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RELAXED EXACT SPECTRAL CONTROLLABILITY OF MEMBRANE SHELLS

By V. VALENTE

ABSTRACT. – For membrane shells, where the exact controllability generally fails, results of relaxed exact controllability using spectral and asymptotic properties of the membrane elastic operator, are given.

1. Introduction

In the framework of the linear thin shell theory [3]-[7], with particular reference to the 2d-membrane shell theory derived by Ciarlet and Lods [1994, 1996], we consider the system of partial differential equations, for the vibrations of a membrane shell surface S, supposed uniformly elliptic with analytic boundary. The operator associated to the membrane deformation energy is a mixed order matrix operator with indices $m_1 \geq m_2 > m_3 = 0$. and it is well known in this case ([11]-[12]) there exists a sequence of eigenvalues going to $+\infty$ while a nonempty essential spectrum is determined by the block of order 0 ($m_3 = 0$).

In § 2 the mathematical model for membrane shells together with the spectral asymptotics are introdused. The exact controllability for membrane shells is formulated in § 3, and an example of non controllability in the regularity spaces is constructed for hemispherical membrane approximation. Results of relaxed spectral exact controllability, in the above spaces, can be established for membrane elastic operators which satisfy the hypotheses of existence and uniqueness theorem by Ciarlet and Sanchez-Palencia [7] (*see* also [5]), as shown in § 4. The section 5 is devoted to relaxed spectral controllability for spherical membranes.

2. The model

Let $\mathcal{E}^3$ be the Euclidean space and let Ω be a bounded open set of boundary Γ of the plane $\mathcal{E}^2$; the middle surface S of an elastic shell is defined by two curvilinear coordinates ξ_1 and ξ_2; it is the image in $\mathcal{E}^3$ of Ω by the map:

$$\eta : (\xi_1, \xi_2) \in \overline{\Omega} \to \mathcal{E}^3;$$

moreover let ξ_3 be the normal distance to S.

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In each point of S two tangent vectors $\mathbf{a}_\alpha = \partial\eta/\partial\xi_\alpha$ $\alpha = 1,2$ and a normal vector $\mathbf{a}_3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|}$ are considered.

In this section the greek indices are taken in the set [1,2] and the latin indices in the set [1,2,3]; the summation convention is used; moreover $f_{,\alpha}$ denotes the partial derivative of f with respect to ξ_α.

The *first fundamental form* $(a_{\alpha\beta})$ and the *second fundamental form* $(b_{\alpha\beta})$ are given by

$$a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta, \qquad \alpha, \beta = 1, 2,$$

$$b_{\alpha\beta} = \mathbf{a}_3 \cdot \mathbf{a}_{\alpha,\beta}, \qquad \alpha, \beta = 1, 2;$$

moreover $(a^{\alpha\beta})$ denotes the inverse matrix of $(a_{\alpha\beta})$ so that the reciprocal basis $\mathbf{a}^\alpha$ is defined by $\mathbf{a}^\alpha = a^{\alpha\beta}\mathbf{a}_\beta$.

Let $\mathbf{v}(\xi_1, \xi_2) = v_1\mathbf{a}^1 + v_2\mathbf{a}^2 + v_3\mathbf{a}^3 = v_i\mathbf{a}^i$ be the displacement vector of the middle surface S, the deformed middle surface is given by $\eta + \mathbf{v}$. In the framework of a linearized theory small displacements $\mathbf{v}$ are considered.

The energy of membrane deformation (also in the general case of anisotropy) is defined by the symmetric form (*see* for example [5][6]):

$$a^m(\mathbf{v}, \tilde{\mathbf{v}}) = \int_S a^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(\mathbf{v})\gamma_{\lambda\mu}(\tilde{\mathbf{v}})\, \mathrm{d}\, S \tag{2.1}$$

where $\mathrm{d}\, S = |\mathbf{a}_1 \times \mathbf{a}_2|\, \mathrm{d}\,\xi_1\, \mathrm{d}\,\xi_2$ and

$$a^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)}\left[a^{\alpha\lambda}a^{\beta\mu} + a^{\alpha\mu}a^{\beta\lambda} + \frac{2\nu}{(1-\nu)}a^{\alpha\beta}a^{\lambda\mu}\right]$$

is the tensor of "elastic moduli", with E and ν Young modulus and Poisson ratio respectively.

The *deformation tensor* of the middle surface $(\gamma_{\alpha\beta}(\mathbf{v}))$ is given by:

$$\gamma_{\alpha\beta}(\mathbf{v}) = \frac{1}{2}(v_{\beta|\alpha} + v_{\alpha|\beta}) - b_{\alpha\beta}v_3,$$

where the bar | denotes the covariante derivative defined by means of the Christoffel symbols $\Gamma^\alpha_{\beta\lambda} = \mathbf{a}^\alpha \cdot \mathbf{a}_{\beta,\lambda}$ and

$$v_{\alpha|\beta} = v_{\alpha,\beta} - \Gamma^\lambda_{\alpha\beta}v_\lambda.$$

Since we are interested to the eigenvalues behavior, we consider the spectral problem

$$a^m(\mathbf{v}, \tilde{\mathbf{v}}) = \int_S a^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(\mathbf{v})\gamma_{\lambda\mu}(\tilde{\mathbf{v}})\, \mathrm{d}\, S = \lambda \int_S (a^{\alpha\beta}v_\alpha\tilde{v}_\beta + v_3\tilde{v}_3)\, \mathrm{d}\, S. \tag{2.2}$$

Let $\mathbf{u}$ be the vector (v_1, v_2) so that $\mathbf{v} = (v_1, v_2, v_3) = (\mathbf{u}, v_3)$ and let $\boldsymbol{\mathcal{U}} \times L^2$, with $\boldsymbol{\mathcal{U}} \subset (H^1(\Omega))^2$), be the space of functions satisfying suitable boundary conditions. The space $\boldsymbol{\mathcal{V}}$ (depending on the form of S) of admissible displacements is:

$$\boldsymbol{\mathcal{V}} = \{\mathbf{v}; \mathbf{u} \in \boldsymbol{\mathcal{U}}, v_3 \in L^2, \quad \mathbf{v} \text{ satisfy the kinematic boundary conditions}\}$$

and $\mathbf{H}$ is the space $(L^2(\Omega))^3$ equipped with the standard scalar product:

$$(\mathbf{v},\tilde{\mathbf{v}})_{\mathbf{H}} = ((\mathbf{v},\tilde{\mathbf{v}})) = \int_S a^{\alpha\beta} v_\alpha \tilde{v}_\beta + v_3\tilde{v}_3 \,\mathrm{d}\,S.$$

It follows that the equilibrium equations are given by:

$$(2.2)_1 \qquad -a^{\alpha\beta\sigma\mu}\gamma_{\sigma\mu}(\mathbf{v})_{|\beta} - \lambda a^{\alpha\beta} v_\beta = 0, \qquad \alpha = 1,2,$$

$$(2.2)_2 \qquad -a^{\alpha\beta\sigma\mu}\gamma_{\sigma\mu}(\mathbf{v}) b_{\alpha\beta} - \lambda v_3 = 0.$$

We can consider one of the following natural boundary conditions:

$$(2.2)_3 \qquad v_\alpha = 0, \qquad \alpha = 1,2,$$

$$(2.2)_4 \qquad a^{\alpha\beta\sigma\mu}\gamma_{\sigma\mu}(\mathbf{v})\nu_\beta = 0, \qquad \alpha = 1,2,$$

where $\boldsymbol{\nu}$ is the unit outward normal vector at the surface at points of ∂S.

In the sequel we write the system $(2.2)_1(2.2)_2$ in the form $\mathbf{A}^m\mathbf{v} = 0$, and the boundary conditions $(2.2)_3(2.2)_4$ respectively in the form $\mathbf{B}\mathbf{v} = 0$ and $\mathbf{C}\mathbf{v} = 0$. Since the component v_3 appears by zero order derivative, while the component v_α appears by first order derivatives, that implies the differential self-adjoint operator $\mathbf{A}^m$ associated to the form a^m is a linear system of differential operator of mixed order with indices $m_3 = 0$, $m_\alpha = 1(\alpha = 1,2)$ indeed:

$$\mathbf{A}^m\mathbf{v} = \begin{pmatrix} \mathbf{A}^m_{\alpha\beta} & \mathbf{A}^m_{\alpha 3} \\ \mathbf{A}^m_{3\beta} & A^m_{33} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ v_3 \end{pmatrix},$$

where $A^m_{33} = a^{\alpha\beta\lambda\mu} b_{\alpha\beta} b_{\lambda\mu}$ is a zero order operator, and $\mathbf{A}^m_{\alpha\beta}$ (resp. $\mathbf{A}^m_{\alpha 3}$, $\mathbf{A}^m_{3\beta}$) is a matrix of differential operators of order $m_\alpha + m_\beta$ (resp. $m_\alpha + m_3$, $m_\beta + m_3$). We denote with $\boldsymbol{\mathcal{A}}^m$ the realization of $\mathbf{A}^m$ defined by the boundary conditions. According to [11] $\sigma_{ess}(\boldsymbol{\mathcal{A}}^m) = \omega \cup \omega_b$, where ω is the set of $\lambda \in \mathbb{R}$ such that $\mathbf{A}^m - \lambda$ is not Douglis-Nirenberg elliptic and ω_b is the set of $\lambda \in \mathbb{R}\backslash\omega$ whose boundary conditions do not satisfy the Shapiro-Lopatinskii condition. Since $m_3 = 0$, $\mathbf{A}^m - \lambda$ is not always Douglis-Nirenberg elliptic and the set ω is non-empty. Following the works of Grubb and Geymonat ([11]-[12]) we find that there exists a sequence of real eigenvalues λ_j^+ of finite multiplicity and disjoint from the essential spectrum, going to $+\infty$. Let λ^* be large enough in order that $\lambda^* \notin \sigma_{ess}(\boldsymbol{\mathcal{A}}^m)$ than the asymptotic behaviour of λ_j^+ is given in first approximation by the asymptotic behaviour of the eigenvalues of $\mathbf{A}^m_{\alpha\beta}$, that is there exists a constant $c(\mathbf{A}^m_{\alpha\beta})$ such that:

$$(2.3) \qquad N(\lambda, \boldsymbol{\mathcal{A}}^m) = \sum_{\lambda^* < \lambda_j^+ < \lambda} 1 = c(\mathbf{A}^m_{\alpha\beta})\lambda^{n/2m_\alpha} + o(\lambda^{n/2m_\alpha}), \quad \lambda \to \infty,$$

with $n = \dim(\Omega) = 2$. In the following we assume that:

(i_1) The surface S is *uniformly elliptic*,

i.e. there exists a positive constant c such that for any $\zeta \in \mathbb{R}^2$, we have:

$$|b_{\alpha\beta}\zeta_\alpha\zeta_\beta| \geq c|\zeta|^2.$$

Moreover S has an *analytic boundary*.

(i_2) The map η is analytic in Ω.

(i_3) The boundary condition satisfies the Shapiro-Lopatinskii condition.

In the above hypotheses Ciarlet-Sanchez Palencia proved [7] for any $\tilde{\mathbf{v}} \in \boldsymbol{V}$ (with kinematic boundary conditions $v_\alpha = 0$ on Γ), the existence and uniqueness of the variational problem:

$$a^m(\mathbf{v}, \tilde{\mathbf{v}}) = ((\mathbf{f}, \tilde{\mathbf{v}})) \quad \mathbf{f} \in \boldsymbol{H}.$$

3. The controllability problem

For sake of simplicity we neglect the effects of angular inertia and take the density of shell material equal to one. The differential equations of (linear) theory of vibrations of thin shells are given by:

$$\ddot{\mathbf{v}} - \mathbf{A}\mathbf{v} = 0 \qquad \mathbf{A} = \Xi^{-1} \cdot \mathbf{A}^m,$$

where Ξ is a block diagonal matrix with elements $(a^{\alpha\beta}, 1)$ In this section the general hypotheses (i_1)-(i_3) are assumed and we consider the following exact controllability problem:

(EC) Given $T > 0$ and an initial state $\{\boldsymbol{\Phi}^0, \boldsymbol{\Phi}^1\}$ find the control function g such that the unique solution $\boldsymbol{\Phi}$ of:

$$\ddot{\boldsymbol{\Phi}} - \mathbf{A}\boldsymbol{\Phi} = 0, \qquad in \ \ Q = \Omega \times (0, T),$$

$$\mathbf{B}\boldsymbol{\Phi} = g, \qquad on \ \ \Sigma = \Gamma \times (0, T),$$

$$\boldsymbol{\Phi}(0) = \boldsymbol{\Phi}^0, \quad \dot{\boldsymbol{\Phi}}(0) = \boldsymbol{\Phi}^1 \quad in \ \ \Omega,$$

satisfies the following conditions:

$$\boldsymbol{\Phi}(T) = 0, \quad \dot{\boldsymbol{\Phi}}(T) = 0 \quad in \ \ \Omega.$$

The system $\mathbf{B}$ is a system of normal boundary conditions *i.e.* there exists a complementary system $\mathbf{C}$ of boundary conditions such that $\{\mathbf{B}, \mathbf{C}\}$ (*see* respectively $(2.2)_3$, $(2.2)_4$) are the reduced Cauchy data of $\mathbf{A}$, or else the following Green formula holds for smooth functions:

$$a(\mathbf{v}, \tilde{\mathbf{v}}) = ((\mathbf{A}\mathbf{v}, \tilde{\mathbf{v}})) + \int_\Gamma \mathbf{C}\mathbf{v} \quad \mathbf{B}\tilde{\mathbf{v}} \,\mathrm{d}\, s.$$

Let us remark that the variational setting for the homogeneous problem associated to (EC) is:

let $T > 0$, $\mathbf{v}^0 \in \boldsymbol{U} \times L^2$ *and* $\mathbf{v}^1 \in \mathbf{H}$ *be given, find a function*

$$\mathbf{v}(t) \in C([0, T]; \boldsymbol{U} \times L^2) \cap C^1([0, T]; \mathbf{H}),$$

such that for any $\tilde{\mathbf{v}} \in \boldsymbol{U} \times L^2$, *we have:*

$$((\ddot{\mathbf{v}}, \tilde{\mathbf{v}})) + a(\mathbf{v}, \tilde{\mathbf{v}}) = 0, \quad \mathbf{v}(0) = \mathbf{v}^0 \; \dot{\mathbf{v}}(0) = \mathbf{v}^1. \tag{3.1}$$

To solve the exact controllability problem, we must look at the existence of solutions of the functional equation:

$$\int_\Sigma \mathbf{C}\boldsymbol{\eta} \quad g(s, t) \,\mathrm{d}\, s \,\mathrm{d}\, t = ((\boldsymbol{\eta}^0, \boldsymbol{\Phi}^1)) - ((\boldsymbol{\eta}^1, \boldsymbol{\Phi}^0)) \tag{3.2}$$

for all initial data of the homogeneous problem such that $\boldsymbol{\eta}^0 \in \boldsymbol{U} \times L^2$ and $\boldsymbol{\eta}^1 \in \mathbf{H}$ with $\mathbf{C}\boldsymbol{\eta} \in L^2(\Sigma)$.

We have the existence of a solution $\boldsymbol{g}$ of (3.2) *if and only if* [8]

$$\left(\int_\Sigma (\mathbf{C}\boldsymbol{\eta})^2 \,\mathrm{d}\, s \,\mathrm{d}\, t \right)^{1/2} \geq const. \| \{\boldsymbol{\eta}^0, \boldsymbol{\eta}^1\} \|_{\boldsymbol{U} \times L^2 \times \mathbf{H}}. \tag{3.3}$$

The Hilbert Uniqueness Method of J.L. Lions [13] [14] gives a constructive result for exact controllability problems. If we put $\boldsymbol{g} = \mathbf{C}\mathbf{v}$, the problem is to find the initial data $\{\mathbf{v}^0, \mathbf{v}^1\}$ which solve the functional equation:

$$\int_\Sigma \mathbf{C}\boldsymbol{\eta} \quad \mathbf{C}\mathbf{v} \,\mathrm{d}\, s \,\mathrm{d}\, t = ((\boldsymbol{\eta}^0, \boldsymbol{\Phi}^1)) - ((\boldsymbol{\eta}^1, \boldsymbol{\Phi}^0)). \tag{3.4}$$

If we show that for T large enough

$$\left(\int_\Sigma (\mathbf{C}\mathbf{v})^2 \,\mathrm{d}\, s \,\mathrm{d}\, t \right)^{1/2} = \| \{\mathbf{v}^0, \mathbf{v}^1\} \|_{\mathbf{F}}$$

defines a *norm* on the set of initial data $\{\mathbf{v}^0, \mathbf{v}^1\}$ of the homogeneous problem, then the controllability problem can be solved and we have exact controllability for any

$\{\mathbf{\Phi}^1, \mathbf{\Phi}^0\} \in \mathbf{F}'$ ($\mathbf{F}$ unknown space). Indeed after introducing the linear operator $\Lambda : \mathbf{F} \to \mathbf{F}'$ we can solve the equation:

$$< \Lambda\{\mathbf{v}^0, \mathbf{v}^1\}, \{\boldsymbol{\eta}^0, \boldsymbol{\eta}^1\} >= ((\boldsymbol{\eta}^0, \mathbf{\Phi}^1)) - ((\boldsymbol{\eta}^1, \mathbf{\Phi}^0))$$

for any $\{\boldsymbol{\eta}^0, \boldsymbol{\eta}^1\}$ in $\mathbf{F}$ and

$$< \Lambda\{\mathbf{v}^0, \mathbf{v}^1\}, \{\mathbf{v}^0, \mathbf{v}^1\} >= \int_\Sigma (\mathbf{C}\mathbf{v})^2 \,\mathrm{d}\, s \,\mathrm{d}\, t.$$

We prove (*see* the following Example 3.1 and [9][10]) that the exact controllability generally fails in $\mathbf{F}' = L^2 \times \boldsymbol{\mathcal{U}}' \times \mathbf{H}$.

Example 3.1. – We consider a spherical membrane shell with opening angle $\frac{\pi}{2}$. Since we consider only axisymmetric deformations, the displacement vector $\mathbf{v}$ has only two components (u, w) and satisfies the equations:

$$\begin{cases} \ddot{u} - \mathcal{L}(u) + (1+\nu)w' = 0, \\ \ddot{w} - \frac{(1+\nu)}{\sin\theta}(u\sin\theta)' + 2(1+\nu)w = 0, \end{cases} \tag{3.5}$$

with $u(0,t) = 0$, $u(\pi/2, t) = 0$, $\mathbf{v}(\theta, 0) = \mathbf{v}^0$, $\dot{\mathbf{v}}(\theta, 0) = \mathbf{v}^1$ and

$$\mathcal{L}(u) = u'' + u'\cot\theta - u(\nu + \cot^2\theta), \quad \nu \in (-1, 1/2),$$

the prime denotes the first derivative with respect to θ. In this case we have:

$$\int_\Sigma (\mathbf{C}\mathbf{v})^2 \,\mathrm{d}\, s \,\mathrm{d}\, t = \int_0^T [u'(\pi/2) - (1+\nu)w(\pi/2)]^2 \,\mathrm{d}\, t$$

and the following Hilbert spaces

$$\mathcal{U} = \left\{ u : \frac{\partial u}{\partial\theta},\ u\cot\theta \in L^2(0, \pi/2; \sin\theta),\ u(0) = u(\pi/2) = 0 \right\},$$
$$\mathbf{H} = L^2(0, \pi/2; \sin\theta) \times L^2(0, \pi/2; \sin\theta),$$

with $((f, g)) = \int_0^{\pi/2} fg \sin\theta \,\mathrm{d}\,\theta$ we denote the scalar product in the Hilbert space $L^2(0, \pi/2; \sin\theta)$.

The eigenvalues $\lambda_n^\pm$ of $\boldsymbol{A}^m$ and the corresponding eigenfunctions $\mathbf{v}_n^\pm = (u_n^\pm, w_n^\pm)$ are given by:

$$\lambda_n^\pm = \frac{1}{2}\{\mu_n + (1+3\nu) \pm \sqrt{(\mu_n + (1+3\nu))^2 - 4(\mu_n - 2)(1-\nu^2)}\}, \tag{3.6}$$

$$u_n^\pm = P_n'(\theta)/\alpha_n^\pm \|P_n\|_0, \qquad w_n^\pm = \frac{\lambda_n^\pm - \mu_n + (1-\nu)}{(1+\nu)\,\alpha_n^\pm} \frac{P_n(\theta)}{\|P_n\|_0}, \tag{3.7}$$

where

$$\alpha_n^\pm = \sqrt{\mu_n + \left(\frac{\lambda_n^\pm - \mu_n(1-\nu)}{(1+\nu)}\right)^2},$$

μ_n and $P_n(\theta)$ eigenvalues and eigenfunctions of

$$P_n'' + P_n' \cot\theta = -\mu_n P_n$$
$$P_n'(0) = P_n'(\pi/2) = 0.$$

We have P_n are the even spherical functions, $\mu_n = 2n(2n+1)$ and it is easy to show that

$$P_n(\pi/2) \neq 0$$

and

$$\lim_{n\to\infty} \frac{P_n(\pi/2)}{\|P_n\|_0} = \frac{2}{\sqrt{\pi}}.$$

Moreover we have the following behaviour:

$$\lim_{n\to\infty} \lambda_n^+ = +\infty, \quad \lim_{n\to\infty} \lambda_n^- = (1-\nu^2),$$

with

$$0 < \lambda_1^- < \lambda_2^- < \ldots < (1-\nu^2) < 2(1+\nu) = \lambda_0^+ < \lambda_1^+ < \lambda_2^+ < \ldots$$

and the following asymptotic estimates hold:

$$\lambda_n^+ = \mu_n + 3\nu + \nu^2 + O\left(\frac{1}{\mu_n}\right), \quad \lambda_n^- = 1 - \nu^2 + O\left(\frac{1}{\mu_n}\right).$$

We can observe that for all n, we obtain

$$u_n^{-\prime}(\pi/2) - (1+\nu)w_n^-(\pi/2) \neq 0$$

and

$$\lim_{n\to\infty} u_n^{-\prime}(\pi/2) - (1+\nu)w_n^-(\pi/2) = 0.$$

Then if we choose the sequence $\{\mathbf{v}_n^0, \mathbf{v}_n^1\} \in (\mathcal{U} \times \mathcal{L}^2) \times \mathbf{H}$ of initial data for the homogeneous problem associated to (3.5), with

$$\mathbf{v}_n^0 = \mathbf{v}_n^-, \qquad \mathbf{v}_n^1 = 0,$$

we have:

$$\|\{\mathbf{v}_n^0, \mathbf{v}_n^1\}\|_{\mathbf{H}\times\mathbf{H}} = 1$$

$$a(\mathbf{v}_n^0, \mathbf{v}_n^0) \to (1-\nu^2) \quad \text{as} \quad n \to \infty,$$

hence, since a^m is continous and coercive in $\mathcal{U} \times L^2$,

$$\|\{\mathbf{v}_n^0, \mathbf{v}_n^1\}\|_{\mathcal{U}\times L^2\times\mathbf{H}} = \|\mathbf{v}_n^0\|_{\mathcal{U}\times L^2} \to const. > 0 \quad \text{as} \quad n \to \infty$$

and for any given $T > 0$

$$\int_0^T [u_n^{-\prime}(\pi/2) - (1+\nu)w_n^-(\pi/2)]^2 \,\mathrm{d}t \to 0 \quad \text{as} \quad n \to \infty.$$

These last two conditions are in contradiction with the necessary (and sufficient) condition for the controllability:

$$\int_0^T [u'(\pi/2,t) - (1+\nu)w(\pi/2,t)]^2 \,\mathrm{d}t \geq c\|\{\mathbf{v}^0, \mathbf{v}^1\}\|^2_{\mathcal{U}\times L^2\times\mathbf{H}}. \tag{3.8}$$

4. The relaxed spectral controllability problem

Let $\mathbf{G}^0$ a closed subspace of $\mathbf{F}' = L^2 \times \mathcal{U}' \times \mathbf{H}$. We say that there is *Relaxed Exact Controllability* if:

(REC) Given $T > 0$ and an initial state $\{\mathbf{\Phi}^0, \mathbf{\Phi}^1\} \in \mathbf{F}' = L^2 \times \mathcal{U}' \times \mathbf{H}$, there exists a control $g \in L^2(\Sigma)$ such that the unique solution $\mathbf{\Phi}$ of:

$$\ddot{\mathbf{\Phi}} - \mathbf{A}\mathbf{\Phi} = 0, \qquad in\ \ Q = \Omega \times (0,T),$$

$$\mathbf{B}\mathbf{\Phi} = g, \qquad on\ \ \Sigma = \Gamma \times (0,T),$$

$$\mathbf{\Phi}(0) = \mathbf{\Phi}^0, \quad \dot{\mathbf{\Phi}}(0) = \mathbf{\Phi}^1, \quad in\ \ \Omega,$$

satisfies the following conditions:

$$\{\mathbf{\Phi}(T), \dot{\mathbf{\Phi}}(T)\} \in \mathbf{G}^0.$$

As in [15] we refer to *Relaxed Spectral Exact Controllability.*

Let γ_j be the eigenvalue λ_j^+ counted without multiplicities and let $\mathbf{v}_{j,h}^+ (h = 1, ...t_j)$ be the eigenfuntions associated to the eigenvalue λ_j^+ of multiplicity t_j. We put $\mathbf{E}_n =$ Span $\{\mathbf{v}_{j,h}^+; 1 \leq h \leq t_j, 1 \leq j \leq n\}$ and $d_j = |\sqrt{\gamma_j} - \sqrt{\gamma_{j-1}}|$, $j \in J = [2, 3, ...n]$ and take $T_n = sup_{\{j \in J\}} \frac{2\pi}{d_j}$

THEOREM 4.1. – *If $\{\mathbf{v}^0, \mathbf{v}^1\} \in \mathbf{G} = (\mathbf{E}_n, \mathbf{E}_n)$ and $\mathbf{v}$ is the unique solution of*

$$\ddot{\mathbf{v}} - \mathbf{A}\mathbf{v} = 0, \qquad in\ \ Q = \Omega \times (0,T),$$

$$\mathbf{B}\mathbf{v} = 0, \qquad on\ \ \Sigma = \Gamma \times (0,T),$$

$$\mathbf{v}(0) = \mathbf{v}^0, \quad \dot{\mathbf{v}}(0) = \mathbf{v}^1, \quad in\ \ \Omega.$$

Then

$$\mathbf{v} = \sum_{-n \leq j \leq n} \mathbf{S}_j exp(\delta_j t)$$

where

$$\delta_j = i\sqrt{\gamma_j}, \quad \delta_{-j} = \overline{\delta}_j \quad and \quad \delta_0 = 0,$$

$$\mathbf{S}_j = \frac{1}{2} \sum_{1 \leq h \leq t_j} (((\mathbf{v}^0, \mathbf{v}_{j,h}^+)) + ((\mathbf{v}^1, \mathbf{v}_{j,h}^+))/i\sqrt{\gamma_j})\mathbf{v}_{j,h}^+, \quad \mathbf{S}_0 = 0, \quad \mathbf{S}_{-j} = \overline{\mathbf{S}}_j.$$

Moreover for any $T \geq T_n$ there exist two constant c_1 and c_2 such that:

$$c_1 \sum_{-n \leq j \leq n} \int_\Sigma (\mathbf{C}\mathbf{S}_j)^2 \leq \int_\Sigma (\mathbf{C}\mathbf{v})^2 \, \mathrm{d}\, s \, \mathrm{d}\, t \leq c_2 \sum_{-n \leq j \leq n} \int_\Sigma (\mathbf{C}\mathbf{S}_j)^2.$$

The proof follows from the results of Ball-Slemrod [2].

THEOREM 4.2. – *If* $\mathbf{v}_{j,h}^{+}$ *is an eigenfunction of the eigenvalue problem:*

$$(\lambda_j^+ - \mathbf{A})\mathbf{v}_{j,h}^+ = 0, \qquad in\ \Omega, \tag{4.1}$$

$$\mathbf{B}\mathbf{v}_{j,h}^+ = 0, \qquad on\ \Gamma \tag{4.2}$$

and the hypotheses $(i_1) - (i_3)$ *of section 2 are assumed, if moreover*

$$\mathbf{C}\mathbf{v}_{j,h}^+ = 0, \qquad on\ \Gamma, \tag{4.3}$$

then $\mathbf{v}_{j,h}^+$ *vanishes in* Ω.

Proof. – The proof follows from the Holmgren Theorem applied to the eigenvalue problem (4.1)-(4.3), that is in explicit form:

$$(4.4)_1 \qquad -a^{\alpha\beta\sigma\mu}\gamma_{\sigma\mu}(\mathbf{v})_{|\beta} - \lambda a^{\alpha\beta}v_\beta = 0, \qquad \alpha = 1,2, \qquad \text{in } \Omega,$$

$$(4.4)_2 \qquad -a^{\alpha\beta\sigma\mu}\gamma_{\sigma\mu}(\mathbf{v})b_{\alpha\beta} - \lambda v_3 = 0, \qquad \text{in } \Omega,$$

$$(4.4)_3 \qquad v_\alpha = 0, \qquad a^{\alpha\beta\sigma\mu}\gamma_{\sigma\mu}(\mathbf{v})\nu_\beta = 0, \qquad \alpha = 1,2, \qquad \text{on } \Gamma.$$

We put $\overline{\gamma}_{\sigma\mu}(\mathbf{u}) = \frac{1}{2}(v_{\mu|\sigma} + v_{\sigma|\mu})$ and from $(4.4)_2$, we obtain

$$(d-\lambda)v_3 = a^{\alpha\beta\sigma\mu}\overline{\gamma}_{\sigma\mu}b_{\alpha\beta} \qquad d = a^{\alpha\beta\sigma\mu}b_{\sigma\mu}b_{\alpha\beta}$$

We can consider the system:

$$(4.5)_1 \qquad -\overline{a}^{\alpha\beta\sigma\mu}\overline{\gamma}_{\sigma\mu}(\mathbf{u})_{|\beta} - \lambda a^{\alpha\beta}v_\beta = 0, \qquad \alpha = 1,2, \qquad \text{in } \Omega,$$

$$(4.5)_2 \qquad v_\alpha = 0, \qquad \overline{a}^{\alpha\beta\sigma\mu}\gamma_{\sigma\mu}(\mathbf{u})\nu_\beta = 0, \qquad \alpha = 1,2, \qquad \text{on } \Gamma,$$

with $\mathbf{u} = (v_1, v_2)$ and $\overline{a}^{\alpha\beta\sigma\mu} = a^{\alpha\beta\sigma\mu} + (\lambda - d)^{-1}a^{\alpha\beta\delta\rho}b_{\delta\rho}b_{\tau\xi}a^{\tau\xi\sigma\mu}$(*see* also [7]).

So we have $\mathbf{A}^m - \lambda_j^+$ is a linear partial differential operator with analytic coefficients and Γ is an analitic surface. Since λ_j^+ does not belong to the *essential spectrum*, i.e $(\lambda_j^+ - \mathbf{A}^m)$ is a Douglis-Nirenberg elliptic operator, we have not real caracteristics hence Γ is a non caracteristic analytic surface. We take $(d - \lambda_j^+) > 0$, that implies λ_j^+ does not belong to the *essential spectrum*, moreover, if we write $(4.5)_1$ in matrix form, as in section 2, we have (*see* [12] Proposition 3.3) that the operator $\mathbf{A}^m_{\alpha\beta} - \mathbf{A}^m_{\alpha 3}(A^m_{33} - \lambda_j^+)^{-1}\mathbf{A}^m_{3\beta}$ is elliptic of the same type as $\mathbf{A}^m_{\alpha\beta}$, so thanks to $(i_1) - (i_2)$, the system $(4.5)_1$ with Cauchy data $(4.5)_2$ equal to zero on Γ, implies $v_\alpha = 0, \quad v_3 = 0 \quad in \quad \Omega.$ □

Now we consider $\mathbf{G} = (\mathbf{E}_n, \mathbf{E}_n)$ the polar set of $\mathbf{G}^0 = (\mathbf{E}_n, \mathbf{E}_n)^0$ in $\mathbf{F} = \boldsymbol{U} \times L^2 \times \mathbf{H}$ *i.e.*

$$\{f^0, f^1\} \in \mathbf{G} \iff ((f^0, g^0)) + ((f^1, g^1)) = 0 \qquad \forall \{g^0, g^1\} \in \mathbf{G}^0.$$

The relaxed exact controllability in $\mathbf{F}'$ follows adapting the HUM method [13][14] to the present situation.

In the next section we deal with the spherical membrane shells. Since in this case the spectrum is known (*see* Example 3.1 of section 3) we can easly prove the above results.

5. Relaxed spectral controllability for spherical membrane shells

For spherical axisymmetric membranes with opening angle $\pi/2$ (*see* (3.5)), the eigenvalues λ_j^+ (*see* (3.6)) are simple, dim(Ω)=1 and moreover lim inf$_{n\to\infty}(\sqrt{\lambda_j^+} - \sqrt{\lambda_{j-1}^+}) = d = 2$. We consider the spaces $\mathbf{E} = Span\{\mathbf{v}_j^+; j = 1, 2, ...\}$ with $\mathbf{v}_j^+ = (u_j^+, w_j^+)$ as defined in (3.7) and $\mathbf{G} = (\mathbf{E}, \mathbf{E})$.

THEOREM 5.1. – *If* $\{\mathbf{v}^0, \mathbf{v}^1\} \in \mathbf{G} = (\mathbf{E}, \mathbf{E})$ *the* $\mathbf{v} = (u, w)$

$$\mathbf{v} = \sum_j \left(((\mathbf{v}^0, \mathbf{v}_j^+)) cos\sqrt{\lambda_j^+} t + ((\mathbf{v}^1, \mathbf{v}_j^+)) sin\sqrt{\lambda_j^+} t / \sqrt{\lambda_j^+} \right) \mathbf{v}_j^+$$

is the unique solution of

$$(5.1) \qquad \begin{cases} \ddot{u} - \mathcal{L}(u) + (1+\nu)w' = 0 \\ \ddot{w} - \frac{(1+\nu)}{\sin\theta}(u \sin\theta)' + 2(1+\nu)w = 0 \end{cases}$$

with boundary conditions

$$u(0, t) = u(\pi/2, t) = 0$$

and initial conditions:

$$\mathbf{v}(0) = \mathbf{v}^0, \quad \dot{\mathbf{v}}(0) = \mathbf{v}^1.$$

We can prove, thanks to the theorem of Ball-Slemrod, there exist two constant c_1 *and* c_2 *such that for any* $T > \pi$.

$$c_1(T) \sum_n \left[((\mathbf{v}^0, \mathbf{v}_n^+))^2 + \frac{((\mathbf{v}^1, \mathbf{v}_n^+))^2}{\lambda_n^+} \right] [{u^+}'_n(\pi/2) - (1+\nu) w_n^+(\pi/2)]^2$$
$$\leq \int_0^T [u'(\pi/2) - (1+\nu) w(\pi/2)]^2 \, \mathrm{d}t$$
$$\leq c_2(T) \sum_n \left[((\mathbf{v}^0, \mathbf{v}_n^+))^2 + \frac{((\mathbf{v}^1, \mathbf{v}_n^+))^2}{\lambda_n^+} \right] [{u^+}'_n(\pi/2) - (1+\nu) w_n^+(\pi/2)]^2.$$

□

Since we know the explicit form of the eigenfunctions and eigenvalues (*see* section 3 example 3.1) we can easy prove the following theorem:

THEOREM 5.2. – *There are two constant* c_3 *and* c_4 *such that:*

$$c_3 \sum_n [\lambda_n^+ ((\mathbf{v}^0, \mathbf{v}_n^+))^2 + ((\mathbf{v}^1, \mathbf{v}_n^+))^2]$$
$$\leq \sum_n \left[((\mathbf{v}^0, \mathbf{v}_n^+))^2 + \frac{((\mathbf{v}^1, \mathbf{v}_n^+))^2}{\lambda_n^+} \right] [{u^+}'_n(\pi/2) - (1+\nu) w_n^+(\pi/2)]^2$$
$$\leq c_4 \sum_n [\lambda_n^+ ((\mathbf{v}^0, \mathbf{v}_n^+))^2 + ((\mathbf{v}^1, \mathbf{v}_n^+))^2]$$

□

and we have:

THEOREM 5.3. – *The norm given in Theorem 5.2 is equivalent to the norm in* $\{\mathcal{U}\times\mathcal{L}^2\}\times\mathbf{H}$: i.e.

$$\sum_n [\lambda_n^+((\mathbf{v}^0,\mathbf{v}_n^+))^2 + ((\mathbf{v}^1,\mathbf{v}_n^+))^2] = \{\|\mathbf{v}^0\|^2_{\mathcal{U}\times\mathcal{L}^2} + \|\mathbf{v}^1\|^2_{\mathbf{H}}\}$$

□

From Theorem 5.1, 5.2 and 5.3 follows *relaxed spectral exact controllability for spherical membranes.*

(REC) *Given* $T>0$ *and an initial state* $\{\mathbf{\Phi}^0,\mathbf{\Phi}^1\}\in\mathbf{F}'=L^2\times\mathcal{U}'\times\mathbf{H}$, *there exists a control* $g(t)\in L^2[0,T]$ *such that the unique solution* $\mathbf{\Phi}=(\phi,\psi)$ *of*

$$(5.2)\qquad \begin{cases} \ddot\phi-\mathcal{L}(\phi)+(1+\nu)\psi'=0\\ \ddot\psi-\frac{(1+\nu)}{\sin\theta}(\phi\sin\theta)'+2(1+\nu)\psi=0\end{cases}$$

in $Q=(0,\pi/2)\times(0,T)$ *with* $\phi(0,t)=0$, $\phi(\pi/2,t)=g(t)$, *and initial conditions*

$$\mathbf{\Phi}(0)=\mathbf{\Phi}^0,\qquad \dot{\mathbf{\Phi}}(0)=\mathbf{\Phi}^1,$$

satisfies the following condition:

$$\{\mathbf{\Phi}(T),\dot{\mathbf{\Phi}}(T)\}\quad\in\mathbf{G}^0.$$

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V. VALENTE
Istituto per le Applicazioni del Calcolo,
C.N.R., Viale del Policlinico,
137, 00161 Roma, Italy.