Available online at www.sciencedirect.com

ScienceDirect

ELSEVIER

Linear Algebra and its Applications 418 (2006) 234–256

LINEAR ALGEBRA AND ITS APPLICATIONS

www.elsevier.com/locate/laa

Zeta functions of line, middle, total graphs of a graph and their coverings

Jin Ho Kwak [a,1], Iwao Sato [b,*,2]

[a] *Combinatorial and Computational Mathematics Center, Department of Mathematics, Pohang University of Science and Technology, Pohang 790-784, Republic of Korea*

[b] *Oyama National College of Technology, Oyama, Tochigi 323-0806, Japan*

Received 3 June 2005; accepted 31 January 2006

Available online 31 March 2006

Submitted by R.A. Brualdi

Abstract

We consider the (Ihara) zeta functions of line graphs, middle graphs and total graphs of a regular graph and their (regular or irregular) covering graphs. Let $L(G)$, $M(G)$ and $T(G)$ denote the line, middle and total graph of G, respectively. We show that the line, middle and total graph of a (regular and irregular, respectively) covering of a graph G is a (regular and irregular, respectively) covering of $L(G)$, $M(G)$ and $T(G)$, respectively. For a regular graph G, we express the zeta functions of the line, middle and total graph of any (regular or irregular) covering of G in terms of the characteristic polynomial of the covering. Also, the complexities of the line, middle and total graph of any (regular or irregular) covering of G are computed. Furthermore, we discuss the L-functions of the line, middle and total graph of a regular graph G.

AMS classification: 05C50; 05C25; 15A15; 15A18

Keywords: Zeta function; Complexity; Graph covering; Line graph; Middle graph; Total graph

* Corresponding author. Tel.: +81 285 20 2176; fax: +81 285 20 2880.

E-mail addresses: jinkwak@postech.ac.kr (J.H. Kwak), isato@oyama-ct.ac.jp (I. Sato).

[1] Supported by Com^{2}MaC-KOSEF in Korea.

[2] Supported by Grant-in-Aid for Science Research (C) in Japan.

0024-3795/$ - see front matter © 2006 Elsevier Inc. All rights reserved.

doi:10.1016/j.laa.2006.01.033

1. Introduction

Throughout this paper graphs and digraphs are assumed to be finite, connected and simple (with no loops and no multiple edges). Let G be a connected (undirected) graph with vertex set $V(G)$ and edge set $E(G)$, and let ν_G and ε_G denote the numbers of vertices and edges of G, respectively. Let $D(G)$ be the arc set of the symmetric digraph corresponding to G. For $e = (u, v) \in D(G)$, let $o(e) = u$ and $t(e) = v$. The inverse arc of e is denoted by e^{-1}. A *path P of length n* in G is a sequence $P = (v_0, v_1, \ldots, v_{n-1}, v_n)$ of $n + 1$ vertices and n arcs (or edges) such that consecutive vertices share an arc (or edge) (we do not require that all vertices are distinct). Also, P is called a (v_0, v_n)*-path*. We say that a path has a *backtracking* if a subsequence of the form $\ldots, x, y, x, \ldots$ appears. A (v, w)-path is called a *cycle* (or *closed path*) if $v = w$.

A cycle C is said to be *reduced* if both C and C^2 have no backtracking. Two cycles $C_1 = (v_1, \ldots, v_m)$ and $C_2 = (w_1, \ldots, w_m)$ are called *equivalent* if there is an integer k such that $w_j = v_{j+k}$ for all j, where the subscripts are modulo m. Let $[C]$ denote the equivalence class which contains a cycle C. Let B^r be the cycle obtained by going r times around a cycle B. Such a cycle is called a *multiple* of B. A cycle C is *prime* if $C \neq B^r$ for any other cycle B and $r \geqslant 2$. Note that each equivalence class of prime, reduced cycles of a graph G corresponds to a unique conjugacy class of the fundamental group $\pi_1(G, v)$ of G at a vertex $v \in V(G)$.

The (*Ihara*) *zeta function* of a graph G is defined as a function of $u \in \mathbf{C}$ with $|u|$ sufficiently small by

$$Z(G, u) = Z_G(u) = \prod_{[C]} \left(1 - u^{|C|}\right)^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G, and $|C|$ is the length of C.

Clearly, the zeta function of a disconnected graph is the product of the zeta functions of its connected components. Zeta functions of graphs were originated from zeta functions of regular graphs by Ihara [16], where their reciprocals are expressed as explicit polynomials. A zeta function of a regular graph G associated to a unitary representation of the fundamental group of G was developed by Sunada [30]. Hashimoto [13] treated multivariable zeta functions of bipartite graphs. Bass [2] generalized Ihara's result on zeta functions of regular graphs to general graphs.

The *adjacency matrix* $\mathbf{A} = \mathbf{A}_G = (a_{ij})$ of G is the $\nu_G \times \nu_G$ matrix with $a_{ij} = 1$ if v_i and v_j are adjacent and $a_{ij} = 0$ otherwise. Let $\mathbf{D} = \mathbf{D}_G$ denote the diagonal matrix whose (i, i)-entry is the degree $\deg_G(v_i)$ of v_i, and $\mathbf{Q} = \mathbf{Q}_G = \mathbf{D} - \mathbf{I}$.

Theorem 1 (Bass). *The reciprocal of the zeta function of G is given by*

$$Z_G(u)^{-1} = (1 - u^2)^{\varepsilon_G - \nu_G} \det(\mathbf{I} - u\mathbf{A}_G + u^2\mathbf{Q}_G). \tag{1}$$

Stark and Terras [29] gave an elementary proof of Theorem 1, and recently Kotani and Sunada [18] gave another proof.

The complexity $\kappa(G)$ of a graph G is the number of spanning trees in G. Hashimoto [14] and Northshield [26] expressed the complexity of a graph as a limit involving its zeta function.

In this paper, we call the matrix $\mathscr{L}_G(u) = \mathbf{I} - u\mathbf{A}_G + u^2\mathbf{Q}_G$ is called the *generalized Laplacian matrix* of G. Note that $\mathscr{L}_G(1)$ is the Laplacian matrix $\mathbf{L}_G$ of G. For a connected graph G, let $f_G(u) = \det \mathscr{L}_G(u)$. Northshield [26] computed $\kappa(G)$ in terms of the generalized Laplacian matrix of G as follows.

Theorem 2 (Northshield). *For a connected graph G*,

$$2(\varepsilon_G - \nu_G)\kappa(G) = f'_G(1), \tag{2}$$

where $f'_G(1)$ *is the derivative of the determinant* $f_G(u) = \det \mathscr{L}_G(u)$ *at* $u = 1$.

The complexities for various graphs were given in [6]. Let $\Phi(G;\lambda) = \det(\lambda\mathbf{I} - \mathbf{A})$ be the *characteristic polynomial* of G.

Theorem 3 [15]. *Let G be a regular graph with valency r. Then the complexity* $\kappa(G)$ *is*

$$\kappa(G) = \frac{1}{\nu_G}\Phi'(G;r). \tag{3}$$

The *line graph* $L(G)$ of a graph G is the graph whose vertex set is the edge set $E(G)$ of G, with two vertices of $L(G)$ being adjacent if and only if the corresponding edges in G have a vertex in common. The *middle graph* $M(G)$ is the graph obtained from G inserting a new vertex into every edge of G and by joining by edges those pairs of these new vertices which lie on adjacent edges of G. Another important graph is a total graph. The *total graph* $T(G)$ is the graph whose vertex set is the union of the vertex set $V(G)$ and the edge set $E(G)$ of G, with two vertices of $T(G)$ being adjacent if and only if the corresponding elements of G are adjacent or incident. There have been lots of work on various properties of line graphs, middle graphs and total graphs of graphs [3,5,6,11,12,25,27,28].

From the definitions, we have

Theorem 4. *The adjacency matrices* $\mathbf{A}_L = \mathbf{A}(L(G))$, $\mathbf{A}_M = \mathbf{A}(M(G))$ *and* $\mathbf{A}_T = \mathbf{A}(T(G))$ *are given as follows*:

$$\mathbf{A}_L = \mathbf{B}\mathbf{B}^t - 2\mathbf{I}_{\varepsilon_G}, \quad \mathbf{A}_M = \begin{bmatrix} \mathbf{A}_L & \mathbf{B} \\ \mathbf{B}^t & \mathbf{0} \end{bmatrix}, \quad \mathbf{A}_T = \begin{bmatrix} \mathbf{A}_L & \mathbf{B} \\ \mathbf{B}^t & \mathbf{A} \end{bmatrix}, \tag{4}$$

where $\mathbf{B} = (b_{ij})$ *is the incidence matrix of* G : $b_{ij} = 1$ *if the edge* e_i *and the vertex* v_j *are incident, and* $b_{ij} = 0$ *otherwise*.

This paper focuses on the following questions: For the line graph $L(G)$, (also for the middle graph $M(G)$ and the total graph $T(G)$)

(A) compute the characteristic polynomial $\Phi(L(G))$, the zeta function $Z_{L(G)}$ and the complexity $\kappa(L(G))$,
(B) determine whether the line graph of a (respectively, regular) covering of a graph G is a (respectively, regular) covering of $L(G)$,
(C) compute the zeta functions and the complexities of coverings of the line graph $L(G)$,
(D) compute the zeta functions and the complexities of line graphs of coverings of G,
(E) decompose the zeta functions of coverings of the line graph into L-functions.

In Section 3, we present the characteristic polynomial, the complexity and the zeta function of the line graph of a regular graph. For any (regular or irregular) covering G^α of a connected graph G derived from a permutation voltage assignment $\alpha : D(G) \to S_n$, we determine a voltage assignment $\beta : D(G) \to S_n$ such that $L(G^\alpha) = L(G)^\beta$. The zeta function of the line graph $L(G^\alpha)$

for a regular graph G is expressed in terms of the characteristic polynomial of G^{α}, and also by using the determinant expression for L-functions of $L(G)$. Furthermore, the complexities of $L(G^{\alpha})$ is computed. In Sections 4 and 5, a parallel work for a middle and a total graph is done respectively. In Section 6, we add some examples.

For a general theory of the representation of groups and graph coverings, the reader is referred to [4,10], respectively. Throughout this paper, let $[n]$ denote the set $\{1, 2, \ldots, n\}$ and let S_n denote the symmetric group on the set $[n]$.

2. Zeta functions and complexities of covering graphs

In this section, we construct a covering of a connected graph G by using a voltage assignment α defined on the arc set $D(G)$, written by G^{α}, and compute the characteristic polynomial $\Phi(G^{\alpha})$, the zeta function $Z(G^{\alpha}, u)$, and the complexity $\kappa(G^{\alpha})$ of the covering G^{α} in terms of the corresponding quantity of the graph G.

Let $N(v) = \{w \in V(G) | (v, w) \in D(G)\}$ denote the neighborhood of a vertex v in G. A graph $\widetilde{G}$ is called a *covering* of G with projection $p : \widetilde{G} \to G$ if there is a surjection $p : V(\widetilde{G}) \to V(G)$ such that $p|_{N(\tilde{v})} : N(\tilde{v}) \to N(v)$ is a bijection for all vertices $v \in V(G)$ and $\tilde{v} \in p^{-1}(v)$. We say that the projection $p : \widetilde{G} \to G$ is an *n-fold covering* of G if p is n-to-one. Two coverings $p_i : \widetilde{G}_i \to G, i = 1, 2$ are said to be *isomorphic* if there exists a graph isomorphism $\Phi : \widetilde{G}_1 \to \widetilde{G}_2$ such that $p_1 = p_2 \circ \Phi$. Such a Φ is called a *covering isomorphism* [19].

A *permutation voltage assignment* (or, *voltage assignment*) of G is a function $\phi : D(G) \to S_n$ with the property that $\phi(e^{-1}) = \phi(e)^{-1}$ for each $e \in D(G)$. The *permutation derived graph* G^{ϕ} is defined as follows: $V(G^{\phi}) = V(G) \times [n]$ and $E(G^{\phi}) = E(G) \times [n]$, so that an edge (e, i) of G^{ϕ} joins a vertex (u, i) to $(v, \phi(e)(i))$ for $e = uv \in D(G)$ and $i = 1, 2, \ldots, n$. The first coordinate projection $p^{\phi} : G^{\phi} \to G$ is an n-fold covering. Following Gross and Tucker [9], every n-fold covering $\widetilde{G}$ of a graph G can be derived from a voltage assignment which assigns the identity element on the directed edges of a fixed spanning tree T of G. We call such a ϕ reduced. That is, for a covering $p : \widetilde{G} \to G$, there exists a reduced voltage assignment ϕ of G such that the derived covering $p^{\phi} : G^{\phi} \to G$ is isomorphic to $p : \widetilde{G} \to G$. Moreover, for a reduced voltage assignment $\phi : D(G) \to S_n$, the derived graph G^{ϕ} is connected if and only if the subgroup of S_n generated by the image of the voltage assignment ϕ acts transitively on the set $[n]$ [10]. Such a voltage assignment is said to be *transitive*.

A covering $p : \widetilde{G} \to G$ is said to be *regular* if there is a subgroup Γ of the automorphism group $\mathrm{Aut}(\widetilde{G})$ of $\widetilde{G}$ acting freely on $\widetilde{G}$ so that the graph G is isomorphic to the quotient graph $\widetilde{G}/\Gamma$, say by h, and the quotient map $\widetilde{G} \to \widetilde{G}/\Gamma$ is the composition $h \circ p$ of p and h. The fiber of an edge or a vertex is its preimage under p.

Let Γ be a finite group. An *ordinary voltage assignment* (or, *Γ-voltage assignment*) of G is a function $\phi : D(G) \to \Gamma$ with the property that $\phi(e^{-1}) = \phi(e)^{-1}$ for each $e \in D(G)$. The values of ϕ are called *voltages*, and Γ is called the *voltage group*. The *ordinary derived graph* $G \times_{\phi} \Gamma$ derived from an ordinary voltage assignment $\phi : D(G) \to \Gamma$ has as its vertex set $V(G) \times \Gamma$, and as its edge set $E(G) \times \Gamma$, so that an edge (e, g) of $G \times_{\phi} \Gamma$ joins a vertex (u, g) to $(v, g\phi(e))$ for $e = uv \in D(G)$ and $g \in \Gamma$. In the (ordinary) derived graph $G \times_{\phi} \Gamma$, a vertex (u, g) is denoted by u_g and an edge (e, g) is denoted by e_g. The first coordinate projection $p_{\phi} : G \times_{\phi} \Gamma \to G$, called the natural projection, commutes with the left multiplication action of the $\phi(e)$ and the right multiplication action of Γ on the fibers, which is free and transitive, so that p_{ϕ} is a regular $|\Gamma|$-fold covering, called simply a *Γ-covering*. Gross and Tucker [9] showed that every finite regular covering of a graph G can be derived from a Γ-voltage assignment, where Γ becomes the covering transformation group.

In [8] and [20], Kwak et al. expressed the zeta function and the complexity of a (regular or irregular) covering of G by using those of G, respectively. The *tensor product* $\mathbf{A} \otimes \mathbf{B}$ of matrices $\mathbf{A}$ and $\mathbf{B}$ is considered as the matrix $\mathbf{A}$ having the element a_{ij} replaced by the matrix $a_{ij}\mathbf{B}$. Set $\mathbf{I}_m$ be the identity matrix of order m.

Theorem 5. *Let G be a connected graph and let $\alpha : D(G) \to S_n$ be a permutation voltage assignment. Let $\Gamma = \langle\{\alpha(u, v)|(u, v) \in D(G)\}\rangle$ be the subgroup of S_n generated by $\{\alpha(u, v)|(u, v) \in D(G)\}$. Let $\rho_1 = \mathbf{1}, \rho_2, \ldots, \rho_k$ be the irreducible representations of Γ with degree $f_1 = 1, f_2, \ldots, f_k$, respectively, and let the permutation representation $P : \Gamma \to \mathrm{GL}(n, \mathbf{C})$ of Γ be decomposed into a direct sum of irreducible representations: say $P = \bigoplus_{\ell=1}^{k} m_\ell \rho_\ell$. For each $\gamma \in \Gamma$, define a matrix $\mathbf{A}_\gamma = \mathbf{A}_\gamma(G) = (a_{uv}^{(\gamma)})$ as follows:*

$$a_{uv}^{(\gamma)} := \begin{cases} 1 & \text{if } (u, v) \in D(G) \text{ and } \alpha(u, v) = \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Then the reciprocal of the zeta function of G^α is

$$Z(G^\alpha, u)^{-1} = (Z(G, u)^{-1})^{m_1} \prod_{i=2}^{k} \left\{ (1 - u^2)^{(\varepsilon_G - \nu_G) f_i} \times \det\left[\mathbf{I}_{\nu_G f_i} - u \sum_{\gamma\in\Gamma} \rho_i(\gamma) \otimes \mathbf{A}_\gamma + u^2(\mathbf{I}_{f_i} \otimes (\mathbf{D} - \mathbf{I}_{\nu_G})) \right] \right\}^{m_i}. \tag{5}$$

Suppose that the n-fold covering G^α of G is connected and that $\varepsilon_G > \nu_G$. Then the complexity of G^α is

$$\kappa(G^\alpha) = \frac{1}{n}\kappa(G) \prod_{k=2}^{\ell} \det\left(\mathbf{I}_{f_k} \otimes \mathbf{D}_G - \sum_{\sigma\in\Gamma} \rho_k(\sigma) \otimes \mathbf{A}_\sigma \right)^{m_k}. \tag{6}$$

For each $\gamma \in \Gamma$, let $\vec{G}_{(\alpha,\gamma)}$ denote the spanning subgraph of the symmetric digraph $\vec{G}$ corresponding to G whose directed edge set is $\alpha^{-1}(\gamma)$. Then the digraph $\vec{G}$ is the edge-disjoint union of spanning subgraphs $\vec{G}_{(\alpha,\gamma)}$, $\gamma \in \Gamma$, and the matrix $\mathbf{A}_\gamma = \mathbf{A}_\gamma(G)$ in Theorem 5 is the adjacency matrix of the digraph $\vec{G}_{(\alpha,\gamma)}$.

Note that the multiplicity m_1 of the irreducible representation $\rho_1 = \mathbf{1}$ is the number of orbits under the action of the group Γ. Thus, if the covering G^α is connected, we have $m_1 = 1$.

As a special case, let the covering G^α of G is a regular covering with a covering transformation group $\mathscr{A}$ so that $G^\phi/\mathscr{A} \cong G$ [10]. Let $\rho_1 = 1, \rho_2, \ldots, \rho_\ell$ be the irreducible representations of $\mathscr{A}$ with degree f_k for each k, where $f_1 = 1$. Then the multiplicity m_k is equal to the degree f_k of ρ_k and the fold number n is $|\mathscr{A}|$, the cardinality of $\mathscr{A}$. Thus we have the same formula as Theorem 5 in [22]:

$$\kappa(G^\phi) = \frac{1}{|\mathscr{A}|}\kappa(G) \prod_{k=2}^{\ell} \det\left(\mathbf{I}_{f_k} \otimes \mathbf{D}_G - \sum_{\sigma\in\Gamma} \rho_k(\sigma) \otimes \mathbf{A}_\sigma \right)^{f_k}.$$

Feng et al. [7] expressed the characteristic polynomial of a (regular or irregular) covering of G in terms of the characteristic polynomial of G. Let $\Phi(\mathbf{F};\lambda)=\det(\lambda\mathbf{I}-\mathbf{F})$ for any square matrix $\mathbf{F}$.

Theorem 6. *Let $G, \mathbf{A}_\sigma, \alpha, \rho_i, f_i, m_i$ be as in Theorem 5. Then the characteristic polynomial of the covering graph G^α is*

$$\Phi(G^\alpha;\lambda)=\Phi(G;\lambda)\cdot\prod_{i=2}^{t}\Phi\left(\sum_{\sigma\in\Gamma}\rho_i(\sigma)\otimes\mathbf{A}_\sigma;\lambda\right)^{m_i}. \tag{7}$$

3. Line graphs of a graph and its covering graphs

For a simplicity of computing, we assume that the base graph G is regular. The characteristic polynomial and the complexity of the line graph $L(G)$ of an r-regular graph G are given as follows [6]:

Theorem 7. *Let G be a connected r-regular graph with ν vertices and ε edges. Then*

$$\Phi(L(G);\lambda)=(\lambda+2)^{\varepsilon-\nu}\Phi(G;\lambda+2-r)\quad \textit{and}\quad \kappa(L(G))=2^{\varepsilon-\nu+1}r^{\varepsilon-\nu-1}\kappa(G). \tag{8}$$

By Bass Theorem, one can get a matrix expression of the zeta function of the line graph $L(G)$ as follows.

Theorem 8. *Let G be a connected graph with ν vertices and ε edges. Then*

$$Z(L(G),u)^{-1}=(1-u^2)^{|E(L(G))|-\varepsilon}\det(\mathbf{I}-u\mathbf{A}_L+u^2(\mathbf{D}_L-\mathbf{I})), \tag{9}$$

where $\mathbf{A}_L=\mathbf{A}(L(G))$ and $\mathbf{D}_L=\mathbf{D}_{L(G)}$.

In particular, if G is regular one can express the reciprocal $Z(L(G),u)^{-1}$ of the zeta function of the line graph $L(G)$ in terms of the characteristic polynomial of G.

Theorem 9. *Let G be a connected r-regular graph with ν vertices and ε edges. Then*

$$\begin{aligned}Z(L(G),u)^{-1}&=(1-u^2)^{(r-2)\varepsilon}u^\nu(1+2u+(2r-3)u^2)^{\varepsilon-\nu}\\&\quad\times\Phi\left(G;\frac{1+(2-r)u+(2r-3)u^2}{u}\right).\end{aligned} \tag{10}$$

Proof. Note that $L(G)$ is $(2r-2)$-regular. By Eq. (1), we have

$$\begin{aligned}Z(L(G),u)^{-1}&=(1-u^2)^{(r-2)\varepsilon}\det(\mathbf{I}_\varepsilon-u\mathbf{A}_L+au^2\mathbf{I}_\varepsilon)\\&=(1-u^2)^{(r-2)\varepsilon}u^\varepsilon\Phi\left(L(G);\frac{1+au^2}{u}\right),\end{aligned}$$

where $a=2r-3$. By Eq. (8),

$$Z(L(G),u)^{-1}=(1-u^2)^{(r-2)\varepsilon}u^\nu(1+2u+au^2)^{\varepsilon-\nu}\Phi\left(G;\frac{1+(2-r)u+au^2}{u}\right).$$

Therefore, Eq. (10) follows. □

In [17], Kotani and Sunada showed that the line graph of a regular covering of a graph G with covering transformation group $\mathscr{A}$ is a regular covering of $L(G)$ with the same covering transformation group $\mathscr{A}$.

Archdeacon et al. [1] showed that the line graph of a covering of a graph G is a covering of $L(G)$.

Theorem 10 (Archdeacon et al.). *Let H be a covering of a graph G, then $L(H)$ is a covering of $L(G)$.*

Let $\alpha : D(G) \to S_n$ a permutation voltage assignment. In the n-fold covering G^α, set $v_g = (v, g)$ and $e_g = (e, g)$, where $v \in V(G), e \in D(G), g \in [n]$. For $e = (u, v) \in D(G)$, the arc e_g emanates from u_g and terminates at $v_{\alpha(e)(g)}$. Note that $e_g^{-1} = (e^{-1})_{\alpha(e)(g)}$.

For $e \in D(G)$, let $[e]$ denote the edge obtained from e by deleting its direction. Then the set $D(L(G))$ of arcs in the line graph $L(G)$ is given by

$$\{(e, f)|e \neq f^{-1}, t(e) = o(f)\},$$

where $o((e, f)) = [e]$ and $t((e, f)) = [f]$. Furthermore, $(e, f)^{-1} = (f^{-1}, e^{-1})$.

Now, we determine a voltage assignment $\beta : D(L(G)) \to S_n$ which derives the covering $L(G^\alpha) \to L(G)$.

Lemma 11. *Let G be a connected graph with ν vertices $v_1, \ldots, v_\nu$, and let $\alpha : D(G) \to S_n$ be a permutation voltage assignment. For each edge $v_i v_j \in E(G)$, let $e_{ij} = (v_i, v_j)$ as an arc. Furthermore, let $\alpha_L : D(L(G)) \to S_n$ be the voltage assignment defined by*

$$\alpha_L([e_{ij}], [e_{jk}]) := \begin{cases} \alpha(e_{ij}) & \text{if } i < j < k, \\ \alpha(e_{jk})\alpha(e_{ij}) & \text{if } i < j \text{ and } j > k, \\ \alpha(e_{jk}) & \text{if } i > j > k, \\ 1 & \text{if } i > j \text{ and } j < k. \end{cases}$$

Then the covering $L(G^\alpha) \to L(G)$ is derived from the voltage assignment α_L.

Proof. At first, note that $((e_{ij})_s)^{-1} = (e_{ji})_{\alpha(e_{ij})(s)}$ for any $s \in [n]$. Let $[(e_{ij})_s] = [(e_{ji})_{\alpha(e_{ij})(s)}] = [e_{ij}]_s$ if $i < j$.

Let $[e_{ij}][e_{jk}] \in E(L(G))$ be any edge of $L(G)$ and $s \in [n]$. Set $v = v_i$, $w = v_j$, $z = v_k$, $x = e_{ij}$ and $y = e_{jk}$. Then we have $x_s = (v_s, w_{\alpha(x)(s)})$, $(x_s)^{-1} = (x^{-1})_{\alpha(x)(s)}$, $y_{\alpha(x)(s)} = (w_{\alpha(x)(s)}, z_{\alpha(y)\alpha(x)(s)})$ and $(y_{\alpha(x)(s)})^{-1} = (y^{-1})_{\alpha(y)\alpha(x)(s)}$. Note that $t(x_s) = o(y_{\alpha(x)(s)})$, i.e., $([x_s], [y_{\alpha(x)(s)}]) \in D(L(G^\alpha))$, We consider four cases.

Case 1. $i < j < k$. In this case, we have $[x_s] = [x]_s$, $[y_{\alpha(x)(s)}] = [y]_{\alpha(x)(s)}$, and so $([x]_s, [y]_{\alpha(x)(s)}) \in D(L(G^\alpha))$. Furthermore, since $\alpha_L([x], [y]) = \alpha(x)$, $([x]_s, [y]_{\alpha(x)(s)}) \in D(L(G)^{\alpha_L})$.

Case 2. $i < j$ and $j > k$. In this case, we have $[x_s] = [x]_s$, $[y_{\alpha(x)(s)}] = [y]_{\alpha(y)\alpha(x)(s)}$, and so $([x]_s, [y]_{\alpha(y)\alpha(x)(s)}) \in D(L(G^\alpha))$. Furthermore, since $\alpha_L([x], [y]) = \alpha(y)\alpha(x)$, $([x]_s, [y]_{\alpha(y)\alpha(x)(s)}) \in D(L(G)^{\alpha_L})$.

Case 3. $i > j > k$. Similarly, we have $[x_s] = [x]_{\alpha(x)(s)}$, $[y_{\alpha(x)(s)}] = [y]_{\alpha(y)\alpha(x)(s)}$, and so $([x]_{\alpha(x)(s)}, [y]_{\alpha(y)\alpha(x)(s)}) \in D(L(G^\alpha))$. Since $\alpha_L([x], [y]) = \alpha(y)$, $([x]_{\alpha(x)(s)}, [y]_{\alpha(y)\alpha(x)(s)}) \in D(L(G)^{\alpha_L})$.

Case 4. $i > j$ and $j < k$. We have $[x_s] = [x]_{\alpha(x)(s)}$, $[y_{\alpha(x)(s)}] = [y]_{\alpha(x)(s)}$, and so $([x]_{\alpha(x)(s)}, [y]_{\alpha(x)(s)}) \in D(L(G^\alpha))$. Since $\alpha_L([x], [y]) = 1$, we have $([x]_{\alpha(x)(s)}, [y]_{\alpha(x)(s)}) \in D(L(G)^{\alpha_L})$. $\square$

Let G be a connected graph with ν vertices $v_1, \ldots, v_\nu$ and ε edges $e_1, \ldots, e_\varepsilon$. For $g \in \Gamma$, the matrix $(\mathbf{A}_L)_g = (a^{(g)}_{ef})$ is defined as follows:

$$a^{(g)}_{ef} := \begin{cases} 1 & \text{if } \alpha_L(e, f) = g \text{ and } (e, f) \in D(L(G)), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{D}_L = (d_{ij})$ be the diagonal matrix with $d_{ii} = \deg_{L(G)} e_i$ and $\mathbf{Q}_L = \mathbf{D}_L - \mathbf{I}_\varepsilon$. By Theorem 5, the decomposition formulas for the zeta function and the complexity of the line graph $L(G^\alpha)$ of a covering G^α of a graph G are obtained as follows.

Theorem 12. *Let G be a connected graph with ν vertices and ε edges, and $\alpha : D(G) \to S_n$ a permutation voltage assignment. Let $\Gamma = \langle\{\alpha(u, v)|(u, v) \in D(G)\}\rangle$ be the subgroup of S_n generated by $\{\alpha(u, v)|(u, v) \in D(G)\}$, and let $P : \Gamma \to \mathrm{GL}(n, \mathbf{C})$ be the permutation representation of Γ such that $P(\gamma) = \mathbf{P}_\gamma$, where $\mathbf{P}_\gamma$ is the permutation matrix of γ.*

Suppose that the n-fold covering G^α of G is connected. Furthermore, let $\rho_1 = 1, \rho_2, \ldots, \rho_t$ be the irreducible representations of Γ, and f_i the degree of ρ_i for each i, where $f_1 = 1$, with the decomposition $P = 1 + m_2\rho_2 + \cdots + m_t\rho_t$ into irreducible representations. Then the reciprocal of the zeta function of $L(G^\alpha)$ is

$$Z(L(G^\alpha), u)^{-1} = Z(L(G), u)^{-1} \times \prod_{i=2}^{t} \left\{ (1-u^2)^{(\varepsilon_L - \varepsilon) f_i} \det\left(\mathbf{I}_{f_i\varepsilon} - u \sum_{g\in\Gamma} \rho_i(g) \otimes (\mathbf{A}_L)_g + u^2 (\mathbf{Q}_L)_{f_i} \right) \right\}^{m_i},$$

where $(\mathbf{Q}_L)_{f_i} = \mathbf{I}_{f_i} \otimes \mathbf{Q}_L$ and $\varepsilon_L = |E(L(G))|$. Suppose that $\varepsilon > \nu$. Then the complexity of $L(G^\alpha)$ is

$$\kappa(L(G^\alpha)) = \frac{1}{n}\kappa(L(G)) \prod_{i=2}^{t} \det\left(\mathbf{I}_{f_i} \otimes \mathbf{D}_L - \sum_{g\in\Gamma} \rho_i(g) \otimes (\mathbf{A}_L)_g \right)^{m_i}.$$

By Theorem 9, one can express the zeta function of the line graph $L(G^\alpha)$ in terms of the characteristic polynomial of G^α when G is regular.

Corollary 13. *Let G be a connected regular graph with valency r, ν vertices and ε edges, and $\alpha : D(G) \to S_n$ a permutation voltage assignment. Suppose that the n-fold covering G^α is connected. Then*

$$Z(L(G^\alpha), u)^{-1} = (1-u^2)^{(r-2)\varepsilon n} u^{\nu n} (1 + 2u + (2r-3)u^2)^{(\varepsilon-\nu)n} \times \Phi\left(G^\alpha; \frac{1 + (2-r)u + (2r-3)u^2}{u} \right) \tag{11}$$

and

$$\kappa(L(G^\alpha)) = 2^{(\varepsilon-\nu)n+1} r^{(\varepsilon-\nu)n-1} \kappa(G^\alpha). \tag{12}$$

Proof. Note that G^α is an r-regular graph. □

Let G be a graph and $\alpha : D(G) \to S_n$ a permutation voltage assignment. The *net voltage* $\alpha(P)$ of each path $P = (v_1, \ldots, v_\ell)$ of G is defined by $\alpha(P) = \alpha(v_1, v_2) \cdots \alpha(v_{\ell-1}, v_\ell)$ [10]. Furthermore, let $\Gamma = \langle\{\alpha(e)|e \in D(G)\}\rangle$, and let ρ be a representation of Γ. The *L-function* of G associated to ρ and α is defined to be the function of $u \in \mathbf{C}$ with $|u|$ sufficiently small as follows.

$$\mathbf{Z}(u,\rho,\alpha)=\mathbf{Z}_G(u,\rho,\alpha)=\prod_{[C]}\det\left(\mathbf{I}_f-\rho(\alpha(C))u^{|C|}\right)^{-1},$$

where $f=\deg\rho$ and $[C]$ runs over all equivalence classes of prime, reduced cycles of G (cf., [13,16,30]).

We give a determinant expression for the L-function of the line graph $L(G)$ for a graph G.

Theorem 14. *Let G be a connected graph with ν vertices and ε edges and $\alpha : D(G)\to S_n$ a permutation voltage assignment. Furthermore, let ρ be a representation of $\Gamma=\langle\{\alpha(e)|e\in D(G)\}\rangle$ with degree f. Suppose that the n-fold covering G^α of G is connected. Then*

$$\mathbf{Z}_{L(G)}(u,\rho,\alpha_L)^{-1}=(1-u^2)^{(\varepsilon_L-\varepsilon)f}\det\left(\mathbf{I}_{f\varepsilon}-u\sum_{g\in\Gamma}\rho(g)\otimes(\mathbf{A}_L)_g+u^2(\mathbf{Q}_L)_f\right).$$

Proof. By Theorem 3 of [21]. □

Corollary 15. *Let G be a connected graph with ν vertices and ε edges, and $\alpha : D(G)\to S_n$ a permutation voltage assignment. Let $\Gamma=\langle\{\alpha(u,v)|(u,v)\in D(G)\}\rangle$ be the subgroup of S_n generated by $\{\alpha(u,v)|(u,v)\in D(G)\}$, and let $P:\Gamma\to \mathrm{GL}(n,\mathbf{C})$ be the permutation representation of Γ such that $P(\gamma)=\mathbf{P}_\gamma$. Suppose that the n-fold covering G^α of G is connected. Furthermore, let $\rho_1=1,\rho_2,\dots,\rho_t$ be the irreducible representations of Γ, and f_i the degree of ρ_i for each i, where $f_1=1$, with the decomposition $P=1+m_2\rho_2+\cdots+m_t\rho_t$ into irreducible representations. Then*

$$Z(L(G^\alpha),u)=\prod_{i=1}^{t}\mathbf{Z}_{L(G)}(u,\rho,\alpha_L)^{m_i}.$$

We express the L-function of the line graph $L(G)$ for a regular graph G in terms of characteristic polynomials.

Theorem 16. *Let G be a connected r-regular graph with ν vertices and ε edges and $\alpha : D(G)\to S_n$ a permutation voltage assignment. Furthermore, let ρ be a representation of $\Gamma=\langle\{\alpha(e)|e\in D(G)\}\rangle$, and f the degree of ρ. Suppose that the n-fold covering G^α of G is connected. Then*

$$\mathbf{Z}_{L(G)}(u,\rho,\alpha_L)^{-1}=(1-u^2)^{(r-2)\varepsilon f}u^{\nu f}(1+2u+(2r-3)u^2)^{(\varepsilon-\nu)f}$$
$$\times\Phi\left(\sum_{g\in\Gamma}\rho(g)\otimes\mathbf{A}_g;\ \frac{1+(2-r)u+(2r-3)u^2}{u}\right).$$

Proof. Similar to the proof of Theorem 4 in [23]. □

By Theorem 16 and Corollary 15, Corollary 13 can be confirmed as follows:

Let $\rho_1=1,\rho_2,\dots,\rho_t$ be the irreducible representations of Γ, and f_i the degree of ρ_i for each i, where $f_1=1$. Assume that the permutation representation $P:\Gamma\to\mathrm{GL}(n,\mathbf{C})$ is decomposed into a direct sum of irreducible representations as $1+m_2\rho_2+\cdots+m_t\rho_t$. By Theorem 16 and Corollary 15, we have

$$\mathbf{Z}(L(G^{\alpha}), u)^{-1} = \prod_{i=1}^{t} \mathbf{Z}_{L(G)}(u, \rho_i, \alpha_L)^{-m_i}$$
$$= \prod_{i=1}^{t} \left\{ (1-u^2)^{(r-2)\varepsilon m_i} u^{\nu m_i} (1+2u+au^2)^{(\varepsilon-\nu)m_i} \right.$$
$$\left. \times \Phi\left(\sum_{g \in \Gamma} \rho_i(g) \otimes \mathbf{A}_g; \frac{1+(2-r)u+au^2}{u} \right) \right\}^{m_i},$$

where $a = 2r-3$. Since $1 + m_2 f_2 + \cdots + m_t f_t = n$,

$$\mathbf{Z}(L(G^{\alpha}), u)^{-1} = (1-u^2)^{(r-2)\varepsilon n} u^{\nu n} (1+2u+au^2)^{(\varepsilon-\nu)n} \Phi\left(G; \frac{1+(2-r)u+au^2}{u}\right)$$
$$\times \prod_{i=2}^{t} \Phi\left(\sum_{g \in \Gamma} \rho_i(g) \otimes \mathbf{A}_g; \frac{1+(2-r)u+au^2}{u} \right)^{m_i}.$$

Now, Corollary 13 follows from Theorem 6.

4. Middle graphs of a graph and its covering graphs

The middle graph $M(G)$ of G is the graph with $V(M(G)) = V(G) \cup E(G)$ and $E(M(G)) = E(L(G)) \cup \{ue | e \in E(G), u \in V(G)$ are incident in $G\}$. Let $V(G) = \{v_1, \ldots, v_\nu\}$. The *end-line graph* G^+ of G is defined as follows: $V(G^+) = \{v_1, \ldots, v_\nu, v'_1, \ldots, v'_\nu\}$ and $E(G^+) = E(G) \cup \{v_1 v'_1, \ldots, v_\nu v'_\nu\}$. Hamada and Yoshimura [12] showed that $M(G) = L(G^+)$.

The characteristic polynomial and the complexity of the middle graph $M(G)$ of an r-regular graph G are given as follows [6,22]:

Theorem 17. *Let G be a connected r-regular graph with ν vertices and ε edges. Then*

$$\Phi(M(G); \lambda) = (\lambda+1)^{\nu} (\lambda+2)^{\varepsilon-\nu} \Phi\left(G; \frac{\lambda^2+(2-r)\lambda-r}{\lambda+1}\right) \tag{13}$$

and

$$\kappa(M(G)) = 2^{\varepsilon-\nu+1}(r+1)^{\varepsilon-1}\kappa(G). \tag{14}$$

As the case of line graph, one can use Bass theorem to get a matrix expression of the zeta function of the middle graph $M(G)$ as follows.

Theorem 18. *Let G be a connected graph with ν vertices and ε edges. Then*

$$Z(M(G), u)^{-1} = (1-u^2)^{|E(M(G))|-\varepsilon_G-\nu_G} \det(\mathbf{I} - u\mathbf{A}_M + u^2(\mathbf{D}_M - \mathbf{I})), \tag{15}$$

where $\mathbf{A}_L = \mathbf{A}(M(G))$ and $\mathbf{D}_M = \mathbf{D}_{M(G)}$.

In particular, if G is regular one can express the reciprocal $Z(M(G), u)^{-1}$ of the zeta function of the middle graph $M(G)$ in terms of the characteristic polynomial of G.

Theorem 19. *Let G be a connected r-regular graph with ν vertices and ε edges. Then*

$$\begin{aligned} &Z(M(G),u)^{-1} \\ &\quad = (1-u^2)^{(r-1)(\varepsilon+\nu)-\varepsilon} u^{\nu}(1+u+(r-1)u^2)^{\nu}(1+2u+(2r-1)u^2)^{\varepsilon-\nu} \\ &\qquad \times \Phi\left(G; \frac{1+(2-r)u+(2r-2)u^2+(r-1)(2-r)u^3+(2r-1)(r-1)u^4}{u(1+u+(r-1)u^2)}\right). \end{aligned} \tag{16}$$

Proof. For a vertex w of $M(G)$, we have

$$\deg w = \begin{cases} r & \text{if } w \in V(G), \\ 2r & \text{if } w \in E(G). \end{cases}$$

Set $a = 2r-1$ and $b = r-1$. By Eq. (1), we have

$$Z(M(G),u)^{-1} = (1-u^2)^{(r-1)(\varepsilon+\nu)-\varepsilon} \det(\mathbf{I}_{\varepsilon+\nu} - u\mathbf{A}_M + u^2\mathbf{Q}_M).$$

The equalities in Eq. (4) imply that

$$\begin{aligned} &\det(\mathbf{I}_{\varepsilon+\nu} - u\mathbf{A}_M + u^2\mathbf{Q}_M) \\ &\quad = \det\begin{bmatrix} (1+au^2)\mathbf{I}_{\varepsilon} - u\mathbf{A}_L & -u\mathbf{B} \\ -u\mathbf{B}^t & (1+bu^2)\mathbf{I}_{\nu} \end{bmatrix} \\ &\quad = \det\begin{bmatrix} (1+au^2)\mathbf{I}_{\varepsilon} - u\mathbf{A}_L - \frac{u^2}{1+bu^2}\mathbf{B}\mathbf{B}^t & -u\mathbf{B} \\ \mathbf{0} & (1+bu^2)\mathbf{I}_{\nu} \end{bmatrix}. \end{aligned}$$

By Eqs. (4) and (8),

$$\begin{aligned} &\det(\mathbf{I}_{\varepsilon+\nu} - u\mathbf{A}_M + u^2\mathbf{Q}_M) \\ &\quad = u^{\varepsilon}(1+bu^2)^{\nu-\varepsilon}(1+u+bu^2)^{\varepsilon}\Phi\left(L(G); \frac{1+(a+b-2)u^2+abu^4}{u(1+u+bu^2)}\right) \\ &\quad = u^{\nu}(1+u+bu^2)^{\nu}(1+2u+au^2)^{\varepsilon-\nu}\Phi \\ &\qquad \times\left(G; \frac{1+(2-r)u+(a+b-r)u^2+b(2-r)u^3+abu^4}{u(1+u+bu^2)}\right). \end{aligned}$$

Thus, we have

$$Z(M(G),u)^{-1} = (1-u^2)^{(r-1)(\varepsilon+\nu)-\varepsilon}u^{\nu}(1+u+bu^2)^{\nu}(1+2u+au^2)^{\varepsilon-\nu}\Phi(G;h(u)),$$

where

$$h(u) = (1+(2-r)u+(a+b-r)u^2+b(2-r)u^3+abu^4)(u(1+u+bu^2))^{-1}.$$

Therefore, the result follows. □

For any permutation voltage assignment $\alpha : D(G) \to S_n$, we show that the middle graph of the n-fold covering G^{α} of G is an n-fold covering of the middle graph $M(G)$ of G. Also, one can determine the voltage assignment which derives the covering $M(G^{\alpha}) \to M(G)$.

Theorem 20. *Let G be a connected graph with ν vertices $v_1, \ldots, v_{\nu}$ and let $\alpha : D(G) \to S_n$ be a permutation voltage assignment. Then $M(G^{\alpha})$ is an n-fold covering of $M(G)$.*

Proof. Recall that $M(G) = L(G^+)$. Now, let $V(G) = \{v_1, \ldots, v_\nu\}$ and $V(G^+) = V(G) \cup \{v'_1, \ldots, v'_\nu\}$. Define a function $\alpha^* : D(G^+) \to S_n$ by

$$\alpha^*(u, v) := \begin{cases} \alpha(u, v) & \text{if } (u, v) \in D(G), \\ 1 & \text{if } uv = v_i v'_i. \end{cases}$$

Then one can show that $(G^\alpha)^+ = (G^+)^{\alpha^*}$, i.e., $M(G^\alpha) = L((G^\alpha)^+) = L((G^+)^{\alpha^*})$. By Lemma 11, $L((G^+)^{\alpha^*})$ is an n-fold covering of $L(G^+)$ and it can be derived from a voltage assignment α^*_L. Therefore, $M(G^\alpha)$ is an n-fold covering of $M(G)$. □

Corollary 21. $M(G^\alpha) = M(G)^{\alpha^*_L}$.

Proof. Since $L(G^\alpha) = L(G)^{\alpha_L}$, $M(G^\alpha) = L((G^+)^{\alpha^*}) = L(G^+)^{\alpha^*_L} = M(G)^{\alpha^*_L}$. □

Mizuno and Sato [22] showed that $M(G^\alpha)$ is a regular covering of $M(G)$ if G^α is a regular covering of G.

We consider the permutation voltage assignment $\alpha^*_L : D(M(G)) \to S_n$. Set $\alpha_M = \alpha^*_L$. Give an order in $V(G^+)$ as $v_1, \ldots, v_\nu, v'_1, \ldots, v'_\nu$. By the definition of α^*, α_M is given as follows:

$$\alpha_M(u, v) = \begin{cases} \alpha_L(u, v) & \text{if } (u, v) \in D(L(G)), \\ 1 & \text{if } u = [e_{ij}],\ v = v_j v'_j \text{ and } i > j, \\ \alpha(e_{ij}) & \text{if } u = [e_{ij}],\ v = v_j v'_j \text{ and } i < j, \end{cases}$$

where $e_{ij} = (v_i, v_j)$.

For $g \in S_n$, the matrix $(\mathbf{A}_M)_g = (a^{(g)}_{uv})$ is defined as follows: $a^{(g)}_{uv} = 1$ if $\alpha_M(u, v) = g$ and $(u, v) \in D(M(G))$, and $a^{(g)}_{uv} = 0$ otherwise. Furthermore, let $\mathbf{D}_M = (d_{ij})$ be the diagonal matrix with $d_{ii} = \deg_{M(G)} e_i (1 \leqslant i \leqslant \varepsilon)$; $d_{ii} = \deg_{M(G)} v_{i-\varepsilon} (\varepsilon + 1 \leqslant i \leqslant \varepsilon + \nu)$, and $\mathbf{Q}_M = \mathbf{D}_M - \mathbf{I}_{\varepsilon+\nu}$, where $V(G) = \{v_1, \ldots, v_\nu\}$ and $E(G) = \{e_1, \ldots, e_\varepsilon\}$. By Theorem 5, the decomposition formulas for the zeta function and the complexity of the middle graph $M(G^\alpha)$ of a covering G^α of a graph G are obtained. Note that $|E(M(G))| > |V(M(G))|$ if $\varepsilon \geqslant \nu$.

Theorem 22. *Let G be a connected graph and let $\alpha : D(G) \to S_n$ be a permutation voltage assignment. Let $\Gamma = \langle\{\alpha(u, v) | (u, v) \in D(G)\}\rangle$ be the subgroup of S_n generated by $\{\alpha(u, v) | (u, v) \in D(G)\}$, and let $P : \Gamma \to \mathrm{GL}(n, \mathbf{C})$ be the permutation representation of Γ. Suppose that the n-fold covering G^α of G is connected. Furthermore, let $\rho_1 = 1, \rho_2, \ldots, \rho_t$ be the irreducible representations of Γ, and f_i the degree of ρ_i for each i, where $f_1 = 1$, with the decomposition $P = 1 + m_2\rho_2 + \cdots + m_t\rho_t$ into irreducible representations. Then the reciprocal of the zeta function of $M(G^\alpha)$ is*

$$Z(M(G^\alpha), u)^{-1} = Z(M(G), u)^{-1} \prod_{i=2}^{t} \left\{ (1 - u^2)^{(\varepsilon_M - \varepsilon_G - \nu_G) f_i} \times \det\left(\mathbf{I} - u \sum_{g \in \Gamma} \rho_i(g) \otimes (\mathbf{A}_M)_g + u^2 (\mathbf{Q}_M)_{f_i} \right) \right\}^{m_i},$$

where $(\mathbf{Q}_M)_{f_i} = \mathbf{I}_{f_i} \otimes \mathbf{Q}_M$ and $\varepsilon_M = |E(M(G))|$. Suppose that $\varepsilon_G \geqslant \nu_G$. Then the complexity of $M(G^\alpha)$ is

$$\kappa(M(G^\alpha)) = \frac{1}{n}\kappa(M(G))\prod_{i=2}^{t}\det\left(\mathbf{I}_{f_i}\otimes\mathbf{D}_M - \sum_{g\in\Gamma}\rho_i(g)\otimes(\mathbf{A}_M)_g\right)^{m_i}.$$

By Theorem 19, one can express the zeta function of the middle graph $M(G^\alpha)$ in terms of the characteristic polynomial of G^α when G is regular.

Corollary 23. *Let G be a connected regular graph with valency r, ν vertices and ε edges, and $\alpha : D(G) \to S_n$ a permutation voltage assignment. Suppose that the n-fold covering G^α of G is connected. Then the reciprocal of the zeta function of $M(G^\alpha)$ is*

$$\begin{aligned} &Z(M(G^\alpha), u)^{-1} \\ &\quad = (1-u^2)^{n(r-1)(\varepsilon+\nu)-\varepsilon n}u^{\nu n}(1+u+(r-1)u^2)^{\nu n}(1+2u+(2r-1)u^2)^{(\varepsilon-\nu)n} \\ &\qquad \times\Phi\left(G^\alpha; \frac{1+(2-r)u+(2r-2)u^2+(r-1)(2-r)u^3+(2r-1)(r-1)u^4}{u(1+u+(r-1)u^2)}\right). \end{aligned} \tag{17}$$

Proof. Note that G^α is r-regular. □

Mizuno and Sato [24] expressed the Bartholdi zeta functions of the line graph and the middle graph of a regular covering of a graph by using the characteristic polynomial of that regular covering.

By Theorem 17, we obtain the following result.

Corollary 24. *Let G be a connected regular graph with valency r, ν vertices and ε edges, and $\alpha : D(G) \to S_n$ a permutation voltage assignment. Suppose that the n-fold covering G^α of G is connected and $\varepsilon \geqslant \nu$. Then the complexity of $M(G^\alpha)$ is*

$$\kappa(M(G^\alpha)) = 2^{(\varepsilon-\nu)n+1}(r+1)^{\varepsilon n-1}\kappa(G^\alpha). \tag{18}$$

Next, we state an alternative formula for the complexity $\kappa(M(G^\alpha))$.

Corollary 25. *Let G be a connected r-regular graph with ν vertices and ε edges, and $\alpha : D(G) \to S_n$ a permutation voltage assignment. Let $\Gamma = \langle\{\alpha(u,v)|(u,v)\in D(G)\}\rangle$ be the subgroup of S_n generated by $\{\alpha(u,v)|(u,v)\in D(G)\}$, and let $P : \Gamma \to \mathrm{GL}(n, \mathbf{C})$ be the permutation representation of Γ such that $P(\gamma) = \mathbf{P}_\gamma$. Suppose that the n-fold covering G^α of G is connected and $\varepsilon \geqslant \nu$. Furthermore, let $\rho_1 = 1, \rho_2, \ldots, \rho_t$ be the irreducible representations of Γ, and f_i the degree of ρ_i for each i, where $f_1 = 1$, with the decomposition $P = 1 + m_2\rho_2 + \cdots + m_t\rho_t$ into irreducible representations. Then the complexity of $M(G^\alpha)$ is*

$$\kappa(M(G^\alpha)) = \frac{1}{n}2^{(\varepsilon-\nu)(n-1)}(r+1)^{\varepsilon(n-1)}\kappa(M(G))\prod_{i=2}^{t}\Phi\left(\sum_{g\in\Gamma}\rho_i(g)\otimes\mathbf{A}_g; r\right)^{m_i}.$$

Proof. By Eqs. (18) and (3), we have

$$\kappa(M(G^\alpha)) = 2^{(\varepsilon-\nu)n+1}(r+1)^{\varepsilon n-1}(1/\nu n)\Phi'(G^\alpha; r).$$

Eq. (7) implies that $\Phi'(G^{\alpha}; r) = \Phi'(G; r) \prod_{i=2}^{t} \Phi(\sum_{g \in \Gamma} \rho_i(g) \otimes \mathbf{A}_g; r)^{m_i}$. By Eqs. (3) and (14), we have

$$\Phi'(G; r) = \nu\kappa(G) = \nu 2^{-(\varepsilon-\nu+1)}(r+1)^{-(\varepsilon-1)}\kappa(M(G)).$$

Therefore, the result follows. □

We give a determinant expression for the L-function of the middle graph $M(G)$ for a graph G.

Theorem 26. *Let G be a connected graph with ν vertices and ε edges and $\alpha : D(G) \to S_n$ a permutation voltage assignment. Furthermore, let ρ be a representation of $\Gamma = \langle\{\alpha(e)|e \in D(G)\}\rangle$, and f the degree of ρ. Suppose that the n-fold covering G^{α} of G is connected. Then*

$$\mathbf{Z}_{M(G)}(u, \rho, \alpha_M)^{-1} = (1-u^2)^{(\varepsilon_M-\varepsilon-\nu)f} \det\left(\mathbf{I} - u\sum_{g \in \Gamma} \rho(g) \otimes (\mathbf{A}_M)_g + u^2(\mathbf{Q}_M)_f\right).$$

Proof. By Theorem 3 of [21]. □

Corollary 27. *Let G be a connected graph with ν vertices and ε edges, and $\alpha : D(G) \to S_n$ a permutation voltage assignment. Let $\Gamma = \langle\{\alpha(u,v)|(u,v) \in D(G)\}\rangle$ be the subgroup of S_n generated by $\{\alpha(u,v)|(u,v) \in D(G)\}$, and let $P : \Gamma \to \mathrm{GL}(n, \mathbf{C})$ be the permutation representation of Γ such that $P(\gamma) = \mathbf{P}_{\gamma}$. Suppose that the n-fold covering G^{α} of G is connected. Furthermore, let $\rho_1 = 1, \rho_2, \ldots, \rho_t$ be the irreducible representations of Γ, and f_i the degree of ρ_i for each i, where $f_1 = 1$, with the decomposition $P = 1 + m_2\rho_2 + \cdots + m_t\rho_t$ into irreducible representations. Then*

$$Z(M(G^{\alpha}), u) = \prod_{i=1}^{t} \mathbf{Z}_{M(G)}(u, \rho_i, \alpha_M)^{m_i}.$$

We express the L-function of the middle graph $M(G)$ for a regular graph G in terms of characteristic polynomials.

Theorem 28. *Let G be a connected r-regular graph with ν vertices and ε edges and $\alpha : D(G) \to S_n$ a permutation voltage assignment. Furthermore, let ρ be a representation of $\Gamma = \langle\{\alpha(e)|e \in D(G)\}\rangle$, and f the degree of ρ. Suppose that the n-fold covering G^{α} of G is connected. Then*

$$\begin{aligned}
&\mathbf{Z}_{M(G)}(u, \rho, \alpha_M)^{-1} \\
&\quad = (1-u^2)^{(r-1)(\varepsilon+\nu)f-\varepsilon f} u^{\nu f}(1+u+(r-1)u^2)^{\nu f}(1+2u+(2r-1)u^2)^{(\varepsilon-\nu)f} \\
&\qquad \times \Phi\left(\sum_{g \in \Gamma} \rho(g) \otimes \mathbf{A}_g; \frac{1+(2-r)u+(2r-2)u^2+(r-1)(2-r)u^3+(2r-1)(r-1)u^4}{u(1+u+(r-1)u^2)}\right).
\end{aligned}$$

Proof. Similar to the proof of Theorem 6 in [23]. □

By Theorem 28 and Corollary 27, Corollary 23 can be confirmed as follows: Let $\rho_1 = 1, \rho_2, \ldots, \rho_t$ be the irreducible representations of Γ, and f_i the degree of ρ_i for each i, where $f_1 = 1$. Let the permutation representation $P : \Gamma \to \mathrm{GL}(n, \mathbf{C})$ be decomposed into a direct sum of irreducible representations as $1 + m_2\rho_2 + \cdots + m_t\rho_t$. By Theorem 28 and Corollary 27, we have

$$\begin{aligned}\mathbf{Z}(M(G^{\alpha}),u)^{-1} &= \prod_{i=1}^{t} \mathbf{Z}_{M(G)}(u,\rho_i,\alpha_M)^{-m_i} \\ &= (1-u^2)^{n(r-1)(\varepsilon+\nu)-\varepsilon n} u^{\nu n}(1+u+bu^2)^{\nu n}(1+2u+au^2)^{(\varepsilon-\nu)n} \\ &\quad \times \Phi(G;h_1(u)) \prod_{i=2}^{t} \Phi\left(\sum_{g\in\Gamma}\rho_i(g)\otimes \mathbf{A}_g; h_1(u)\right)^{m_i},\end{aligned}$$

where $a=2r-1$, $b=r-1$ and $h_1(u)=\frac{1+(2-r)u+(a+b-r)u^2+b(2-r)u^3+abu^4}{u(1+u+bu^2)}$. Now, Corollary 23 follows from Theorem 6.

5. Total graphs of a graph and its covering graphs

The characteristic polynomial and the complexity of the total graph $T(G)$ of an r-regular graph G are given as follows. We denote the set of all eigenvalues of G by Spec$G$.

Theorem 29. *Let G be a connected regular graph with valency r, ν vertices and ε edges, and* Spec$G=\{\lambda_1=r,\lambda_2,\ldots,\lambda_\nu\}$. *Then*

$$\Phi(T(G);\lambda) = (\lambda+2)^{\varepsilon-\nu}\prod_{j=1}^{\nu}\left(\lambda^2-(2\lambda_j+r-2)\lambda+\lambda_j^2+(r-3)\lambda_j-r\right), \tag{19}$$

and

$$\kappa(T(G)) = \frac{1}{\nu}2^{\varepsilon-\nu+1}(r+1)^{\varepsilon-\nu}\prod_{j=2}^{\nu}(\lambda_j-r)(\lambda_j-2r-3). \tag{20}$$

Proof. Note that $T(G)$ is a $2r$-regular and Eq. (19) comes from Theorem 2.20 of [6]. By Theorem 3, we have

$$\kappa(T(G)) = \frac{1}{\varepsilon+\nu}\Phi'(T(G);2r).$$

But, we have

$$\Phi(T(G);\lambda) = (\lambda+2)^{\varepsilon-\nu}\prod_{j=1}^{\nu}(\lambda^2-(2\lambda_j+r-2)\lambda+\lambda_j^2+(r-3)\lambda_j-r),$$

where Spec$G=\{\lambda_1=r,\lambda_2,\ldots,\lambda_\nu\}$. In the case of $\lambda_j=r$, $\lambda^2-(2\lambda_j+r-2)\lambda+\lambda_j^2+(r-3)\lambda_j-r=(\lambda-2r)(\lambda-r+2)$. Since the multiplicity of $\lambda_j=r$ is 1, we set

$$\Phi(T(G);\lambda) = (\lambda-2r)k(\lambda),$$

and then

$$\Phi'(T(G);\lambda) = k(\lambda)+(\lambda-2r)k'(\lambda),$$

Thus,

$$\Phi'(T(G);2r) = k(2r) = 2^{\varepsilon-\nu}(r+1)^{\varepsilon-\nu}(r+2)\prod_{j=2}^{\nu}(\lambda_j-r)(\lambda_j-2r-3).$$

Since $\varepsilon+\nu=\nu(r+2)/2$, the result follows. □

Theorem 30. *Let G be a connected regular graph with valency r, ν vertices and ε edges, and* $\mathrm{Spec}G = \{\lambda_1 = r, \lambda_2, \dots, \lambda_\nu\}$. *Then*

$$Z(T(G), u)^{-1} = (1-u^2)^{(\varepsilon+\nu)(r-1)}(1+2u+(2r-1)u^2)^{\varepsilon-\nu} \prod_{j=1}^{\nu} \Big\{(2r-1)^2u^4 - (2r-1)(2\lambda_j + r - 2)u^3 + (\lambda_j^2 + (r-3)\lambda_j + 3r - 2)u^2 - (2\lambda_j + r - 2)u + 1\Big\}. \tag{21}$$

Proof. Note that $T(G)$ is $2r$-regular. Set $a = 2r - 1$. By Eq. (1), we have

$$Z(T(G), u)^{-1} = (1-u^2)^{(\varepsilon+\nu)(r-1)} \det(\mathbf{I}_{\varepsilon+\nu} - u\mathbf{A}_T + au^2\mathbf{I}_{\varepsilon+\nu}) = (1-u^2)^{(\varepsilon+\nu)(r-1)} u^{\varepsilon+\nu} \Phi\left(T(G); \frac{1+au^2}{u}\right).$$

From Eq. (19), we have

$$u^{\varepsilon+\nu}\Phi\left(T(G); \frac{1+au^2}{u}\right) = u^{\varepsilon+\nu}\left(\frac{1+au^2}{u} + 2\right)^{\varepsilon-\nu} \prod_{i=1}^{\nu} \left\{ Bigg(\frac{1+au^2}{u}\right)^2 - (2\lambda_i + r - 2)\left(\frac{1+au^2}{u}\right) + \lambda_i^2 + (r-3)\lambda_i - r \Bigg\}$$
$$= (1+2u+au^2)^{\varepsilon-\nu} \prod_{i=1}^{\nu} \Big\{a^2u^4 - a(2\lambda_i + r - 2)u^3 + (\lambda_i^2 + (r-3)\lambda_i - r + 2a)u^2 - (2\lambda_i + r - 2)u + 1\Big\},$$

where $\mathrm{Spec}G = \{\lambda_1 = r, \lambda_2, \dots, \lambda_\nu\}$. Thus, we have

$$Z(T(G), u)^{-1} = (1-u^2)^{(\varepsilon+\nu)(r-1)}(1+2u+au^2)^{\varepsilon-\nu} \prod_{i=1}^{\nu} h(u, \lambda_i), \tag{22}$$

where $h(u, \lambda) = a^2u^4 - a(2\lambda + r - 2)u^3 + (\lambda^2 + (r-3)\lambda - r + 2a)u^2 - (2\lambda + r - 2)u + 1$. □

First we show that the total graph $T(G^\alpha)$ of a covering G^α of G is a covering of the total graph $T(G)$ of G. For two graphs G and H, let $G \cup H$ be the graph with vertex set $V(G \cup H) = V(G) \cup V(H)$ and edge set $E(G \cup H) = E(G) \cup E(H)$.

Theorem 31. *Let G be a connected graph and let $\alpha : D(G) \to S_n$ be a permutation voltage assignment. Then the total graph $T(G^\alpha)$ of G^α is an n-fold covering of the total graph $T(G)$ of G.*

Proof. At first, note that $T(G) = M(G) \cup G$. Thus, we have $T(G^\alpha) = M(G^\alpha) \cup G^\alpha = M(G)^{\alpha_M} \cup G^\alpha$. Define a function $\alpha_T : D(T(G)) \to S_n$ by

$$\alpha_T(u,v) := \begin{cases} \alpha_M(u,v) & \text{if } (u,v) \in D(M(G)), \\ \alpha(u,v) & \text{if } (u,v) \in D(G). \end{cases}$$

Then it follows that $T(G^\alpha) = M(G)^{\alpha_T} \cup G^{\alpha_T} = (M(G) \cup G)^{\alpha_T} = T(G)^{\alpha_T}$. □

Mizuno and Sato [22] showed that $T(G^\alpha)$ is a regular covering of $T(G)$ if G^α is a regular covering of G.

We consider the permutation voltage assignment $\alpha_T : D(T(G)) \to S_n$. For $g \in S_n$, the matrix $(\mathbf{A}_T)_g = (a_{uv}^{(g)})$ is defined as follows: $a_{uv}^{(g)} = 1$ if $\alpha_T(u,v) = g$ and $(u,v) \in D(T(G))$, and $a_{uv}^{(g)} = 0$ otherwise. Furthermore, let $\mathbf{D}_T = (d_{ij})$ be the diagonal matrix with $d_{ii} = \deg_{T(G)} e_i (1 \leqslant i \leqslant \varepsilon)$; $d_{ii} = \deg_{T(G)} v_{i-\varepsilon} (\varepsilon + 1 \leqslant i \leqslant \varepsilon + \nu)$, and $\mathbf{Q}_T = \mathbf{D}_T - \mathbf{I}_{\varepsilon+\nu}$, where $V(G) = \{v_1, \ldots, v_\nu\}$ and $E(G) = \{e_1, \ldots, e_\varepsilon\}$.

By Theorem 5, the decomposition formulas for the zeta function and the complexity of the total graph $T(G^\alpha)$ of a covering G^α of a graph G are obtained. Note that $|E(T(G))| > |V(T(G))|$ if $\varepsilon \geqslant \nu$.

Theorem 32. *Let G be a connected graph with ν vertices and ε edges, and $\alpha : D(G) \to S_n$ a permutation voltage assignment. Let $\Gamma = \langle \{\alpha(u,v) | (u,v) \in D(G)\} \rangle$ be the subgroup of S_n generated by $\{\alpha(u,v) | (u,v) \in D(G)\}$, and let $P : \Gamma \to \mathrm{GL}(n, \mathbf{C})$ be the permutation representation of Γ such that $P(\gamma) = \mathbf{P}_\gamma$. Suppose that the n-fold covering G^α of G is connected. Furthermore, let $\rho_1 = 1, \rho_2, \ldots, \rho_t$ be the irreducible representations of Γ, and f_i the degree of ρ_i for each i, where $f_1 = 1$, with the decomposition $P = 1 + m_2\rho_2 + \cdots + m_t\rho_t$ into irreducible representations. Then the reciprocal of the zeta function of $T(G^\alpha)$ is*

$$Z(T(G^\alpha), u)^{-1} = Z(T(G), u)^{-1} \prod_{i=2}^{t} \Bigg\{ (1-u^2)^{(\varepsilon_T - \varepsilon - \nu) f_i} \times \det\Bigg(\mathbf{I} - u \sum_{g \in \Gamma} \rho_i(g) \otimes (\mathbf{A}_T)_g + u^2 (\mathbf{Q}_T)_{f_i} \Bigg) \Bigg\}^{m_i},$$

where $(\mathbf{Q}_T)_{f_i} = \mathbf{I}_{f_i} \otimes \mathbf{Q}_T$ and $\varepsilon_T = |E(T(G))|$. Suppose that $\varepsilon \geqslant \nu$. Then the complexity of $T(G^\alpha)$ is

$$\kappa(T(G^\alpha)) = \frac{1}{n} \kappa(T(G)) \prod_{i=2}^{t} \det\Bigg(\mathbf{I}_{f_i} \otimes \mathbf{D}_T - \sum_{g \in \Gamma} \rho_i(g) \otimes (\mathbf{A}_T)_g \Bigg)^{m_i}.$$

One can express the zeta function of the total graph $T(G^\alpha)$ in terms of the eigenvalues of G^α when G is regular. Recall that Spec$G$ is a subfamily of SpecG^α by Theorem 6.

Corollary 33. *Let G be a connected regular graph with valency r, ν vertices and ε edges, and let $\alpha : D(G) \to S_n$ be a permutation voltage assignment. Furthermore, let* Spec$G^\alpha = \{\lambda_1 = r, \lambda_2, \ldots, \lambda_\nu, \lambda_{\nu+1}, \ldots, \lambda_{n\nu}\}$, *where* Spec$G = \{\lambda_1, \ldots, \lambda_\nu\}$. *Suppose that the n-fold covering G^α of G is connected. Then*

$$Z(T(G^\alpha), u)^{-1} = (1-u^2)^{\nu(r+2)(r-1)(n-1)/2} (1 + 2u + (2r-1)u^2)^{\nu(r-2)(n-1)/2} Z(T(G), u)^{-1}$$

$$\times \prod_{j=\nu+1}^{n\nu} \Big\{(2r-1)^2u^4 - (2r-1)(2\lambda_j + r - 2)u^3$$
$$+(\lambda_j^2 + (r-3)\lambda_j + 3r - 2)u^2 - (2\lambda_j + r - 2)u + 1\Big\}.$$

Proof. Again $T(G^\alpha)$ is $2r$-regular. Set $a = 2r - 1$ and let

$$\mathrm{Spec} G^\alpha = \{\lambda_1 = r, \lambda_2, \ldots, \lambda_\nu, \lambda_{\nu+1}, \ldots, \lambda_{2\nu}, \ldots, \lambda_{n\nu}\}.$$

Then by Eq (22), we have

$$Z(T(G^\alpha), u)^{-1}$$
$$= (1-u^2)^{(\varepsilon+\nu)(r-1)n}(1 + 2u + au^2)^{(\varepsilon-\nu)n} \prod_{i=1}^{\nu} h(u, \lambda_i) \prod_{j=\nu+1}^{n\nu} h(u, \lambda_j)$$
$$= (1-u^2)^{(\varepsilon+\nu)(r-1)(n-1)}(1 + 2u + au^2)^{(\varepsilon-\nu)(n-1)} Z(T(G), u)^{-1} \prod_{j=\nu+1}^{n\nu} h(u, \lambda_j),$$

where $h(u, \lambda) = a^2u^4 - a(2\lambda + r - 2)u^3 + (\lambda^2 + (r-3)\lambda - r + 2a)u^2 - (2\lambda + r - 2)u + 1.$ □

We give a determinant expression for the L-function of the total graph $T(G)$ for a graph G.

Theorem 34. *Let G be a connected graph with ν vertices and ε edges and $\alpha : D(G) \to S_n$ a permutation voltage assignment. Furthermore, let ρ be a representation of $\Gamma = \langle\{\alpha(e) | e \in D(G)\}\rangle$ of degree f. Suppose that the n-fold covering G^α of G is connected. Then*

$$\mathbf{Z}_{T(G)}(u, \rho, \alpha_T)^{-1} = (1-u^2)^{(\varepsilon_T - \varepsilon - \nu)f} \det\left(\mathbf{I} - u\sum_{g\in\Gamma} \rho(g) \otimes (\mathbf{A}_T)_g + u^2(\mathbf{Q}_T)_f\right).$$

Proof. By Theorem 3 of [21]. □

Corollary 35. *Let G be a connected graph with ν vertices and ε edges, and $\alpha : D(G) \to S_n$ a permutation voltage assignment. Let $\Gamma = \langle\{\alpha(u, v) | (u, v) \in D(G)\}\rangle$ be the subgroup of S_n generated by $\{\alpha(u, v) | (u, v) \in D(G)\}$, and let $P : \Gamma \to \mathrm{GL}(n, \mathbf{C})$ be the permutation representation of Γ such that $P(\gamma) = \mathbf{P}_\gamma$. Suppose that the n-fold covering G^α of G is connected. Furthermore, let $\rho_1 = 1, \rho_2, \ldots, \rho_t$ be the irreducible representations of Γ, and f_i the degree of ρ_i for each i, where $f_1 = 1$, with the decomposition $P = 1 + m_2\rho_2 + \cdots + m_t\rho_t$ into irreducible representations. Then*

$$Z(T(G^\alpha), u) = \prod_{i=1}^{t} \mathbf{Z}_{T(G)}(u, \rho_i, \alpha_T)^{m_i}.$$

We express the L-function of the total graph $T(G)$ for a regular graph G in terms of eigenvalues of some matrix.

Theorem 36. *Let G be a connected r-regular graph with ν vertices and ε edges and $\alpha : D(G) \to S_n$ a permutation voltage assignment. Furthermore, let ρ be a representation of $\Gamma = \langle\{\alpha(e) | e \in D(G)\}\rangle$ of degree f. Let* Spec $\sum_{g\in\Gamma} \rho(g) \otimes \mathbf{A}_g = \{\lambda_1, \ldots, \lambda_{\nu f}\}$ *be the family of all eigenvalues of the matrix $\sum_{g\in\Gamma} \rho(g) \otimes \mathbf{A}_g$. Suppose that the n-fold covering G^α of G is connected. Then*

$$\begin{aligned}&\mathbf{Z}_{T(G)}(u, \rho, \alpha_T)^{-1}\\ &\quad = (1-u^2)^{(r-1)(\varepsilon+\nu)f}(1+2u+(2r-1)u^2)^{(\varepsilon-\nu)f}\\ &\qquad \times \prod_{j=1}^{\nu f}\Big\{(2r-1)^2u^4-(2r-1)(2\lambda_j+r-2)u^3+(\lambda_j^2+(r-3)\lambda_j+3r-2)u^2\\ &\qquad -(2\lambda_j+r-2)u+1\Big\}.\end{aligned}$$

Proof. At first, $T(G)$ is a $2r$-regular graph. By Theorem 34, we have

$$\mathbf{Z}_{T(G)}(u, \rho, \alpha_T)^{-1} = (1-u^2)^{(r-1)(\varepsilon+\nu)f} \det\left(\mathbf{I}_{(\varepsilon+\nu)f} - u\sum_{g\in\Gamma}\rho(g)\otimes(\mathbf{A}_T)_g + u^2 a\mathbf{I}_{(\varepsilon+\nu)f}\right),$$

where $a = 2r-1$. But, we have

$$\begin{aligned}&\det(\mathbf{I}_{(\varepsilon+\nu)f} - u\sum_{g\in\Gamma}\rho(g)\otimes(\mathbf{A}_T)_g + u^2 a\mathbf{I}_{(\varepsilon+\nu)f})\\ &\quad = u^{(\varepsilon+\nu)f}\det\left(\frac{1+au^2}{u}\mathbf{I}_{(\varepsilon+\nu)f} - \sum_{g\in\Gamma}(\mathbf{A}_T)_g\otimes\rho(g)\right).\end{aligned}$$

Now, let $V(G) = \{v_1, \ldots, v_\nu\}$ and $E(G) = \{e_1, \ldots, e_\varepsilon\}$. Furthermore, let $\mathbf{B}_\rho$ be the $\varepsilon f \times \nu f$ matrix defined as follows:

$$(\mathbf{B}_\rho)_{ij} := \begin{cases}\mathbf{I}_f & \text{if } e_i = (v_j, v_k) \text{ and } j < k,\\ \rho(\alpha(e_{kj})) & \text{if } e_i = (v_k, v_j) \text{ and } j > k,\\ \mathbf{0}_f & \text{otherwise,}\end{cases}$$

where $(\mathbf{B}_\rho)_{ij}$ is the (i, j)-block of $\mathbf{B}_\rho$. Then we have

$$\mathbf{B}_\rho\overline{\mathbf{B}}_\rho^t = \sum_{g\in\Gamma}(\mathbf{A}_L)_g\otimes\rho(g) + 2\mathbf{I}_{\varepsilon f} \tag{23}$$

and

$$\overline{\mathbf{B}}_\rho^t\mathbf{B}_\rho = \sum_{g\in\Gamma}\mathbf{A}_g\otimes\rho(g) + r\mathbf{I}_{\nu f}, \tag{24}$$

where $\overline{\mathbf{B}}_\rho^t$ is the conjugate transpose of $\mathbf{B}_\rho$. By (4), we have

$$\sum_{g\in\Gamma}(\mathbf{A}_T)_g\otimes\rho(g) = \begin{bmatrix}\sum_{g\in\Gamma}(\mathbf{A}_L)_g\otimes\rho(g) & \mathbf{B}_\rho\\ \overline{\mathbf{B}}_\rho^t & \sum_{g\in\Gamma}\mathbf{A}_g\otimes\rho(g)\end{bmatrix}, \tag{25}$$

where

$$\mathbf{A}_T = \begin{bmatrix} \mathbf{A}_L & \mathbf{B} \\ \mathbf{B}^t & \mathbf{A} \end{bmatrix}.$$

By (23)–(25), we have

$$\begin{aligned} h(u) &:= \det(\mathbf{I}_{(\varepsilon+\nu)f} - u\sum_{g\in\Gamma}\rho(g)\otimes(\mathbf{A}_T)_g + u^2(\mathbf{Q}_T)_f) \\ &= \det(\mathbf{I}_{(\varepsilon+\nu)f} - u\sum_{g\in\Gamma}(\mathbf{A}_T)_g\otimes\rho(g) + u^2\mathbf{Q}_T\otimes\mathbf{I}_f) \\ &= u^{(\varepsilon+\nu)f}\det\begin{bmatrix} \frac{1+au^2}{u}\mathbf{I}_{\varepsilon f} - \mathbf{B}_\rho\overline{\mathbf{B}}^t_\rho + 2\mathbf{I}_{\varepsilon f} & -\mathbf{B}_\rho \\ -\overline{\mathbf{B}}^t_\rho & \frac{1+au^2}{u}\mathbf{I}_{\nu f} - \overline{\mathbf{B}}^t_\rho\mathbf{B}_\rho + r\mathbf{I}_{\nu f} \end{bmatrix}. \end{aligned}$$

Let $b = \frac{1+au^2}{u}$. Then we have

$$\begin{aligned} h(u) &= u^{(\varepsilon+\nu)f}\det\begin{bmatrix} (b+2)\mathbf{I}_{\varepsilon f} - \mathbf{B}_\rho\overline{\mathbf{B}}^t_\rho & -\mathbf{B}_\rho \\ -\overline{\mathbf{B}}^t_\rho & (b+r)\mathbf{I}_{\nu f} - \overline{\mathbf{B}}^t_\rho\mathbf{B}_\rho \end{bmatrix} \\ &= u^{(\varepsilon+\nu)f}\det\begin{bmatrix} (b+2)\mathbf{I}_{\varepsilon f} & -\mathbf{B}_\rho \\ -(b+r+1)\overline{\mathbf{B}}^t_\rho + \overline{\mathbf{B}}^t_\rho\mathbf{B}_\rho\overline{\mathbf{B}}^t_\rho & (b+r)\mathbf{I}_{\nu f} - \overline{\mathbf{B}}^t_\rho\mathbf{B}_\rho \end{bmatrix} \\ &= u^{(\varepsilon+\nu)f}\det\begin{bmatrix} (b+2)\mathbf{I}_{\varepsilon f} & \mathbf{0} \\ * & -\frac{b+r+1}{b+2}\overline{\mathbf{B}}^t_\rho\mathbf{B}_\rho + \frac{1}{b+2}\overline{\mathbf{B}}^t_\rho\mathbf{B}_\rho\overline{\mathbf{B}}^t_\rho\mathbf{B}_\rho + (b+r)\mathbf{I}_{\nu f} - \overline{\mathbf{B}}^t_\rho\mathbf{B}_\rho \end{bmatrix}. \end{aligned}$$

Let $\mathbf{A}_\rho = \sum_{g\in\Gamma}\mathbf{A}_g\otimes\rho(g)$. By (24), we have

$$\begin{aligned} h(u) &= u^{(\varepsilon+\nu)f}\det\begin{bmatrix} (b+2)\mathbf{I}_{\varepsilon f} & \mathbf{0} \\ * & \frac{1}{b+2}\{\mathbf{A}^2_\rho - (2b-r+3)\mathbf{A}_\rho + (b^2-br+2b-r)\mathbf{I}_{\nu f}\} \end{bmatrix} \\ &= u^{(\varepsilon+\nu)f}(b+2)^{(\varepsilon-\nu)f}\det(\mathbf{A}^2_\rho - (2b-r+3)\mathbf{A}_\rho + (b^2-br+2b-r)\mathbf{I}_{\nu f}). \end{aligned}$$

Now, let

$$\mathrm{Spec}\mathbf{A}_\rho = \{\lambda_1, \ldots, \lambda_{\nu f}\}.$$

Then we have

$$\begin{aligned} &u^{2\nu f}\det(\mathbf{A}^2_\rho - (2b-r+3)\mathbf{A}_\rho + (b^2-br+2b-r)\mathbf{I}_{\nu f}) \\ &\quad = \prod_{j=1}^{\nu f}\left\{a^2u^4 - a(2\lambda_j+r-2)u^3 + (\lambda_j^2 + (r-3)\lambda_j + 2a - r)u^2 - (2\lambda_j+r-2)u + 1\right\}, \end{aligned}$$

and so, the result follows. □

As the cases of line and middle graphs, one can derive Corollary 33 from Theorem 36 and Corollary 35. The details are omitted.

6. Examples

Let $G = K_3$ be the complete graph with three vertices v_1, v_2, v_3 and let $\alpha : D(K_3) \to S_3$ be the permutation voltage assignment defined by $\alpha(v_1, v_2) = (12)$, $\alpha(v_1, v_3) = (23)$ and $\alpha(v_2, v_3) = 1$.

Then, the 3-fold covering K_3^α is the cycle graph C_9 with nine vertices. Furthermore, we have $\alpha_L([e_{12}],[e_{23}]) = \alpha(v_1,v_2) = (12)$; $\alpha_L([e_{23}],[e_{31}]) = \alpha(v_3,v_1)\alpha(v_2,v_3) = (23)$; $\alpha_L([e_{31}],[e_{12}]) = 1$. Since $L(K_3^\alpha) = L(C_9) = C_9$ and $L(K_3) = K_3$, it is certain that $L(K_3^\alpha) = L(K_3)^{\alpha_L}$.

The prime, reduced cycles of K_3 are C and C^{-1}, where $C = (v_1, v_2, v_3, v_1)$. Thus, $Z(K_3,u)^{-1} = (1-u^3)^2$. Similarly, we have $Z(L(K_3^\alpha),u)^{-1} = (1-u^9)^2$.

Next, we have $\Gamma = \langle (12),(23)\rangle = S_3$. And S_3 has three irreducible representations $\rho_1 = 1$, ρ_2 (the sign representation) and ρ_3 with degrees $f_1 = f_2 = 1$ and $f_3 = 2$, respectively. The representation ρ_3 is given by

$$\rho_3(1) = \mathbf{I}_2, \quad \rho_3((123)) = \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{bmatrix}, \quad \rho_3((132)) = \begin{bmatrix} \zeta^2 & 0 \\ 0 & \zeta \end{bmatrix},$$
$$\rho_3((12)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \rho_3((23)) = \begin{bmatrix} 0 & \zeta \\ \zeta^2 & 0 \end{bmatrix}, \quad \rho_3((13)) = \begin{bmatrix} 0 & \zeta^2 \\ \zeta & 0 \end{bmatrix},$$

where $\zeta = \exp\frac{2\pi\sqrt{-1}}{3} = \frac{-1+\sqrt{-3}}{2}$. Let $P : \Gamma \to \mathrm{GL}(3,\mathbf{C})$ be the permutation representation of Γ such that $P(\gamma) = \mathbf{P}_\gamma$. Then we have $P = 1 + \rho_3$. Let $\rho = \rho_3$. By Eq. (7), we have

$$\begin{aligned}\Phi(K_3^\alpha;\lambda) &= \Phi(K_3;\lambda)\Phi\left(\sum_{g\in S_3}\rho(g)\otimes \mathbf{A}_g;\lambda\right)\\ &= \det\begin{bmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{bmatrix} \cdot \det\begin{bmatrix} \lambda & 0 & 0 & 0 & -1 & -\zeta \\ 0 & \lambda & -1 & -1 & 0 & 0 \\ 0 & -1 & \lambda & -\zeta & 0 & 0 \\ 0 & -1 & -\zeta^2 & \lambda & 0 & 0 \\ -1 & 0 & 0 & 0 & \lambda & -1 \\ -\zeta^2 & 0 & 0 & 0 & -1 & \lambda \end{bmatrix}\\ &= (\lambda^3 - 3\lambda - 2)(\lambda^3 - 3\lambda + 1)^2.\end{aligned}$$

By Eq. (11), we have

$$Z(L(K_3^\alpha),u)^{-1} = u^9\Phi\left(K_3^\alpha;\frac{1+u^2}{u}\right) = (1-u^3)^2(1+u^3+u^6)^2 = (1-u^9)^2$$

as shown already.

Let $V(M(K_3)) = \{[v_1],[v_2],[v_3],[e_{12}],[e_{23}],[e_{31}]\}$, where we set $[v_i] = v_i v_i'$, $i = 1,2,3$. Then, the permutation voltage assignment $\alpha_M : D(M(K_3)) \to S_3$ is defined as follows:

$$\begin{aligned}&\alpha_M(u.v) = \alpha_L(u,v),\ (u.v) \in D(L(K_3)); \quad \alpha_M([v_2],[e_{12}]) = \alpha(v_1,v_2)^{-1} = (12);\\ &\alpha_M([v_3],[e_{23}]) = \alpha_M([v_1],[e_{13}]) = \alpha_M([v_1],[e_{12}]) = \alpha_M([v_2],[e_{23}]) = 1;\\ &\alpha_M([v_3],[e_{13}]) = (23).\end{aligned}$$

It is certain that $M(K_3^\alpha) = M(C_9) = M(K_3)^{\alpha_M}$. By Eq. (17), we have

$$\begin{aligned}&Z(M(K_3^\alpha),u)^{-1}\\ &\quad = (1-u^2)^9 u^9\left(1+u+u^2\right)^9\Phi\left(K_3^\alpha;\frac{1+2u^2+3u^4}{u+u^2+u^3}\right)\\ &\quad = (1-u^2)^9(1+u+3u^2+u^3+3u^4)(1-u)\end{aligned}$$

$$\times(1+u^2-6u^3-4u^4-13u^5-6u^6-9u^7)$$
$$\times(1-u+3u^2-8u^3+10u^4-20u^5+21u^6-36u^7+22u^8-32u^9$$
$$+39u^{10}-9u^{11}+27u^{12})^2.$$

Furthermore, noting $\operatorname{Spec} K_3=\{2,-1,-1\}$, one can have by Eq. (21)

$$Z(T(K_3),u)^{-1}=(1-u^2)^6(1-4u+6u^2-12u^3+9u^4)(1+2u+6u^2+6u^3+9u^4)^2.$$

Since $K_3^\alpha=C_9$, the eigenvalues of K_3^α are given as follows [6]:

$$\operatorname{Spec} K_3^\alpha=\{2,-1,-1\}\cup\left\{2\cos\frac{2j\pi}{9}\,\middle|\,1\leqslant j\leqslant 8;\ j\neq 3,6\right\}.$$

By Corollary 33, we have

$$\begin{aligned}
&Z(T(K_3^\alpha),u)^{-1}\\
&=(1-u^2)^{12}Z(T(K_3),u)^{-1}\\
&\quad\times\prod_{j=1;j\neq 3,6}^{8}\left\{9u^4-12\cos\frac{2j\pi}{9}u^3+\left(4\cos^2\frac{2j\pi}{9}-2\cos\frac{2j\pi}{9}+4\right)u^2\right.\\
&\quad\left.-4\cos\frac{2j\pi}{9}u+1\right\}\\
&=(1-u^2)^{18}(1-4u+6u^2-12u^3+9u^4)(1+2u+6u^2+6u^3+9u^4)^2\\
&\quad\times\prod_{j=1;j\neq 3,6}^{8}\left\{9u^4-12\cos\frac{2j\pi}{9}u^3+\left(4\cos^2\frac{2j\pi}{9}-2\cos\frac{2j\pi}{9}+4\right)u^2\right.\\
&\quad\left.-4\cos\frac{2j\pi}{9}u+1\right\}.
\end{aligned}$$

As the other example, let $G=K_4$ be the complete graph with four vertices v_1,v_2,v_3,v_4. Let $\alpha: D(K_4)\to S_3$ be the permutation voltage assignment defined by $\alpha(v_1,v_2)=(12)$, $\alpha(v_1,v_3)=(23)$ and $\alpha(v_1,v_4)=\alpha(v_2,v_3)=\alpha(v_2,v_4)=\alpha(v_3,v_4)=1$. The complexity of K_4 is $4^{4-2}=16$ [6]. By Eq. (6), we have

$$\kappa((K_4)^\alpha)=\frac{1}{3}\kappa(G)\det\begin{bmatrix}
3 & 0 & 0 & -1 & 0 & -1 & -\zeta & 0\\
0 & 3 & -1 & -1 & -1 & 0 & 0 & 0\\
0 & -1 & 3 & -1 & -\zeta & 0 & 0 & 0\\
-1 & -1 & -1 & 3 & 0 & 0 & 0 & 0\\
0 & -1 & -\zeta^2 & 0 & 3 & 0 & 0 & -1\\
-1 & 0 & 0 & 0 & 0 & 3 & -1 & -1\\
-\zeta^2 & 0 & 0 & 0 & 0 & -1 & 3 & -1\\
0 & 0 & 0 & 0 & -1 & -1 & -1 & 3
\end{bmatrix}$$
$$=\frac{1}{3}\cdot 4^2\cdot 720=3840,$$

where $\zeta=\frac{-1+\sqrt{-3}}{2}$. By Eq. (18), we have $\kappa(M(K_4^\alpha))=2^7 4^{17}\kappa(K_4^\alpha)=2^7 4^{17}\cdot 3840$.

References

[1] D. Archdeacon, J. Lee, M.Y. Sohn, Line graphs of covering graphs are covering graphs, Bull. Korean Math. Soc. 37 (2000) 487–491.
[2] H. Bass, The Ihara–Selberg zeta function of a tree lattice, Internat. J. Math. 3 (1992) 717–797.
[3] L. Branković, D. Cvetković, The eigenspace of the eigenvalue -2 in generalized line graphs and a problem in security of statistical databases, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. 14 (2003) 37–48.
[4] M. Burrow, Representation Theory of Finite Group, Academic Press, New York, 1965.
[5] Y. Chen, Properties of spectra of graphs and line graphs, Appl. Math. J. Chinese Univ. Ser. B 17 (2002) 371–376.
[6] D.M. Cvetković, M. Doob, H. Sachs, Spectra of Graphs, Academic Press, New York, 1979.
[7] R. Feng, J.H. Kwak, J. Lee, Characteristic polynomials of graph coverings, Bull. Austral. Math. Soc. 69 (2004) 133–136.
[8] R. Feng, J.H. Kwak, Zeta functions of graph bundles, J. Korean Math. Soc., to appear.
[9] J.L. Gross, T.W. Tucker, Generating all graph coverings by permutation voltage assignments, Discrete Math. 18 (1977) 273–283.
[10] J.L. Gross, T.W. Tucker, Topological Graph Theory, Wiley-Interscience, New York, 1987.
[11] R.P. Gupta, Independence and covering numbers of line graphs and total graphs, in: Proof Techniques in Graph Theory, Proc. Second Ann Arbor Graph Theory Conf., Ann Arbor, MI, 1968, Academic Press, New York, 1969, pp. 61–62.
[12] T. Hamada, I. Yoshimura, Traversability and connectivity of the middle graph of a graph, Discrete Math. 14 (1976) 247–256.
[13] K. Hashimoto, Zeta Functions of Finite Graphs and Representations of p-Adic Groups, Adv. Stud. Pure Math., Academic Press, New York, 1989, pp. 211–280.
[14] K. Hashimoto, On the zeta- and L-functions of finite graphs, Internat. J. Math. 1 (1990) 381–396.
[15] H. Hutschenreuther, Einfacher Beweis des Matrix-Gerűst-Satzes der Netzwerktheorie, Wiss. Z. TH. Ilmenau 13 (1967) 403–404.
[16] Y. Ihara, On discrete subgroups of the two by two projective linear group over p-adic fields, J. Math. Soc. Japan 18 (1966) 219–235.
[17] M. Kotani, T. Sunada, Jacobian tori associated with a finite graph and its abelian covering graphs, Adv. Appl. Math. 24 (2000) 89–110.
[18] M. Kotani, T. Sunada, Zeta functions of finite graphs, J. Math. Sci. Univ. Tokyo 7 (2000) 7–25.
[19] J.H. Kwak, J. Lee, Enumeration of graph coverings, surface branched coverings and related group theory, in: S. Hong, J.H. Kwak, K.H. Kim, F.W. Raush (Eds.), Combinatorial and Computational Mathematics: Present and Future, Papers from the workshop held at the Pohang University, Pohang, February 2000, World Scientific, Singapore, 2001, pp. 97–161.
[20] J.H. Kwak, Y.S. Park, I. Sato, Weighted complexities of graph products and bundles, European J. Combin., to appear.
[21] H. Mizuno, I. Sato, Zeta functions of graph coverings, J. Combin. Theory Ser. B 80 (2000) 247–257.
[22] H. Mizuno, I. Sato, Zeta functions for images of graph coverings by some operations, in: Proceedings of the Workshop on Graph Theory and Related Topics, Sendai, 1999, Interdiscip. Inform. Sci. 7 (2001) 53–60.
[23] H. Mizuno, I. Sato, L-functions for images of graph coverings by some operations, Discrete Math. 256 (2002) 335–347.
[24] H. Mizuno, I. Sato, Bartholdi zeta functions of line graphs and middle graphs of graph coverings, Discrete Math. 292 (2005) 143–157.
[25] M. Nihei, Algebraic connectivity of the line graph, the middle graph and the total graph of a regular graph, Ars Combin. 69 (2003) 215–221.
[26] S. Northshield, A note on the zeta function of a graph, J. Combin. Theory Ser. B 74 (1998) 408–410.
[27] P. Pakoński, G. Tanner, K. Życzkowski, Families of line-graphs and their quantization, J. Statist. Phys. 111 (2003) 1331–1352.
[28] I. Sato, Subtori of Jacobian tori associated with middle graphs of abelian regular covering graphs, Far East J. Appl. Math. 13 (2003) 181–193.
[29] H.M. Stark, A.A. Terras, Zeta functions of finite graphs and coverings, Adv. Math. 121 (1996) 124–165.
[30] T. Sunada, L-functions in geometry and some applications, in: Lecture Notes in Math., vol. 1201, Springer-Verlag, New York, 1986, pp. 266–284.