examples of the phenomena we have been discussing. The following lemma is useful for narrowing the possibilities for graphs of cocycles. The notation is as usual.

LEMMA 4.1. *Assume G is abelian and S is a* DVR. *Let f: $G \times G \to S^\#$ be a cocycle. Then $v(f(\sigma, \tau)) = v(f(\tau, \sigma))$ for all $\sigma, \tau \in G$. In particular, if H is the subgroup of G associated to f, then $\sigma H \leqslant \sigma\tau H$ if and only if $\tau H \leqslant \sigma\tau H$.*

Proof. It is easy to see that the second statement follows from the first. Since f is a cocycle, there is a positive integer n such that $f^n \sim 1$ over K; that is, there is a one-cochain α: $G \to K^x$ such that $f^n(\sigma, \tau) = \alpha(\sigma)\,\alpha^\sigma(\tau)/\alpha(\sigma\tau)$ for all $\sigma, \tau \in G$. But then $v(f^n(\sigma, \tau)) = v(\alpha(\sigma)) + v(\alpha^\sigma(\tau)) - v(\alpha(\sigma\tau)) = v(\alpha(\sigma)) + v(\alpha(\tau)) - v(\alpha(\tau\sigma)) = v(\alpha^\tau(\sigma)) + v(\alpha(\tau)) - v(\alpha(\tau\sigma)) = v(f^n(\tau, \sigma))$. Hence $v(f(\sigma, \tau)) = v(f(\tau, \sigma))$. ▌

We now proceed to the examples:

EXAMPLE 4.2. Let $G = \langle\sigma\rangle$, the cyclic group of order four and assume S is a DVR (e.g., $R = \mathbb{C}[[x]]$ and $S = \mathbb{C}[[y]]$, where $y^4 = x - 1$). It is not difficult to write down all the graphs (i.e., partial orderings) on coset spaces G/H satisfying

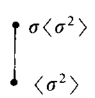
(1) H is the unique minimal element,

(2) The partial ordering is lower subtractive, and

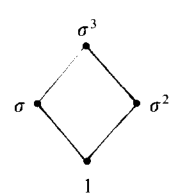
(3) If σ, $\tau \in G$, then $\sigma H \leqslant \sigma\tau H$ if and only if $\tau H \leqslant \sigma\tau H$. They are as follows:

(A_1)

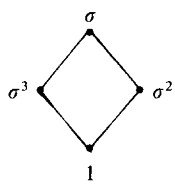
(A_2)



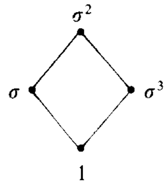
(A_3)



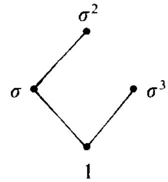
(A_4)



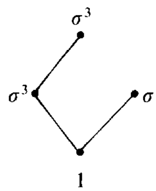
(A_5)



(A_6)

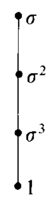


(A_7)

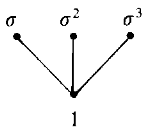


(A_8)

(A_9)



(A_{10})



To see that these graphs actually arise as the graphs of cocycles, proceed as follows. First note that it suffices to show A_1, A_2, A_5, A_8, and A_9 arise because the others can be obtained as products of these. But A_1, A_2, A_8, and A_9 arise by the remarks subsequent to Theorem 2.3. To see that A_5 arises we will find an appropriate crossed product order as a subalgebra of $M_4(R) = A_f$, where f: $G \times G \to S^{\#}$ is the identity cocycle. Let $A_f = \coprod_{i=0} Sx_{\sigma^i}$ as usual and let $y_\sigma = \pi x_\sigma$, $y_{\sigma^2} = \pi^2 x_{\sigma^2}$, $y_{\sigma^3} = \pi x_{\sigma^3}$. One easily checks that $S \oplus Sy_\sigma \oplus Sy_{\sigma^2} \oplus Sy_{\sigma^3}$ is a subalgebra of A_f and the graph of the corresponding cocycle is A_5.

Let $\tilde{g}$ denote the cocycle we just found. It is given in the following:

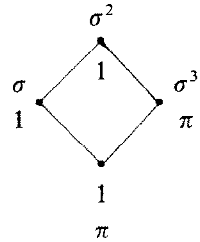
$\tilde{g}$	1	σ	σ^2	σ^3
1	1	1	1	1
σ	1	1	π^2	π^2
σ^2	1	π^2	π^4	π^2
σ^3	1	π^2	π^2	1

Let g: $G \times G \to S^{\#}$ be given by

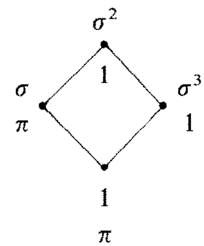
g	1	σ	σ^2	σ^3
1	1	1	1	1
σ	1	1	π	π
σ^2	1	π	π^2	π
σ^3	1	π	π	1

Then $g^2 = \tilde{g}$ and g is itself a cocycle. The algebra A_g is quite interesting because it is not maximal but it is "irreducible": it is not the product of other non-Azumaya crossed product orders. We want to determine the ideals of $A_g = \coprod_\sigma Sx_\sigma$ by the methods of Section Two. We first determine the weighted graphs. By Lemma 4.1 the left and right graphs are the same. They are

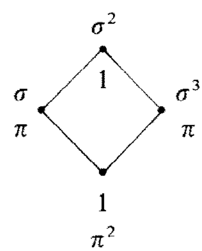
σ (left and right)



σ^3 (left and right)



σ^2 (left and right)



Let $A = A_g$. By Proposition 2.5, $Ax_\sigma A = \coprod_{i=0}^{3} \pi^{k_i} Sx_{\sigma^i}$, where $k_i = \min_{\tau \in G}\{v(f(\sigma, \sigma^{-1}\tau)) + v(f(\sigma^i\tau^{-1}, \tau))\}$. It is then easy to compute that $k_0 = 1$, $k_1 = 0$, and $k_3 = 1$, so $Ax_\sigma A = \pi S \oplus \pi Sx_\sigma \oplus Sx_{\sigma^2} \oplus \pi Sx_{\sigma^3}$. Similarly $Ax_{\sigma^2}A = \pi^2 S \oplus \pi Sx_\sigma \oplus Sx_{\sigma^2} \oplus \pi Sx_{\sigma^3}$ and $Ax_{\sigma^3}A = \pi S \oplus Sx_\sigma \oplus Sx_{\sigma^2} \oplus \pi Sx_{\sigma^3}$. By the remarks preceding Proposition 2.5, the ideals of A_g are obtained as sums of the form $\pi^{k_0}A + \pi^{k_1}Ax_\sigma A + \pi^{k_2}Ax_{\sigma^2}A + \pi^{k_3}Ax_{\sigma^3}A$, where each k_i is a nonnegative integer.

In this particular example, though, one can proceed more simply. From

Lemma 4.1 it follows that $Ax_{\sigma^i} = x_{\sigma^i}A$ for all i and thus $Ax_{\sigma^i}A = Ax_{\sigma^i}$ for all i. This makes these ideals much easier to compute. Note that $\operatorname{rad}(A_g) = S \oplus \pi S x_\sigma \oplus \pi S x_{\sigma^2} \oplus \pi S x_{\sigma^3} = Ax_\sigma + Ax_\sigma{}^3$.

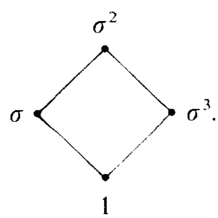
EXAMPLE 4.3. We again take $G = \langle \sigma \rangle$ the cyclic group of order four but now assume S has exactly two maximal ideals, $M_1 = (\pi_1)$ and $M_2 = (\pi_2)$ with $\sigma(\pi_1) = \pi_2$, $D_{M_1} = \langle \sigma^2 \rangle = D_{M_2}$. Let $D_i = D_{M_i}$, $i = 1, 2$. Consider the cocycle $\bar{f}$: $D_1 \times D_1 \to S^{\#}_{M_1}$ given by

$\bar{f}$	1	σ^2
1	1	1
σ^2	1	π_1

Then $A_{\bar{f}}$ is a maximal order (Theorem 2.3) and $\bar{f}$ has graph $\genfrac{}{}{0pt}{}{\sigma^2}{1}$. As described in Section 3, we lift $\bar{f}$ to f_1: $G \times G \to S^{\#}$ using the coset decomposition $G = D_1 \cup \sigma D_1 = D_1 \cup D_1 \sigma^3$. The cocycle f_1 is given by

f_1	1	σ	σ^2	σ^3
1	1	1	1	1
σ	1	1	π_1	π_1
σ^2	1	π_1	π	π_2
σ^3	1	π_2	π_2	π_1

The graph of f_1 is



There are functions ϕ_i: $G \to D_i$ which can be tabulated as follows:

	ϕ_1	ϕ_2
1	1	1
σ	σ^2	1
σ^2	σ^2	σ^2
σ^3	1	σ^2

In this particular example we get an isomorphism of partially ordered sets $\phi_1 \times \phi_2 \colon G \to D_1 \times D_2$.

By the general theory A_{f_1} is a maximal order. Moreover, under the one-to-one correspondence between ideals of A_{f_1} and ideals of $A_{\bar{f}}$, the ideal $A_{f_1} x_{\sigma^2} A_{f_1}$ corresponds to $A_{\bar{f}} x_{\sigma^2} A_{\bar{f}} = \text{rad}(A_{\bar{f}})$. Hence $\text{rad}(A_{f_1}) = A_{f_1} x_{\sigma^2} A_{f_1}$. But it is easy to see that $A_{f_1} x_{\sigma^2} = x_{\sigma^2} A_{f_1}$ and so $\text{rad}(A_{f_1}) = A_{f_1} x_{\sigma^2} = A_{\sigma^2} A_{f_1}$.

If we consider the other allowable coset decomposition, $G = D_1 \cup \sigma^3 D_1 = D_1 \cup D_1 \sigma$, then we obtain the cocycle $f_2 \colon G \times G \to S^{\#}$ given by

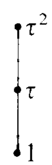
f_2	1	σ	σ^2	σ^3
1	1	1	1	1
σ	1	1	π_2	π_2
σ^2	1	π_2	π	π_1
σ^3	1	π_1	π_1	π_2

The graph of f_2 is the same as that of f_1, but the functions $\phi_i \colon G \to D_i$ are now switched. Again, A_{f_2} is a maximal order and in fact $A_{f_1} \cong A_{f_2}$ as R-algebras. However, we know there is no isomorphism $A_{f_1} \to A_{f_2}$ which is the identity on S.

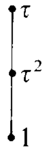
EXAMPLE 4.4. Let $G = S_3$, the symmetric group on three letters. Let $\sigma = (1, 2)$, $\tau = (1, 2, 3)$. Assume S has exactly two maximal ideals $M_1 = (\pi_1)$ and $M_2 = (\pi_2)$, so that $D_{M_1} = \langle \tau \rangle = D_{M_2}$. Let $D_i = D_{M_i}$. Let $\bar{f} \colon D_1 \times D_1 \to S^{\#}_{M_1}$ be the cocycle given by

$\bar{f}$	1	τ	τ^2
1	1	1	1
τ	1	1	π
τ^2	1	π	π

The graph of $\bar{f}$ is



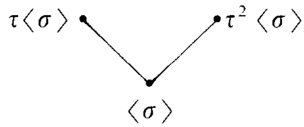
Hence $A_{\bar{f}}$ is a maximal order. We lift $\bar{f}$ to $f \colon G \times G \to S^{\#}$ using the coset decomposition $G = D_1 \cup \sigma D_1 = D_1 \cup D_1 \sigma$. Then A_f is maximal. In particular, A_f is primary so the ordering on D_2 is obtained by conjugating the ordering on D_1 by σ. Hence the graph of f restricted to $D_2 \times D_2$ is



The map ϕ_1 is determined from the coset decomposition $G = D_1 \cup D_1\sigma$ and because $M_2 = M_1^\sigma$, the general theory tells us that $\phi_2(x) = \sigma\phi_1(x)\sigma$ for all $x \in G$. We tabulate the results:

	ϕ_1	ϕ_2
1	1	1
σ	1	1
τ	τ	τ
τ^2	τ^2	τ^2
$\sigma\tau$	τ^2	τ^2
$\sigma\tau^2$	τ	τ

Thus, $\phi_1 = \phi_2$, but of course the ordering on D_1 is different from that on D_2. From this table we see that the associtated subgroup for f is $\langle\sigma\rangle$ and the graph of f is

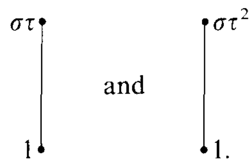


One interesting aspect of this example is that $\mathrm{rad}(A_f) = A_f(x_\tau + x_{\tau^2})$ and $\mathrm{rad}(A_f)$ cannot be expressed as $A_f x_g$ for any choice of $g \in G$. If the other allowable coset decompositions are used, the effect is to replace $\langle\sigma\rangle$ in the graph of f by $\langle\sigma\tau\rangle$ or $\langle\sigma\tau^2\rangle$.

EXAMPLE 4.5. For the final example, let $G = S_3$ but now assume S has exactly three maximal ideals M_1, M_2, and M_3. Let $M_i = (\pi_i)$. Assume $\tau(M_1) = M_2$, $\tau(M_2) = M_3$, $\sigma(M_1) = M_1$, $\sigma(M_2) = M_3$. Then $D_{M_1} = \langle\sigma\rangle$, $D_{M_2} = \langle\sigma\tau\rangle$ and $D_{M_3} = \langle\sigma\tau^2\rangle$. Let $D_i = D_{M_i}$. Let $\bar{f}$: $D_1 \times D_1 \to S_{M_1}^{\#}$ be given by

$\bar{f}$	1	σ
1	1	1
σ	1	π_1

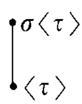
The crossed-product order $A_{\bar{f}}$ is maximal. As before we can lift $\bar{f}$ to G in various ways to obtain maximal orders. First consider the coset decomposition $G = D_1 \cup \tau D_1 \cup \tau^2 D_1 = D_1 \cup D_1 \tau^2 \cup D_1 \tau$. If f_1 is the cocycle determined by this decomposition, then the orderings it induces on D_2 and D_3 are obtained by conjugating the ordering on D_1 by τ and τ^2, respectively. The graphs are



We can compute the functions ϕ_i: $G \to D_i$ as described in the last example (e.g., $\phi_2(x) = \phi_{M_1^\tau}(x) = \tau\phi_1(\tau^{-1}x)\tau^{-1}$ for all $x \in G$). We tabulate the results:

	ϕ_1	ϕ_2	ϕ_3
1	1	1	1
σ	σ	$\sigma\tau$	$\sigma\tau^2$
τ	1	1	1
τ^2	1	1	1
$\sigma\tau$	σ	$\sigma\tau$	$\sigma\tau^2$
$\sigma\tau^2$	σ	$\sigma\tau$	$\sigma\tau^2$

The graph of f_1 can be determined from this table. It is

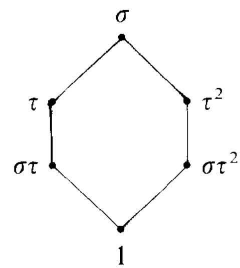


In particular $\langle\tau\rangle$ is the associated subgroup for f_1.

A more interesting cocycle, call it f_2, arises from the coset decomposition $G = D_1 \cup \sigma\tau D_1 \cup \sigma\tau^2 D_1 = D_1 \cup D_1 \sigma\tau \cup D_1 \sigma\tau^2$. In this case the orders on D_2 and D_3 are as for f_1 and the functions ϕ_i: $G \to D_i$ are given as follows:

	ϕ_1	ϕ_2	ϕ_3
1	1	1	1
σ	σ	$\sigma\tau$	$\sigma\tau^2$
τ	σ	$\sigma\tau$	1
τ	σ	1	$\sigma\tau^2$
$\sigma\tau$	1	$\sigma\tau$	1
$\sigma\tau^2$	1	1	$\sigma\tau^2$

The associated subgroup for f_2 is trivial and the graph for f_2 is



The cocycle itself is given by

f_2	1	σ	τ	τ^2	$\sigma\tau$	$\sigma\tau^2$
1	1	1	1	1	1	1
σ	1	π	$\pi_1\pi_3$	$\pi_1\pi_2$	π_3	π_2
τ	1	$\pi_1\pi_2$	π_2	$\pi_1\pi_2$	1	π_1
τ^2	1	$\pi_1\pi_3$	$\pi_1\pi_3$	π_3	π_1	1
$\sigma\tau$	1	π_2	π_2	1	π_2	1
$\sigma\tau^2$	1	π_3	1	π_3	1	π_3

By the one-to-one correspondence between ideals of A_{f_2} and $A_{\bar{f}}$, we see that $\mathrm{rad}(A_{f_2}) = A_{f_2} x_\sigma A_{f_2}$. But from the table one can check that $A_{f_2} x_\sigma = x_\sigma A_{f_2}$ and so $\mathrm{rad}(A_{f_2}) = A_{f_2} x_\sigma$.

This example is interesting because the graphs of f_1 and f_2 are quite different and yet we know by the theory that $A_{f_1} \cong A_{f_2}$ as R-algebras. The other cocycles on $G \times G$ obtained from $\bar{f}$ (there are $[D_1 : H_1]^{[G:D_1]-1} = 4$ in all) can be found by conjugating the graph of f_2 by τ and τ^2.

REFERENCES

1. M. Auslander and A. Brumer, Brauer groups of discrete valuation rings, *Nederl. Akad. Wetensch. Proc. Soc. A* **71** (1968), 286–296.
2. M. Auslander and O. Goldman, Maximal orders, *Trans. Amer. Math. Soc.* **97** (1960), 1–24.
3. M. Auslander and O. Goldman, The Brauer group of a commutative ring, *Trans. Amer. Math. Soc.* **97** (1960), 367–409.
4. F. DeMeyer and E. Ingraham, "Separable Algebras over Commutative Rings," Lecture Notes in Mathematics, No. 181, Springer-Verlag, Berlin, 1971.
5. D. Haile, R. Larson, and M. Sweedler, A new invariant for ℂ over ℝ: Almost invertible cohomology theory and the classification of idempotent cohomology classes and algebras by partially ordered sets with a Galois action, *Amer. J. Math.* **105** (3) (1983), 689–814.
6. M. Harada, Some criteria for hereditarity of crossed products, *Osaka J. Math.* **1** (1964), 69–80.
7. I. Reiner, "Maximal Orders," Academic Press, New York, 1975.