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A generalization of Quillen's small object argument

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Abstract

We generalize the small object argument in order to allow for its application to proper classes of maps (as opposed to sets of maps in Quillen's small object argument). The necessity of such a generalization arose with appearance of several important examples of model categories which were proven to be non-cofibrantly generated [J. Adámek, H. Herrlich, J. Rosický, W. Tholen, Weak factorization systems and topological functors, *Appl. Categ. Structures* 10 (3) (2002) 237–249 [2]; Papers in honour of the seventieth birthday of Professor Heinrich Kleisli (Fribourg, 2000); B. Chorny, The model category of maps of spaces is not cofibrantly generated, *Proc. Amer. Math. Soc.* 131 (2003) 2255–2259; J.D. Christensen, M. Hovey, Quillen model structures for relative homological algebra, *Math. Proc. Cambridge Philos. Soc.* 133 (2) (2002) 261–293; D.C. Isaksen, A model structure on the category of pro-simplicial sets, *Trans. Amer. Math. Soc.* 353 (2001) 2805–2841]. Our current approach allows for construction of functorial factorizations and localizations in the equivariant model structures on diagrams of spaces [E.D. Farjoun, Homotopy theories for diagrams of spaces, *Proc. Amer. Math. Soc.* 101 (1987) 181–189] and diagrams of chain complexes. We also formulate a non-functorial version of the argument, which applies in two different model structures on the category of pro-spaces [D.A. Edwards, H.M. Hastings, Čech and Steenrod homotopy theories with applications to geometric topology, *Lecture Notes in Mathematics*, vol. 542, Springer, Berlin, 1976; D.C. Isaksen, A model structure on the category of pro-simplicial sets, *Trans. Amer. Math. Soc.* 353 (2001) 2805–2841].

The examples above suggest a natural extension of the framework of cofibrantly generated model categories. We introduce the concept of a class-cofibrantly generated model category, which is a

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model category generated by classes of cofibrations and trivial cofibrations satisfying some reasonable assumptions.

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1. Introduction

Quillen's definition of a model category has been slightly revised over the last decade. The changes applied to the first axiom MC1 requiring the existence of all finite limits and colimits, and to the last axiom MC5 requiring the existence of factorizations. The modern approaches to the subject [15,17] demand the existence of all small limits and colimits in MC1. This gives some technical advantages while treating transfinite constructions, such as localizations, in model categories. The modern version of the axiom MC5 requires the factorizations to be functorial. Unfortunately we will not be able to accept this stronger form of MC5, since we will be interested also in model categories for which the existence of functorial factorizations is unknown.

The most widely known model category without functorial factorizations is the category of pro-spaces or, more generally, of pro-objects (in the sense of Grothendieck) in a proper model category \mathcal{C} [11,23] and its Bousfield localization modelling the étale homotopy theory [4,18]. We introduce a new construction of factorizations in these model categories, but our construction still lacks functoriality.

The main tool for the construction of (functorial) factorizations in model categories and localizations thereof is Quillen's small object argument [15,17,26]. However, in its original form, the argument is applicable neither to the category of diagrams with the equivariant model structure [10], nor to pro-categories, since it allows for the application in cofibrantly generated model categories only. We propose here a generalization which may be used in a wider class of model categories. The collections of generating cofibrations and generating trivial cofibrations may now form proper classes, satisfying the conditions of the following theorem:

Theorem 1.1 (*The generalized small object argument*). *Suppose \mathcal{C} is a category containing all small colimits, and I is a class of maps in \mathcal{C} satisfying the following conditions:*

- (1) *There exists a cardinal κ , such that each element $A \in \text{dom}(I)$ is κ -small relative to I -cof.*
- (2) *For every map $f \in \text{Map } \mathcal{C}$ there exists a (resp., functorially assigned) map $S(f) \in I$ -cof equipped with a (resp., natural) morphism of maps $t_f : S(f) \rightarrow f$, such that any morphism of maps $i \rightarrow f$ with $i \in I$ factors through the (resp., natural) map t_f .*

Then there is a (resp., functorial) factorization (γ, δ) on \mathcal{C} such that, for all morphisms f in \mathcal{C} , the map $\gamma(f)$ is in I -cof and the map $\delta(f)$ is in I -inj.

Remark 1.2. Note that the theorem above contains two sets of conditions and two statements: about existence of functorial and non-functorial factorizations. In the rest of the

paper we refer to these statements as to the functorial and the non-functorial versions of the argument respectively. The proof of the theorem is given only for the functorial version. The adaptation of the proof for the non-functorial version may be achieved by removing the verification of the functoriality.

We say that a class I of maps in a category \mathcal{C} permits the (resp., functorial) generalized small object argument if it satisfies conditions (1) and (2) of Theorem 1.1.

This theorem is the second attempt by the author to generalize the small object argument. The previous version appeared in the study of the equivariant localizations of diagrams of spaces [6]. The specific properties of the equivariant model category of D -shaped diagrams of spaces and also the non-functorial factorization technique developed by Dror Farjoun in [10] suggested a rather complicated technical notion of instrumentation. It is essentially a straightforward “functorialization” of Dror Farjoun’s ideas, which contain, implicitly, the non-functorial version of the generalized small object argument. The classes of generating cofibrations and generating trivial cofibrations of diagrams satisfy the conditions of instrumentation, but it is difficult (or impossible) to verify these conditions in other model categories. The conditions of Theorem 1.1 on the class I of maps are easier to handle and also more general than those of instrumentation, as we explain in Section 3.

This paper shows also that two rather different homotopy theories of pro-spaces and of diagrams of spaces fit into a certain joint framework. In order to describe the similarity between them let us give the following

Definition 1.3. A model category \mathcal{C} is called *class-cofibrantly generated* if

- (1) there exists a class I of maps in \mathcal{C} (called a class of *generating cofibrations*) that permits (either functorial or non-functorial version of) the generalized small object argument and such that a map is a trivial fibration if and only if it has the right lifting property with respect to every element of I , and
- (2) there exists a class J of maps in \mathcal{C} (called a class of *generating trivial cofibrations*) that permits (either functorial or non-functorial version of) the generalized small object argument and such that a map is a fibration if and only if it has the right lifting property with respect to every element of J .

The categorical dual to a class-cofibrantly generated model category is called *class-fibrantly generated*.

The purpose of this paper is to give several non-trivial examples of class-cofibrantly and class-fibrantly generated model categories. One of the model categories we discuss here is new, others are classical and thus we only give a new construction of factorizations, applying the current version of the small object argument.

In particular, we show that the equivariant model structure on the diagrams of spaces is class-cofibrantly generated, construct the equivariant model structure on the diagrams of chain complexes and prove that both known model structures on the category of pro-spaces are class-fibrantly generated.

Although the non-functorial version of the argument proves a weaker result, it is useful in many model categories, where the existence of functorial factorizations is still an open

question. Recent applications of the non-functorial version of the generalized small object argument include new model structures on pro-spaces [20], pro-spectra [22] and on small diagrams over a large category [7].

The applications of Quillen’s small object argument are not limited to abstract homotopy theory. A similar argument is used, for example, in the theory of categories to construct reflections in a locally presentable category with respect to a small orthogonality class [3, 1.36]. Recently another generalization of the small object argument was considered by the category theorists Adámek et al. [1]. Their version of the argument applies to the “injective subcategory problem” in locally ranked categories—a generalization of the notion of a locally presentable category which includes topological spaces. We hope that our generalization of the small object argument will be applicable to the “orthogonal subcategory problem” and “injective subcategory problem” with respect to some reasonable classes of morphisms.

The rest of the paper is organized as follows: Section 2 is devoted to the proof of the generalized small object argument. The non-functorial version of the argument is left for the reader. Next, we review some of our previous results about the diagrams of spaces in Section 3 and show how they fit into the newly established framework. We extend this approach to the diagrams of chain complexes in Section 4. After providing the necessary preliminaries on pro-categories in Section 5 we apply the generalized cosmall object argument in Section 6.

2. Proof of the generalized small object argument

Proof of Theorem 1.1. Given a cardinal κ such that every domain of I is κ -small relative to I -cof, we let λ be a κ -filtered ordinal (i.e., λ is a limit ordinal and, if $A \subset \lambda$ and $|A| \leq \kappa$, then $\sup A < \lambda$).

To any map $f : X \rightarrow Y$ we will associate a functor $Z^f : \lambda \rightarrow \mathcal{C}$ such that $Z_0^f = X$, and a natural transformation $\rho^f : Z^f \rightarrow Y$ factoring f , i.e., for each $\beta < \lambda$ the triangle

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow & \nearrow \rho_\beta^f \\ & Z_\beta^f & \end{array}$$

is commutative. Each map $i_\beta^f : Z_\beta^f \rightarrow Z_{\beta+1}^f$ will be a pushout of a map of the form $S(f)$, i.e., $i_\beta^f \in I$ -cof, since I -cof is closed under pushouts.

We will define Z^f and $\rho^f : Z^f \rightarrow Y$ by transfinite induction, beginning with $Z_0^f = X$ and $\rho_0^f = f$. If we have defined Z_α^f and ρ_α^f for all $\alpha < \beta$ for some limit ordinal β , define $Z_\beta^f = \text{colim}_{\alpha < \beta} Z_\alpha^f$, and define ρ_β^f to be the map induced, naturally, by the ρ_α^f . Having defined Z_β^f and ρ_β^f , we define $Z_{\beta+1}^f$ and $\rho_{\beta+1}^f$ as follows. Consider the natural map $\iota(\rho_\beta^f) : S(\rho_\beta^f) \rightarrow$

ρ_β^f , i.e., the following commutative square:

$$\begin{array}{ccc} A & \xrightarrow{\text{dom}(t(\rho_\beta^f))} & Z_\beta^f \\ S(\rho_\beta^f) \downarrow & & \rho_\beta^f \downarrow \\ B & \xrightarrow{\text{codom}(t(\rho_\beta^f))} & Y. \end{array}$$

Define $Z_{\beta+1}^f$ to be the pushout of this diagram and define $\rho_{\beta+1}^f$ to be the map naturally induced by ρ_β^f .

For each morphism $g = (g^1, g^2): f_1 \rightarrow f_2$ in the category $\text{Map } \mathcal{C}$, i.e., for each commutative square

$$\begin{array}{ccc} X_1 & \xrightarrow{g^1} & X_2 \\ f_1 \downarrow & & f_2 \downarrow \\ Y_1 & \xrightarrow{g^2} & Y_2 \end{array}$$

we define a natural transformation $\xi^g: Z^{f_1} \rightarrow Z^{f_2}$ by transfinite induction over small ordinals, beginning with $\xi_0^g = g^1$. If we have defined ξ_α^g for all $\alpha < \beta$ for some limit ordinal β , define $\xi_\beta^g = \text{colim}_{\alpha < \beta} \xi_\alpha^g$. Having defined ξ_β^g , we define $\xi_{\beta+1}^g: Z_{\beta+1}^{f_1} \rightarrow Z_{\beta+1}^{f_2}$ to be *natural* map induced by $g_\beta = (\xi_\beta^g, g^2): \rho_\beta^{f_1} \rightarrow \rho_\beta^{f_2}$, namely the *unique* map between the pushouts of the horizontal lines of the following diagram which preserves its commutativity:

$$\begin{array}{ccccc} B_1 & \xleftarrow{S(\rho_\beta^{f_1})} & A_1 & \xrightarrow{\text{dom}(t(\rho_\beta^{f_1}))} & Z_\beta^{f_1} \\ h_2 \downarrow & & h_1 \downarrow & & \downarrow \xi_\beta^g \\ B_2 & \xleftarrow{S(\rho_\beta^{f_2})} & A_2 & \xrightarrow{\text{dom}(t(\rho_\beta^{f_2}))} & Z_\beta^{f_2}. \end{array}$$

In this diagram $(h_1, h_2) = S(g_\beta)$. The commutativity of the diagram follows readily, since S is a functor and t is a natural transformation.

The required functorial factorization (γ, δ) is obtained when we reach the limit ordinal λ in the course of our induction. Then we define $\gamma(f): X \rightarrow Z_\lambda^f$ to be the (transfinite) composition of the pushouts, and $\delta(f) = \rho_\lambda^f: Z_\lambda^f \rightarrow Y$. $\gamma(f) \in I\text{-cof}$ since $I\text{-cof}$ is closed under transfinite compositions.

To complete the definition of the functorial factorization (see [17, 1.1.1], [16, 1.1.1]) we need to define for each morphism $g: f_1 \rightarrow f_2$ a natural map $(\gamma, \delta)^g: Z_\lambda^{f_1} \rightarrow Z_\lambda^{f_2}$ which makes the appropriate diagram commutative. Take $(\gamma, \delta)^g = \xi_\lambda^g$.

It remains to show that $\delta(f) = \rho_\lambda^f$ has the right lifting property with respect to I . To see this, suppose we have a commutative square as follows:

$$\begin{array}{ccc} C & \xrightarrow{h'} & Z_\lambda^f \\ l \downarrow & & \downarrow \rho_\lambda^f \\ D & \xrightarrow{k'} & Y \end{array}$$

where l is a map of I . Due to the first condition of the theorem the object C is κ -small relative to I -cof, i.e., there is an ordinal $\beta < \lambda$ such that h' is the composite $C \xrightarrow{h_\beta} Z_\beta^f \rightarrow Z_\lambda^f$. Hence we obtain the following commutative diagram:

$$\begin{array}{ccc} C & \xrightarrow{h_\beta} & Z_\beta^f \\ l \downarrow & & \downarrow \rho_\beta^f \\ & & Z_\lambda^f \\ D & \xrightarrow{k'} & Y \end{array}$$

$\rho_\lambda^f \downarrow$

The second condition of the theorem implies that there exists a factorization in the category $\text{Map } \mathcal{C}$ of the map (h_β, k') through $t(\rho_\beta^f)$ which is a map of maps with domain $S(\rho_\beta^f) : A \rightarrow B$ and range ρ_β^f , i.e., there is a commutative diagram

$$\begin{array}{ccccc} & & h_\beta & & \\ & & \curvearrowright & & \\ C & \longrightarrow & A & \xrightarrow{h} & Z_\beta^f \\ l \downarrow & & \downarrow S(\rho_\beta^f) & & \downarrow \rho_\beta^f \\ D & \longrightarrow & B & \xrightarrow{k} & Y \\ & & \downarrow \rho_\lambda^f & & \downarrow \rho_\lambda^f \\ & & Y & & Y \end{array}$$

k'

where $(h, k) = t(\rho_\beta^f)$.

By construction, there is a map $B \xrightarrow{k_\beta} Z_{\beta+1}^f$ such that $k_\beta S(\rho_\beta^f) = i_\beta^f h$ and $k = \rho_{\beta+1}^f k_\beta$, where i_β^f is the map $Z_\beta^f \rightarrow Z_{\beta+1}^f$. The composition $D \rightarrow B \xrightarrow{k_\beta} Z_{\beta+1}^f \rightarrow Z_\lambda^f$ is the required lift in the initial commutative square. \square

Remark 2.1. In all the applications we have in mind, the map $S(f)$ is a coproduct of maps from I . Hence the construction above provides us with the factorization of any map f into an I -cellular map followed by an I -injective map, as in the classical construction. But we prefer to leave the formulation of conditions on the class I of maps in the present (simpler) form, since we hope that they will be useful elsewhere and we do not see a big advantage in I -cellular maps instead of I -cofibrations.

3. Example: instrumented classes of maps permit the generalized small object argument

Let us recall, in an informal manner, the notion of instrumentation introduced in [6]. Instrumentation for a class I of maps in a category \mathcal{C} is a formalization of the following functorial version of the classical cosolution-set condition: for any morphism f in \mathcal{C} there is a *naturally* assigned set of maps $\mathcal{I}(f) = \{i \rightarrow f \mid i \in I\}$, such that for any morphism of maps $j \rightarrow f$ with $j \in I$ there exists a factorization $j \rightarrow i \rightarrow f$ with $(i \rightarrow f) \in \mathcal{I}(f)$. Additionally, every domain of a map in I is κ -small with respect to I -cell for some fixed cardinal κ .

Proposition 3.1. *Any instrumented class of maps I in a category \mathcal{C} permits the generalized small object argument.*

Proof. The first condition of Theorem 1.1 is satisfied because of the same assumption for instrumented classes of maps.

Instrumentation gives rise to the augmented functor S in the following way:

$$S(f) = \coprod \text{dom}(\mathcal{I}(f)) = \coprod \{i \mid (i \rightarrow f) \in \mathcal{I}(f)\}.$$

Naturality of \mathcal{I} ensures the functoriality of S . The augmentation $t_f: S(f) \rightarrow f$ exists since every i is equipped with a map into f , hence their coproduct is naturally mapped into f . Certainly $S(f) \in I\text{-cof}$, and the factorization property follows from the similar property of instrumentation. \square

Instrumented classes of maps were applied to the study of *equivariant* model structures on diagrams of spaces. Let D be a small category. It was essentially shown in [6] that the category of D -shaped diagrams of spaces (by the category of spaces we mean here either the category of simplicial sets, or the category of compactly generated topological spaces with the standard simplicial model structure) is class-cofibrantly generated. In this section, we show that the classes of generating cofibrations and generating trivial cofibrations permit (the new version of) the generalized small-object argument.

Let us recall first the definition of the equivariant model structure on \mathcal{S}^D initially introduced in [10]. We use the word *collection* to denote a set or a proper class with respect to some fixed universe \mathfrak{U} . A D -diagram Q of spaces is called an *orbit* if $\text{colim}_D Q = *$. We denote by \mathcal{O}_D the collection of all orbits of D (which is not necessarily a set). For any diagram \underline{W} and

a map $f : \underline{X} \rightarrow \underline{Y}$, there is an induced map of simplicial sets $\text{map}(\underline{W}, f) : \text{map}(\underline{W}, \underline{X}) \rightarrow \text{map}(\underline{W}, \underline{Y})$; see [9] for details.

Definition 3.2. In the equivariant model structure on \mathcal{S}^D a morphism $f : \underline{X} \rightarrow \underline{Y}$ is a

- *weakequivalence* if and only if $\text{map}(\underline{Q}, f)$ is a weak equivalence of spaces for any orbit \underline{Q}
- *fibration* if and only if $\text{map}(\underline{Q}, f)$ is a fibration of spaces for any orbit \underline{Q} ;
- *cofibration* if and only if it has the left lifting property with respect to any trivial fibration.

The standard axioms of simplicial model categories were verified in [10] for the equivariant model structure on \mathcal{S}^D . Functorial factorizations were constructed in [6]. In that construction we used a different version of the generalized small-object argument, which applied only to instrumented classes of maps. Therefore Proposition 3.1 shows that Theorem 1.1 generalizes the version of the small-object argument that appeared in [6].

Let $I = \{ \underline{Q} \otimes \partial \Delta^n \hookrightarrow \underline{Q} \otimes \Delta^n \mid \underline{Q} \in \mathcal{O}_D, n \geq 0 \}$ and $J = \{ \underline{Q} \otimes A_k^n \xrightarrow{\sim} \underline{Q} \otimes \Delta^n \mid \underline{Q} \in \mathcal{O}_D, n \geq k \geq 0 \}$ be two classes of maps in \mathcal{S}^D . If the index category D is such that \mathcal{O}_D is a set (this happens, for example, when D is a group), then the collections I and J are sets of cofibrations and the equivariant model structure on \mathcal{S}^D is cofibrantly generated with I equal to the set of generating cofibrations and J equal to the set of generating trivial cofibrations. But usually I and J are proper classes of maps, and it was shown in [6] that they form classes of generating cofibrations and generating trivial cofibrations in the equivariant model structure on the D -shaped diagrams of spaces, which is class-cofibrantly generated. Proposition 3.1 implies that the classes I and J permit the functorial version of the generalized small object argument, since the classes I and J are instrumented.

4. Equivariant model structure on the diagrams of chain complexes

Let R be a unitary ring. The category $\text{Ch}(R)$ of (unbounded) chain complexes of R -modules carries a cofibrantly generated model structure with weak equivalences being quasi-isomorphisms and fibrations being levelwise surjections of chain complexes [14,17]. The purpose of this section is to extend this model structure to the category of diagrams of chain complexes with *equivariant* weak equivalences (as opposed to objectwise).

Let \mathcal{A} be a complete and cocomplete abelian category and \mathcal{P} be a *projective class* in \mathcal{A} ; see [8] for the definition. Then, under certain conditions on \mathcal{P} , the category of chain complexes in \mathcal{A} carries the relative model structure: a map $f : A \rightarrow B$ is a weak equivalence or a fibration if the induced map $\text{hom}(P, f) : \text{hom}(P, A) \rightarrow \text{hom}(P, B)$ is a weak equivalence or a fibration in the standard model structure on $\text{Ch}(\mathbb{Z})$. We exhibit below a projective class in the category \mathcal{A} of D -shaped diagrams of R -modules which satisfies the technical conditions that ensure the existence of the relative model structure.

To fix notation, assume that the differentials lower degree. For an object X of $\text{Ch}(R)$, the suspension ΣX of X has $(\Sigma X)_n = X_{n-1}$ and $d_{\Sigma X} = -d_X$. Given an R -module M , by abuse of notation we denote by M the chain complex concentrated in degree 0; $D^k M$ denotes the

(contractible) complex such that $(D^k M)_n = M$ if $n = k$ or $n = k - 1$ but $(D^k P) = 0$ for other values on n , and whose differential is the identity in degree k .

Let D be a small category. A D -diagram of sets \underline{T} is called an orbit if, as before, $\text{colim}_D \underline{T} = *$. And let \mathcal{O}_D be the collection of all D -orbits. For every orbit \underline{T} we associate a diagram of free R -modules $P_{\underline{T}} = R(\underline{T})$. Recall that the free functor is the left adjoint of the forgetful functor $U(\cdot)$.

Let $\mathcal{P}' = \{P_{\underline{T}} \mid \forall \underline{T} \in \mathcal{O}_D\}$, and let \mathcal{P} be the projective class determined by \mathcal{P}' , i.e., if \mathcal{E} is the class of \mathcal{P}' -epic maps, then \mathcal{P} is precisely the class of all diagrams P of chain complexes such that each map in \mathcal{E} is P -epic. In other words, \mathcal{P}' is a collection of enough projectives for \mathcal{P} .

The same argument as in [6, 3.1] shows that $P_{\underline{T}}$ are \aleph_0 -small relative to split monomorphisms with \mathcal{P} -projective cokernel. Therefore, the projective class \mathcal{P} has enough \aleph_0 -small projectives.

In order to conclude that the \mathcal{P} -relative model structure on the category $\text{Ch}(\mathcal{A})$ exists it suffices, by [8, 2.2(B)], to prove that \mathcal{P} -resolutions can be chosen functorially. We suggest the following construction: for every diagram of R -modules $\underline{X} \in \mathcal{A}$, let $\mathcal{O}_{\underline{X}} = \{*\times_{U(\text{colim}_D \underline{X})} U \underline{X} \mid x : * \rightarrow U(\text{colim}_D \underline{X})\}$ be the set of orbits over each point $x \in U(\text{colim}_D \underline{X})$. Every element $\underline{T}_x \in \mathcal{O}_{\underline{X}}$ is equipped with the projection map $\pi_x : \underline{T}_x \rightarrow U \underline{X}$. Consider the collection of adjoint maps $\varphi_x : P_{\underline{T}_x} \rightarrow \underline{X}$, for all $x \in U(\text{colim}_D \underline{X})$. We define the \mathcal{P} -resolution functor $P \underline{X} = \bigoplus_{x:*\rightarrow U(\text{colim}_D \underline{X})} P_{\underline{T}_x}$ and the canonical \mathcal{P} -epic map $\varepsilon_{\underline{X}} : P \underline{X} \rightarrow \underline{X}$ is induced by the maps φ_x .

We have to verify that $\varepsilon_{\underline{X}}$ is indeed \mathcal{P} -epic. It suffices to show that $\varepsilon_{\underline{X}}$ is \mathcal{P}' -epic, i.e., for every map $P_{\underline{T}} \rightarrow \underline{X}$ we must construct a factorization $P_{\underline{T}} \rightarrow P \underline{X} \rightarrow \underline{X}$. By adjointness this is equivalent to factorization of the map $\underline{T} \rightarrow U \underline{X}$ through $UP \underline{X}$. The later factorization exists by construction of $P \underline{X}$: denote the induced map $\text{colim}_D \underline{T} = * \rightarrow U(\text{colim}_D \underline{X})$ by x , then there exists a factorization $\underline{T} \rightarrow \underline{T}_x \rightarrow U \underline{X}$ which can be further refined to $\underline{T} \rightarrow \underline{X} \rightarrow UP_{\underline{T}_x} \rightarrow UP \underline{X} \rightarrow U \underline{X}$, where the second map is the unit of the adjunction and the third map is the inclusion.

This finishes the proof that there exist the equivariant model structure on the category of diagrams of chain complexes. But our main motivation in this example is to find another class-cofibrantly generated model category. This model category is generated by classes $I = \{\Sigma^{n-1} P_{\underline{T}} \rightarrow D^n P_{\underline{T}} \mid n \in \mathbb{Z}, P_{\underline{T}} \in \mathcal{P}'\}$ and $J = \{0 \rightarrow D^n P_{\underline{T}} \mid n \in \mathbb{Z}, P_{\underline{T}} \in \mathcal{P}'\}$ of generating cofibrations and generating trivial cofibrations respectively. Lemma [8, 5.5] implies that \mathcal{P} -relative fibrations are those maps that have the right lifting property with respect to J , and \mathcal{P} -relative trivial fibrations are those maps that have the right lifting property with respect to I .

We only need to show that the classes I and J satisfy the functorial version of the generalized small object argument. The domains of the elements of J are obviously \aleph_0 -small. The domains of the elements of I are \aleph_0 -small by Lemma [8, 4.3] since every element of \mathcal{P}' is \aleph_0 -small. It remains to verify the existence of an augmented functorial construction which associates to every map $f : X \rightarrow Y$ between diagram of chain complexes (=chain complexes of diagrams of R -modules) a (trivial) cofibration $F(f)$ with a natural map $t : F(f) \rightarrow f$, such that for every element of $i \in I (j \in J)$, any map $i \rightarrow f$ factors through t . We will give the construction for I . The construction for J is similar.

Given $i : \Sigma^{n-1} P_T \rightarrow D^n P_T$, the maps from i to f are in bijective correspondence with commutative squares:

$$\begin{array}{ccc} P_T & \longrightarrow & X_{n-1} \\ \downarrow & & \downarrow f_{n-1} \\ Y_n & \xrightarrow{d} & Y_{n-1}. \end{array}$$

Let $W_n = X_{n-1} \times_{Y_{n-1}} Y_n$, then the commutative squares above are in bijective correspondence with the maps $P_T \rightarrow W_n$ which are in bijective correspondence with the maps $T \rightarrow U W_n$, by adjointness. Let \mathcal{O}_{W_n} be the set of orbits over each point in $U W_n$. Then we define

$$F(f) = \bigoplus_{n \in \mathbb{Z}} \left(\bigoplus_{Q \in \mathcal{O}_{W_n}} \begin{array}{c} \Sigma^{n-1} P_Q \\ \downarrow \\ D^n P_Q \end{array} \right).$$

The augmentation map is induced by the maps $P_Q \rightarrow W_n$ and the factorization property is readily verified. The functoriality of the constructions follows from the naturality on each step.

We have shown that the equivariant model structure on the category of diagrams of chain complexes is class-cofibrantly generated. One can show, by an argument similar to [5] that this model structure is not cofibrantly generated.

5. Preliminaries on pro-categories

Definition 5.1. A small, non-empty category I is *cofiltering* if for every pair of objects i and j there exists an object k together with maps $k \rightarrow i$ and $k \rightarrow j$; and for every pair of morphisms $i \xrightarrow{f} j$ there exists a map $h : k \rightarrow i$ with $fh = gh$. A diagram is said to be *cofiltering* if its indexing category is so.

For a category \mathcal{C} , the category **pro- \mathcal{C}** has objects all cofiltering diagrams in \mathcal{C} , and the set of morphisms from a pro-object X indexed by a cofiltering J into a pro-object Y indexed by a cofiltering I is given by the following formula:

$$\text{Hom}_{\text{pro-}\mathcal{C}}(X, Y) = \lim_i \text{colim}_j \text{Hom}_{\mathcal{C}}(X_j, Y_i).$$

A *pro-object* (*pro-morphism*) is an object (morphism) of **pro- \mathcal{C}** . The structure maps of diagrams which represent pro-objects are also called *bonding* maps and denoted by $b_{i_1, i_2} : Y_{i_1} \rightarrow Y_{i_2}$.

A *constant* pro-object (pro-morphism) is an object of **pro- \mathcal{C}** (pro-Map \mathcal{C}) indexed by the category with one object and no non-identity morphisms.

This definition of morphisms in a pro-category requires, perhaps, some clarification. By definition, a pro-map $f : X \rightarrow Y$ is a compatible collection of maps $\{f_i : X \rightarrow Y_i \mid i \in I, f_i \in \text{colim}_j \text{Hom}_{\mathcal{C}}(X_j, Y_i)\}$. By construction of direct limits in the category of sets, $\text{colim}_j \text{Hom}_{\mathcal{C}}(X_j, Y_i)$ is the set of equivalence classes of the elements of $\coprod_j \text{Hom}_{\mathcal{C}}(X_j, Y_i)$. If for each equivalence class f_i we choose a representative $f_{j,i} : X_j \rightarrow Y_i$, then we obtain a representative of the pro-morphism f . This motivates the following alternative characterization of the morphisms in $\text{pro-}\mathcal{C}$ (cf. [11, 2.1.2]):

Definition 5.2. A *representative* of a morphism from a pro-object X indexed by a cofiltering I to a pro-object Y indexed by a cofiltering K is a function $\theta : K \rightarrow I$ (not necessarily order-preserving) and morphisms $f_k : X_{\theta(k)} \rightarrow Y_k$ in \mathcal{C} for each $k \in K$ such that if there exists a map $k' \rightarrow k$ in K , then for some $i \in I$ equipped with maps $i \rightarrow \theta(k)$ and $i \rightarrow \theta(k')$ the following diagram commutes:

$$\begin{array}{ccccc} X_i & \xrightarrow{b_{i,\theta(k')}} & X_{\theta(k')} & \xrightarrow{f_{k'}} & Y_{k'} \\ & \searrow^{b_{i,\theta(k)}} & & & \downarrow b_{k',k} \\ & & X_{\theta(k)} & \xrightarrow{f_k} & Y_k \end{array}$$

A representative $(\phi, \{g_k\})$ *rarefies* the representative $(\theta, \{f_k\})$ if for every $k \in K$, $\phi(k) \geq \theta(k)$ and there exists a bonding map $b_{\phi(k),\theta(k)} : X_{\phi(k)} \rightarrow X_{\theta(k)}$ with $g_k = f_k \circ b_{\phi(k),\theta(k)}$.

A representative $(\theta, \{f_k\})$ is called *strict* [12, p. 36] if θ is a functor and the maps $\{f_k : X_{\theta(k)} \rightarrow Y_k \mid k \in K\}$ constitute a natural transformation $f : X \circ \theta \rightarrow Y$. In other words, the representative of a morphism is strict if all f_k fit into commutative squares of the following form:

$$\begin{array}{ccc} X_{\theta(k')} & \xrightarrow{f_{k'}} & Y_{k'} \\ b_{\theta(k'),\theta(k)} \downarrow & & \downarrow b_{k',k} \\ X_{\theta(k)} & \xrightarrow{f_k} & Y_k \end{array}$$

A strict representative $(\theta, \{f_k\})$ is called *levelwise* if the domain and range of f are indexed by the same indexing category I and $\theta = \text{Id}_I$.

Remark 5.3. Not every pro-map has a levelwise representative, but every pro-map may be reindexed, up to pro-isomorphism, into a pro-map equipped with levelwise representative (see, e.g., [4, A.3.2]). Corollary 5.6 gives a brief proof of existence of functorial levelwise replacement. In [21] Dan Isaksen proves that the original construction by Artin-Mazur is functorial.

The proof of the following standard proposition may be found, for example, in [24, Chapter 1, Section 1].

Proposition 5.4. *Two representatives $(\theta, \{f_k\})$ and $(\theta', \{f'_k\})$ are called equivalent if there exists a representative $(\theta'', \{f''_k\})$ which rarefies both of them. This relation between representatives is an equivalence relation. The equivalence classes of representatives of morphisms from a pro-object X into a pro-object Y are in natural bijective correspondence with the elements of $\text{Hom}_{\text{pro-}\mathcal{C}}(X, Y)$.*

An important technique in pro-categories is reindexing. We use two types of reindexing: for maps and for objects. Their crucial property is functoriality.

The following theorem, proven in [25], will provide us with a functorial choice of a levelwise representative of a pro-morphism:

Theorem 5.5 (C.V. Meyer). *Let \mathcal{C} be any category, and let*

$$F : \text{pro}(\text{Map } \mathcal{C}) \rightarrow \text{Map}(\text{pro-}\mathcal{C})$$

be the obvious functor. Then F is fully faithful and essentially surjective, i.e., the categories $\text{pro}(\text{Map } \mathcal{C})$ and $\text{Map}(\text{pro-}\mathcal{C})$ are equivalent.

Fix once and for all the functors which induce the equivalences:

$$F : \text{pro}(\text{Map } \mathcal{C}) \xrightarrow{\sim} \text{Map}(\text{pro-}\mathcal{C}) : G.$$

Beware that the pro-objects of [25] are indexed by cofiltered categories that are not necessarily small. In this paper we consider only small indexing categories. Nevertheless, Theorem 5.5 is still true, as explained in [19, 3.1, 3.5].

Corollary 5.6. *There exists a functor $L : \text{Map}(\text{pro-}\mathcal{C}) \rightarrow \text{Map}(\text{pro-}\mathcal{C})$ naturally isomorphic to the identity satisfying the following property. For every $f \in \text{Map}(\text{pro-}\mathcal{C})$ the domain and the range of $L(f)$ are indexed by the same indexing category I and there exists a strict representative $(\theta, \{f_k\})$ of f with $\theta = \text{Id}_I$. In other words, f has a levelwise representative.*

Proof. Take $L = FG$. \square

We are going to use inductive arguments, therefore we need a *functorial* reindexing result that produces pro-objects indexed by *cofinite strongly directed sets*

Definition 5.7. A partially ordered set I is *directed* if for any $i, i' \in I$ there exists $i'' \in I$ with $i'' \geq i$ and $i'' \geq i'$; I is called *strongly directed* if $i \geq i'$ and $i \leq i'$ implies $i = i'$. A directed set I is called *cofinite* if every $i \in I$ has only a finite number of predecessors.

The following theorem [11, 2.1.6], [24, p. 15, Theorem 4] supplies us with the required reindexing for pro-objects:

Theorem 5.8. *There exists a functor $M : \text{pro-}\mathcal{C} \rightarrow \text{pro-}\mathcal{C}$ (called the Mardešić functor), naturally equivalent to the identity, such that $M(X)$ is indexed by a cofinite strongly directed set for every X in $\text{pro-}\mathcal{C}$.*

The following proposition (see [24, Lemma 2, p. 9] for the proof) allows us to assume that whenever our pro-objects are indexed by cofinite strongly directed sets we may choose a strict representatives for each morphism.

Proposition 5.9. *Let $f : X \rightarrow Y$ be a pro-map. If X and Y are indexed by directed sets and the indexing category of Y is cofinite and strongly directed, then there exists a strict representative $(\theta, \{f_k\})$ of f .*

6. Applications of the generalized small object argument: Factorizations in $\text{pro-}\mathcal{C}$

We discuss in this section the application of Theorem 1.1, or more precisely of its dual, to the two different model categories on the category of pro-spaces. The strict model structure was constructed by Edwards and Hastings [11]. Weak equivalences and cofibrations are *essentially levelwise* in the strict model structure, i.e., levelwise, up to a reindexing. The properness of \mathcal{C} is the only condition required for the existence of the strict model structure on $\text{pro-}\mathcal{C}$ [23]. The localization of the strict model structure on the category of pro-(simplicial sets) with respect to the class of maps rendered into strict weak equivalences by the functor P (the functor which replaces a space with its Postnikov tower) was constructed by Isaksen [18]. Weak equivalences in Isaksen's model structure generalize those of Artin-Mazur [4] and Grossman [13].

We recall from [17, 2.1.7] that given a class M of maps in a model category, $M\text{-proj}$ is the class of maps which have the left lifting property with respect to every map in M .

Theorem 6.1. *Let \mathcal{C} be a proper model category. Then the strict model structure on $\text{pro-}\mathcal{C}$ is class-fibrantly generated by the classes M of constant fibrations and N of constant trivial fibrations. The classes of maps M and N admit the non-functorial version of the generalized cosmall object argument.*

Proof. The class of trivial cofibrations equals $M\text{-proj}$ and the class of cofibrations between pro-objects equals $N\text{-proj}$ [23, 5.5]. Therefore, we only have to prove that the classes M and N satisfy the non-functorial version of the generalized cosmall object argument. It was shown in [19] that every constant pro-object is countably cosmall.

In order to apply the generalized cosmall object argument on the class M , we need to construct a map $S : \text{Map } \mathcal{C} \rightarrow \text{Map } \mathcal{C}$ equipped with a morphism of maps $t : \text{Id}_{\text{Map } \mathcal{C}} \rightarrow S$ such that for every $f \in \text{Map } \mathcal{C}$, $S(f)$ is an M -fibration and any morphism of maps $f \rightarrow g$ with $g \in M$ factors through $t(f) : f \rightarrow S(f)$.

We start by replacing f with an isomorphic levelwise representative $\{f_i\}_{i \in I}$ for some cofiltering I ; Corollary 5.6 allows to do it in a functorial way. Next, for every $i \in I$, we factorize f_i into a trivial cofibration q_i followed by a fibration p_i using the factorizations of the model category \mathcal{C} . Then define $S(f) = \times_i p_i$.

For every $i \in I$ there exists a pro-map $\varphi_i : \{X_i\} \rightarrow X_i$ defined by the strict representative $\{\theta(i)=i, \{f_i=\text{Id}_{X_i} : X_i \rightarrow X_i\}\}$. The same is true for $\{Y_i\}$. These maps define the canonical maps $\{X_i\} \rightarrow \times_i X_i$ and $\{Y_i\} \rightarrow \times_i Y_i$. Composition of the first map with $\times_i q_i$ finishes the definition of the morphism of maps $t_{\{f_i\}} : \{f_i\} \rightarrow S(\{f_i\})$.

L is defined to be the class of all co- n -fibrations (see [18, Definition 3.2]) between constant pro-objects for all $n \in \mathbb{N}$.

Proposition 6.2. *The class of trivial cofibrations equals L -proj.*

Proof. Every element of L is a strong fibration [18, Definition 6.5]; therefore every trivial cofibration has the left lifting property with respect to L by [18, Proposition 14.5]. Conversely, if a map i has the left lifting property with respect to L , then i has the left lifting property with respect to the class L' of all retracts of L -cocell complexes. By [23, Proposition 5.2] L' contains all strong fibrations. But then [18, Proposition 6.6] implies that L' contains all fibrations. Therefore, i must be a trivial cofibration. \square

Theorem 6.3. *Isaksen's model structure on pro- \mathcal{S} is class-fibrantly generated with classes L and N of generating fibrations and generating trivial fibrations, respectively.*

Proof. It suffices to construct a factorization of every morphism of pro- \mathcal{S} into a trivial cofibration followed by a fibration. We apply the same construction as in Theorem 6.1 to the class L , except for the factorizations of the levelwise representation $\{f_i\}$.

Apply first the Mardešić functor of Theorem 5.8 in order to guarantee that our pro-system is indexed by a cofinite strongly directed set. Since the Mardešić functor is naturally isomorphic to the identity, we abuse notation and keep calling the indexing category I . We construct the factorizations of the maps f_i by induction on the number $n(i)$ of predecessors of i and factor f_i into an $n(i)$ -cofibration q_i followed by a co- $n(i)$ -fibration, which is possible by [18, Proposition 3.3]. This is an ordinary induction on the set of natural numbers, since I is now cofinite.

For any element $g : A \rightarrow B$ of L , there is a number $n \in \mathbb{N}$ such that g is a co- n -fibration. We may always enlarge k such that $n(k) \geq n$ and hence q_k will be an n -cofibration by [18, Lemma 3.6]. Finally, the lift in the commutative square in the diagram in the proof of Theorem 6.1 exists by [18, Definition 3.2]. \square

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