On locally compact Hausdorff spaces with finite metrizability number

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Abstract

The metrizability number $m(X)$ of a space $X$ is the smallest cardinal number $\kappa$ such that $X$ can be represented as a union of $\kappa$ many metrizable subspaces. In this paper, we study compact Hausdorff spaces with finite metrizability number. Our main result is the following representation theorem: If $X$ is a locally compact Hausdorff space with $m(X) = n < \omega$, then for each $k, 1 \leq k < n$, $X$ can be represented as $X = G \cup F$, where $G$ is an open dense subspace, $F = X \setminus G$, $m(G) = k$, and $m(F) = n - k$. © 2001 Elsevier Science B.V. All rights reserved.

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The metrizability number $m(X)$ of a space $X$ is the smallest cardinal number $\kappa$ such that $X$ can be represented as a union of $\kappa$ many metrizable subspaces. In [5–7], we studied the metrizability number and related cardinal invariants for the class of compact Hausdorff spaces. The aim of this paper is to study compact (in general, locally compact) Hausdorff spaces with finite metrizability number. Typical examples of nonmetrizable compact Hausdorff spaces with finite metrizability number are the one-point compactification of an uncountable discrete space, the Alexandroff duplicate of the unit segment, and the one-point compactification of the space $\Psi$ (cf. [3, 5I]). Our main goal is to prove Theorem 6, below, and some of its consequences. We first prove the following lemmas.

Lemma 1. A locally compact Hausdorff space which is a union of countably many dense (or open) metrizable subspaces is metrizable.

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Proof. Any such space has a σ-disjoint (hence, point-countable) base. Furthermore, any locally compact Hausdorff space with a point-countable base is metrizable (see [4, Section 7]). □

Lemma 2. A locally compact Hausdorff space with finite metrizability number contains an open dense metrizable subspace.

Proof. Let \( X \) be a locally compact Hausdorff space such that \( m(X) < \omega \). We first prove, by induction on \( m(X) \), that \( X \) contains a non-empty open metrizable subspace.

Clearly, this is true if \( m(X) = 1 \). Let \( m(X) = n \), where \( 1 < n < \omega \), and suppose that all locally compact Hausdorff spaces of metrizability number less than \( n \) contain non-empty open metrizable subspaces. Let

\[
X = M_1 \cup M_2 \cup \cdots \cup M_n,
\]

where each \( M_i \) is metrizable. Since \( X \) is not metrizable, by Lemma 1, for some \( j \), \( M_j \) is not dense in \( X \). Let \( U \) be a non-empty open subset of \( X \) such that \( U \cap M_j = \emptyset \). Since \( m(U) \leq n - 1 \) and \( U \) is a locally compact Hausdorff subspace, by the inductive hypothesis, \( U \) contains a non-empty open metrizable subspace.

Let \( D \) be a maximal disjoint family of non-empty open metrizable subspaces of \( X \). Then \( \bigcup D \) is the required dense open metrizable subspace of \( X \). □

Lemma 3. Let \( X \) be a locally compact Hausdorff space, and let \( X = \bigcup \{ M_i : i < \omega \} \), where each \( M_i \) is metrizable. Then \( \bigcap \{ \text{cl} M_i : i < \omega \} \) is metrizable.

Proof. Let \( F = \bigcap \{ \text{cl} M_i : i < \omega \} \). For each \( i \), let \( B_i \) be a σ-discrete base of \( M_i \). For each \( V \in B_i \), fix an open subset \( U(V) \) of \( \text{cl} M_i \) such that \( V = U(V) \cap M_i \). If \( B_i = \{ U(V) : V \in B_i \} \), then \( B_i \) is a base in \( \text{cl} M_i \) at each point of \( M_i \). Also, \( B_i \) is σ-disjoint. Let \( B = \bigcup \{ B_i : i < \omega \} \). Since \( F \subseteq \bigcup \{ M_i : i < \omega \} \), the family \( \{ U \cap F : U \in B \} \) is a σ-disjoint (hence point-countable) base of \( F \). Since \( F \) is locally compact, \( F \) is metrizable (see [4, Section 7]). □

Let \( X \) be a locally compact Hausdorff space, and let \( m(X) = n \), where \( 2 \leq n < \omega \). Let \( X = \bigcup \{ M_i : i = 1, \ldots, n \} \), where each \( M_i \) is metrizable. We may assume, without loss of generality, that \( M_i \cap M_j = \emptyset \) whenever \( i \neq j \).

For each non empty subset \( A \) of the set \( \{ 1, \ldots, n \} \), let

\[
Y(A) = \bigcap \{ \text{cl} M_i : i \in A \}.
\]

For each \( k \in \{ 1, \ldots, n \} \), let

\[
Z_k = \bigcup \{ Y(A) : |A| = n - k + 1 \}.
\]

Clearly, each \( Z_k \) is a closed subspace of \( X \). Also, \( Z_1 = \bigcap \{ \text{cl} M_i : i = 1, \ldots, n \} \), \( Z_n = \bigcup \{ \text{cl} M_i : i = 1, \ldots, n \} = X \), and \( Z_1 \subseteq Z_2 \subseteq \cdots \subseteq Z_n \).

Lemma 4. For each \( k = 1, 2, \ldots, n - 1 \), \( Z_{k+1} \setminus Z_k \) is metrizable.
Proof. Clearly, \( Z_{k+1} \setminus Z_k = \bigcup \{ Y(A) \setminus Z_k : |A| = n - k \} \). First, let us show that for each \( A \subseteq \{ 1, \ldots, n \} \), with \(|A| = n - k \), \( Y(A) \setminus Z_k \) is metrizable. Towards this end, let us notice that \( Y(A) \setminus Z_k = Y(A) \setminus \bigcup \{ cl M_i : i \notin A \} \). If \( W = X \setminus \bigcup \{ cl M_i : i \notin A \} \), then \( W \) is a locally compact Hausdorff space and \( W = \bigcup \{ W \cap M_i : i \in A \} \). By Lemma 3, \( \bigcap \{ cl W (W \cap M_i) : i \in A \} \) is metrizable. Since \( Y(A) \setminus Z_k \subseteq \bigcap \{ cl W (W \cap M_i) : i \in A \} \), \( Y(A) \setminus Z_k \) is metrizable.

Now, let \( A \) and \( B \) be distinct subsets of \( \{ 1, \ldots, n \} \) such that \(|A| = |B| = n - k \). Then \( A \setminus B \neq \emptyset \) and \( B \setminus A \neq \emptyset \). Let \( i \in A \setminus B \). Then \( cl(Y(A) \setminus Z_k) \subseteq cl M_i \) and \((Y(B) \setminus Z_k) \cap cl M_i = \emptyset \). Thus \( cl(Y(A) \setminus Z_k) \cap (Y(B) \setminus Z_k) = \emptyset \). Similarly, \( cl(Y(B) \setminus Z_k) \cap (Y(A) \setminus Z_k) = \emptyset \). This shows that \( Z_{k+1} \setminus Z_k = \bigoplus \{ Y(A) \setminus Z_k : |A| = n - k \} \). As a disjoint sum of metrizable spaces, \( Z_{k+1} \setminus Z_k \) is metrizable. \( \square \)

Lemma 5. For each \( k = 1, \ldots, n \), \( m(Z_k) = k \) and \( m(X \setminus Z_k) = n - k \).

Proof. (i) First, we show by induction on \( k \) that \( m(Z_k) \leq k \).

Since \( Z_1 = \bigcap \{ cl M_i : i = 1, \ldots, n \} \), by Lemma 3, \( Z_1 \) is metrizable, i.e., \( m(Z_1) = 1 \). Suppose that \( k < n \) and that \( m(Z_k) \leq k \). By Lemma 4, \( Z_{k+1} \setminus Z_k \) is metrizable and since \( Z_{k+1} = Z_k \cup (Z_{k+1} \setminus Z_k) \), \( m(Z_{k+1}) \leq k + 1 \).

(ii) Next, we show by induction on \( k \) that \( m(Z_{k}) \geq k \).

Since \( Z_n = X \), \( m(Z_n) = n \). Suppose that \( k < n - 1 \) and that \( m(Z_{n-k}) \geq n - k \). By Lemma 4, \( Z_{n-k} \setminus Z_{n-(k+1)} \) is metrizable and since \( Z_{n-k} = Z_{n-(k+1)} \cup (Z_{n-k} \setminus Z_{n-(k+1)}) \), \( m(Z_{n-(k+1)}) \geq n - k + 1 \).

It follows from (i) and (ii), above, that \( m(Z_k) = k \) for each \( k = 1, \ldots, n \).

(iii) For each \( k = 1, \ldots, n \), \( X \setminus Z_k = (Z_n \setminus Z_{n-1}) \cup (Z_{n-1} \setminus Z_{n-2}) \cup \cdots \cup (Z_{k+1} \setminus Z_k) \).

Hence, by Lemma 4, \( m(X \setminus Z_k) \leq n - k \). Since \( m(X) = n \) and \( m(Z_k) = k \), \( m(X \setminus Z_k) \geq n - k \). Thus \( m(X \setminus Z_k) = n - k \). \( \square \)

Theorem 6. If \( X \) is a locally compact Hausdorff space with \( m(X) = n, 2 \leq n < \omega \), then for each \( k, 1 \leq k < n \), \( X \) can be represented as \( X = G \cup F \), where \( G \) is an open dense subspace of \( X \), \( F \cap G = \emptyset \), \( m(G) = k \), and \( m(F) = n - k \).

Proof. Let \( W \) be a dense open metrizable subspace of \( X \) (such a subspace exists because of Lemma 2). We set \( G = W \cup (X \setminus Z_{n-k}) \).

Since \( (X \setminus Z_{n-k}) \subseteq G \) and, by Lemma 5, \( m(X \setminus Z_{n-k}) = k \), \( m(G) \geq k \).

Since \( G = W \cup (X \setminus Z_{n-1}) \cup (Z_{n-1} \setminus Z_{n-2}) \cup \cdots \cup (Z_{n-k+1} \setminus Z_{n-k}) = (W \cup (X \setminus Z_{n-1})) \cup (Z_{n-1} \setminus Z_{n-2}) \cup \cdots \cup (Z_{n-k+1} \setminus Z_{n-k}) \) and, by Lemma 1, \( W \cup (X \setminus Z_{n-1}) \) is metrizable, \( m(G) \leq k \). Thus \( m(G) = k \).

Let \( F = X \setminus G \). Since \( F \subseteq Z_{n-k} \), \( m(F) \leq n - k \). Since \( m(X) = n \) and \( m(G) = k \), \( m(F) = n - k \). \( \square \)

Corollary 7. If \( X \) is a locally compact Hausdorff space with \( m(X) = n, 2 \leq n < \omega \), then \( X \) can be represented as \( X = G \cup F \), where \( G \) is an open dense metrizable subspace of \( X \), \( F \cap G = \emptyset \), and \( m(F) = n - 1 \).
Corollary 8. If $X$ is a locally compact Hausdorff space with $m(X) = n$, $2 \leq n < \omega$, then $X$ can be represented as $X = G \cup F$, where $G$ is an open dense subspace of $X$ with $m(G) = n - 1$, $F \cap G = \emptyset$, and $F$ is metrizable.

Definition 9 (cf. [8]). Given a cardinal function $\varphi$ and a space $X$, the $\varphi$-spectrum of $X$, denoted by $Sp(\varphi, X)$, is defined as $Sp(\varphi, X) = \{\varphi(F) : F = \text{cl} F \subseteq X \text{ and } |F| \geq \omega\}$.

Corollary 10. If $X$ is a locally compact Hausdorff space with $m(X) = n$, $n < \omega$, then $Sp(m, X) = \{1, \ldots, n\}$.

Proof. This is an immediate consequence of Theorem 6.

Let us observe that if $m(X) > \omega$, then the above result may not hold even for compact Hausdorff spaces. For example, $Sp(m, \omega_1 + 1) = \{1, \omega_1\}$, $Sp(m, \beta \mathbb{N}) = \{2^{2\omega}\}$, and if $X$ is the top and the bottom of the lexicographic square, then $Sp(m, X) = \{1, 2^\omega\}$ (see [6, Examples 4, 5, 8]).

Theorem 11. If $X$ is a locally compact Hausdorff space with $m(X) = n$, $n < \omega$, and $Y$ is a perfect image of $X$, then $m(Y) \leq m(X)$.

Proof. The proof is by induction on $n$.

If $n = 1$ (i.e., if $X$ is metrizable), then any perfect image of $X$ is metrizable (cf. [2, Theorem 4.4.15]).

Assume the theorem holds for all locally compact Hausdorff spaces of metrizability number less than $n$. Let $f : X \to Y$ be a perfect onto map. By Corollary 7, $X$ can be represented as $X = G \cup F$, where $G$ is an open dense metrizable subspace of $X$, $F \cap G = \emptyset$, and $m(F) = n - 1$. By the inductive hypothesis, $m(f(F)) \leq n - 1$. Let $U = X \setminus f^{-1}(f(F))$. Then the restriction of the function $f$ to $U$ is a perfect map onto the space $f(U)$. Since $U \subseteq G$ and $G$ is metrizable, $f(U)$ is metrizable. Since $Y = f(U) \cup f(F)$, $m(Y) \leq n$.

Corollary 12. If $X$ is a compact Hausdorff space with $m(X) < \omega$ and $Y$ is a continuous image of $X$, then $m(Y) \leq m(X)$.

Theorem 13. Let $X$ be a locally compact Hausdorff space and let $\{F_i : i = 1, \ldots, k\}$ be a finite family of closed subsets of $X$ such that $m(F_i) \leq n < \omega$, for each $i = 1, \ldots, k$. Then $m(\bigcup\{F_i : i = 1, \ldots, k\}) \leq n$.

Proof. Let $Y = \bigcup\{F_i : i = 1, \ldots, k\}$ and let $Z = \bigoplus\{F_i \times \{i\} : i = 1, \ldots, k\}$. Then $Z$ is a locally compact Hausdorff space and $m(Z) \leq n$. Let $f : Z \to Y$ be the natural projection map, i.e., $f(x, i) = x$. Then $f$ is a perfect map onto $Y$. By Theorem 11, $m(Y) \leq m(Z) \leq n$. 


Definition 14. Let $X$ be a topological space and let $p \in X$. The metrizability order of $p$ in $X$, $mo(p, X)$, is defined as

$$mo(p, X) = \inf \{ m(U) : U \text{ is a neighborhood of } p \text{ in } X \}.$$ 

Theorem 15. Let $X$ be a compact Hausdorff space and let $m(X) = n < \omega$. Then there exists a point $p \in X$ such that $mo(p, X) = n$. Moreover, the subspace $Z = \{ x \in X : mo(x, X) = n \}$ is metrizable.

Proof. To prove the first part of the theorem, i.e., that $Z \neq \emptyset$, assume the contrary. Then for each $x \in X$ there would exist an open neighborhood $U_x$ of $x$ such that $m(\text{cl} U_x) \leq n - 1$. Since there would be finitely many such open sets covering the entire space $X$, by Theorem 13, $m(X) \leq n - 1$. This is a contradiction.

To prove the second part of the theorem, i.e., that $Z$ is metrizable, let $X$ be represented as $X = G \cup F$, where $G$ is dense open with $m(G) = n - 1$, $G \cap F = \emptyset$, and $F$ is metrizable (cf. Corollary 8). Since $G$ is open and $m(G) = n - 1$, $Z \cap G = \emptyset$. Hence $Z \subseteq F$. ♦

Lemma 16. Let $X$ be a compact Hausdorff space such that $m(X) = 2$. Then $X$ can be represented as $X = A \cup B$, where $A$ is dense open and metrizable, and $B = \{ p \in X : mo(p, X) = 2 \}$ is non-empty closed and metrizable.

Proof. By Theorem 6, $X$ can be represented as $X = G \cup F$, where $G$ is dense open metrizable and $F$ is closed metrizable and $G \cap F = \emptyset$. Let $A = \{ x \in X : x \text{ has a metrizable neighborhood in } X \}$ and $B = \{ x \in X : mo(x, X) = 2 \}$. Then $A$ is open in $X$, $B$ is closed in $X$, and $X = A \cup B$. Also, $B \subseteq F$ and $G \subseteq A$. By Theorem 15, $B$ is non-empty and since $B$ is a subspace of $F$, $B$ is metrizable. Since $F$ has a countable base, $A \cap F$ can be covered by countably many open metrizable subspaces of $X$. Hence, by Lemma 1, $A = G \cup (A \cap F)$ is metrizable. ♦

Example 17. Consider the space $\Psi = N \cup M$ (cf. [3, 51]), where $M$ is a maximal almost disjoint family of subsets of the set $N$ of natural numbers. Then $\Psi$ is a locally compact Hausdorff space and $m(\Psi) = 2$. Also, each point $x \in \Psi$ has a metrizable neighborhood. Thus Theorem 15 cannot be generalized to locally compact Hausdorff spaces.

Example 18. Let $X = \Psi \cup \{ x^* \}$ be the one-point compactification of $\Psi$. Then $m(X) = 3$. Also, $X = \bigcup \{ F_i : i = 1, 2, \ldots \}$, where for each $i$, $F_i = \{ i \} \cup M \cup \{ x^* \}$ is a closed subspace of $X$ with $m(F_i) = 2$. Thus Theorem 13 cannot be generalized to infinite families of closed sets.

There is even more striking example. In Theorem 30, below, we show that there exists a compact Hausdorff separable space $X$ such that $w(X) = \omega_1$, $X$ can be represented as
Then locally compact Hausdorff spaces of metrizability number less than

\[ m(Y) < \omega. \]

**Theorem 19.** Let \( X \) and \( Y \) be locally compact Hausdorff spaces such that \( m(X) = n = m(Y) < \omega \). Then \( m(X \times Y) \leq n(n+1)/2 \).

**Proof.** The proof is by induction on \( n \).

Clearly, the theorem holds for \( n = 1 \). Assume \( n \geq 2 \) and that the theorem holds for all locally compact Hausdorff spaces of metrizability number less than \( n \). By Corollary 7, \( X \) can be represented as \( X = A \cup B \), where \( A \) is dense open and metrizable, and \( B \) is closed and disjoint from \( A \) and \( m(B) = n - 1 \). Let \( Y = C \cup D \) constitute a similar representation of the space \( Y \). Then the space \( X \times Y \) can be represented as follows:

\[ X \times Y = (A \times C) \cup (A \times D) \cup (B \times C) \cup (B \times D). \]

The subspace \( A \times C \) is metrizable. We shall show that the metrizability number of the subspace \((A \times D) \cup (B \times C)\) is equal to \( n - 1 \).

To this end, let us notice that \( m(A \times D) = n - 1 = m(B \times C) \). Also, \( \text{cl}(A \times D) \cap (B \times C) = \emptyset = \text{cl}(B \times C) \cap (A \times D) \). Therefore \( A \times D \) and \( B \times C \) are disjoint closed-open subspaces of the space \((A \times D) \cup (B \times C)\). Thus \( m((A \times D) \cup (B \times C)) = n - 1 \).

In consequence, \( m(X \times Y) \leq 1 + (n - 1) + m(B \times D) \). By the induction hypothesis, \( m(B \times D) \leq (n - 1)n/2 \). Hence \( m(X \times Y) \leq 1 + (n - 1) + (n - 1)n/2 = n(n + 1)/2 \).

**Theorem 20.** If \( X \) is a compact Hausdorff space, \( Y \) is a locally compact Hausdorff space, \( m(X) < \omega \), and \( m(Y) < \omega \), then \( m(X \times Y) \geq m(X) + m(Y) - 1 \).

**Proof.** Let \( m = m(X) \), \( n = m(Y) \), and \( k = m(X \times Y) \). Clearly, \( k \geq m \). By Theorem 6, \( X \times Y \) can be represented as \( X \times Y = G \cup F \), where \( G \) is an open dense subspace of \( X \times Y \), \( F \cap G = \emptyset \), \( m(G) = m - 1 \), and \( m(F) = k - m + 1 \). By Theorem 15, there exists a point \( p \) in \( X \) such that \( \text{mo}(p, X) = m \). Then \( \{p\} \times Y \) consists only of points whose metrizability order with respect to the space \( X \times Y \) is at least \( m \). Hence \( \{p\} \times Y \cap G = \emptyset \). Since \( \{p\} \times Y \subseteq F \), \( m(F) \geq m(\{y\} \times Y) = n \), i.e., \( k - m + 1 \geq n \). Thus \( m(X \times Y) \geq m + n - 1 \).

**Corollary 21.** Let \( X \) be a compact Hausdorff space with \( m(X) = n < \omega \). Then, for each \( k = 1, 2, \ldots, m(X^k) \geq k(n - 1) + 1 \).

**Proof.** The proof, by induction on \( k \), is straightforward.

**Corollary 22.** If \( X \) is a compact Hausdorff space, and \( Y \) is a locally compact Hausdorff space such that \( m(X) = 2 = m(Y) \), then \( m(X \times Y) = 3 \).

**Proof.** The corollary follows immediately from Theorems 19 and 20.

**Example 23.** Let \( X \) be the set \( \omega_1 \) topologized as follows. Each successor ordinal is isolated. For each limit ordinal \( \alpha \), let \( \alpha_1, \alpha_2, \ldots \) be an increasing sequence of successor
ordinals converging to $\alpha$. A base of neighborhoods at $\alpha$ consists of the sets of the form $\{\alpha\} \cup \{\alpha_i; i \geq n\}$, where $n < \omega$. Then $X$ is the so-called ladder system space on $\omega_1$. The space $X$ is first countable locally compact Hausdorff, and $m(X) = 2$. In [1], it is shown that, for each $n = 1, 2, \ldots, m(X^n) = 2$. Thus the above two corollaries cannot be generalized to locally compact Hausdorff spaces.

Example 24. Let $X$ be any ladder system space on $\omega_1$ and let $Y$ be the one-point compactification of $X$. Then $m(Y) = 3$. Indeed, if it is not the case, then $m(Y) = 2$. By Lemma 16, $Y$ can be represented as $Y = A \cup B$, where $A$ is dense open and metrizable, and $B = \{p \in Y: mo(p, Y) = 2\}$ is non-empty closed and metrizable. Clearly, $A$ contains all successor ordinals and, since the subspace $L$ of limit ordinals is discrete, $B \cap L$ is countable. Hence $A$ contains all successor ordinals and all but countably many limit ordinals. This contradicts the metrizability of $A$. Now, if $Z$ is the one-point compactification of the space $X \times X$, then, by a similar argument, $m(Z) = 3$. The space $Y \times Y$ is another compactification of the space $X \times X$, but $m(Y \times Y) \geq 5$, according to Theorem 20.

Lemma 25. Let $X$ be a compact Hausdorff space of finite metrizability number. If there exists an uncountable point-countable family $G$ of open subsets of $X$ such that for each $U \in G$, $m(U) \geq k$, then $m(X) \geq k + 1$.

Proof. Let $m(X) = n$, and let $X = G \cup F$, where $G$ is open dense in $X$, $F = X \setminus G$, $m(G) = n - 1$, and $m(F) = 1$. Since $F$ is compact and metrizable, $F$ intersects at most countably many members of $G$. Hence there exists $U \in G$ such that $U \subseteq G$. Therefore $m(U) \leq n - 1$, i.e., $k \leq n - 1$. Thus $n \geq k + 1$. $\Box$

Lemma 26. Let $Y$ be a locally compact Hausdorff separable space, and let $X = Y \cup Z$ be a Hausdorff compactification of $Y$, where $Z = X \setminus Y$, and $m(X) < \omega$. If there exists an uncountable point-countable family $G$ of open subsets of $Z$ such that for each $U \in G$, $m(U) \geq k$, then $m(X) \geq k + 2$.

Proof. Let $m(X) = n$, and let $X = G \cup F$, where $G$ is open dense in $X$, $F = X \setminus G$, $m(G) = 1$, and $m(F) = n - 1$. Since $X$ is separable, $G$ is separable. Hence $G$ has a countable base. Therefore $G \cap Z$ is separable. Hence $G \cap Z$ intersects at most countably many members of $G$. Thus uncountably many members of $G$ are contained in $Z \setminus G$. Since $Z \setminus G$ is compact, by the preceding lemma, $m(Z \setminus G) \geq k + 1$. Thus $n - 1 \geq k + 1$. Hence $n \geq k + 2$. $\Box$

The following is an obvious consequence of Lemma 25.

Corollary 27. Let $X$ be a compact Hausdorff space such that $m(X) = n < \omega$. If $G$ is a point-countable family of open subsets of $X$ such that for each $U \in G$, $m(U) = n$, then $G$ must be countable.
Example 28. Let $X$ be a compact Hausdorff space such that $m(X) = n < \omega$. Let $Z$ be the one-point compactification of the disjoint sum of uncountably many copies of $\omega$. Then, by Lemma 25, it follows that $m(Z) = n + 1$.

The aim of the sequel is to construct an example of a compact Hausdorff space $X$ such that $m(X) = \omega$ and $X$ can be represented as a union of an increasing sequence of length $\omega$ of closed subspaces each of metrizability number 2. In what follows, we will adopt the following notation.

- Let $D^*$ denote the one-point compactification of a discrete space $D$ of cardinality $\omega_1$.
- Given a compact Hausdorff space $S$ of weight $\omega_0$, let $bN(S)$ denote a Hausdorff compactification of the discrete space $N$ of non-negative integers such that $bN(S) \setminus N = S$ (see [2, 3.12.18(c)]).
- Let $\mathcal{P}$ denote the class of all compact Hausdorff spaces $X$ such that $X$ can be represented as $X = \bigcup \{F_i : i < \omega\}$, where for each $i$, $F_i$ is closed in $X$, $F_i \subseteq F_{i+1}$, and $m(F_i) = 2$. Note that if $X \in \mathcal{P}$ and $w(X) = \omega_1$, then $bN(X) \in \mathcal{P}$.

Proposition 29. For every positive integer $n \geq 3$, there exists a compact separable Hausdorff space $X$ such that $w(X) = \omega_1$, $m(X) = n$, and $X \in \mathcal{P}$.

Proof. Since $D^* \in \mathcal{P}$, $bN(D^*) \in \mathcal{P}$ and, by Lemma 26, $m(bN(D^*)) = 3$.

Suppose that $n \geq 3$ and that we have constructed a compact Hausdorff space $X$ of weight $\omega_1$ such that the following conditions are satisfied:

(a) $X = Y \cup \{x^*\}$, $m(X) = n$, $m(Y) = n - 1$;

(b) $X = \bigcup \{F_i : i < \omega\}$, where for each $i$, $F_i$ is closed in $X$, $F_i \subseteq F_{i+1}$, $m(F_i) = 2$, and $F_i = W_i \cup \{x^*\}$ with $W_i$ being metrizable.

(To get such a representation for the space $bN(D^*)$, set $Y = N \cup D$ and $F_i = \{0, 1, 2, \ldots, i\} \cup D^*$.)

For each $\alpha < \omega_1$, let $X(\alpha) = X$, and let $Y(\alpha)$, $F_i(\alpha)$, $W_i(\alpha)$, and $x^*(\alpha)$ denote, respectively, the copies of $Y$, $F_i$, $W_i$, and $x^*$ in the space $X(\alpha)$.

Let $Z = \bigoplus \{X(\alpha) : \alpha < \omega_1\} \cup \{z^*\}$ be the one-point compactification of the disjoint sum $\bigoplus \{X(\alpha) : \alpha < \omega_1\} \cup \{z^*\}$ to a single point $s^*$, and let $q : Z \to S$ be the quotient map. Clearly, $S$ is a space of weight $\omega_1$. Since the space $S$ is homeomorphic to the space $\bigoplus \{Y(\alpha) : \alpha < \omega_1\} \cup \{s^*\}$, $S$ satisfies the condition (a), above. Let us set $K_i = q(\bigoplus \{F_i(\alpha) : \alpha < \omega_1\} \cup \{z^*\})$. Then $S = \bigcup \{K_i : i < \omega\}$, where for each $i$, $K_i$ is closed in $S$ and $K_i \subseteq K_{i+1}$. Let us notice that, for each $i$, $K_i$ is homeomorphic to $\bigoplus \{W_i(\alpha) : \alpha < \omega_1\} \cup \{s^*\}$. Since, for each $\alpha < \omega_1$, $W_i(\alpha)$ is metrizable, $m(K_i) = 2$. Thus $S$ satisfies also the condition (b), above.

Let us consider the space $bN(S)$. It is a compact separable Hausdorff space. We shall show that $bN(S)$ has a representation satisfying the conditions (a) and (b) and that $m(bN(S)) = n + 1$.

To this end, let us notice that $bN(S)$ is homeomorphic to $N \cup \bigoplus \{Y(\alpha) : \alpha < \omega_1\} \cup \{s^*\}$. Since, for each $\alpha < \omega_1$, $m(Y(\alpha)) = n - 1$, it follows, by Lemma 26, that $m(bN(S)) =$
In consequence, the metrizability number of the subspace \( N \cup \bigoplus \{ Y(\alpha) : \alpha < \omega_1 \} \) must be equal to \( n \). Thus \( bN(S) \) satisfies the condition (a). Setting \( F_i = \{0, 1, 2, \ldots, i\} \cup K_i \) we get (b) as well.

Thus, by induction on \( n \), the proof of the proposition is complete.

**Theorem 30.** There exists a compact Hausdorff separable space \( X \) such that \( w(X) = \omega_1 \), \( m(X) = \omega \), and \( X \in \mathcal{P} \).

**Proof.** By the preceding proposition, for each integer \( n \geq 3 \), let \( X_n \) be a compact separable Hausdorff space such that \( w(X_n) = \omega_1 \), \( m(X_n) = n \), and \( X_n \in \mathcal{P} \). Let \( X \) be the one-point compactification of the disjoint sum \( \bigoplus \{ X_n : n = 3, 4, \ldots \} \). Then \( X \) is the desired space.

**References**