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Nearly Subnormal Operators and Moment Problems

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We use separation-of-cones techniques and ideas from multivariable operator theory to show that polynomial hyponormality does not imply subnormality for Hilbert space operators. As an application, we obtain a new result in the theory of power moments in two dimensions. © 1993 Academic Press, Inc.

1. INTRODUCTION

Several classes of Hilbert spaces operators are defined around the notion of a normal operator; the corresponding theories, although often radically distinct, invite us to make comparisons among them. Two typical examples of such classes are those of subnormal and hyponormal operators. They were introduced in 1950 by P. R. Halmos [Hal 1] in an attempt to extend the basic facts of the spectral theory of normal operators. Soon it was discovered that each of these classes has its own, surprisingly rich, collection of phenomena. It is not the purpose of the present paper to discuss these distinctions; as a by-product of the main result below, however, one can assert that the gap between hyponormal and subnormal operators is even larger than previously thought. We prove in the sequel that there are intermediate classes of Hilbert space operators which deserve attention in the future.

While a normal operator is modeled as multiplication by the independent variable on a sum of L^2-spaces associated with positive Borel

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measures compactly supported in $\mathbb{C}$, a *subnormal* operator (i.e., one which is the restriction of a normal operator to a closed invariant subspace) corresponds to the same multiplier acting on subspaces of L^2 generated by analytic functions. These relations extend much further; for instance, the basic results in the classical theory of moments of planar measures can be derived from the spectral theorem for normal operators (see [Akh, Lan]). On the other hand, the theory of subnormal operators has nontrivial applications to one-variable complex function theory (see [Con]).

A bounded operator T on a Hilbert space $\mathscr{H}$ is said to be *hyponormal* if $T^*T \geqslant TT^*$. This inequality is satisfied by all subnormal operators (as a straightforward matrix calculation shows), but the converse is not true. The functional models, and consequently the refined structure theory of hyponormal operators (in the spirit of the above mentioned two other examples), were completed unexpectedly late (in the early eighties), thanks to the contributions of many mathematicians (see the monographs [Cla, MP, Xia). One such model is given by singular integral operators on the real line with kernels of Cauchy type.

In the sequel, we confine our discussion to one aspect of the distinction between subnormality and hyponormality; namely, it is immediate that subnormality is preserved under polynomial calculus while hyponormality is not (the latter statement takes a bit of proving). A hyponormal operator which does remain hyponormal under such a functional calculus is called *polynomially hyponormal.* A natural question, going back to the pioneering ages of the two theories, is whether polynomial hyponormality coincides with subnormality.

The aim of the present paper is to answer this question in the negative; that is, we show that there exist polynomially hyponormal operators which are not subnormal. (An announcement of our main result appears in [CuP].) The idea of the proof is to extend the intrinsic connection between subnormal operators and classical moment problems in the plane to classes of nearly subnormal operators (as for instance polynomially hyponormal ones) and moment problems for certain linear functionals not necessarily represented by measures. This is possible due to a remarkably simple and useful "dictionary" developed by J. Agler [Ag 2]. Agler's idea is to associate with every cyclic contractive operator a linear functional acting on $\mathbb{C}[z, \bar{z}]$ via a non-commutative functional calculus which translates near subnormality notions into positivity on special cones of polynomials. Working at the level of linear functionals on the space of polynomials in two real variables, and adapting some refined techniques due to G. Cassier [Cas] originating in moment problems in $\mathbb{R}^n$, we exhibit the desired example.

Earlier results leading to our solution of the above mentioned question are numerous in the literature, and we briefly recall some of them here. The question appears to have arisen early on in the study of subnormality and hyponormality; it is generally believed that it circulated as an open problem or as a conjecture in the early 1960's, and perhaps even earlier. However, the first occurrence in the literature seems to be in a paper of A. Joshi [Jos2], published in 1975. The problem appears indirectly in a few articles on hyponormality and subnormality written in the 1950's and 1960's, which generally point towards a negative answer to this question. For instance, P. R. Halmos gave in [Hal1] the first example of a hyponormal operator T such that T^2 is not hyponormal, and J. Stampfli exhibited in [Sta] a non-subnormal hyponormal operator T such that T^n is subnormal for every $n \geqslant 2$. Stampfli also proved that if $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ is a sequence of positive numbers whose associated unilateral weighted shift W_α is subnormal (here $\mathscr{H} = l^2(\mathbb{Z}_+)$), then α cannot have two equal weights without being *flat*, i.e., $\alpha_1 = \alpha_2 = ...$; thus, a plausible way to construct a polynomially hyponormal shift which is not subnormal would be to have a sequence of weights of the form $\alpha_0 < \cdots < \alpha_k = \alpha_{k+1} = \cdots$. By varying the weights $\alpha_0, ..., \alpha_k$, one might be able to produce a non-subnormal polynomially hyponormal shift. A. Joshi [Jos1] proved that if $\alpha_0 = \alpha_1 < \alpha_2 = \alpha_3 = \cdots$, then W_α cannot be quadratically hyponormal, and P. Fan [Fan] established that if $\alpha_0 = \alpha_1 = 1$ and $\alpha_2 = \alpha_3 = \cdots = 2$, then $W_\alpha + sW_\alpha^2$ is not hyponormal for $0 < s < 1/\sqrt{5}$. These last two results were later subsumed in a more general fact, found in [Cu1]: A quadratically hyponormal shift with three equal weights must be flat, therefore subnormal (two equal weights and quadratic hyponormality, however, do not force flatness). It was also found that Stampfli's result does not really require subnormality, but merely 2-hyponormality [Cu1, Corollary 5], a notion introduced in [At] and [CMX], and intimately related to multivariable operator theory. Precisely, a commuting k-tuple $T = (T_1, ..., T_k)$ of operators on $\mathscr{H}$ is said to be *joinlty hyponormal* if the *joint commutator*

$$[T^*, T] := \begin{pmatrix} [T_1^*, T_1] & \cdots & [T_k^*, T_1] \\ \cdots & & \cdots \\ [T_1^*, T_k] & \cdots & [T_k^*, T_k] \end{pmatrix}$$

is a positive operator on $\mathscr{H} \oplus \cdots \oplus \mathscr{H}$. An operator T is said to be *k-hyponormal* if $(T, ..., T^k)$ is jointly hyponormal. The well known Bram–Halmos criterion for subnormality (see [Con, II.1.9]) establishes that an operator T is subnormal if and only if T is k-hyponormal for every $k \geqslant 1$ (cf. [CMX, Cu1, Cu2]). In particular, a subnormal operator is always 2-hyponormal, while the converse is false [Cu1, Proposition 7]. It should be remarked that for every $k \geqslant 1$, the k-hyponormality of T does imply that T is *weakly k-hyponormal*, i.e., $p(T)$ hyponormal for every $p \in \mathbb{C}[z]$ of degree at most k (with converse false). Thus, we have two staircases of notions, one climbing from hyponormality to subnormality, while the other spans from hyponormality to polynomial hyponormality (or weak "∞-hyponormality"). Although, step by step, the former staircase is higher than the latter, it was conceivable that in the limit they could both reach the same height, namely subnormality. The results about weighted shifts indicated, however, that the relation between polynomial hyponormality and 2-hyponormality deserved more attention. As we mentioned before, we now know that "weakly ∞-hyponormal" does not imply 2-hyponormal.

In [McCP], the authors reduced the proof of the existence of a non-subnormal polynomially hyponormal operator to the class of unilateral shifts. This led to a rather detailed investigation of k-hyponormality and quadratic hyponormality for unilateral weighted shifts [Cu1, Cu2, CF1, CF2], with an emphasis on characterizations and model theory. An important consequence of the results in those works is the occurrence of a large gap between quadratic hyponormality and 2-hyponormality, which made it all the more plausible to try to build examples where polynomial hyponormality and 2-hyponormality could be separated. The proof of our main result follows this philosophy; however, it relies heavily on a detailed investigation of certain polynomial cones (in the spirit of the classical theory of moments) and, incidentally, it requires that certain linear functionals do *not* arise from weighted shifts. (The reader is alerted to the fact that k-hyponormality is called strong k-hyponormality in [McCP].)

So far we have described how a problem in operator theory can be studied using tools from the classical theory of moments. Our techniques, however, will also allow us to obtain a simplification of the main result in [Cas] for the power moment problem in two dimensions. The simplification consists in imposing only a marginal positivity condition on the moments of a measure μ, in order to have supp(μ) contained in a prescribed subalgebraic compact set. Similar results were known only for discs [Atz] and ellipsoids [McG]. For the general case, the operator theoretic point of view developed in this paper seems to be essential.

2. Polynomial Hyponormal Operators Are Not Subnormal

The main result of this section is the following.

Theorem 2.1. *There exists a polynomially hyponormal operator which is not 2-hyponormal.*

Since every subnormal operator is 2-hyponormal (by the Bram–Halmos criterion), Theorem 2.1 implies at once the following consequence.

COROLLARY 2.2. *There exists a polynomially hyponormal operator which is not subnormal.*

Our first aim is to recall the basic facts of Agler's dictionary and then to transform Theorem 2.1 into a positive statement concerning linear functionals on the space of polynomials in two real variables.

Let T be a contraction acting on a Hilbert space $\mathscr{H}$, and let $p \in \mathbb{C}[z, \bar{z}]$ be a polynomial in z and $\bar{z}$, $p(z, \bar{z}) = \sum_{m,n} a_{mn} z^m \bar{z}^n$. Set $p(T, T^*) := \sum_{m,n} a_{mn} T^{*n} T^m$. The assignment $p \to p(T, T^*)$ defines an ordered functional calculus, frequently referred to as the *hereditary functional calculus* [Ag2]. Given $\gamma \in \mathscr{H}$, we now define $\Lambda_T: \mathbb{C}[z, \bar{z}] \to \mathbb{C}$ by $\Lambda_T(p) := (p(T, T^*)\gamma, \gamma)$, $p \in \mathbb{C}[z, \bar{z}]$. The functional Λ_T satisfies the following two properties (i) $\Lambda_T(p\bar{p}) \geqslant 0$, and (ii) $\Lambda_T((1 - z\bar{z}) p\bar{p}) \geqslant 0$, for every $p \in \mathbb{C}[z]$. Conversely, if $\Lambda: \mathbb{C}[z, \bar{z}] \to \mathbb{C}$ is a linear functional satisfying (i) and (ii), and if $\mathscr{N} := \{p \in \mathbb{C}[z]: \Lambda(p\bar{p}) = 0\}$, then multiplication by z on $\mathbb{C}[z]$ induces a contraction T with cyclic vector $1 + \mathscr{N}$ on the Hilbert space completion of the quotient $\mathbb{C}[z]/\mathscr{N}$ under the inner product $\langle p, q \rangle := \Lambda(p\bar{q})$, $p, q \in \mathbb{C}[z]$; that is, $\Lambda = \Lambda_T$. This construction transfers notions associated with operator positivity to positivity for linear functionals on spaces of polynomials. For instance, the subnormality of T on the cyclic subspace generated by γ is detected by the positivity of Λ_T on $\{|p(z, \bar{z})|^2 : p(z, \bar{z}) \in \mathbb{C}[z, \bar{z}]\}$, while k-hyponormality and polynomial hyponormality require that Λ_T be positive on other cones. Specifically, one can show that the map $(T, \gamma) \to \Lambda_T$ establishes a one-to-one correspondence between the unitary equivalence classes of k-hyponormal contractions with fixed cyclic vector γ and the linear functionals on $\mathbb{C}[z, \bar{z}]$ which are positive on the cone

$$\mathscr{S}^k := \operatorname{co}\left\{(1 - |z|^2)\,|p|^2 + \left|\sum_{i=0}^{k} q_i \bar{z}^i\right|^2 : p, q_0, ..., q_k \in \mathbb{C}[z]\right\}$$

(cf. [McCP]). For instance, we know that T is 2-hyponormal if and only if

$$M_2(T) := \begin{pmatrix} I & T^* & T^{*2} \\ T & T^*T & T^2T \\ T^2 & T^*T^2 & T^{*2}T^2 \end{pmatrix} \geqslant 0$$

(relative to the usual identification of operators on $\mathscr{H} \oplus \mathscr{H} \oplus \mathscr{H}$ with 3×3 matrices of operators on $\mathscr{H}$), and this is equivalent to the condition

$$\left(\begin{pmatrix} I & T^* & T^{*2} \\ T & T^*T & T^{*2}T \\ T^2 & T^*T^2 & T^{*2}T^2 \end{pmatrix} \begin{pmatrix} p_0(T)\gamma \\ p_1(T)\gamma \\ p_2(T)\gamma \end{pmatrix}, \begin{pmatrix} p_0(T)\gamma \\ p_1(T)\gamma \\ p_2(T)\gamma \end{pmatrix} \right) \geqslant 0,$$

which in turn is equivalent to

$$(|p_0+p_1\bar{z}+p_2\bar{z}^2|^2\,(T, T^*)\gamma, \gamma)\geqslant 0$$

or

$$\Lambda_T(|p_0+p_1\bar{z}+p_2\bar{z}^2|^2)\geqslant 0$$

$(p_0, p_1, p_2\in\mathbb{C}[z])$. Similarly, T is polynomially hyponormal if and only if

$$\begin{aligned}&\begin{pmatrix} I & r(T)^* \\ r(T) & r(T)^*\,r(T)\end{pmatrix}\geqslant 0\\ &\Leftrightarrow\left(\begin{pmatrix} I & r(T)^* \\ r(T) & r(T)^*\,r(T)\end{pmatrix}\begin{pmatrix} p(T)\gamma \\ q(T)\gamma\end{pmatrix}, \begin{pmatrix} p(T)\gamma \\ q(T)\gamma\end{pmatrix}\right)\geqslant 0\\ &\Leftrightarrow(|p+q\bar{r}|^2\,(T, T^*)\gamma, \gamma)\geqslant 0\\ &\Leftrightarrow\Lambda_T(|p+q\bar{r}|^2)\geqslant 0\end{aligned}$$

$(p, q, r\in\mathbb{C}[z])$, which shows at once that the polynomial hyponormality of T may be expressed as the positivity of the functional Λ_T on the associated cone

$$\mathscr{W}:=\operatorname{co}\{(1-|z|^2)\,|p|^2+|s+q\bar{r}|^2 : p, q, r, s\in\mathbb{C}[z]\}.$$

Our ploy is the following: We first construct a polynomial $p(z, \bar{z})\in\mathscr{S}^2$ and a linear functional Λ on $\mathbb{C}[z, \bar{z}]$ such that properties (i) and (ii) above hold, and which satisfies in addition the conditions $\Lambda(p)<0$ and $\Lambda|_{\mathscr{W}}\geqslant 0$. Using [McCP, Theorem 2.4], we then know that $\Lambda=\Lambda_T$ for some contraction T. Finally, by the previous considerations, such a T is polynomially hyponormal and not 2-hyponormal.

To begin, we must introduce some notation. For $m\geqslant 0$, let $\mathbb{C}[z, \bar{z}]_m$ denote the set of polynomials in z and $\bar{z}$ of total degree at most m, let $\mathbb{C}[z, \bar{z}]^{\mathrm{h}}$ denote the set of homogeneous polynomials, let $\mathbb{C}[z, \bar{z}]^{\mathrm{h}}_m$ denote the set of homogeneous polynomials of degree m, and let $\mathbb{R}[x, y]_m$ (resp., $\mathbb{R}[x, y]^{\mathrm{h}}_m$) denote the real polynomials (resp., homogeneous polynomials) of degree at most m (resp., equal to m). Observe that $\{p(z, \bar{z})\in\mathbb{C}[z, \bar{z}]^{\mathrm{h}}_m : p=\bar{p}\}=\mathbb{R}[x, y]^{\mathrm{h}}_m$, via the usual identification $(z+\bar{z})/2=x$, $(z-\bar{z})/2i=y$. In addition, we introduce the auxiliary cone

$$\Gamma:=\operatorname{co}\{|p(z)+q(z)\,\overline{r(z)}|^2 : p, q, r\in\mathbb{C}[z]\}\subseteq\mathscr{W},$$

and for a cone $K\subseteq\mathbb{C}[z, \bar{z}]$ and $m\geqslant 0$, we let $K_m:=K\cap\mathbb{C}[z, \bar{z}]_m$ and $K^{\mathrm{h}}_m:=K\cap\mathbb{C}[z, \bar{z}]^{\mathrm{h}}_m$.

Although Γ is simpler to work with than $\mathscr{W}$, the functional to be constructed must separate $\mathscr{W}$ from a polynomial in $\mathscr{S}^2$; however, as we see below, both cones contain the same homogeneous polynomials of degree 4, and this fact will be crucial in our technique. The functional Λ is first defined on $\mathbb{R}[x, y]_4^{\mathrm{h}}$ so that it separates Γ_4^{h} from a polynomial in $\mathscr{S}^2$, and is then extended to all of $\mathbb{R}[x, y]$, one step at a time, by keeping it positive along $\mathscr{W}$.

LEMMA 2.3. (i) *For* $m \geqslant 0$, $\mathbb{R}[x, y]_m = \mathscr{W}_m - \mathscr{W}_m$, *so in particular* $\operatorname{int}.\mathscr{W}_m \neq \phi$.

(ii) *For* $m \geqslant 0$ *even*, $\mathbb{R}[x, y]_m = \Gamma_m - \Gamma_m = \mathscr{W}_m - \mathscr{W}_m$.

(iii) *For* $m \geqslant 0$ *even*, $\mathbb{R}[x, y]_m^{\mathrm{h}} = \Gamma_m^{\mathrm{h}} - \Gamma_m^{\mathrm{h}}$ *(and thus* $\operatorname{int}.\Gamma_m^{\mathrm{h}} \neq \phi$*).*

Proof. Note that

$$\{\operatorname{Re}(z^{n+j}\bar{z}^k), \operatorname{Im}(z^{n+j}\bar{z}^k) : j, k \geqslant 0, j+k \leqslant n\}$$

spans $\mathbb{R}[x, y]_{2n}$. Now observe that

$$|z^n + \alpha z^j \bar{z}^k|^2 - |z^n - \alpha z^j \bar{z}^k|^2 = 4 \operatorname{Re}(\bar{\alpha} z^{n+k} \bar{z}^j) \qquad (\text{all } i, j, k \in \mathbb{Z}_+, \alpha \in \mathbb{C}).$$

By taking $\alpha = 1$, i, it follows at once that $\operatorname{Re}(z^{n+k}\bar{z}^j)$, $\operatorname{Im}(z^{n+k}\bar{z}^j) \in \Gamma_{2n} - \Gamma_{2n} \subseteq \mathscr{W}_{2n} - \mathscr{W}_{2n}$ whenever $j+k \leqslant n$. Assume that m is even, say $m = 2n$. By taking $j+k=n$, we see that $\mathbb{R}[x, y]_{2n}^{\mathrm{h}} = \Gamma_{2n}^{\mathrm{h}} - \Gamma_{2n}^{\mathrm{h}} \subseteq \mathscr{W}_{2n} - \mathscr{W}_{2n}$, since the dimension of $\mathbb{R}[x, y]_{2n}^{\mathrm{h}}$ is $2n+1$ and the polynomials $\operatorname{Re}(z^n\bar{z}^n)$, $\operatorname{Re}(z^{n+1}\bar{z}^{n-1})$, $\operatorname{Im}(z^{n+1}\bar{z}^{n-1})$, ..., $\operatorname{Re}(z^{2n})$, $\operatorname{Im}(z^{2n})$ are linearly independent. When m is odd (say $m = 2n-1$) the analogous argument fails, since the degree of $|z^n + \alpha z^j \bar{z}^k|^2$ is $2n$. However, we can use the polynomial $(1 - |z|^2)$ to "lower the degree" of $|z^n + \alpha z^j \bar{z}^k|^2$ as follows. Observe that

$$\begin{aligned}
&[|z^n + \alpha z^j \bar{z}^k|^2 + (1 - |z|^2)\,|z|^{2n-2}] - [|z^n - \alpha z^j \bar{z}^k|^2 + (1 - |z|^2)\,|z|^{2n-2})] \\
&\quad = 4\operatorname{Re}(\bar{\alpha} z^{n+k} \bar{z}^j)
\end{aligned}$$

(all $i, j, k \in \mathbb{Z}_+$, $\alpha \in \mathbb{C}$), from which we obtain that $\mathbb{R}[x, y]_m = \mathscr{W}_m - \mathscr{W}_m$ also in this case (although the homogeneity property may be lost). ∎

LEMMA 2.4. *Γ_4 is generated by polynomials of the form*

$$|c_0 + c_1 z + c_2 \bar{z} + c_3 z^2 + c_4 z\bar{z} + c_5 \bar{z}^2|^2,$$

where $c_4 = 0$ *or* $c_5 = 0$.

Proof. First, observe that Γ_4 is generated by polynomials of the form $|p + q\bar{r}|^2$, where p, q, r are polynomials in z of degree at most 2. Write $p(z) = p_0 + p_1 z + p_2 z^2$, $q(z) = q_0 + q_1 z + q_2 z^2$, and $r(z) = r_0 + r_1 z + r_2 z^2$. A calculation now shows that we must necessarily have $q_1\bar{r}_2 = q_2\bar{r}_1 = q_2\bar{r}_2 = 0$, and that $c_4 = q_1\bar{r}_1$ and $c_5 = q_0\bar{r}_2$. The result now follows easily. ∎

LEMMA 2.5. *Let $p(z, \bar{z}) := |-\sqrt{2}\,|z|^2 + z^2 + \bar{z}^2|^2 \in \mathscr{S}^2$. Then there exists a (real) linear functional Λ_4^{h} on $\mathbb{R}[x, y]_4^{\mathrm{h}}$ such that $\Lambda_4^{\mathrm{h}}(p) < 0$, and $\Lambda_4^{\mathrm{h}}|_{\mathrm{int}.\Gamma_4^{\mathrm{h}}} > 0$.*

Proof. Consider the linear functional Λ_4^{h} on $\mathbb{C}[z, \bar{z}]_4^{\mathrm{h}}$ given (on generators) by $\Lambda_4^{\mathrm{h}}(z^4) = \Lambda_4^{\mathrm{h}}(\bar{z}^4) = 0$, $\Lambda_4^{\mathrm{h}}(z^2\,|z|^2) = \Lambda_4^{\mathrm{h}}(\bar{z}^2\,|z|^2) = b$, and $\Lambda_4^{\mathrm{h}}(|z|^4) = 1$, where $b \in \mathbb{R}$. For $c_3, c_4, c_5 \in \mathbb{C}$, we then have

$$\begin{aligned}
&\Lambda_4^{\mathrm{h}}(|c_3 z^2 + c_4 z\bar{z} + c_5 \bar{z}^2|^2) \\
&\quad = \lambda_4^{\mathrm{h}}(c_3\bar{c}_3 z^2\bar{z}^2 + c_3\bar{c}_4 z^3\bar{z} + c_3\bar{c}_5 z^4 + c_4\bar{c}_3 z\bar{z}^3 + c_4\bar{c}_4 z^2\bar{z}^2 \\
&\qquad + c_4\bar{c}_5 z^3\bar{z} + c_5\bar{c}_3\bar{z}^4 + c_5\bar{c}_4 z\bar{z}^3 + c_5\bar{c}_5 z^2\bar{z}^2) \\
&\quad = \left\langle \begin{pmatrix} 1 & b & 0 \\ b & 1 & b \\ 0 & b & 1 \end{pmatrix} \begin{pmatrix} c_3 \\ c_4 \\ c_5 \end{pmatrix}, \begin{pmatrix} c_3 \\ c_4 \\ c_5 \end{pmatrix} \right\rangle.
\end{aligned}$$

Thus, the positivity of Λ_4^{h} on $\mathscr{S}^2$ is controlled by the matrix

$$A := \begin{pmatrix} 1 & b & 0 \\ b & 1 & b \\ 0 & b & 1 \end{pmatrix},$$

whose eigenvalues are 1, $1 + \sqrt{2}\,b$, and $1 - \sqrt{2}\,b$. By taking $b > 1/\sqrt{2}$, we obtain a negative eigenvalue, with eigenvector $(1, -\sqrt{2}, 1)$, so that $\Lambda_4^{\mathrm{h}}(p) < 0$. On the other hand, the compression of A to the subspaces $\{(c_3, 0, c_5)\colon c_3, c_5 \in \mathbb{C}\}$ and $\{(c_3, c_4, 0)\colon c_3, c_4 \in \mathbb{C}\}$ is positive definite, provided $b < 1$. Looking at the restriction of Λ_4^{h} to $\mathbb{R}[x, y]_4^{\mathrm{h}}$ and using Lemma 2.4, we complete the proof. ∎

LEMMA 2.6. *$\mathscr{W}_4^{\mathrm{h}} \subseteq \Gamma_4$, that is, the $(1 - |z|^2)\sum_j |s_j(z)|^2$ component of an homogeneous polynomial of total degree 4 can be eliminated.*

Proof. Let $f = \sum_j |p_i + q_i\bar{r}_i|^2 + (1 - |z|^2)\cdot\sum_j |s_j|^2 \in \mathscr{W} \cap \mathbb{C}[z, \bar{z}]_4^{\mathrm{h}}$ (recall that $p_i, q_i, r_i, s_j \in \mathbb{C}[z]$). Of course we can always arrange for r_i to vanish at the origin (all i), so that

$$f = \sum_i |p_i + \bar{z}q_i\bar{\tilde{r}}_i|^2 + (1 - |z|^2)\cdot\sum_j |s_j|^2,$$

where $\tilde{r}_i \in \mathbb{C}[z]$. Since f is homogeneous of degree 4, its constant term must be zero; i.e.,

$$\sum_i |p_i(0)|^2 + \sum_j |s_j(0)|^2 = 0.$$

It follows that each p_i and each s_j must vanish at the origin, and therefore

$$f=\sum_i |z\tilde{p}_i+\bar{z}q_i\bar{r}_i|^2+|z|^2(1-|z|^2)\cdot\sum_j |\tilde{s}_j|^2,$$

where $\tilde{p}_i, \tilde{s}_j \in \mathbb{C}[z]$. We now look at the coefficient of $|z|^2$, which must again be zero. We have

$$\sum_i \{|\tilde{p}_i(0)|^2+q_i(0)\,\bar{r}_i(0)|^2\}+\sum_j |\tilde{s}_j(0)|^2=0,$$

which implies that $\tilde{p}_i=z\tilde{\tilde{p}}_i$, $\tilde{s}_j=z\tilde{\tilde{s}}_j$ and that for each i, $q_i=z\tilde{q}_i$ or $\tilde{r}_i=z\tilde{\tilde{r}}_i$. Split the collection of i indices into two sub-collections, those indices k for which $q_k=z\tilde{q}_k$, and the rest, denoted l. We see that

$$f=\sum_k |z^2\tilde{\tilde{p}}_k+z\bar{z}\tilde{q}_k\bar{r}_k|^2+\sum_l |z^2\tilde{\tilde{p}}_l+\bar{z}^2q_l\bar{\tilde{\tilde{r}}}_l|^2+|z|^4(1-|z|^2)\cdot\sum_j |\tilde{\tilde{s}}_j|^2.$$

If we now recall again that f is homogeneous of degree 4, we see that the contribution $-|z|^6\cdot\sum_j |\tilde{\tilde{s}}_j|^2$ must be offset by a similar expression coming out of the first summations, and that

$$f=\sum_k |a_k z^2+b_k z\bar{z}|^2+\sum_l |c_l z^2+d_l\bar{z}^2|^2+\sum_j |e_j|^2\,|z|^4,$$

where $a_k:=\tilde{\tilde{p}}_k(0)$, $b_k:=\tilde{q}_k(0)\,\bar{r}_k(0)$, $c_l:=\tilde{\tilde{p}}_l(0)$, $d_l:=q_l(0)\,\bar{\tilde{\tilde{r}}}_l(0)$, and $e_j:=\tilde{\tilde{s}}_j(0)$. It is now clear that the $(1-|z|^2)$ component has disappeared, so $f\in\Gamma$. ∎

In the next lemma we use the separation theorem for convex sets in finite dimensional spaces (as stated, for instance, in [CoC, I.3.1.3]), by following the idea in [Cas, Théorème 4].

LEMMA 2.7. *The linear functional constructed in Lemma 2.5 can be extended to a linear functional Λ on $\mathbb{C}[z,\bar{z}]$, maintaining the positivity on $\mathscr{W}$.*

Proof. According to Lemmas 2.3, 2.5, and 2.6, and by the separation theorem alluded to before, there exists an extension Λ_4 to $\mathbb{R}[x,y]_4$ satisfying $\Lambda_4|_{\text{int.}(\mathscr{W}_4)}>0$. Next, we proceed one step at a time, first extending Λ_4 to $\mathbb{R}[x,y]_5$ (and maintaining the strict positivity on int.$(\mathscr{W}_5)$), then to $\mathbb{R}[x,y]_6$, etc.; this allows us to obtain a linear functional Λ on $\mathbb{R}[x,y]$ which is non-negative on $\mathscr{W}$, and preserving $\Lambda(p)<0$. To complete the proof, we complexify Λ, which then acts on $\mathbb{C}[z,\bar{z}]$. ∎

The functional Λ of Lemma 2.7 separates the polynomial $p \in \mathcal{S}^2$ from the cone $\mathcal{W}$. This completes the proof of Theorem 2.1.

By combining [McCP, Theorem 3.4] and Corollary 2.2, we can obtain the following result.

COROLLARY 2.8. *There exists a unilateral weighted shift that is polynomially hyponormal but not subnormal.*

Remark 2.9. The proof of Theorem 3.4 in [McCP] does not allow us to keep track of the cone $\mathcal{S}^2$, and therefore we do not know if the unilateral shift associated to our example is necessarily not 2-hyponormal. It is therefore still an open question whether the implication "polynomially hyponormal $\Rightarrow$ 2-hyponormal" can be disproved with a weighted shift.

Since the spectrum of a weighted shift is polynomially convex, it also follows that there exists an analytically hyponormal operator (i.e., with respect to the analytic functional calculus) which is not subnormal.

We end this section with a result on approximation of polynomials.

COROLLARY 2.10. *The polynomial* $|-\sqrt{2}\,|z|^2 + z^2 + \bar{z}^2|^2$ *is not the uniform limit (on* $\bar{\mathbb{D}}$*) of polynomials of the form* $\sum_i |p_i + q_i \bar{r}_i|^2 + (1-|z|^2) \cdot \sum_j |s_j|^2$, $p_i, q_i, r_i, s_i \in \mathbb{C}[z]$.

3. CONNECTIONS WITH THE CLASSICAL THEORY OF MOMENTS

Ever since it was discovered at the beginning of this century that extensions of positive linear functionals on cones of real polynomials can solve the classical problems of moments, both topics have been increasingly and fruitfully studied. One of the last achievements in this direction, the modern dilation and extension theory of Hilbert space operators, is much reminiscent of its function–theoretic origins, especially of moment problems. Conversely, much progress in function theory of one or several variables has been recently made by using essentially the geometry of Hilbert spaces, e.g., the application of the subnormality criteria of Bram–Halmos and of Agler to moment problems; cf. [Atz, AP, McG]. In this section we discuss an interplay between two-parameter power moment problems and some classes of Hilbert space operators which are nearly normal, in the spirit of the preceding sections.

When compared with the classical theories on the line, the multiparameter power moment problems are much less known. For instance, even on $\mathbb{R}^2$, it is still an open question whether the usual power moment problem is equivalent to (possibly many) positive definiteness conditions. However, the difference between moment problems on $\mathbb{R}^2$ and on compact subsets of $\mathbb{R}^2$ is sensible. A series of positive results have been recently obtained by a variety of methods, cf. [Atz, AP, BeM, Cas, McG, Sch]. Below we focus on the results due to Cassier, by restricting ourselves to a class of subalgebraic compact subsets of $\mathbb{R}^2$. Quite specifically, by taking advantage of the complex coordinate on $\mathbb{R}^2$, and by using Agler's dilation theorem, we simplify the main result in [Cas, Théorème 5]. Then we prove that this new result is optimal in some sense, and on the other hand we derive from it a boundedness criterion for formally subnormal, closed operators.

Let $K \subseteq \mathbb{C}$ be a compact set. The K-problem of moments consists of finding necessary and sufficient conditions on a double sequence of complex numbers $a = (a_{mn})_{m,n=0}^{\infty}$ to be represented as

$$a_{mn} = \int z^m \bar{z}^n \, d\mu(z) \qquad (m, n \in \mathbb{Z}_+), \tag{1}$$

where μ is a (finite) positive Borel measure on K.

For K a disc or a region bounded by a curve like $Ax^p + By^q - 1 = 0$, $A, B > 0$, there are simple and not unexpected solutions to this problem; cf. [Atz, McG]. More generally, Cassier has extended these results to a broad class of subalgebraic compact subsets of $\mathbb{C}$, and more recently K. Schmüdgen has completely solved that problem [Sch]. In order to state the main results, we need first some terminology and notation.

A compact set $K \subseteq \mathbb{C}$ is called *admissible* if there exists a real polynomial $P \in \mathbb{R}[x, y]$ with the leading homogeneous part strictly negative on $\mathbb{R}^2 \setminus \{0\}$ and such that $K = P^{-1}(\mathbb{R}_+)$; in other words, K is a subalgebraic subset of $\mathbb{C}$ given by a single inequality condition.

For $Q \in \mathbb{C}[z, \bar{z}]$, $Q(z, \bar{z}) = \sum_{i,j} c_{ij} z^i \bar{z}^j$, we let Qa denote the double sequence given $(Qa)_{mn} := \sum_{i,j} c_{ij} a_{m+i,n+j}$. For an admissible compact set $K = P^{-1}(\mathbb{R}_+)$, Cassier proved in [Cas, Théorème 5] that problem (1) is solvable if and only if the following two kernels are positive definite:

$$(a_{m+q,n+p})_{(m,n),(p,q) \in \mathbb{Z}_+^2} \geqslant 0 \quad \text{and} \quad ((Pa)_{m+q,n+p})_{(m,n),(p,q) \in \mathbb{Z}_+^2} \geqslant 0. \tag{2}$$

As shown by Atzmon [Atz], in the particular case of a disc, i.e.,

$$P(x, y) = c^2 - (x-h)^2 - (y-k)^2,$$

the second positivity condition in (2) can be reduced from $\mathbb{Z}_+^2$ to the marginal subsemigroup $\mathbb{Z}_+ \times \{0\} \subseteq \mathbb{Z}_+^2$:

$$(a_{m+q,n+p})_{(m,n),(p,q) \in \mathbb{Z}_+^2} \geqslant 0 \quad \text{and} \quad ((Pa)_{m,p})_{m,p \in \mathbb{Z}_+} \geqslant 0. \tag{3}$$

Our next aim is to prove that this phenomenon holds for any admissible compact set. First, the equivalent formulation in terms of linear functionals is needed.

As we saw in Section 2, a sequence of complex numbers $a=(a_{mn})_{m,n\in\mathbb{Z}_+}$ can be identified with a linear functional $\lambda=\lambda_a$ on the space of polynomials $\mathbb{C}[\bar{z}, z]$ by

$$\lambda(z^m\bar{z}^n)=a_{mn}, \qquad m, n\in\mathbb{Z}_+.$$

The basic observation is that $(a_{m+q,n+p})_{(m,n),(p,q)\in\mathbb{Z}_+^2}$ is a positive kernel if and only if $\lambda(|r|^2)\geqslant 0$ for any $r\in\mathbb{C}[z,\bar{z}]$. Moreover, a is a K-moment sequence as in (1) if and only if $\lambda(p)\geqslant 0$ for any $p\in\mathbb{R}[x,y]$ with $p|_K\geqslant 0$ (see, for instance, [Cas]).

Thus, for an admissible compact set $K=P^{-1}(\mathbb{R}_+)$, one distinguishes several convex cones of polynomials which are significant for the K-problem of moments:

$$P_+(K):=\{p\in\mathbb{R}[x,y]: P|_K\geqslant 0\},$$
$$S(K):=\mathrm{co}\{|r|^2+P\,|q|^2: r, q\in\mathbb{C}[z,\bar{z}]\},$$

and

$$T(K):=\mathrm{co}\{|r|^2+P\,|q|^2: r\in\mathbb{C}[z,\bar{z}], q\in\mathbb{C}[z]\}.$$

The double sequence a satisfies (1), (2), or (3) if and only if $\lambda_a|_{P_+(K)}\geqslant 0$, $\lambda_a|_{S(K)}\geqslant 0$, or $\lambda_a|_{T(K)}\geqslant 0$, respectively.

THEOREM 3.1. *Let $K=P^{-1}(\mathbb{R}_+)$ be an admissible compact subset of $\mathbb{C}$ and let $a=(a_{mn})_{m,n\in\mathbb{Z}_+}$ be a double sequence of complex numbers. Then the K-problem of moments with data a is solvable if and only if condition* (3) *is satisfied.*

Proof. We have to prove that a linear functional λ on $\mathbb{C}[z,\bar{z}]$ which is non-negative on the cone $T(K)$ has the same property on $P_+(K)$. By adapting "Lemme Fondamental" and "Lemme 5" of [Cas] to the cone $T(K)$, one observes that there exists a positive constant M such that $M-|z|^2\in T(K)$. Then $(M-|z|^2)\,|q|^2\in T(K)$ for every $q\in\mathbb{C}[z]$, so that the functional λ is non-negative on the cone

$$T(K):=\mathrm{co}\{|r|^2+(M-|z|^2)\,|q|^2: r\in\mathbb{C}[z,\bar{z}], q\in\mathbb{C}[z]\}.$$

In view of Agler's theorem, there exists then a subnormal operator S with cyclic vector γ and $\|S\|\leqslant M^{1/2}$, which represents this functional:

$$\lambda(r(z,\bar{z}))=\langle r(S,S^*)\gamma,\gamma\rangle, \qquad r\in\mathbb{C}[z,\bar{z}].$$

By assumption, one knows in addition that $\lambda(P\,|q|^2)\geqslant 0$ for $q\in\mathbb{C}[z]$, which corresponds to the operator inequality $P(S, S^*)\geqslant 0$. According to [McG, Proposition 1], the minimal normal extension of S has the spectrum contained in K, whence there exists a positive Borel measure μ on K which represents the linear functional λ:

$$\lambda(r)=\int r\,d\mu, \qquad r\in\mathbb{C}[z,\bar{z}]. \quad \blacksquare$$

At this moment we know that a functional that is non-negative on the cone $T(K)$ is automatically continuous in the uniform norm on K, and it is non-negative on the larger cone $P_+(K)$. In particular, it follows that $T(K)$ is dense in $P_+(K)$ in the uniform norm on K.

Theorem 2.1 above shows that the positivity of a functional is not necessarily transferred from the cone

$$T_0(K):=\operatorname{co}\{|p+q\bar{r}|^2+P\,|s|^2 : p, q, r, s\in\mathbb{C}[z]\}$$

to $T(K)$, and a fortiori to $P_+(K)$, for any disc K. Next we prove the same fact for an arbitrary admissible compact subset K with nonempty interior.

PROPOSITION 3.2. *Let K be an admissible compact subset of $\mathbb{C}$. If* $\operatorname{int} K\neq\phi$, *then there exists a homogeneous polynomial of degree 4 in $T(K)$ that can be separated from $T_0(K)$ by a real hyperplane.*

Proof. By using a suitable translation, we may assume that $0\in\operatorname{int} K$, and that $P(0)>0$. Let $Q:=P-P(0)$. The proof of Theorem 2.1 can be repeated for $P(0)+Q$ instead of $1-|z|^2$. More exactly, the polynomial $q_0(z,\bar{z}):=|-\sqrt{2}\,z\bar{z}+z^2+\bar{z}^2|^2$ does not belong to $T_0(K)^{\mathrm{h}}_4$. By repeating the extension construction, one finds a functional Λ on $\mathbb{R}[x,y]$ with $\lambda(q_0)<0$ and $\lambda|_{T_0(K)}\geqslant 0$, as desired. $\blacksquare$

This shows that we should not expect the K-problem of moments to be resolved by a condition simpler than (3).

Let $\mathbf{T}$ be the unit circle and let $A:\mathbb{C}[z,\bar{z}]\to\mathbb{C}[\rho]$ be the average operator

$$A(r)(\rho):=\frac{1}{2\pi i}\int_{\mathbf{T}} r(tz,\bar{t}\bar{z})\,\frac{dt}{t} \qquad (r\in\mathbb{C}[z,\bar{z}]),$$

where $\rho:=|z|^2$; that is, $A(z^m\bar{z}^n)=0$ if $m\neq n$ and $A(|z|^{2n})=\rho^n$. In the case of $K=\bar{\mathbb{D}}$, the closed unit disc, Proposition 3.2 above and Proposition 3.2 in [McCP] show that the images of the cones $T_0(K)$ and $T(K)$ through A can be separated by a hyperplane in $\mathbb{R}[\rho]$. Since every non-negative linear functional on $AT_0(K)$ is automatically continuous in the uniform norm on

$[0, 1]$, and by computing the generic element in $AT_0(K)$, one can thus establish the following corollary (see [McCP] for details).

COROLLARY 3.3. *The convex cone*

$$\operatorname{co}\left\{\sum_{i\geqslant 0}\rho^i\left|b_i+\sum_{j\geqslant 0}a_{i+j}c_j\rho^j\right|^2+\sum_{i\geqslant 1}\rho^i\left|\sum_{j\geqslant 0}a_jc_{i+j}\rho^j\right|^2\right.$$
$$\left.+(1-\rho)\cdot\sum_{i\geqslant 0}\rho^i\,|d_i|^2,\, a_i, b_i, c_i, d_i\in\mathbb{C}\right\}$$

is not uniformly dense in the set of all non-negative polynomials on $[0, 1]$. *(The sums under the convex hull symbol are all taken to be finite.)*

We notice that every non-negative polynomial on $[0, 1]$ is a finite convex combination of elements of the form $p^2+\rho q^2+(1-\rho)\,r^2$, where $p, q, r\in\mathbb{R}[\rho]$ (cf. [Akh].)

By reversing Agler's dictionary, Theorem 3.1 above provides a nontrivial boundedness criterion for formally subnormal, a priori unbounded operators. There are a number of subnormality notions for unbounded closed operators (cf. [StSz]). For us, a *formally subnormal* operator with cyclic vector γ is a densely defined closed operator S on $\mathscr{H}$, such that (i) $\gamma\in\operatorname{Dom}(S^n)$ for every $n\geqslant 0$; (ii) the vectors $\gamma, S\gamma, S^2\gamma, \ldots$, span $\mathscr{H}$; and (iii) the Bram–Halmos condition on these vectors is satisfied; that is,

$$\langle S^i\gamma_j, S^j\gamma_i\rangle_{i,j\in\mathbb{Z}_+}$$

is a positive definite kernel for any $\gamma_i\in\mathscr{D}:=\mathbb{C}\langle\gamma, S\gamma, S^2\gamma, \ldots\rangle$. Examples of such operators can easily be constructed from positive measures on $\mathbb{C}$ which have all moments finite.

For any polynomial $r(z, \bar{z})=\sum c_{ij}z^i\bar{z}^j$ we write

$$\langle r(S, S^*)\xi, \xi\rangle := \sum c_{ij}\langle S^i\xi, S^j\xi\rangle, \qquad \xi\in\mathscr{D}.$$

Because $\mathscr{D}$ is dense in $\mathscr{H}$, one may speak unambiguously of positivity for an operator like $r(S, S^*)$.

PROPOSITION 3.4. *Let S be a formally subnormal operator with cyclic vector γ and let $\mathscr{D}:=\mathbb{C}\langle S^n\gamma : n\geqslant 0\rangle$. If $K=P^{-1}([0, +\infty))$ is an admissible compact subset of $\mathbb{C}$ and $P(S, S^*)\geqslant 0$, then the operator S is bounded subnormal and its minimal normal extension has spectrum contained in K.*

Proof. The functional $\Lambda(r):=\langle r(S, S^*)\gamma, \gamma\rangle$, $r\in\mathbb{C}[z, \bar{z}]$, is non-negative on the cone $T(K)$ by assumption. We now simply repeat the proof of Theorem 3.1. ∎

We stated the last proposition separately because it seems to be difficult (and so even more interesting) to find an operator-theoretic proof of it. For instance, if $P(z, \bar{z}) := -|z|^{2n} + Q(z, \bar{z})$, with degree of Q less than $2n$, the inequality $P(S, S^*) \geqslant 0$ is equivalent to

$$\|S^n \xi\|^2 \leqslant \sum_{i+j<2n} c_{ij} \langle S^i \xi, S^j \xi \rangle \qquad (\xi \in \mathscr{D}),$$

for certain coefficients $c_{ij} \in \mathbb{C}$. Even in this apparently simple case the boundedness of S is not at hand, but in turn it follows from Cassier's techniques.

Theorem 3.1 can easily be generalized to $\mathbb{C}^n$, by using again "Lemme Fondamental" and "Lemme 5" in [Cas] in conjunction with the multi-dimensional analogue of the Bram–Halmos criterion for subnormality (see for instance [AP, Proposition 0]). Specifically, let $z = (z_1, ..., z_n)$ denote the complex coordinates in $\mathbb{C}^n$ and let $P(z)$ be an admissible polynomial, that is

$$P(z) = -a_1(\operatorname{Re} z_1)^{2p} - b_1(\operatorname{Im} z_1)^{2p} - \cdots - a_n(\operatorname{Re} z_n)^{2p} - b_n(\operatorname{Im} z_n)^{2p}$$
$$+ Q(z) + \text{lower degree terms},$$

where Q is an homogeneous sum of degree $2p$ of squares of absolute values of elements in $\mathbb{C}[z, \bar{z}]$, $a_i > 0$, $b_i > 0$, $1 \leqslant i \leqslant n$.

If we now define the cones $S(K)$ and $T(K)$ as before, with $K := P^{-1}([0, +\infty))$, an argument identical to the one used in the proof of Theorem 3.1 shows that a functional λ on $\mathbb{C}[z, \bar{z}]$ is represented by a positive measure supported on K if and only if $\lambda|_{T(K)} \geqslant 0$.

This observation extends the results of [AP] and [McG]. However, we do not know whether a similar reduction (i.e., $\lambda|_{S(K)} \geqslant 0 \Leftrightarrow \lambda|_{T(K)} \geqslant 0$) is valid on an arbitrary sub-algebraic subset of $\mathbb{C}^n$.

4. Concluding Remarks and Open Problems

As a consequence of the main result of Section 2, we know that the class of polynomially hyponormal operators is distinct from the class of subnormal operators. A natural question now is how much this distinction is reflected by the properties of polynomially hyponormal operators. In this section we pose some problems which may help answer this question.

Problem 4.1. Are polynomially hyponormal operators reflexive? Do they at least have nontrivial invariant subspaces?

It is known that subnormal operators are reflexive, and that a large part of hyponormal operators have invariant subspaces (see [Con, MP, Tho]). An answer to Problem 4.1 would necessarily shed light on the dilation theory (and implicitly on the desired functional models) for polynomially hyponormal operators. We mention that rationally hyponormal operators do have nontrivial subspaces: If $R(\sigma(T)) = C(\sigma(T))$, T is a von Neumann operator, so the main result in [Ag 1] applies; if $R(\sigma(T)) \neq C(\sigma(T))$, then [Br] can be used.

Problem 4.2. Are the classes of polynomially hyponormal, rationally (with n distinct poles) hyponormal, and analytically hyponormal operators all different?

The method of proving Theorem 2.1 above was essentially based on the analysis of linear functionals on convex cones of polynomials. A possible approach to Problem 4.2 would consist of extending this analysis to cones of Laurent series or analytic functions defined on more complicated domains.

An important invariant for operators with trace-class self-commutator is the principal function introduced by J. D. Pincus (see [Cla, MP, Xia]). It is known that this function is integer-valued for subnormal operators ([CP]).

Problem 4.3. Let T be a polynomially hyponormal operator with trace-class self-commutator. Is the principal function of T integer-valued?

The rather involved proof in [CP] relies, in an essential way, on the properties of the minimal normal extension of a subnormal operator. A solution to Problem 4.3 would automatically lead to a better understanding of R. Carey and J. D. Pincus' result.

Problem 4.4. Classify the polynomially hyponormal operators with finite-rank self-commutator.

The subnormal operators with finite-rank self-commutator have recently been classified by R. Olin *et al.* [OOT], and independently by D. Xia [Xia2]. This classification is remarkably rigid; a positive answer to Problem 4.3 would probably suggest that the same will be true of polynomially hyponormal operators.

Problem 4.5. What is the dilation and extension theory for polynomially hyponormal operators?

Problem 4.5 asks for specific functional models for polynomially hyponormal operators, and for the identification of function-theoretic phenomena which pertain exclusively to that class. Along the same line of thought, and dealing with a well-investigated collection of operators, we can formulate the next problem. Recall first that, according to a theorem of C. Berger, a weighted shift W_α is subnormal if and only if the Hankel forms $(\gamma_{i+j})_{i,j=0}^{\infty}$ and $(\gamma_{i+j+1})_{i,j=0}^{\infty}$ are positive semi-definite, where $\{\gamma_i\}$ is the sequence of moments of α (cf. [Cu2, Remark after Corollary 5.4]).

Problem 4.6. Is there an analogue of Berger's Theorem for polynomially hyponormal weighted shifts? Alternatively, is there a matricial characterization of polynomial hyponormality for weighted shifts which parallels the above mentioned one for subnormal shifts?

Even when it could be argued that the class of polynomially hyponormal operators remains a bit artificial (mainly due to the lack of concrete nontrivial examples), its relevance in the study of both hyponormal and subnormal operators is now apparent; this gives at least one reason for pursuing a detailed investigation of polynomial hyponormality.

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