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Homeomorphisms of unimodal inverse limit spaces with a non-recurrent critical point ☆

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Let T be a tent map with the slope strictly between $\sqrt{2}$ and 2. Suppose that the critical point of T is not recurrent. Let K denote the inverse limit space obtained by using T repeatedly as the bonding map. We prove that every homeomorphism of K to itself is isotopic to some power of the natural shift homeomorphism.

1. Introduction

In the last fifteen years inverse limits of unimodal maps have been studied extensively. One of the main problems in the field of study is to classify all such spaces based upon the dynamics of the particular unimodal map that generates the inverse limit space. There are many known topological invariants in this class of spaces such as endpoints [3,6], folding points [7,10,13], asymptotic arc-components [8], and complicated subcontinua [1,5,9]. The main conjecture is due to W.T. Ingram:

Ingram's conjecture. *Let T_s and T_t be tent maps with slopes s and t respectively. Then $\varprojlim\{[0,1],T_s\}$ is homeomorphic with $\varprojlim\{[0,1],T_t\}$ if and only if $s=t$.*

Ingram's conjecture has been proved in many special cases. If T_s is a tent map with a periodic critical point of period n and T_t is a tent map with a periodic critical point of period n' then Barge and Martin proved that the core of $\varprojlim\{[0,1],T_s\}$ has n endpoints and the core of $\varprojlim\{[0,1],T_t\}$ has n' endpoints [3]. Hence if $n \neq n'$ then $\varprojlim\{[0,1],T_s\}$ is not homeomorphic

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with $\varprojlim\{[0,1],T_t\}$. Bruin extended this by introducing the notion of folding points, and showing that if T_s and T_t have preperiodic critical points of order n and n' respectively then $\varprojlim\{[0,1],T_s\}$ and $\varprojlim\{[0,1],T_t\}$ have n and n' many folding points respectively [7]. Hence if $n \neq n'$ then the associated inverse limit spaces are not homeomorphic. Barge and Diamond proved the conjecture in the case that T_s is one of the three tent maps with a periodic critical point of period 5 [2]. In two papers Kailhofer proved Ingram's conjecture in the case that the periodic point is periodic [11] and [12]. Subsequently, Block, Jakimovik, Kailhofer, and Keesling gave a simplified proof [4]. Independently Štimac proved in her dissertation research that Ingram's conjecture holds in the case that the critical point is periodic [16], and then she extended her work to the case that the critical point is preperiodic [15]. Recently Raines and Štimac have proved the Ingram's conjecture in the case that the critical point is non-recurrent [14].

The focus of this paper is the case that the inverse limit is induced by a tent map with a non-recurrent critical point. Here we describe the structure of the isotopy classes of the set of homeomorphisms of such a space to itself by showing:

Main Theorem. *Let T be a tent map with slope strictly between $\sqrt{2}$ and 2 and suppose that the critical point of T is not recurrent. Let $h:\varprojlim\{[0,1],T\} \to \varprojlim\{[0,1],T\}$ be a homeomorphism. Then there is an integer k such that h is isotopic to σ^k.*

By σ we mean the natural shift homeomorphism

$$\sigma(x_0,x_1,\ldots) = \big(T(x_0),x_0,x_1,\ldots\big)$$

on $\varprojlim\{[0,1],T\}$.

Block, Jakimovik, Kailhofer and Keesling have proved this result in the case that the critical point is periodic [4]. The main difference between the periodic case and the non-recurrent case is that in the periodic case there are only finitely many folding points (points $\bar{x} \in \varprojlim\{[0,1],T\}$ with the property that there is no neighborhood of $\bar{x}$ homeomorphic to the product of a zero-dimensional set and an arc), and all of these folding points are in fact endpoints. In the case we consider, the non-recurrent case, there are no endpoints, but we have (perhaps uncountably many) folding points. These folding points present the main difficulty in proving our result.

2. Definitions and preliminary lemmas

Let $T = T_s:[0,1] \to [0,1]$ be a tent map with slope, s, strictly between $\sqrt{2}$ and 2 such that the critical point of T, $\frac{1}{2}$, is non-recurrent. Let K denote the inverse limit space, $K = \varprojlim\{[0,1],T\}$. For each non-negative integer n, let $\pi_n:K \to [0,1]$ denote the natural projection given by $\pi_n(x_0,x_1,\ldots) = x_n$. Then $\pi_n = T \circ \pi_{n+1}$ for each n. Let C_0 denote the arc-component of the endpoint, $(0,0,0,\ldots)$ of K. We call C_0 the *tail* of K. Let X denote the complement of C_0 in K. We call X the *core* of K. Note that $X = \varprojlim\{[T^2(\frac{1}{2}),T(\frac{1}{2})],T\}$.

Lemma 2.1. *Let C be a composant of X. Then C is an arc-component of X. Moreover, there is a continuous bijection $\varphi_C:\mathbb{R} \to C$ from the real line onto C.*

This result is true because the critical point is non-recurrent. It may not be true in other circumstances.

Lemma 2.2. *C_0 is a ray, i.e., there is a homeomorphism $\varphi:[0,\infty) \to C_0$ onto C_0. Moreover, C_0 is an open dense subset of K.*

This is a well-known fact for any value of s with $1 < s < 2$.

Lemma 2.3. *The arc-components of K are precisely the composants of X and the ray C_0.*

This follows from Lemmas 2.1 and 2.2.

Lemma 2.4. *Suppose that A is a continuum and either $A \subset C_0$ or A is a proper subset of X. Then for n sufficiently large, $\pi_n|_A$ is a homeomorphism. In particular, A is an arc or a point.*

Proof. If $A \subset C_0$, then the conclusion is well known and does not depend on the hypothesis that $\frac{1}{2}$ is not recurrent under T. So, we assume that $A \subset X$. If there exists a positive integer N such that $\frac{1}{2} \notin \pi_k(A)$ for all $k \geqslant N$, then $n = N$ satisfies the conclusion of the lemma.

Suppose $\frac{1}{2} \in \pi_j(A)$ for infinitely many j. There is an open interval $V \subset [0,1]$ with $\frac{1}{2} \in V$ such that there are no elements of $\{T(\frac{1}{2}),T^2(\frac{1}{2}),\ldots\}$ in V. Now for each k, $\pi_k(A)$ is an arc. As $k \to \infty$, the length of $\pi_k(A)$ goes to zero. This is because the restriction of T to $[T^2(\frac{1}{2}),T(\frac{1}{2})]$ is locally eventually onto. Hence, for some n, $\pi_n(A)$ is a subset of V. The conclusion follows. □

Following [4], we define the $\bar{d}$-metric on arc-components. Let C be some arc-component of K and let $\bar{x}, \bar{y} \in C$. Choose $n \in \mathbb{N}$ to be large enough such that if A is the arc with endpoints $\bar{x}$ and $\bar{y}$ in C then $\pi_n|_A$ is a homeomorphism. Then define

$$\bar{d}(\bar{x}, \bar{y}) = s^n \left|\pi_n(\bar{x}) - \pi_n(\bar{y})\right|$$

(recall that s is the slope of the tent map we are using as the bonding map for K). Notice that for all $m \geqslant n$ we have

$$\bar{d}(\bar{x}, \bar{y}) = s^m \left|\pi_m(\bar{x}) - \pi_m(\bar{y})\right|.$$

We let $\bar{\ell}$ denote the length of an arc under the $\bar{d}$-metric.

Lemma 2.5. *Let $\bar{x}, \bar{y} \in K$. Let j be a positive integer. Then*

$$\left|\pi_{j+1}(\bar{x}) - \pi_{j+1}(\bar{y})\right| \geqslant \frac{|\pi_j(\bar{x}) - \pi_j(\bar{y})|}{s}.$$

Proof. If $\pi_{j+1}(\bar{x}), \pi_{j+1}(\bar{y})$ are on the same side of $\frac{1}{2}$, then equality holds. Suppose that $\pi_{j+1}(\bar{x}), \pi_{j+1}(\bar{y})$ are on opposite sides of $\frac{1}{2}$. Then

$$\begin{aligned}\left|\pi_j(\bar{x}) - \pi_j(\bar{y})\right| &\leqslant \max\left\{1 - \pi_j(\bar{x}), 1 - \pi_j(\bar{y})\right\} = s \cdot \max\left\{\left|\pi_{j+1}(\bar{x}) - \frac{1}{2}\right|, \left|\pi_{j+1}(\bar{y}) - \frac{1}{2}\right|\right\} \\ &\leqslant s \cdot \left|\pi_{j+1}(\bar{x}) - \pi_{j+1}(\bar{y})\right|.\end{aligned}$$

This proves the lemma. □

Lemma 2.6. *Let C be an arc-component of K. Let $\bar{x}, \bar{y} \in C$. Then for any positive integer j we have*

$$\bar{d}(\bar{x}, \bar{y}) \geqslant \left|\pi_j(\bar{x}) - \pi_j(\bar{y})\right| \cdot s^j.$$

Proof. For some integer $k > j$ we have

$$\bar{d}(\bar{x}, \bar{y}) = \left|\pi_k(\bar{x}) - \pi_k(\bar{y})\right| \cdot s^k.$$

If $n = k - 1$ we see that by Lemma 2.5,

$$\bar{d}(\bar{x}, \bar{y}) \geqslant \left|\pi_n(\bar{x}) - \pi_n(\bar{y})\right| \cdot s^n.$$

The conclusion follows by repeating this argument. □

Lemma 2.7. *Let D be a subcontinuum of K. Suppose that there exists a real number M such that $\bar{d}(\bar{x}, \bar{y}) \leqslant M$ whenever $\bar{x} \in D \cap C_0$ and $\bar{y} \in D \cap C_0$. Then either $D \subset C_0$ or $D \subset X$.*

Proof. Suppose $D \cap C_0 \neq \emptyset$. The boundedness of $D \cap C_0$ in the $\bar{d}$-metric implies that there is a maximum element of $D \cap C_0$. This implies that $D \subset C_0$. If $D \cap C_0 = \emptyset$, then $D \subset X$. □

Let $F \subseteq K$ be the set of folding points for K, i.e. $\bar{x} \in F$ if and only if $\pi_n(\bar{x}) = x_n \in \omega(\frac{1}{2})$ for all $n \in \mathbb{N}$. Here $\omega(\frac{1}{2})$ is the set of ω-limit points of $\frac{1}{2}$ under T. In [13] it was shown that $\bar{x} \in F$ if and only if every neighborhood of $\bar{x}$ is not homeomorphic to the product of a zero-dimensional set and an open arc.

Lemma 2.8. *Suppose that A is an arc in K which contains no folding points. Then there is a positive integer k such that there are no points of the closure of the orbit of $\frac{1}{2}$ under T in $\pi_k(A)$.*

Proof. A and F are disjoint compact subsets of K. Hence, there is a positive integer j such that $\pi_j(A) \cap \pi_j(F) = \emptyset$. It follows that there are only finitely many elements of the orbit of $\frac{1}{2}$ under T in A. So, for some positive integer n, $A \cap \{T^n(\frac{1}{2}), T^{n+1}(\frac{1}{2}), T^{n+2}(\frac{1}{2}), \ldots\} = \emptyset$. Set $k = j + n$. The conclusion follows. □

Lemma 2.9. *Let $\{A_i\}_{i \in \mathbb{N}}$ be a sequence of arcs in K. Suppose that B is a subcontinuum of K and $A_i \to B$ in the Hausdorff metric. Suppose also that there is an $M > 0$ such that $\bar{\ell}(A_i) \leqslant M$ for all i. Then B is an arc or a point in K and $\bar{\ell}(B) \leqslant M$.*

Proof. We claim that $\bar{d}(\bar{x}, \bar{y}) \leqslant M$ whenever $\bar{x}, \bar{y}$ are in $B \cap C_0$. Proceeding by contradiction, suppose that $\bar{d}(\bar{x}, \bar{y}) > M$. For some j, $s^j \cdot |\pi_j(\bar{x}) - \pi_j(\bar{y})| = \bar{d}(\bar{x}, \bar{y})$. There exist $\bar{x}^n, \bar{y}^n \in A_n$ such that $\bar{x}^n \to \bar{x}$ and $\bar{y}^n \to \bar{y}$. Hence $\pi_j(\bar{x}^n) \to \pi_j(\bar{x})$ and $\pi_j(\bar{y}^n) \to \pi_j(\bar{y})$. It follows that for n sufficiently large, $s^j \cdot |\pi_j(\bar{x}^n) - \pi_j(\bar{y}^n)| > M$. But $\bar{d}(\bar{x}^n, \bar{y}^n) \geqslant s^j \cdot |\pi_j(\bar{x}^n) - \pi_j(\bar{y}^n)|$ by Lemma 2.6. This contradicts $\bar{\ell}(A_n) \leqslant M$. This proves the claim.

By Lemma 2.7, either $B \subset C_0$ or $B \subset X$. Moreover, as in the first part of the proof, we see that for any composant C of X, $\bar{d}(\bar{x}, \bar{y}) \leqslant M$ whenever $\bar{x}$ and $\bar{y}$ are in $B \cap C$. It follows that $B \neq X$. By Lemma 2.4 B is an arc or a point and, by the previous argument, $\bar{\ell}(B) \leqslant M$. □

Suppose that

$$h : K \to K$$

is a homeomorphism. Then we have $h(C_0) = C_0$.

We let $\mathbb{Z}_+$ denote the set of non-negative integers, $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. We let $\mathbb{N}$ denote the set of positive integers, $\mathbb{N} = \{1, 2, 3, \ldots\}$. Let $p \in \mathbb{Z}_+$. Define the point $\bar{x} \in K$ to be a *p-point* if there exists $l \in \mathbb{Z}_+$ such that $\pi_{p+l}(\bar{x}) = \frac{1}{2}$. We call such l the *p-level* of the p-point $\bar{x}$. Let $E_{p,l}$ be the set of all p-points of p-level l in K, and let E_p be the set of all p-points in K.

Let $S \in \mathbb{N}$ be large enough to satisfy the conditions from [14, Remark 4.7]. These conditions are quite technical and will mostly not be important in this paper. A few of the implications, however, of $n \geqslant S$ will be important, and we mention them below. We write $\mathcal{C} \prec \mathcal{D}$ if the chaining $\mathcal{C}$ refines the chaining $\mathcal{D}$, and we define the *mesh* of $\mathcal{C}$ to be the largest diameter of any of its links. In [14], we construct a sequence of chainings of K, $\{\mathcal{C}_{r,k}\}_{r \in \mathbb{Z}_+, k \geqslant S}$. Let $r \in \mathbb{Z}_+$ and $k \geqslant S$. Let $V_{r,k}$ be the collection of all $\{0, 1\}$ words of length $k + r + 1$ that can occur as an initial segment of an itinerary of a point $x \in [0, 1]$ under the map T. We can use the parity-lexicographic ordering to order this finite set

$$V_{k,r} = \{v^i_{k,r}\}^j_{i=1}.$$

Let $I^i_{k,r}$ be the interval of points in $[0, 1]$ whose itinerary begins with $v^i_{k,r}$. Let

$$L^i_{k,r} = \operatorname{conv}\left(I^i_{k,r} \cup I^{i+1}_{k,r}\right),$$

where conv J stands for the interior of the convex hull of J. Define $\ell^i_{k,r} \subseteq K$ to be

$$\ell^i_{k,r} = \pi^{-1}_k\left(L^i_{k,r}\right).$$

Notice that this is the set of points in the inverse limit that have the property that the initial segment of the kth coordinate is either $v^i_{k,r}$ or $v^{i+1}_{k,r}$. Then we let

$$\mathcal{C}_{k,r} = \left\{\ell^i_{k,r} \colon 1 \leqslant i \leqslant j\right\}.$$

It was shown in [14] that this collection of chains satisfies:

(1) $\mathcal{C}_{q,m} \prec \mathcal{C}_{p,n}$ provided $q \geqslant p$ and $m \geqslant n$ [14, Lemma 2.22];
(2) the mesh of $\mathcal{C}_{q,m}$ goes to zero as $q, m \to \infty$;
(3) each p-point, $\bar{x}$, is contained in a link of $\mathcal{C}_{p,n}$, $\ell^x_{p,n}$, such that if A is the arc-component of $\ell^x_{p,n}$ that contains $\bar{x}$ then $A \cap E_p = \{\bar{x}\}$ [14, Remark 3.2].

Let p and n be given (with $n \geqslant S$), and choose $q > p$ and $m > n$ such that

$$h(\mathcal{C}_{q,m}) \prec \mathcal{C}_{p,n}.$$

Let $\bar{x} \in E_q \cap C_0$. It was shown that if A is the arc-component of a link of $\mathcal{C}_{p,n}$ which contains $h(\bar{x})$ then there is a unique p-point, $\bar{z}$, in A [14, Lemma 3.3]. An 'adjusted' map, $h_{q,p}$ was defined, which maps $\bar{x}$ to this p-point, $\bar{z}$. The map $h_{q,p}$ was extended in a natural monotonic way on the arcs between adjacent q-points in C_0 (see [14, Definition 3.4]).

It was shown in [14, Theorems 3.14 and 3.18] that for $q, p \in \mathbb{Z}_+$ and $m, n \geqslant S$ such that

$$h(\mathcal{C}_{q,m}) \prec \mathcal{C}_{p,n},$$

there exists $a \in \mathbb{N}$ such that for every $j \in \mathbb{N}$,

$$h_{q,p}(E_{q,j} \cap C_0) = E_{p,j+a} \cap C_0 = E_{p+a,j} \cap C_0.$$

Therefore, for $b = p + a - q$ we have that

$$h_{q,p}|_{E_q \cap C_0} = \sigma^b|_{E_q \cap C_0}. \tag{1}$$

The next lemma shows that $h_{q,p}$ is (mostly) independent of q.

Lemma 2.10. *Let $q_1, p_1 \in \mathbb{Z}_+$, $m_1, n_1 \in \mathbb{N}$ with $m_1, n_1 \geqslant S$ such that*

$$h(\mathcal{C}_{q_1,m_1}) \prec \mathcal{C}_{p_1,n_1} \prec h(\mathcal{C}_{q,m}) \prec \mathcal{C}_{p,n}.$$

Then $h_{q_1,p_1}(\bar{x}) = h_{q,p}(\bar{x})$ for all $\bar{x} \in E_{q_1} \cap C_0$.

Proof. Let $\bar{x} \in E_{q_1} \cap C_0$. Since $h(\mathcal{C}_{q_1,m_1}) \prec h(\mathcal{C}_{q,m})$ we see that $\mathcal{C}_{q_1,m_1} \prec \mathcal{C}_{q,m}$ and so $q_1 \geqslant q$ and $m_1 \geqslant m$. This implies that $E_{q_1} \subseteq E_q$. Let $\ell^x_{q_1,m_1}$ be a link of $\mathcal{C}_{q_1,m_1}$ containing $\bar{x}$ and let A_1 be the arc-component containing $\bar{x}$. Then $A_1 \subseteq A$ where A is the arc-component of the link of $\mathcal{C}_{q,m}$ containing $\bar{x}$. Let B_1 be the arc-component of $\mathcal{C}_{p_1,n_1}$ containing $h(\bar{x})$ and notice then that $B_1 \subseteq B$ where B is the arc-component of a link of $\mathcal{C}_{p,n}$ containing $h(\bar{x})$. Let $\bar{y} \in E_p$ be the unique p-point in B. Then by definition of $h_{q,p}$ we have $h_{q,p}(\bar{x}) = \bar{y}$. Since $B_1 \subseteq B$ we then must have that $h_{q_1,p_1}(\bar{x}) = \bar{y} = h_{q,p}(\bar{x})$. □

Lemma 2.11. *Let $\bar{x} \in F$ then $h(\bar{x}) = \sigma^b(\bar{x})$.*

Proof. Let $\bar{x} \in F$, and let $\bar{y} = h(\bar{x})$. Since h is a homeomorphism, $\bar{y} \in F$. Let $p_1, q_1 \in \mathbb{Z}_+$ and $m_1, n_1 \geqslant S$ such that

$$h(\mathcal{C}_{q_1,m_1}) \prec \mathcal{C}_{p_1,n_1} \prec h(\mathcal{C}_{q,m}) \prec \mathcal{C}_{p,n}$$

and recursively define $p_j, q_j \in \mathbb{Z}_+$ and $m_j, n_j \geqslant S$ such that

$$h(\mathcal{C}_{q_j,m_j}) \prec \mathcal{C}_{p_j,n_j} \prec h(\mathcal{C}_{q_{j-1},m_{j-1}}) \prec \mathcal{C}_{p_{j-1},n_{j-1}}.$$

For each $j \in \mathbb{N}$, let $\ell^x_{q_j,m_j}$ be a link of $\mathcal{C}_{q_j,m_j}$ which contains $\bar{x}$ and let $\ell^y_{p_j,n_j}$ be a link of $\mathcal{C}_{p_j,n_j}$ which contains $h(\ell^x_{q_j,m_j})$. Define $\bar{z}^j \in E_{q_j} \cap C_0$ such that $\bar{z}^j \in \ell^x_{q_j,m_j}$. Then we must have:

(1) $\bar{z}^j \to \bar{x}$ as $j \to \infty$;
(2) $h(\bar{z}^j) \in \ell^y_{p_j,n_j}$ and hence by (1) and Lemma 2.10, $\sigma^b(\bar{z}^j) = h_{q,p}(\bar{z}^j) = h_{q_j,p_j}(\bar{z}^j) \in \ell^y_{p_j,n_j}$;
(3) since the mesh of $\mathcal{C}_{p_j,n_j}$ goes to zero, $\sigma^b(\bar{z}^j) = h_{q_j,p_j}(\bar{z}^j) \to \bar{y}$.

Thus

$$h(\bar{x}) = \sigma^b(\bar{x}). \qquad \square$$

Lemma 2.12. *Let C be an arc-component of K, and let $\bar{z} \in C$, then $h(\bar{z}) \in \sigma^b(C)$.*

Proof. Let $\bar{x} \in C \cap E_q$. Then there is a sequence of points $\bar{x}^n \in C_0 \cap E_q$ such that $\bar{x}^n \to \bar{x}$. Since $h_{q,p}(\bar{x}^n) = \sigma^b(\bar{x}^n)$ we see that

$$\bar{d}\big(h(\bar{x}^n), \sigma^b(\bar{x}^n)\big) < 2\epsilon$$

where $\epsilon > 0$ is the mesh of the chaining $\mathcal{C}_{p,n}$. Let A_n be the arc in C_0 with endpoints $\sigma^b(\bar{x}^n)$ and $h(\bar{x}^n)$ if these points are distinct. Otherwise let A_n be the singleton set $\{\sigma^b(\bar{x}^n)\}$. By passing to a subsequence, we may assume that $A_n \to B$ in the Hausdorff metric. Since $\bar{\ell}(A_n) < 2\epsilon$ for each n, it follows from Lemma 2.9 that B is an arc or a point that contains $\sigma^b(\bar{x})$ and $h(\bar{x})$. Thus $h(\bar{x}) \in \sigma^b(C)$. Since h is a homeomorphism, this shows that $h(\bar{z}) \in \sigma^b(C)$. □

Let C be an arc-component of X with $C \neq C_0$. Fix a continuous bijection $\varphi : \mathbb{R} \to C$ as in Lemma 2.1. We define an order $<$ on C by $\bar{x} < \bar{y}$ if and only if $\varphi^{-1}(\bar{x}) < \varphi^{-1}(\bar{y})$. In a similar way, using Lemma 2.2, we define an order $<$ on C_0.

Lemma 2.13. *Let C be an arc-component of K and let $\bar{x} < \bar{y}$ in C. Then $h(\bar{x}) < h(\bar{y})$ if and only if $\sigma^b(\bar{x}) < \sigma^b(\bar{y})$.*

Proof. Since h is a homeomorphism of C onto $h(C)$, it is either order-preserving or reversing. Let $\bar{x}^i$ and $\bar{x}^{i+1}$ be adjacent q-points in C with $\bar{x}^i < \bar{x}^{i+1}$. By [14] we know that $h(\bar{x}^i)$ and $\sigma^b(\bar{x}^i)$ are on the same arc-component of a link of $\mathcal{C}_{p,n}$, A_i. Also $h(\bar{x}^{i+1})$ and $\sigma^b(\bar{x}^{i+1})$ are on the same arc component of a link of $\mathcal{C}_{p,n}$, A_{i+1}. It is clear that every point of A_i is less than every point of A_{i+1} or every point of A_i is greater than every point of A_{i+1} depending upon whether σ^b is order-preserving or reversing. Hence $h(\bar{x}^i) < h(\bar{x}^{i+1})$ if and only if $\sigma^b(\bar{x}) < \sigma^b(\bar{y})$. □

Lemma 2.14. *There is a real number $M > 0$ such that $\bar{d}(\sigma^b(\bar{z}), h(\bar{z})) \leqslant M$ for all $\bar{z} \in K$.*

Proof. Let $\epsilon > 0$ be the mesh of $\mathcal{C}_{p,n}$. Then notice that the $\bar{d}$-length of an arc-component of a link of $\mathcal{C}_{p,n}$ is at most 2ϵ since these arc-components contain at most one p-point. Let $\bar{x}^i$ and $\bar{x}^{i+1}$ be adjacent q-points in C with $\bar{x}^i \leqslant \bar{z} < \bar{x}^{i+1}$. Then, without loss of generality, $h(\bar{x}^i) \leqslant h(\bar{z}) < h(\bar{x}^{i+1})$ and $\sigma^b(\bar{x}^i) \leqslant \sigma^b(\bar{z}) < \sigma^b(\bar{x}^{i+1})$. By [14], $\sigma^b(\bar{x}^i)$ and $h(\bar{x}^i)$ are on the same arc-component of a link of $\mathcal{C}_{p,n}$, and the same is true for $\sigma^b(\bar{x}^{i+1})$ and $h(\bar{x}^{i+1})$. Let

$$\bar{a} = \min\big\{\sigma^b\big(\bar{x}^i\big), h\big(\bar{x}^i\big)\big\}$$

and let

$$\bar{b} = \max\big\{\sigma^b\big(\bar{x}^{i+1}\big), h\big(\bar{x}^{i+1}\big)\big\}.$$

Let B be the arc with endpoints $\bar{a}$ and $\bar{b}$. We see that both $\sigma^b(\bar{z})$ and $h(\bar{z})$ are in B. Moreover, the length of B is less than or equal to the length of the arc from $\sigma^b(\bar{x}^i)$ to $\sigma^b(\bar{x}^{i+1})$ plus the lengths of the arc-components of a link of $\mathcal{C}_{p,n}$ which contains a p-point. That is to say

$$\bar{d}\big(\sigma^b(\bar{z}), h(\bar{z})\big) \leqslant \bar{l}(B) \leqslant s^p + 4\epsilon. \qquad \square$$

3. Isotopy

In the previous section we found an integer b such that h and σ^b permute the composants of the core X in the same way. Moreover, we established several results which give a stronger connection between these two homeomorphisms. Our goal is to show that these two homeomorphisms are isotopic. We will first show that the desired conclusion holds in the case that $b = 0$.

So, we assume now that $h : K \to K$ is a homeomorphism satisfying the following properties.

Property 1. If $\bar{x} \in F$ then $h(\bar{x}) = \bar{x}$.

Property 2. Each arc-component C of K is mapped to itself by h in an order-preserving way.

Property 3. There is a real number $M > 0$ such that $\bar{d}(\bar{z}, h(\bar{z})) \leqslant M$ for all $\bar{z} \in K$.

Lemma 3.1. *Let E be a real number. There exists a positive integer p such that if $\bar{w}, \bar{y} \in K$ are distinct p-points on the same arc-component of K, then $\bar{d}(\bar{w}, \bar{y}) \geqslant E$.*

Proof. Since $\frac{1}{2}$ is not recurrent under T, there is a $\gamma > 0$ such that for each positive integer i we have $|T^i(\frac{1}{2}) - \frac{1}{2}| > \gamma$. There exists a positive integer p such that $s^p \cdot \gamma \geqslant E$. Let $\bar{w}, \bar{y} \in K$ be distinct p-points on the same arc-component of K. There are positive integers $j \geqslant p$ and $k \geqslant p$ such that $\pi_j(\bar{w}) = \frac{1}{2}$ and $\pi_k(\bar{y}) = \frac{1}{2}$. We have two cases.

Case 1. $j \neq k$.

We may assume that $j < k$. Then $\pi_j(\bar{w}) = \frac{1}{2}$ and $\pi_j(\bar{y}) = T^{(k-j)}(\frac{1}{2})$. Hence, $|\pi_j(\bar{w}) - \pi_j(\bar{y})| > \gamma$. By Lemma 2.6,

$$\bar{d}(\bar{w}, \bar{y}) \geqslant s^j \cdot \gamma \geqslant E.$$

Case 2. $j = k$.

For some integer $n > j$ we have $\pi_n(\bar{w}) \neq \pi_n(\bar{y})$. We may assume that $n > j$ is the least integer with this property. Then $\pi_n(\bar{w})$ and $\pi_n(\bar{y})$ are on opposite sides of $\frac{1}{2}$ as they are inverse images of the same point under T. Thus, by Lemma 2.5, we have

$$\big|\pi_n(\bar{w}) - \pi_n(\bar{y})\big| = 2 \cdot \left|\pi_n(\bar{w}) - \frac{1}{2}\right| \geqslant 2 \cdot \frac{\gamma}{s^{(n-j)}}.$$

Finally, by Lemma 2.6, we have

$$\bar{d}(\bar{w}, \bar{y}) \geqslant s^n \cdot \big|\pi_n(\bar{w}) - \pi_n(\bar{y})\big| \geqslant 2 \cdot \gamma \cdot s^j \geqslant E. \qquad \square$$

Theorem 3.2. *Let $\bar{x} \in X$, and let $(\bar{x}^n)$ be a sequence of points in K which converges to $\bar{x}$. Let A_n denote the arc with endpoints $\bar{x}^n$ and $h(\bar{x}^n)$ if $\bar{x}^n$ and $h(\bar{x}^n)$ are distinct, or the singleton set $\{\bar{x}^n\}$ if $h(\bar{x}^n) = \bar{x}^n$. Let A denote the arc with endpoints $\bar{x}$ and $h(\bar{x})$ if $\bar{x}$ and $h(\bar{x})$ are distinct, or the singleton set $\{\bar{x}\}$ if $h(\bar{x}) = \bar{x}$. Then $A_n \to A$ in the Hausdorff metric.*

Proof. We consider two cases:

Case 1. $h(\bar{x}) = \bar{x}$.

By passing to a subsequence we may assume that the sequence (A_n) converges in the Hausdorff metric to some continuum B which contains $\{\bar{x}\}$. Proceeding by contradiction, we suppose that $\{\bar{x}\}$ is a proper subset of B.

It follows from Lemma 2.9 that B is an arc. Moreover, at least one endpoint of B which we call $\bar{z}$ is not $\bar{x}$.

We claim that $\bar{z} \in F$. To see this, suppose $\bar{z} \notin F$. There is a positive integer k and an open interval $V \subset [0,1]$ such that $\pi_k(\bar{z}) \in V$ and no point of V lies in the closure of the orbit of $\frac{1}{2}$ under T. By Lemma 2.4 we may assume in addition that the restriction of π_k to B is a homeomorphism. Moreover, $\pi_k(A_n) \to \pi_k(B)$ and $\pi_k(z)$ is an endpoint of $\pi_k(B)$. Without loss of generality we may assume that

$$\pi_k(\bar{x}) = \pi_k\big(h(\bar{x})\big) < \pi_k(\bar{z}).$$

Set $\alpha = \pi_k(\bar{z}) - \pi_k(\bar{x})$. Then for n sufficiently large, we have that $\pi_k(\bar{x}^n) < (\pi_k(\bar{x}) + \frac{\alpha}{3})$ and $\pi_k(h(\bar{x}^n)) < (\pi_k(\bar{x}) + \frac{\alpha}{3})$. Also, for n sufficiently large there is a point $\bar{y}^n$ on the arc A_n with $\pi_k(\bar{y}^n) > (\pi_k(\bar{z}) - \frac{\alpha}{3})$. Hence, for n sufficiently large, the point $\bar{t}^n \in A_n$ such that $\pi_k(\bar{t}^n)$ is the maximum value must be a k-point with a non-zero k-level. Thus, if $V = (a, b)$ then $b \in \pi_k(A_n)$ for n sufficiently large. It follows that $b \in \pi_k(B)$. This is a contradiction as $b > z$ and z is the right endpoint of $\pi_k(B)$. So the claim is established.

Choose a positive integer p large enough that the restriction of π_p to B is a homeomorphism, and $\bar{d}(w, y) > M$ whenever $w, y \in K$ are distinct p-points on the same arc-component of K. This integer p exists by Lemmas 2.4 and 3.1.

Let $\delta = |\pi_p(\bar{x}) - \pi_p(\bar{z})|$. Then $\delta > 0$. By passing to a subsequence again, we may assume that for each positive integer n

$$\big|\pi_p(\bar{x}) - \pi_p\big(\bar{x}^n\big)\big| < \frac{\delta}{3}$$

and

$$\big|\pi_p(\bar{x}) - \pi_p\big(h\big(\bar{x}^n\big)\big)\big| < \frac{\delta}{3}.$$

Since $\pi_p(\bar{x}) \neq \pi_p(\bar{z})$ we may assume without loss of generality that $\pi_p(\bar{x}) < \pi_p(\bar{z})$. Now, for each n there is a point $\bar{t}^n$ on A_n such that $\bar{t}^n \to \bar{z}$. Then $\pi_p(\bar{t}^n) \to \pi_p(\bar{z})$. By passing to a subsequence again, we may assume that for each n

$$\big|\pi_p(\bar{z}) - \pi_p\big(\bar{t}^n\big)\big| < \frac{\delta}{3}.$$

Moreover, we may choose $\bar{t}^n$ such that $\pi_p(\bar{t}^n) \geqslant \pi_p(\bar{y})$ for all $\bar{y} \in A_n$. It follows that $\bar{t}^n$ is a p-point for each n.

Now, for each n we have that $\bar{t}^n$ and $h(\bar{t}^n)$ lie on the same arc-component of K, and there is no p-point on the arc B_n which joins them (other than the endpoint $\bar{t}^n$). Hence, the restriction of π_p to B_n is a homeomorphism. Moreover, since h is order-preserving on each arc-component, we must have $h(\bar{x}^n) \in B_n$ for each n. It follows that for each n

$$\pi_p\big(h\big(\bar{t}^n\big)\big) < \pi_p(\bar{x}) + \frac{\delta}{3}.$$

This contradicts the fact that $\pi_p(h(\bar{t}^n)) \to \pi_p(h(\bar{z})) = \pi_p(\bar{z})$.

Case 2. $h(\bar{x}) \neq \bar{x}$.

By passing to a subsequence we may assume the sequence (A_n) converges in the Hausdorff metric to some continuum B which contains A. Proceeding by contradiction, we suppose that A is a proper subset of B.

Then as in Case 1, B is an arc, and at least one endpoint of B which we call $\bar{z}$ is neither $\bar{x}$ nor $h(\bar{x})$. Also, as in Case 1, we see that $\bar{z} \in F$.

Again, we may choose a positive integer p large enough that the restriction of π_p to B is a homeomorphism, and $\bar{d}(w, y) > M$ whenever $\bar{w}, \bar{y} \in K$ are distinct p-points on the same arc-component of K.

Let δ be the smaller of $|\pi_p(\bar{x}) - \pi_p(\bar{z})|$ and $|\pi_p(h(\bar{x})) - \pi_p(\bar{z})|$. Then $\delta > 0$. By passing to a subsequence again, we may assume that for each positive integer n

$$\big|\pi_p(\bar{x}) - \pi_p\big(\bar{x}^n\big)\big| < \frac{\delta}{3}$$

and

$$\big|\pi_p\big(h(\bar{x})\big) - \pi_p\big(h\big(\bar{x}^n\big)\big)\big| < \frac{\delta}{3}.$$

Observe that the three points $\pi_p(\bar{x})$, $\pi_p(h(\bar{x}))$, $\pi_p(\bar{z})$ are distinct, and $\pi_p(\bar{z})$ does not lie on the interval joining the other two points. Hence, we may construct $\bar{t}^n$ as in Case 1, and obtain a contradiction in the same way. □

Theorem 3.3. *The homeomorphism h is isotopic to the identity map of K to itself.*

Proof. We define an isotopy H as follows. Let $\bar{x} \in K$, and let $t \in [0, 1]$. If $h(\bar{x}) = \bar{x}$ set $H(\bar{x}, t) = \bar{x}$. Suppose $h(\bar{x}) \neq \bar{x}$. Then there is a unique arc $A \subset K$ with endpoints $\bar{x}$ and $h(\bar{x})$. Also, there is a positive integer k such that if g denotes the restriction of π_k to A, then g is a homeomorphism. Set

$$H(\bar{x}, t) = g^{-1}\big((1 - t) \cdot g\big(h(\bar{x})\big) + t \cdot g(\bar{x})\big).$$

Then H is well defined. We show that H is continuous at $(\bar{x}, t)$. We have three cases.

Case 1. $\bar{x}$ is in the tail.

Since the tail is open in K it is easy to see that H is continuous at $(\bar{x}, t)$.

Case 2. $\bar{x} \in X$ and $h(\bar{x}) = \bar{x}$.

In this case it follows immediately from Theorem 3.2 that H is continuous at $(\bar{x}, t)$.

Case 3. $\bar{x} \in X$ and $h(\bar{x}) \neq \bar{x}$.

Suppose $(\bar{x}^n, t_n) \to (\bar{x}, t)$. Let A be the arc with endpoints $\bar{x}$ and $h(\bar{x})$. By Properties 1 and 2, there are no folding points in A. By Lemma 2.8, there is a positive integer k such that there are no points of the closure of the orbit of $\frac{1}{2}$ under T in $\pi_k(A)$.

Claim. $\pi_k(H(\bar{x}^n, t_n)) \to \pi_k(H(\bar{x}, t))$.

To prove the claim, let A_n be as in the statement of Theorem 3.2. By Theorem 3.2, $A_n \to A$, and hence $\pi_k(A_n) \to \pi_k(A)$. Let W be an open interval with $\pi_k(A) \subset W$ such that there are no points of the closure of the orbit of $\frac{1}{2}$ under T in W. Then $\pi_k(A_n) \subset W$ for all n sufficiently large. It follows that

$$\pi_k\big(H\big(\bar{x}^n, t_n\big)\big) = (1 - t_n) \cdot \pi_k\big(h\big(\bar{x}^n\big)\big) + t_n \cdot \pi_k\big(\bar{x}^n\big)$$

for all n sufficiently large, and

$$\pi_k\big(H(\bar{x}, t)\big) = (1 - t) \cdot \pi_k\big(h(\bar{x})\big) + t \cdot \pi_k(\bar{x}).$$

This establishes the claim.

Now we observe that the proof of the claim is valid for all sufficiently large k. Thus, the claim holds for all k. This implies that $H(\bar{x}^n, t_n) \to H(\bar{x}, t)$ and the continuity is established.

Finally, consider the function h_t defined by $h_t(\bar{x}) = H(\bar{x}, t)$. Then $h_0 = h$ and h_1 is the identity map of K to itself. Now, fix $t \in [0, 1]$. We show that h_t is injective.

Suppose that $h_t(\bar{x}) = h_t(\bar{y})$, for some $\bar{x}, \bar{y} \in K$. Then $\bar{x}$, $\bar{y}$ are on the same arc-component of K. First, suppose that $h(\bar{x}) = \bar{x}$. If $\bar{y} > \bar{x}$, then $h(\bar{y}) > \bar{x}$ by Property 2. Moreover, $h_t(\bar{x}) = \bar{x}$ while $h_t(\bar{y})$ is between $\bar{y}$ and $h(\bar{y})$. This contradicts $h_t(\bar{x}) = h_t(\bar{y})$. Similarly, if $\bar{y} < \bar{x}$, then we obtain a contradiction. Thus, $\bar{y} = \bar{x}$. So, we may assume that $h(\bar{x}) \neq \bar{x}$ and $h(\bar{y}) \neq \bar{y}$.

Similarly, if $\bar{x} \neq \bar{y}$ and h has a fixed point between $\bar{x}$ and $\bar{y}$, then we obtain a contradiction.

So suppose, without loss of generality that $\bar{x} < h(\bar{x})$ and $\bar{y} < h(\bar{y})$ (if one maps to a larger point, and the other maps to a smaller point we quickly get a contradiction). Then let $\bar{a} = \min\{\bar{x}, \bar{y}\}$, and let $\bar{b} = \max\{h(\bar{x}), h(\bar{y})\}$. Let A be the arc with endpoints $\bar{a}$, $\bar{b}$. Let m be chosen such that the restriction of π_m to A is a homeomorphism. Then it is an easy algebra exercise to see that we must have $\pi_m(\bar{x}) = \pi_m(\bar{y})$. Since the restriction of π_m to A is a homeomorphism, this implies that $\bar{x} = \bar{y}$. Thus, h_t is injective.

It is similarly easy to show that h_t is surjective. Thus, H is an isotopy. □

We are now ready to prove the main result of the paper as the following corollary.

Corollary 3.4. *Let T be a tent map with a non-recurrent critical point. Let*

$$h : \varprojlim\big\{[0, 1], T\big\} \to \varprojlim\big\{[0, 1], T\big\}$$

be a homeomorphism. Then there is an integer k such that h is isotopic to σ^k.

Proof. Let k be the integer b defined in Section 2. Let $h' = \sigma^{-b} \circ h$. We show that h' satisfies Properties 1–3 at the beginning of this section.

Property 1. Let $\bar{x} \in F$. Then by Lemma 2.11, $h(\bar{x}) = \sigma^b(\bar{x})$. Thus, $h'(\bar{x}) = (\sigma^{-b} \circ \sigma^b)(\bar{x}) = \bar{x}$.

Property 2. Let C be an arc-component of K. By Lemma 2.12, $h(C) = \sigma^b(C)$. Hence $h'(C) = C$. Moreover, C is mapped to itself by h' in an order-preserving way by Lemma 2.13.

Property 3. By Lemma 2.14, $\bar{d}(\sigma^b(\bar{z}), h(\bar{z})) \leqslant M$ for all $\bar{z} \in K$. In general $\bar{d}(\sigma^{-1}(\bar{x}), \sigma^{-1}(\bar{y})) = s \cdot \bar{d}(\bar{x}, \bar{y})$. Thus, $\bar{d}(\bar{x}, h'(\bar{x})) \leqslant M \cdot s^b$.

By Theorem 3.3, the homeomorphism h' is isotopic to the identity. It follows that h is isotopic to σ^b. □

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