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The edge fault-diameter of Cartesian graph bundles[☆]

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ABSTRACT

A Cartesian graph bundle is a generalization of a graph covering and a Cartesian graph product. Let G be a k_G-edge connected graph and $\bar{\mathcal{D}}_c(G)$ be the largest diameter of subgraphs of G obtained by deleting $c < k_G$ edges. We prove that $\bar{\mathcal{D}}_{a+b+1}(G) \leq \bar{\mathcal{D}}_a(F) + \bar{\mathcal{D}}_b(B) + 1$ if G is a graph bundle with fibre F over base B, $a < k_F$, and $b < k_B$. As an auxiliary result we prove that the edge-connectivity of graph bundle G is at least $k_F + k_B$.

1. Introduction

In the design of large interconnection networks several factors have to be taken into account. A usual constraint is that each processor can be connected to a limited number of other processors and the delays in communication must not be too long. Extensively studied network topologies in this context include graph products and bundles. For example the meshes, tori, hypercubes and some of their generalizations are Cartesian products. It is less known that some well known topologies are Cartesian graph bundles, i.e. some twisted hypercubes [5,8] and multiplicative circulant graphs [15]. Other graph products, sometimes under different names, have been studied as interesting communication network topologies [4,12,15].

Furthermore, an interconnection network should be fault-tolerant. Since nodes or links of a network do not always work, if some nodes or links are faulty, some information may not be transmitted by some of these nodes, links. Therefore the (edge) fault-diameter has been determined

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for many important networks recently [6,7,11,17]. The concept of the fault-diameter of Cartesian product graphs was first described in [10], but the upper bound was wrong, as shown by Xu, Xu and Hou who corrected the mistake [17]. An upper bound for the fault-diameter of Cartesian graph products and bundles was given in [1,2]. Also an upper bound for the edge fault-diameter of Cartesian graph products was given in [3].

In this paper we generalize the result of [3] to Cartesian graph bundles. As a k-edge connected graph remains connected if up to $k-1$ edges are missing, we study the diameter of a graph with any permitted number of edges deleted. We show that the edge-connectivity of Cartesian graph bundle G with fibre F over the base graph B, is at least $k_F + k_B$ and we give an upper bound for the edge fault-diameter of Cartesian graph bundles in terms of edge fault-diameters of the fibre and the base graph. We also show that the bounds are tight.

2. Preliminaries

Throughout the paper we will use the following definitions and notation.

Definition 2.1. A *simple graph* $G = (V, E)$ is determined by a *vertex set* $V = V(G)$ and a set $E = E(G)$ of (unordered) pairs of vertices, called the set of *edges*. As usual, we will use the shorthand notation uv for edge $\{u, v\}$.

Two graphs are *isomorphic* if there is a bijection between the vertex sets that preserves adjacency and nonadjacency.

Definition 2.2. Let G_1 and G_2 be graphs. The *Cartesian product* of graphs G_1 and G_2, $G = G_1 \Box G_2$, is defined on the vertex set $V(G_1) \times V(G_2)$. Vertices (u_1, v_1) and (u_2, v_2) are adjacent if either $u_1u_2 \in E(G_1)$ and $v_1 = v_2$ or $v_1v_2 \in E(G_2)$ and $u_1 = u_2$.

For further reading on graph products we recommend [9]. Graph bundles were first studied in [13].

Definition 2.3. Let B and F be graphs. A graph G is a *Cartesian graph bundle with fibre F over the base graph B* if there is a *graph map* $p : G \to B$ such that for each vertex $v \in V(B)$, $p^{-1}(\{v\})$ is isomorphic to F, and for each edge $e = uv \in E(B)$, $p^{-1}(\{e\})$ is isomorphic to $F \Box K_2$.

More precisely, the mapping $p : G \to B$ maps graph elements of G to graph elements of B, i.e. $p : V(G) \cup E(G) \to V(B) \cup E(B)$. In particular, here we also assume that the vertices of G are mapped to vertices of B and the edges of G are mapped either to vertices or to edges of B. We say an edge $e \in E(G)$ is *degenerate* if $p(e)$ is a vertex. Otherwise we call it *nondegenerate*. The mapping p will also be called the *projection* (of the bundle G to its base B). Note that each edge $e = uv \in E(B)$ naturally induces an isomorphism $\varphi_e : p^{-1}(\{u\}) \to p^{-1}(\{v\})$ between two fibres. It may be of interest to note that while it is well known that a graph can have only one representation as a product (up to isomorphism and up to the order of factors) [9], there may be many different graph bundle representations of the same graph [22]. Here we assume that the bundle representation is given. We wish to note that in some cases finding a representation of G as a graph bundle can be done in polynomial time [18–23]. For example, one of the easy classes is that of the Cartesian graph bundles over a triangle-free base [18].

Let G be a Cartesian graph bundle with fibre F over the base graph B. The *fibre of vertex* $x \in V(G)$ is denoted by F_x, formally, $F_x = p^{-1}(\{p(x)\})$. We will also use the notation $F(u)$ for the fibre of the vertex $u \in V(B)$, i.e. $F(u) = p^{-1}(\{u\})$. Note that $F_x = F(p(x))$.

Remark 2.4. For a later reference note that a graph bundle over a tree T (as a base graph) with fibre F is isomorphic to the Cartesian product $T \Box F$ (not difficult to see; appears already in [14]), i.e. we can assume that all isomorphisms φ_e are identities.

Example 2.5. Let $F = K_2$ and $B = C_3$. In Fig. 1 we see two nonisomorphic bundles with fibre F over the base graph B. Informally, one can say that bundles are "twisted products".

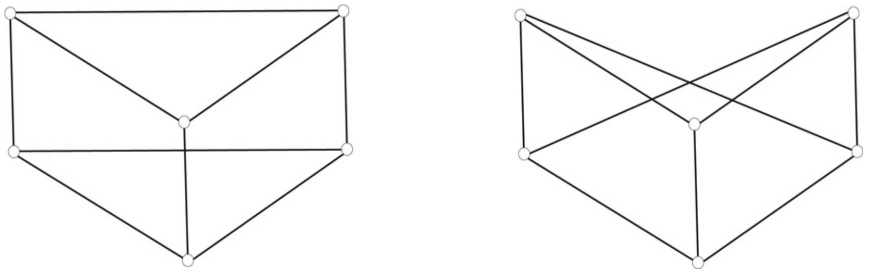

Fig. 1. Nonisomorphic bundles from Example 2.5.

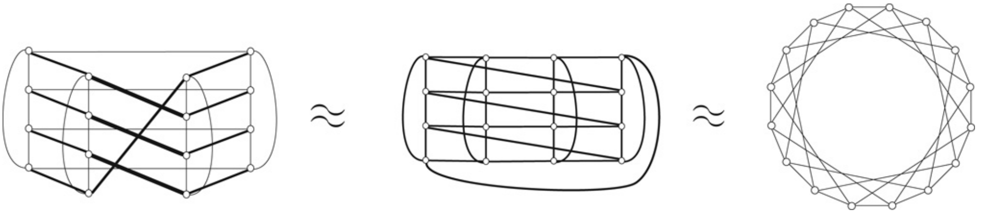

Fig. 2. Twisted torus: Cartesian graph bundle with fibre C_4 over base C_4.

Example 2.6. It is less known that graph bundles also appear as computer topologies. A well known example is the twisted torus in Fig. 2. The Cartesian graph bundle with fibre C_4 over base C_4 is the ILIAC IV architecture.

Definition 2.7. A *walk* between x and y is a sequence of vertices and edges $v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k$ where $x = v_0, y = v_k$, and $e_i = v_{i-1}v_i$ for each i. A walk with all vertices distinct is called a *path*, and the vertices v_0 and v_k are called the *endpoints* of the path. The *length* of a path P, denoted by $\ell(P)$, is the number of edges in P. The *distance* between vertices x and y, denoted by $d_G(x, y)$, is the length of a shortest path between x and y in G. The *diameter* of a graph G, $d(G)$, is the maximum distance between any two vertices in G.

A path P in G, defined by a sequence $x = v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k = y$, can alternatively be seen as a subgraph of G with $V(P) = \{v_0, v_1, v_2, \ldots, v_k\}$ and $E(P) = \{e_1, e_2, \ldots, e_k\}$. Note that the reverse sequence gives rise to the same subgraph. Hence we use P for a path either from x to y or from y to x.

Definition 2.8. The *edge-connectivity* of a graph G, $\lambda(G)$, is the minimum cardinality over all edge-separating sets in G. A graph G is said to be *k-edge connected*, if $\lambda(G) \geq k$.

One of Menger's theorems (see, for example, [16], page 229) reads:

Theorem 2.9 (*Menger*). *Let G be a connected graph, and let s and t be vertices of G. Then the maximum number of edge-disjoint paths from s to t is equal to the minimum number of edges separating s from t.*

The following well known corollary easily follows.

Corollary 2.10. *Let G be a k-edge connected graph and δ_G be its minimum degree. Then $\delta_G \geq k$.*

Let G be a graph, $x, y \in V(G)$ distinct vertices, P a path from x to y in G, and $z \in V(P) \setminus \{x, y\}$. We will use $x \xrightarrow{P} z$ to denote the subpath $\tilde{P} \subseteq P$ from x to z. If z is adjacent to x in P, we will simply use $x \rightarrow z$.

Definition 2.11. Let G be a graph and $X \subseteq E(G)$. A path P from a vertex x to a vertex y *avoids* X in G if $E(P) \cap X = \emptyset$.

Definition 2.12. Let G be a Cartesian graph bundle with fibre F over a base graph B, $u, v \in V(B)$ be distinct vertices, P be a path from u to v in B, and $x \in F(u)$. Then $\tilde{P}(x)$ is the path from $x \in F(u)$ to a vertex in $F(v)$ such that $p(\tilde{P}(x)) = P$ and $\ell(\tilde{P}(x)) = \ell(P)$. We call $\tilde{P}(x)$ a *lift* of the path P to the vertex $x \in V(G)$.

Remark 2.13. Let G be a Cartesian graph bundle with fibre F over a base graph B, $u, v \in V(B)$ be distinct vertices, and P be a path from u to v in B. Then it is easy to see that (1) and (2) below hold.

(1) If P_1 and P_2 are lifts of P to the same vertex $x \in V(G)$, then $P_1 = P_2$.
(2) Let $x, x' \in F(u)$. Then $\tilde{P}(x)$ and $\tilde{P}(x')$ have different endpoints in $F(v)$ and are edge-disjoint if and only if $x \neq x'$.

3. Edge-connectivity of Cartesian graph bundles

Theorem 3.1. *Let F and B be k_F-edge connected and k_B-edge connected graphs respectively, and G a Cartesian graph bundle with fibre F over the base graph B. Let $\lambda(G)$ be the edge-connectivity of G. Then $\lambda(G) \geq k_F + k_B$.*

Proof. Let $p : G \to B$ be the projection such that for each vertex $v \in V(B)$, $p^{-1}(\{v\})$ is isomorphic to F, and for each edge $e = uv \in E(B)$, $p^{-1}(\{e\})$ induces an isomorphism $\varphi_e : F(u) \to F(v)$. Let x and y be two distinct vertices in G. By Theorem 2.9 it is enough to construct $k_F + k_B$ edge-disjoint paths from x to y in G. We will distinguish two cases.

Case 1. Suppose $p(x) \neq p(y)$. As B is k_B-edge connected, there are k_B pairwise edge-disjoint paths $P_1, P_2, \ldots, P_{k_B}$ from $p(x)$ to $p(y)$ in B, and let P_1 be one of the shortest paths among $P_1, P_2, \ldots, P_{k_B}$.

(1) First we shall construct $k_F + 1$ edge-disjoint paths from x to y in G with degenerate edges in $F_x \cup F_y$ and nondegenerate edges in $p^{-1}(P_1)$.

Note that $\varphi_{e_k} \circ \varphi_{e_{k-1}} \circ \cdots \circ \varphi_{e_1}$, where $e_1, e_2, \ldots, e_k$ is the sequence of all edges of the path P_1 from $p(x)$ to $p(y)$, is an isomorphism from F_x to F_y. Denote this isomorphism by φ_{P_1}.

Now let $x' \in F_y$ be the endpoint of the path $\tilde{P}_1(x)$ in F_y.

(a) If $x' = y$, then $\tilde{P}_1(x)$ is the path from x to y in G. As F_x is k_F-edge connected, there are at least k_F neighbors of x in F_x. Denote these neighbors by u_i, $i = 1, 2, \ldots, k_F$. The endpoints u_i' in F_y of the paths $\tilde{P}_1(u_i)$ are mutually distinct vertices; all of them are neighbors of y in F_y. Therefore there are k_F edge-disjoint paths

$$x \to u_i \overset{\tilde{P}_1(u_i)}{\to} u_i' \to y,$$

$i = 1, 2, \ldots, k_F$. This way we constructed $k_F + 1$ edge-disjoint paths from x to y.

(b) If $x' \neq y$, then let $y' \in F_x$ be the endpoint of the path $\tilde{P}_1(y)$ lying in F_x. As F_x is k_F-edge connected, there are k_F edge-disjoint paths $Q_1, Q_2, \ldots, Q_{k_F}$ from x to y' in F_x. Without loss of generality, let Q_1 be one of the shortest paths among them. Let $Q_i' = \varphi_{P_1}(Q_i)$ for each $i = 1, 2, \ldots, k_F$. First we construct two edge-disjoint paths between x and y in G as

$$y \overset{\tilde{P}_1(y)}{\to} y' \overset{Q_1}{\to} x,$$

and

$$x \overset{\tilde{P}_1(x)}{\to} x' \overset{Q_1'}{\to} y.$$

Now let $q_2, q_3, \ldots, q_{k_F}$ be the neighbors of x in F_x such that $q_i \in Q_i$ for each $i = 2, 3, \ldots, k_F$. Note that for each i, $q_i \neq y'$. We construct additional $k_F - 1$ edge-disjoint paths as

$$x \to q_i \overset{\tilde{P}_1(q_i)}{\to} \varphi_{P_1}(q_i) \overset{Q_i'}{\to} y,$$

$i = 2, 3, \ldots, k_F$.

This way we constructed $k_F + 1$ edge-disjoint paths from x to y.

(2) Additional $k_B - 1$ paths can be constructed by taking $k_B - 1$ neighbors $r_2, r_3, \ldots, r_{k_B}$ of $p(y)$ in B such that $r_i \in P_i$ for each $i = 2, 3, \ldots, k_B$. As P_1 is one of the shortest paths between $p(x)$ and $p(y)$, therefore $r_i \neq p(x)$ for each $i = 2, 3, \ldots, k_B$, and the fibres $F(r_i)$ are disjoint with fibres F_x and F_y. Let $r_i' \in F(r_i) \cap \tilde{P}_i(x)$. There is a path R_i from r_i' to $\varphi_{\{p(y),r_i\}}(y)$ within fibre $F(r_i)$ for each $i = 2, 3, \ldots, k_B$. Then the paths

$$x \xrightarrow{\tilde{P}_i(x)} r_i' \xrightarrow{R_i} \varphi_{\{p(y),r_i\}}(y) \to y$$

are pairwise edge-disjoint, and are edge-disjoint with the $k_F + 1$ paths constructed before.

This way we constructed $k_F + k_B$ edge-disjoint paths from x to y in G.

Case 2. If $p(x) = p(y)$ then there are k_F edge-disjoint paths from x to y within the fibre F_x. Additional k_B paths can be constructed by taking $k_B \leq \delta_B$ neighbors of $p(x)$ in B, say $r_1, r_2, \ldots, r_{k_B}$, and observing that the paths

$$x \to \varphi_{\{p(x),r_i\}}(x) \xrightarrow{R_i} \varphi_{\{p(x),r_i\}}(y) \to y,$$

where R_i is a path from $\varphi_{\{p(x),r_i\}}(x)$ to $\varphi_{\{p(x),r_i\}}(y)$ in $F(r_i)$, and $i = 1, 2, \ldots, k_B$, are pairwise edge-disjoint and are edge-disjoint with all paths in F_x. This way we constructed $k_F + k_B$ edge-disjoint paths from x to y in G. $\square$

The following statement easily follows:

Corollary 3.2. *Let G_1 and G_2 be k_1-edge connected and k_2-edge connected graphs, respectively. Then the Cartesian product $G_1 \Box G_2$ is at least $(k_1 + k_2)$-edge connected.*

Proof. $B = G_2$ and $F = G_1$. $\square$

4. The edge fault-diameter of Cartesian graph bundles

Our main result is an upper bound for the edge fault-diameter of Cartesian graph bundles in terms of edge fault-diameters of the fibre and the base graph.

Definition 4.1. Let G be a k-edge connected graph and $0 \leq a < k$. Then we define the *a-edge fault-diameter* of G as

$$\bar{\mathcal{D}}_a(G) = \max\{d(G \setminus X) \mid X \subseteq E(G), |X| = a\}.$$

Note that $\bar{\mathcal{D}}_a(G)$ is the largest diameter among subgraphs of G with a edges deleted; hence $\bar{\mathcal{D}}_0(G)$ is just the diameter of G, $d(G)$. For $a \geq \lambda(G)$, the a-edge fault-diameter of G does not exist. In other words, $\bar{\mathcal{D}}_a(G) = \infty$ as some of the graphs are not edge connected.

We will use the following technical lemma in the proof of Theorem 4.3. The lemma follows from the upper bound for the Cartesian product of graphs [3]. As the proof is short we decided to write it out for completeness.

Lemma 4.2. *Let $G = Q \Box F$ be the Cartesian product of a path Q with vertices $V(Q) = \{v_0, v_1, \ldots, v_k\}$ and a graph F with $\bar{\mathcal{D}}_a(F) < \infty$. Let s and t be vertices of G with coordinates $s = (s_1 = v_0, s_2)$ and $t = (t_1 = v_k, t_2)$ and let $X \subseteq E(G)$ be a set of edges with $|X| \leq a+1$. Then $d_{G\setminus X}(s, t) \leq \bar{\mathcal{D}}_a(F) + \ell(Q) + 1$.*

Proof. Let Q be a path, $V(Q) = \{v_0, v_1, \ldots, v_k\}$, $G = Q \Box F$, $k_F \geq a + 1$. Let $s \in F(v_0)$ and $t \in F(v_k)$ be vertices of G, and let $X \subseteq E(G)$ be a set of edges with $|X| \leq a + 1$.

We distinguish two cases.

First, if $|F(v_k) \cap X| = a + 1$ then $|F(v_0) \cap X| = 0$ and $\tilde{Q}(t)$ avoids X. Therefore there is a path R from s to the endpoint of the path $\tilde{Q}(t)$ within fibre $F(v_0)$ of length $\ell(R) \leq \bar{\mathcal{D}}_0(F) \leq \bar{\mathcal{D}}_a(F)$, and $\ell(\tilde{Q}(t)) = \ell(Q)$. Therefore $d_{G\setminus X}(s, t) \leq \bar{\mathcal{D}}_a(F) + \ell(Q) + 1$.

Second, let $|F(v_k) \cap X| \le a$. As F is at least $(a+1)$-edge connected, there are at least $a+1$ neighbors of s in $F(v_0)$. Denote the neighbors by u_i, $i = 1, 2, \ldots, a+1$. Among the $a+2$ edge-disjoint paths from s to vertices u_i' ($i = 1, 2, \ldots, a+2$) in $F(v_k)$, constructed as

$$s \to u_i \overset{\tilde{Q}(u_i)}{\to} u_i'$$

of length $1 + \ell(Q)$, and the path

$$s \overset{\tilde{Q}(s)}{\to} s' = u_{a+2}'$$

of length $\ell(Q)$, at least one avoids X. Without loss of generality, say

$$P_1 : s \to u_1 \overset{\tilde{Q}(u_1)}{\to} u_1'$$

avoids X. As $|F(v_k) \cap X| \le a$, there is a path R in $F(v_k)$ avoiding X from u_1' to t of length $\ell(R) \le \bar{\mathcal{D}}_a(F)$. Therefore there is a path

$$P : s \overset{P_1}{\to} u_1' \overset{R}{\to} t$$

from s to t of length $\ell(P) \le 1 + \ell(Q) + \bar{\mathcal{D}}_a(F)$, and hence $d_{G\setminus X}(s,t) \le \bar{\mathcal{D}}_a(F) + \ell(Q) + 1$. □

Now we give the main result of this paper.

Theorem 4.3. *Let F and B be k_F-edge connected and k_B-edge connected graphs respectively,* $0 \le a < k_F$, $0 \le b < k_B$, *and G a Cartesian bundle with fibre F over the base graph B. Then*

$$\bar{\mathcal{D}}_{a+b+1}(G) \le \bar{\mathcal{D}}_a(F) + \bar{\mathcal{D}}_b(B) + 1.$$

Proof. Let $k = a + b + 2$. By Theorem 3.1, $\lambda(G) \ge k$; hence $\bar{\mathcal{D}}_{k-1}(G)$ is well defined. Let δ_F be the minimum degree of F and δ_B be the minimum degree of B. Recall that $\delta_F \ge \lambda(F) > a$ and $\delta_B \ge \lambda(B) > b$. Let $X \subseteq E(G)$ be such that $|X| = k - 1 = a + b + 1$, and $x, y \in V(G)$ be two distinct vertices. We shall construct a path P from x to y in $G \setminus X$, with length $\ell(P) \le \bar{\mathcal{D}}_a(F) + \bar{\mathcal{D}}_b(B) + 1$.

As before, let $p : G \to B$ be the projection from G to its base B, so $p(X) \subseteq V(B) \cup E(B)$. Denote the set of degenerate edges in X by X_D, and the set of nondegenerate edges by X_N, $X = X_D \cup X_N$, $p(X_D) \subseteq V(B)$ and $p(X_N) \subseteq E(B)$. Let $|X_D| = a_0$ and $|X_N| = b_0$. Then $a_0 + b_0 = a + b + 1$.

(1) We first assume that x and y are in distinct fibres, i.e. $p(x) \ne p(y)$. We now distinguish two cases.

Case 1. If $b_0 < b$, then there is a path Q between $p(x)$ and $p(y)$ in B that avoids $p(X_N)$ of length $\ell(Q) \le \bar{\mathcal{D}}_{b_0}(B) \le \bar{\mathcal{D}}_b(B)$. Let $y' \in F_x$ be the endpoint of the path $\tilde{Q}(y)$.

If $|F_x \cap X_D| \le a$, then there is a path R from x to y' within fibre F_x that avoids X_D of length $\ell(R) \le \bar{\mathcal{D}}_a(F)$. Therefore there is a path P from x to y in $G \setminus X$

$$P : y \overset{\tilde{Q}(y)}{\to} y' \overset{R}{\to} x$$

of length $\ell(P) \le \bar{\mathcal{D}}_b(B) + \bar{\mathcal{D}}_a(F)$.

If $|F_x \cap X_D| \ge a + 1$, then $|(G \setminus F_x) \cap X_D| \le a_0 - (a+1) = b - b_0$, so outside F_x we have at most $b - b_0$ degenerate edges of X. As B is $(b+1)$-edge connected, and $b_0 < b$, there are at least $b + 1 - b_0$ neighbors of $p(x)$ such that the edges from $p(x)$ to these neighbors avoid $p(X_N)$. As there are more such neighbors than degenerate edges of X outside F_x ($b + 1 - b_0 > b - b_0$), there is a neighbor u of $p(x)$ in B such that $|F(u) \cap X_D| = 0$ and $e = \{p(x), u\} \notin p(X_N)$. As $b_0 < b$, there is a path Q from u to $p(y)$ in B that avoids $p(X_N)$ of length $\ell(Q) \le \bar{\mathcal{D}}_{b_0}(B) \le \bar{\mathcal{D}}_b(B)$. Let $u' \in F(u)$ be the endpoint of the path $\tilde{Q}(y)$. As $|F(u) \cap X_D| = 0$, there is a path R from $\varphi_e(x)$ to u' within $F(u)$ of length $\ell(R) \le \bar{\mathcal{D}}_0(F) \le \bar{\mathcal{D}}_a(F)$. Therefore there is a path P from y to x in $G \setminus X$

$$P : y \overset{\tilde{Q}(y)}{\to} u' \overset{R}{\to} \varphi_e(x) \to x$$

of length $\ell(P) \le \bar{\mathcal{D}}_b(B) + \bar{\mathcal{D}}_a(F) + 1$. Note that if $y \in F(u)$, P has length $\ell(P) \le \bar{\mathcal{D}}_a(F) + 1$.

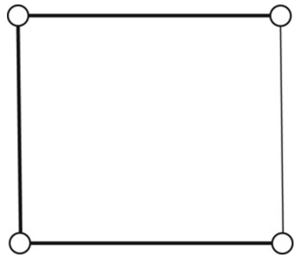

Fig. 3. $G = P_2 \Box P_2$ with one faulty link.

Case 2. Let $b_0 \geq b$. First we choose b edges in X_N, $\{e_1, e_2, \ldots, e_b\} \subseteq X_N$. Then there is a path Q from $p(x)$ to $p(y)$ in B, that avoids $p(\{e_1, e_2, \ldots, e_b\})$, with length $\ell(Q) \leq \bar{\mathcal{D}}_b(B)$. Therefore the subgraph $p^{-1}(Q)$, which is isomorphic to $Q \Box F$ by Remark 2.4, intersects X in at most $b_0 - b$ nondegenerate edges and a_0 degenerate edges of X. Therefore $|p^{-1}(Q) \cap X| \leq b_0 - b + a_0 = a + 1$. By Lemma 4.2, there is a path P from x to y with length $\ell(P) \leq \bar{\mathcal{D}}_a(F) + \ell(Q) + 1 \leq \bar{\mathcal{D}}_a(F) + \bar{\mathcal{D}}_b(B) + 1$.

(2) To complete the proof, we have to consider the case where x and y are in the same fibre, i.e. $p(x) = p(y)$, and $F_x = F_y$.

If $|F_x \cap X_D| \leq a$ then there is a path of length at most $\bar{\mathcal{D}}_a(F)$ within the fibre.

If $|F_x \cap X_D| \geq a + 1$, then $|X_N| = b_0 \leq b$. Therefore, as before, there is a neighbor u of $p(x)$ in B such that $|F(u) \cap X_D| = 0$ and $e = \{p(x), u\} \notin p(X_N)$. We may construct the path P as

$$P : x \to \varphi_e(x) \xrightarrow{R} \varphi_e(y) \to y,$$

and $\ell(P) \leq 1 + \bar{\mathcal{D}}_0(F) + 1 \leq \bar{\mathcal{D}}_b(B) + \bar{\mathcal{D}}_a(F) + 1$. $\square$

The next example shows that the bound in Theorem 4.3 is tight.

Example 4.4. Let $G = P_2 \Box P_2$; see Fig. 3. G is a graph bundle with fibre $F = P_2$ over the base graph $B = P_2$. Then for $a = b = 0$ we have

$$\bar{\mathcal{D}}_{a+b+1}(G) = 3,$$
$$\bar{\mathcal{D}}_b(B) + \bar{\mathcal{D}}_a(F) + 1 = 1 + 1 + 1 = 3.$$

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