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C-Polynomial Approximation of H^p and $\mathscr{H}^p$ Functions

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1. INTRODUCTION AND RESULTS

Let C denote the unit circle and D its interior. Polynomials whose zeros lie on C will be called C-polynomials. It is known that every zero free bounded holomorphic function in D can be boundedly approximated in D by C-polynomials [1, 3]. In this paper, we present some results on C-polynomial approximation for other classes of functions.

Throughout this note, we take $p \geqslant 1$ and let H^p be the Hardy space with norm $\| \ \|_p$, and $\mathscr{H}^p$ the class of functions f holomorphic in D such that

$$| f |_p = \left\{ \frac{1}{\pi} \iint_D | f(x + iy)|^p \, dx \, dy \right\}^{1/p} < \infty.$$

Both H^p and $\mathscr{H}^p$ are Banach spaces. For H^p, we have a result similar to that for bounded holomorphic functions, namely

THEOREM 1. *If $f \in H^p$ is zero free in D, then there exist ˏ -polynomials P_n that converge to f uniformly on each compact subset of D and satisfy $\| P_n \|_p \leqslant 2 \| f \|_p$ for all n.*

Here, uniform approximation on compact subsets of D cannot be replaced by approximation in H^p as can be seen from the following

THEOREM 2. *If the P_n are C-polynomials such that $\| P_n - f \|_p \to 0$, then f is either a C-polynomial or the zero function.*

However, we have a stronger result for the space $\mathscr{H}^p$.

THEOREM 3. *If $f \in \mathscr{H}^p$ is zero free in D, then there exist C-polynomials P_n such that $| P_n - f |_p \to 0$.*

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2. Proofs of the Theorems

The proofs of Theorems 1 and 3 depend on the following result of I. Schur [2], which is also stated and used in [3].

LEMMA. *Let*

$$P_n(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n\,,$$
$$\tilde{P}_n(z) = \bar{a}_0 z^n + \bar{a}_1 z^{n-1} + \cdots + \bar{a}_n\,,$$
$$P_n{}^*(z) = z^n \tilde{P}_n(z^{-1}) \qquad \text{and} \qquad Q_{n,m} = P_n + z^m P_n{}^*.$$

If P_n does not vanish in D, then the $Q_{n,m}$ are C-polynomials for all $m = 0, 1, 2, \ldots$, and

$$|P_n{}^*(z)| \leqslant |P_n(z)| \qquad \text{for } |z| \leqslant 1. \tag{1}$$

Now let $f \in H^p$ be zero free in D and, for $0 < r < 1$, let $f_r(z) = f(rz)$. Then $\|f_r - f\|_p \to 0$ as $r \uparrow 1$, and since each f_r is holomorphic and zero free for $|z| < 1/r$, it can be approximated uniformly on $\bar{D}$ by polynomials P_n which do not vanish in D. Hence, f can be approximated in H^p by polynomials P_n which do not vanish in D. Let $P_n{}^*$ and $Q_{n,m}$ be as in the lemma. Then by (1), we have, for all m,

$$\begin{aligned} \|Q_{n,m} - f\|_p &\leqslant \|P_n - f\|_p + \|z^m P_n{}^*\|_p \\ &= \|P_n - f\|_p + \|P_n{}^*\|_p \\ &\leqslant \|P_n - f\|_p + \|P_n\|_p \to \|f\|_p\,. \end{aligned} \tag{2}$$

On each compact subset K of D, the $P_n{}^*$ are uniformly bounded, and hence, $z^m P_n{}^* \to 0$ uniformly on K as $m \to \infty$, independent of n. Thus, the $Q_{n,m}$ approximate f uniformly on K as n and $m \to \infty$. Hence, we can choose a single sequence from $Q_{n,m}$ to approximate f uniformly on every compact subset of D. By (2) we can divide the elements of this sequence by suitable positive numbers tending to one to get a sequence R_k of C-polynomials such that $R_k \to f$ uniformly on each compact subset of D and $\|R_k\|_p \leqslant 2\|f\|_p$ for all k. This completes the proof of Theorem 1.

If $f \in \mathscr{H}^p$ is zero free in D, then by the same argument as above, we obtain C-polynomials $Q_{n,m} = P_n + z^m P_n{}^*$ such that

$$|P_n - f|_p \to 0$$

and

$$\begin{aligned} |Q_{n,m} - f|_p &\leqslant |P_n - f|_p + |z^m P_n{}^*|_p \\ &\leqslant |P_n - f|_p + |z^m P_n|_p\,. \end{aligned}$$

For each $0 < \rho < 1$,

$$\| z^m P_n \|_p \leqslant \max_{|z| \leqslant \rho} | z^m P_n(z)| + \left\{ \frac{1}{\pi} \iint_{\rho < |z| < 1} | P_n |^p \right\}^{1/p}.$$

Let $\epsilon > 0$ be given. Since $\| P_n - f \|_p \to 0$ and

$$\lim_{\rho \to 1} \iint_{\rho < |z| < 1} |f|^p = 0,$$

we can choose $1 - \rho > 0$ so small that

$$\left\{ \frac{1}{\pi} \iint_{\rho < |z| < 1} | P_n |^p \right\}^{1/p} < \epsilon$$

for all sufficiently large n. For this fixed ρ, the polynomials P_n are uniformly bounded on $| z | \leqslant \rho$, and hence,

$$\max_{|z| \leqslant \rho} | z^m P_n(z)| \to 0$$

as $m \to \infty$, independent of n. By choosing a suitable sequence of C-polynomials from $Q_{n,m}$, we complete the proof of Theorem 3.

To prove Theorem 2, let f be holomorphic in D and let the P_n be C-polynomials such that $\| P_n - f \|_p \to 0$. Then $f \in H^p$, and since $P_n \to f$ uniformly on compact subsets of D, f must be zero free in D, unless $f \equiv 0$. Hence, we can assume, without loss of generality, that $P_n(0) = f(0) = 1$ for all n. Suppose that f is not a C-polynomial. In order to approximate f, the degrees of the polynomials P_n must then tend to infinity. Thus, we can write

$$P_n(z) = \prod_{k=1}^{n} (1 - z e^{-i\theta_{n,k}})$$

where $n = n_j \to \infty$. Let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

By Hölder's inequality, it can be shown that

$$\left| \prod_{k=1}^{n} (- e^{-i\theta_{n,k}}) - a_n \right| = \left| \frac{1}{2\pi i} \int_{|z|=r} \frac{P_n(z) - f(z)}{z^{n+1}} \, dz \right| \leqslant \| P_n - f \|_p .$$

Since $f \in H^p$, $a_n \to 0$ by the Riemann-Lebesgue Theorem. This is a contradiction since $\| P_n - f \|_p \to 0$ but $| \prod_{k=1}^{n} (- e^{-i\theta_{n,k}})| = 1$ for all n.

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