Reconstruction of a Radially Symmetric Potential from Two Spectral Sequences

William Rundell

Department of Mathematics, Texas A&M University, College Station, Texas 77843

and

Paul E. Sacks

Department of Mathematics, Iowa State University, Ames, Iowa 50011

Submitted by Joyce R. McLaughlin

Received May 24, 1999

We consider the problem of determining a radially symmetric potential in the three-dimensional Schrödinger equation from eigenvalues associated with two different angular-momentum quantum numbers. This leads to a singular eigenvalue problem for which there are no known general uniqueness results for the most reasonable conjectures. We are able to give strong evidence for uniqueness in some cases and we discuss a computational solution method together with some supporting analysis. One application we have in mind is the determination of certain physical parameters in the standard model of the sun constructed from eigenvalue data. © 2001 Elsevier Science

1. INTRODUCTION

Consider the eigenvalue problem in the unit ball of $\mathbb{R}^3$,

$$-\Delta U + q(X)U = \lambda U \quad |X| < 1, \quad U(X) = 0 |X| = 1,$$

(1.1)

with a central potential $q(X) = q(|X|)$. By looking for solutions in the separated form

$$U(r, \theta, \phi) = \frac{\psi(r)}{r} Y_l^m(\theta, \phi),$$

(1.2)

1 This author was partially supported through NSF Grant DMS-9906985.
in which $X = (r, \theta, \phi)$ are spherical coordinates in $\mathbb{R}^3$ and $Y^m_l$ are spherical harmonics, we get the ordinary differential equation for $\psi$,

$$
\psi'' + \left( \lambda - q(r) - \frac{l(l + 1)}{r^2} \right) \psi = 0 \quad 0 < r < 1
$$

(1.3)

for $l = 0, 1, 2, \ldots$, supplemented by the regularity condition

$$
\psi(r) = O(r) \quad r \to 0.
$$

(1.5)

For fixed $l$, the problem in (1.3)–(1.5) has a countable sequence of eigenvalues, which we denote by $\lambda_{l,n}$, $n = 1, 2, \ldots$. In this paper we are interested in the inverse spectral problem of reconstructing the potential $q(r)$ from knowledge of the eigenvalues $\lambda_{l,n}$. We expect that the problem would be highly overdetermined in this form (although we know of no proof that this is the case), so instead we will actually seek to recover the potential from certain subsets of the eigenvalues. When norming constant information is available, some uniqueness results have been proved in [GR, C]. The uniqueness result of [NSU] may also be specialized to the case of a radial potential.

The primary problem of interest in this paper is the case when the spectral data consists of two sequences of eigenvalues $\{\lambda_{l_1,n}\}_{n=1}^{\infty}$ for two distinct choices of angular order $l_1, l_2$. Some progress toward a uniqueness proof for this situation may be found in Corollary 1.2 of Carlson and Shubin [CS], in which it is established that the corresponding isospectral set, i.e., the set of potentials sharing such data, is locally of finite dimension, under the additional assumption that $l_2 - l_1$ is an odd integer. On the other hand, the set of potentials sharing the eigenvalues for any single value of $l$ will be locally of infinite dimension ([PT, GR, C]). See also [H] for a related inverse spectral problem in which eigenvalues for two angular orders are used in a somewhat different way.

In this paper we focus on several issues related to the results in [CS]:

• Can we make any more precise statements about the local dimension of the isospectral set; for example, is it ever zero?

• What can be said in the case that $l_2 - l_1$ is even?

• Are there computational methods which can be developed?

Our main results may be summarized as follows: We explain how, in a suitable linearized sense, uniqueness for the inverse spectral problem is equivalent to showing that a certain collection of functions depending on $(l_1, l_2)$ (see (2.17) or (4.3) below) is complete in $L^2(0, 1)$. We then proceed to prove the required completeness property for several specific small integer choices of $(l_1, l_2)$, when linearization is done with respect to
$q = 0$. The simplest case of even $l_2 - l_1$, namely $l_1 = 0, l_2 = 2$, is included. We have been unable to give a proof which is valid for an arbitrary pair $(l_1, l_2)$, although it is likely that our approach can be made to work for some other special cases. The main technical tool is the exploitation of a class of linear integral operators which essentially map squared eigenfunctions for one $l$ value to those for another $l$ value. Properties of these operators are developed in Section 3. If uniqueness cannot be proved, it may still be of interest to estimate the dimension of the solution set, and this may also be done using properties of these operators, but we will not pursue that direction here. Finally, we briefly indicate how these results can be applied to the problem of a numerical solution of the inverse spectral problem.

We conclude this introduction by mentioning one other very interesting and closely related problem which has partly motivated this work, the problem of determining coefficients which characterize the interior of the sun from observations of its oscillations. These are the natural frequencies $\{\lambda_{l,n}\}$ coming from an equation

$$\psi'' + \left(\frac{\lambda}{c^2(r)} - Q(r, l, \lambda) - \frac{l(l+1)}{r^2}\right)\psi = 0 \quad (1.6)$$

modeling normal modes of acoustic vibrations; see e.g. [G]. Two fundamental quantities are the propagation speed for acoustic waves $c(r)$ and the density $\rho(r)$. In (1.6) $\psi$ is the radial part of the wave function and $Q$ has the form $Q(r, l, \lambda) = \frac{l(l+1)}{r}C_1(\rho, c) + C_2(\rho)$. Thus $Q$ is ultimately a functional of $c, \rho$ only, but in the most general form has a reciprocal dependence on $\lambda$. It may be reasonable for certain purposes to set the term $C_1$ equal to zero, giving a more conventional type of equation of the form

$$\psi'' + \left(\frac{\lambda}{c^2(r)} - q(r) - \frac{l(l+1)}{r^2}\right)\psi = 0 \quad (1.7)$$

The simplest case is when $c$ is known and constant (but a travel-time transformation could be used if $c$ was known and not constant) and we have to determine $q$ (and hence $\rho(r)$) from spectral data; that is, we have Eq. (1.3).

There is a large amount of data available ([GO]) consisting of $\lambda_{l,n}$ for $l, n$ up to about (1000, 30). The effective accuracy of the data decreases with increasing values of $l$ and also for very small values of $l$. No other spectral information of sufficient accuracy seems readily available. We are therefore forced to consider using multiple values of $l$ in order to obtain reconstructions.

## 2. THE FORWARD MAP AND ITS DERIVATIVE

First let us recall some well-known results about (1.3) with $l = 0$ (e.g., [B, PT]). If we define the map $q \mapsto \lambda_n(q)$, the $n$th eigenvalue of (1.3) subject
to the Dirichlet boundary conditions $\psi(0) = \psi(1) = 0$, then $\lambda_n(q) = \lambda_n(\hat{q})$ implies that

\[ \int_0^1 (\hat{q}(x) - q(x))\psi_n(x)\hat{\psi}_n(x)\,dx = 0, \quad (2.1) \]

where $\psi_n, \hat{\psi}_n$ are eigenfunctions for $q, \hat{q}$. If $\zeta = \hat{q} - q$ then in the limit as $\zeta \to 0$ we get

\[ \int_0^1 \zeta(x)\psi_n^2(x)\,dx = 0 \quad (2.2) \]

or, what amounts to the same thing, the Frechet derivative of $\lambda_n(q)$ is (with $\psi_n$ now normalized in $L^2(0,1)$)

\[ D_q\lambda_n(q)\zeta = \int_0^1 \zeta(x)\psi_n^2(x)\,dx. \quad (2.3) \]

See e.g. Theorem 2.3 of [PT].

From the eigenvalue asymptotics

\[ \lambda_n(q) = (n\pi)^2 + \int_0^1 q(x)\,dx + a_n \quad \{a_n\} \in l^2 \quad (2.4) \]

we obtain the additional orthogonality relationship that if $\lambda_n(q) = \lambda_n(q + \zeta)$ for all $n$ then

\[ \int_0^1 \zeta(x)\,dx = 0. \quad (2.5) \]

The well-known theorem ([B; PT, Theorem 3.3]) that a symmetric function $q$ is uniquely determined by $\{\lambda_n(q)\}_{n=1}^{\infty}$ may be thought of as the fact that the set of functions $\{1, \psi_n^2\}$ is complete in the even subspace of $L^2(0,1)$. If we add a second spectral sequence $\mu_n(q)$, the $n$th eigenvalue of (1.3) with boundary conditions $\psi(0) = 0, \psi'(1) + h\psi(1) = 0$ and eigenfunction $\phi_n$, we get a second set of orthogonality relationships,

\[ \int_0^1 \zeta(x)\phi_n^2(x)\,dx = 0, \quad (2.6) \]

and it can be shown that

\[ \{1, \psi_n^2, \phi_n^2\}_{n=1}^{\infty} \quad (2.7) \]

is complete in all of $L^2(0,1)$; i.e., a linearized uniqueness result holds for the problem of determining $q \in L^2(0,1)$ from the eigenvalue data $\{\lambda_n(q), \mu_n(q)\}_{n=1}^{\infty}$. Of course, one knows in this case ([B]) that uniqueness holds globally, not just in the linearized sense.
In [GR] a similar fact is derived and used in the singular case \( l = 1 \); namely, if \( \lambda_{l,n}(q) \) denotes the \( n \)th eigenvalue of (1.3)–(1.5), then

\[
D_q \lambda_{l,n}(q) \xi = \int_0^1 \xi(x) \psi_{l,n}^2(x) \, dx,
\]

(2.8)

where again \( \psi_{l,n} \) is the \( n \)th normalized eigenfunction. Clearly we expect that for any \( l \), if we define \( \lambda_{l,n}(q) \) to be \( n \)th eigenvalue of (1.3)–(1.5) with normalized eigenfunction \( \psi_{l,n} \), then

\[
D_q \lambda_{l,n}(q) \xi = \int_0^1 \xi(x) \psi_{l,n}^2(x) \, dx.
\]

(2.9)

Now let \( \Lambda \) denote some subset of the indices \((l,n)\). The condition \( \lambda_{l,n}(q) = \lambda_{l,n}(\hat{q}) \) for all \((l,n) \in \Lambda\) implies, in the limit of small \( \zeta = \hat{q} - q \), that \( \zeta \) is orthogonal to the subspace of \( L^2(0,1) \) spanned by \( \{\psi_{l,n}^2\}_{(l,n) \in \Lambda} \).

In the best case this subspace is all of \( L^2(0,1) \), in which case we have a uniqueness result for the linearized inverse spectral problem. A more general formulation of the linearized inverse spectral problem is as follows:

**Problem.** Find the subspace of \( L^2(0,1) \) spanned by \( \{\psi_{l,n}^2\}_{(l,n) \in \Lambda} \).

Although not stated in quite these terms, the principal result of [CS] amounts to the statement that this subspace always has finite codimension in the case that \( \Lambda = \Lambda_{l_1,l_2} = \{(l,n) : n = 1, 2, \ldots, l = l_1 \text{ or } l_2 \} \text{ if } l_1 - l_2 \text{ is odd.} \)

Asymptotics of the eigenvalues are (see [GR] for \( l = 1 \), [CS] or [AS, 9.5.12] for the specific case of \( q = 0 \))

\[
\lambda_{l,n} = \left( n + \frac{l}{2} \right)^2 \pi^2 + \int_0^1 q(x) \, dx - l(l+1) + r_{l,n}, \quad \sum_{n=1}^\infty r_{l,n}^2 < \infty, \quad (2.10)
\]

or, for later reference,

\[
\sqrt{\lambda_{l,n}} = \left( n + \frac{l}{2} \right) \pi + \frac{\int_0^1 q(x) \, dx - l(l+1)}{(2n+l)\pi} + \beta_{l,n}, \quad \sum_{n=1}^\infty \eta\beta_{l,n}^2 < \infty.
\]

(2.11)

So again the mean value \( \int_0^1 q(x) \, dx \) is determined by the eigenvalue sequence for any one \( l \) value. If the spectral data contains all of the eigenvalues for any fixed \( l \) value then we should really look at the span of \( \{1, \psi_{l,n}^2\}_{(l,n) \in \Lambda} \).

For technical convenience, we will actually work with a somewhat different mapping, defined as follows. For given \( q \) there exists a solution (see the proof of Proposition 2.1 below) \( \Psi(x, \lambda, q) \) of (1.3) satisfying the normalization condition

\[
\lim_{x \to 0} \frac{\Psi(x, \lambda, q)}{x^{l+1}} = 1.
\]

(2.12)
Now set

\[ F(\lambda, \mu, n) = \{ \Psi_1(1, \mu, n, q) \}_{(l,n) \in \Lambda}. \]  

(2.13)

Given spectral data \( \{\lambda_{l,n}, (l, n) \in \Lambda\} \), the inverse spectral problem is then equivalent to that of solving

\[ F(\lambda, \mu, n) = 0. \]  

(2.14)

All further considerations in this paper are restricted to the case of small potentials, i.e., linearization of \( F(\lambda, \mu, n) \) at \( q = 0 \). The following is proved in Appendix A.

**Proposition 2.1.** If \( \lambda_{l,n,0} \) denotes the eigenvalue of (1.3)–(1.5) when \( q \equiv 0 \) then

\[ D_q F(0, \lambda_{l,n,0}) \zeta = \left\{ c_{l,n} \int_0^1 x^2 j_l^2 \left( \sqrt{\lambda_{l,n,0} x} \right) \zeta(x) dx \right\}_{l,n \in \Lambda} \]  

(2.15)

for some \( c_{l,n} \neq 0 \).

Here \( j_l, y_l \) are the spherical Bessel functions with standard normalizations ([AS]). The exact value of \( c_{l,n} \) may be found in the proof. We see therefore that invertibility of this linear operator is again equivalent to a certain completeness result. In fact, it is not hard to check that when \( q \equiv 0 \) we have \( \psi_{l,n}(x) = C x j_l(\sqrt{\lambda_{l,n,0} x}) \), so that (not surprisingly) uniqueness holds for \( D_q F(0, \lambda_{l,n,0}) \zeta = 0 \) exactly if it holds for the system

\[ D_q \lambda_{l,n,0}(0) \zeta = 0, \quad (l, n) \in \Lambda. \]  

(2.16)

Specializing to the case \( \Lambda = \Lambda_{l_1, l_2} \) and taking into account the earlier remarks about including the condition that \( \zeta \) have zero mean, we propose the following conjecture, whose verification would immediately imply a uniqueness result for the inverse spectral problem linearized at \( q = 0 \) in either of the formulations discussed above. (For notational convenience we define \( \phi_l(x) = x j_l(x) \).)

**Conjecture.** For any nonnegative integers \( l_1 \) and \( l_2 \) the set of functions

\[ \left\{ 1, \phi_{l_1}(\sqrt{\lambda_{l_1,n,0} x}), \phi_{l_2}(\sqrt{\lambda_{l_2,n,0} x}) \right\} \]  

(2.17)

is complete in \( L^2(0, 1) \).

This paper does not contain a proof of this conjecture, but we will make some progress toward its resolution. The precise results we prove below actually pertain to the case when \( \lambda_{l,n,0} \) is replaced with \( (n + \frac{l}{2})^2 \pi^2 \), i.e., the leading term in its asymptotic expansion. Thus these are not strictly results about the linearization of \( F(\lambda, \mu, n) \), but instead ones which we believe reveal the essential structure of the problem and which furthermore seem quite
adequate for computational purposes. In some cases we will also discuss perturbation arguments which may then be used to draw conclusions about the span of the functions (2.17).

The factor \(l/2\) in the leading term of the asymptotic expansion (2.11) will be critical in providing new information as we change the value of \(l\). It will also be a source of difficulty since it means that certain low frequencies are omitted from the information in the case \(l > 0\). It is precisely this factor that will limit us from obtaining the complete proof to the above conjecture.

3. SOME IMPORTANT OPERATORS AND THEIR PROPERTIES

Our general strategy involves the exploitation of a family of linear operators \(T_l\) which have the special property that \(T_l[\cos(2\sqrt{\lambda}x)] = 2\phi_l(\sqrt{\lambda}x) - 1\); i.e., it essentially maps squared eigenfunctions for \(l = 0\) to those for \(l > 0\). In the case \(l = 1\), these operators were used in connection with a somewhat different inverse spectral problem in [GR]; they also played a role in the analysis of [CS]. By means of these operators, a set of orthogonality relations for \(\zeta\) with respect to the functions \(\phi_l(\sqrt{\lambda}x)\) can be transformed into orthogonality relations for \(T_l[\zeta]\) with respect to \(\cos 2\sqrt{\lambda}x\). The more transparent nature of these latter conditions will allow us, in some cases, to conclude the desired completeness property.

The proofs of all results stated in this section may be found in Appendix B.

**Lemma 3.1.** For each positive integer \(l\), define the operator \(S_l: L^2(0, 1) \to L^2(0, 1)\) by

\[
S_l[f](x) = f(x) - 4lx^{2l-1} \int_x^1 \frac{f(s)}{s^{2l}} \, ds.
\]

We then have the following properties:

(i) The adjoint of \(S_l\) is

\[
S_l^*[g](x) = g(x) - 4lx^{-2l} \int_0^x s^{2l-1} g(s) \, ds.
\]

(ii) The family \(S_l\) pairwise commutes: \(S_{l_1} S_{l_2} = S_{l_2} S_{l_1}\) for any \(l_1, l_2\).

(iii) \(S_l\) is bounded on \(L^2(0, 1)\).

(iv) \(S_l\) is one to one and

\[
S_l^{-1}[g] = g(x) - 4lx^{-2l-1} \int_x^1 s^{2l} g(s) \, ds.
\]
(v) The function \( \{x^{2l}\} \) is the only element in the null space of \( S_l^* \), and for \( \psi \in L^2(0, 1) \) and \( n = 0, 1, \ldots \),
\[
\int_0^1 x^n S_l[\psi](x) \, dx = \frac{n - 2l}{n + 2l} \int_0^1 x^n \psi(x) \, dx. \tag{3.4}
\]

(vi) The functions \( \phi_l \) satisfy
\[
\phi_l = -S_l^*[\phi_{l-1}]. \tag{3.5}
\]

(vii) If \( g = S_l[f] \) then \( f \) and \( g \) are related by the equation
\[
f^{(2l)}(x) + \frac{4l}{x} f^{(2l-1)}(x) = g^{(2l)}(x). \tag{3.6}
\]

In particular, note Property (vi) which allows us to step from squares of eigenfunctions corresponding to a given \( l \) to those corresponding to \( l - 1 \). By chaining together these step operators we should be able to achieve our aim of transferring information about inner products with the squares of eigenfunctions to inner products with those of \( l = 0 \). This is made precise by the following lemma.

**Lemma 3.2.** For each \( l = 1, 2, \ldots \) define the operators \( T_l \) by
\[
T_l = (-1)^{l-1} S_l S_{l-1} \cdots S_1. \tag{3.7}
\]

Then \( T_l \) is a bounded, one-to-one linear operator on \( L^2(0, 1) \) such that
\[
\int_0^1 2\phi_l(\sqrt{\lambda}x) - 1)\zeta(x) \, dx = \int_0^1 \cos(2\sqrt{\lambda}x) T_l[\zeta](x) \, dx \tag{3.8}
\]
for any \( \zeta \in L^2(0, 1) \) and \( \lambda \geq 0 \). That is, the adjoint operators satisfy \( T_l^*[\cos 2\sqrt{\lambda}x] = 2\phi_l(\sqrt{\lambda}x) - 1 \). The null space of \( T_l^* \) is the span of the functions \( \{x^2, x^4, \ldots, x^{2l}\} \).

It is straightforward to compute explicit expressions for the operators \( T_l \) using (3.7); for example,
\[
T_2[\zeta](x) = -\zeta(x) - 12x \int_x^1 \frac{\zeta(t)}{t^2} \, dt + 24x^3 \int_x^1 \frac{\zeta(t)}{t^4} \, dt,
\]
\[
T_3[\zeta](x) = \zeta(x) - 24x \int_x^1 \frac{\zeta(t)}{t^2} \, dt + 120x^3 \int_x^1 \frac{\zeta(t)}{t^4} \, dt - 120x^5 \int_x^1 \frac{\zeta(t)}{t^6} \, dt. \tag{3.9}
\]

As a final preparation for our approach, we state two more results which will be fundamental later on. Since we now have a means of relating the functions \( \phi_l(\sqrt{\lambda}x) \) back to \( \cos 2\sqrt{\lambda}x \), parity issues will come back into play. Throughout this paper we will use the terms even and odd functions on the
interval $[0, 1]$. By these we simply mean functions obeying the relations $f(1-x) = f(x)$ and $f(1-x) = -f(x)$ respectively. A function $f(x)$ is said to have definite parity if it is either an even or an odd function on $[0, 1]$. We denote by $P_e$ and $P_o$ the projection operators in $L^2(0, 1)$ onto the subspaces of even and odd functions, respectively.

**Lemma 3.3.** Suppose that $f, g \in C^\infty([0, 1])$ have definite parity. If $S_{l+m} \cdots S_l[f] = g$ for some $l$ and $m$, then $f$ and $g$ and all their derivatives must vanish at $x = 0, 1$.

**Lemma 3.4.** Suppose that $f, g \in L^2([0, 1])$ have definite parity. If $S_{l+m} \cdots S_l[f] = g$ for some $l$ and $m = 0, 1$, then $f = g = 0$.

## 4. OUTLINE OF THE GENERAL STRATEGY

As indicated in the Introduction we are interested in the inverse problem of recovering the potential $q \in L^2(0, 1)$ from eigenvalue data $\{\lambda_{l,n}\}_{(l,n) \in \Lambda}$ where the index set

$$\Lambda = \Lambda_{l_1, l_2} := \{(l, n); l = l_1, l_2, n = 1, 2, \ldots\}$$

(4.1)

for two distinct nonnegative integers $l_1, l_2$. The related linear problem, according to the discussion of Section 2, is to show that the set of functions

$$\left\{1, \phi_l \left( \sqrt{\lambda_{l,n}} x \right) \right\}_{(l,n) \in \Lambda_{l_1, l_2}}$$

(4.2)

is complete in $L^2(0, 1)$. We will actually make the further approximation $\sqrt{\lambda_{l,n}} \approx (n + \frac{1}{2})\pi$ from (2.12) and seek instead to prove that the set

$$\Phi_{l_1, l_2} = \left\{1, \phi_l \left( \left( n + \frac{l}{2} \right) \pi x \right) \right\}_{(l,n) \in \Lambda_{l_1, l_2}}$$

(4.3)

is complete.

Let us consider the case where the spectral data consist of the eigenvalues from the values $l = 0$ and $l = 1$. The assumption that $\zeta$ is orthogonal to the set $\Phi_{0,1}$ implies that $\int_0^1 \cos(2n\pi t)\zeta(t) dt = 0$ and $\int_0^1 \phi_1((n + \frac{1}{2})\pi t)\zeta(t) dt = 0$. The first of these shows that $\zeta$ is odd, while the second leads, using Lemma 3.1, to the condition

$$\int_0^1 \cos((2n + 1)\pi x)\Gamma_1[\zeta](x) dx = 0, \quad n = 1, 2, \ldots$$

(4.4)
From this we see that $T_1[\zeta]$ must be a linear combination of $\cos(\pi x)$ and an even function,

$$T_1[\zeta] = \zeta(x) - 4x \int_x^1 \frac{\zeta(s)}{s^2} ds = \chi(x) + \epsilon \cos \pi x,$$

$$\epsilon \in \mathbb{R}, \quad \chi(x) = \chi(1-x).$$  \hfill (4.5)

We must show that the only odd solution of (4.5) is $\zeta = 0$.

In the case where we have data from $l = 1$ and $l = 2$ we no longer have the parity assumption on $\zeta$, but instead we have the two equations

$$T_1[\zeta] = \zeta(x) - 4x \int_x^1 \frac{\zeta(s)}{s^2} ds = \chi_e(x) + \epsilon_1 \cos \pi x$$

and

$$T_2[\zeta] = \zeta(x) + 12x \int_x^1 \frac{\zeta(t)}{t^2} dt - 24x^3 \int_x^1 \frac{\zeta(t)}{t^4} dt$$

$$= \chi_o(x) + \epsilon_0 + \epsilon_2 \cos(2\pi x),$$  \hfill (4.6)

where $\epsilon_i \in \mathbb{R}$ for $i = 0, 1, 2$, $\chi_e$ is even, and $\chi_o$ is odd.

The general picture is now evident; we obtain pairs of equations of the form

$$T_l[\zeta](x) = \begin{cases} 
\chi_e(x) + \sum_{k=1}^{l} \epsilon_k \cos k \pi x & \text{if } l \text{ is odd} \\
\chi_o(x) + \sum_{k=0}^{l} \epsilon_k \cos k \pi x & \text{if } l \text{ is even,}
\end{cases}$$  \hfill (4.7)

where $\epsilon_k \in \mathbb{R}$ for $k = 0, 1, \ldots, l$, $\chi_e$ is even, and $\chi_o$ is odd.

If we have spectral information for the two values $l_1 < l_2$ then we can express this with two equations

$$T_{l_1}[\zeta] = f, \quad T_{l_2}[\zeta] = g$$  \hfill (4.8)

or by

$$S_{l_2} \cdots S_{l_1+1}[f] = g.$$  \hfill (4.9)

an equation containing two functions $\chi$ of known parity and a total of at most $(l_1 + l_2 + 3)/2$ unknown constants $\epsilon_k$.

The first step is to show that in each equation or pair of equations that all of the constants $\epsilon_k = 0$. If we are able to reduce to the homogeneous case then (4.9) becomes an equation of the type studied in Lemmas 3.3 and 3.4, mapping functions of definite parity to functions of definite parity. When Lemma 3.4 is applicable, i.e., if $l_2 - l_1 = 1, 2$, we conclude that $\zeta = 0$. In any case, the use of Lemma 3.4 immediately tells us that the dimension of the orthogonal complement of $\Phi_{l_1,l_2}$ is finite and in fact is at most equal to the sum of the number of unknown parameters, that is, $(l_1 + l_2 + 3)/2$. 

 radially symmetric potential reconstruction 363
5. UNIQUENESS FOR THE CASE \( l = 0, 1 \)

Our aim is to shown that the only odd solution of (4.5) is \( \zeta = 0 \). We give two quite different proofs of this since the ideas will be of value when we consider higher values of \( l \).

**Proof.** The first idea is to differentiate (4.5) twice to get

\[
\zeta''(x) + \frac{4}{x} \zeta'(x) = \chi''(x) - \epsilon \pi^2 \cos(\pi x). \tag{5.1}
\]

We may eliminate the \( \chi \) term by projecting this equation onto the subspace of odd functions in \( L^2(0, 1) \), giving

\[
\zeta''(x) + \left( \frac{2}{x} - \frac{2}{1-x} \right) \zeta'(x) = -\epsilon \pi^2 \cos(\pi x). \tag{5.2}
\]

The solution of (5.2) may be found explicitly. Integrating once gives

\[
x^2(1-x)^2 \zeta'(x) = -\epsilon \pi^2 \int_0^x (1-s)^2 \cos(\pi s) \, ds + C. \tag{5.3}
\]

In order that \( \zeta \in L^2(0, 1) \) we must have \( C = 0 \). Integrating a second time leads to

\[
\zeta(x) = -\epsilon \pi^2 \int_0^x \int_{\xi}^x \frac{s^2(1-s)^2}{\xi^2(1-\xi)^2} \cos(\pi s) \, ds \, d\xi + C, \tag{5.4}
\]

where in this case \( C = 0 \) because of \( \int_0^1 \zeta(x) \, dx = 0 \).

Return now to Eq. (4.5) and evaluate at \( x = 0 \) and \( x = 1 \) (using the fact that \( \lim_{x \to 0} x^3 \int_x^1 \frac{\zeta(s)}{s^2} \, ds = \zeta(0) \)) to give \( \zeta(1) = \chi(1) - \epsilon \) and \(-3\zeta(0) = \chi(0) + \epsilon \). Since \( \zeta \) is odd and \( \chi \) is even, this pair of equations implies that \( \zeta(0) = -\epsilon \) and of course we must have \( \zeta(\frac{1}{2}) = 0 \). If \( \epsilon > 0 \) we then must have \( \zeta > 0 \) at some point of \((0, \frac{1}{2})\), which is inconsistent with (5.3). Similarly, \( \epsilon < 0 \) is impossible, so we conclude that \( \zeta \equiv 0 \).

**Proof.** For a second proof, we denote by \( R \) the operator \( R = P_o T_1 P_o \) (recall that \( P_o \) is the projection onto the odd subspace of \( L^2(0, 1) \)). If a nontrivial solution \( \zeta \) of the basic equation (4.5) exists then it also satisfies \( R \zeta = \epsilon \cos \pi x \). We proceed to derive a contradiction by proving that \( \cos \pi x \) cannot be in the range of \( R \). To do this, it is enough to show that there exists a \( g \) in the null space of \( R^* \) such that \( \int_0^1 g(x) \cos \pi x \, dx \neq 0 \). We have \( R^* = P_o T_1^* P_{o} \), hence \( R^* g = 0 \) is equivalent to an equation

\[
g(x) - \frac{2}{x^2} \int_0^x s g(s) \, ds + \frac{2}{(1-x)^2} \int_0^{1-x} s g(s) \, ds = 0, \quad g(x) = -g(1-x). \tag{5.5}
\]
Since the singularity is logarithmic at $x = \frac{1}{2}$, $g(x) = \sum c_n(x - \frac{1}{2})^{2n+1}$. A recursion relation for the coefficients may be derived and analytically solved to give $c_n = -4^n(n+1)/(4n^2 - 1)$. The resulting series can be summed explicitly to give

$$g(x) = \frac{3}{4}(x - \frac{1}{2}) - \frac{1}{8}(6x^2 - 6x + 1) \ln \left( \frac{x}{1-x} \right).$$

(5.6)

Since the singularity is logarithmic at $x = 0, 1$, $g \in L^2(0, 1)$ and is also an admissible solution of (5.5). Finally, to show that $g$ and $\cos \pi x$ are not orthogonal we consider the sum

$$\int_0^1 g(x) \cos \pi x \, dx = -\sum_{n=0}^{\infty} \frac{4^n(n+1)}{4n^2 - 1} \int_0^1 (x - \frac{1}{2})^{2n+1} \cos \pi x \, dx.$$  

(5.7)

The $n = 0$ term is $-2/\pi^2$ while for $n \geq 1$ the term may be estimated, after an obvious change of variables and using $|\sin x| \leq |x|$, as

$$\frac{n+1}{2(4n^2-1)} \int_0^1 x^{2n+1} \sin \frac{\pi x}{2} \, dx \leq \frac{\pi(n+1)}{4(4n^2-1)(2n+3)}.$$  

(5.8)

Since $\sum_{n=1}^{\infty} ((n+1)/(4n^2 - 1)(2n + 3)) = \frac{\pi^2}{2}$, we find $\int_0^1 g(x) \cos \pi x \, dx \leq \frac{\pi^2}{16} - 2/\pi^2 < 0$. This shows that $\epsilon = 0$, and now Lemma 3.4 shows that the only solution to $T_1[\xi] = \chi$ is that $\xi$ is zero.

We have therefore proven the following.

**Proposition 5.1.** The set $\Phi_{0,1}$ is complete in $L^2(0, 1)$. In particular, the operator (2.15) with $\Lambda = \Lambda_{0,1}$ and $\lambda_{i, n, 0}$ replaced with $(n + \frac{1}{2})^2 \pi^2$ is one to one on $\{\xi \in L^2(0, 1): \int_0^1 \xi = 0\}$.

From the above argument we also obtain

**Corollary 5.2.** The functions $\{P_{\sigma} \phi_i((n + \frac{1}{2}) \pi x)\}_{n=1}^{\infty}$ are linearly independent, and in consequence the functions in $\Phi_{0,1}$ are a basis of $L^2(0, 1)$.

**Proof.** From Lemma 3.2 we obtain $T_1[\cos 2\sqrt{\lambda} x] = 2\phi_1(\sqrt{\lambda} x) - 1$. Hence the equation $\sum_{n=1}^{\infty} c_n P_{\sigma} \phi_i((n + \frac{1}{2}) \pi x) = 0$ is equivalent to $P_{\sigma} T_1^{*} (\sum_{n=1}^{\infty} c_n \cos(2n + 1) \pi x) = 0$. Since the sum is already odd, it must be in the null space of $R^*$, and so by the above discussion

$$\sum_{n=1}^{\infty} c_n \cos(2n + 1) \pi x = Cg(x)$$

(5.9)

for some constant $C$ where $g$ is the function in (5.6). On the other hand, such a representation is impossible unless $C = 0$ since we have shown that $g$ is not orthogonal to $\cos \pi x$. Thus we must also have $c_n = 0$ for $n = 1, 2, \ldots$
It remains to show that the functions $\Phi_{0,1}$ are linearly independent. If
\begin{equation}
\sum_{n=1}^{\infty} c_n \phi_1((n + \frac{1}{2})\pi x) + d_0 + \sum_{n=1}^{\infty} d_n \phi_0(n\pi x) = 0,
\end{equation}
then by applying $P_0$ to both sides, recalling that $\phi_0(n\pi x) = \sin^2 n\pi x$ is even, we find that $P_0(\sum_{n=1}^{\infty} c_n \phi_1((n + \frac{1}{2})\pi x)) = 0$. Hence, as above, $c_n = 0$ and subsequently $d_n = 0$ for each $n = 0, 1, \ldots$.

As remarked earlier, Proposition 5.1 does not correspond precisely to uniqueness for the linearized problem of interest, due to the fact that $\lambda_{l,n,0}$ in (2.15) has been replaced by $(n + \frac{1}{2})^2 \pi^2$, the leading term in its asymptotic expansion. We might also wish to have the same conclusion when $\lambda_{l,n,0}$ is replaced with $\lambda_{l,n}$, the prescribed eigenvalue data. Assuming as always that $q$ is a small potential, these sequences will not differ much from each other, so that it is natural to try a perturbation argument.

First, one may check that $|\phi_1'| \leq C_1$ where for example $C_0 = 1$, $C_1 \approx 1.03$. Then
\begin{equation}
\| \phi_1((n + \frac{1}{2})\pi x) - \phi_1(\sqrt{\mu_{l,n}x}) \|_{L^2(0,1)} \leq C_1 |(n + \frac{1}{2}) - \sqrt{\mu_{l,n}}|.
\end{equation}
From this it follows from the Krein–Rutman–Milman Theorem ([Y, p. 39]) that there exists $\varepsilon_0 > 0$ such that the set $\{1, \phi_{l,n}(\sqrt{\mu_{l,n}x})\}_{n \geq 1}$, for $l = 0, 1$, is complete in $L^2(0,1)$ provided that
\begin{equation}
\sum_{n=1}^{\infty} |\sqrt{\mu_{0,n}} - n\pi| + |\sqrt{\mu_{1,n}} - (n + \frac{1}{2})\pi| < \varepsilon_0.
\end{equation}

It remains of interest to see if a perturbation argument can be applied in the case $\mu_{l,n} = \lambda_{l,n,0}$, and in this case (5.12) will not be applicable, because according to (2.11) the differences appearing in the sum decay only like $O(n^{-1})$. Set
\begin{equation}
\begin{align*}
f_n(x) &= 2\phi_0(n\pi x) - 1, \\ g_n(x) &= 2\phi_1((n + \frac{1}{2})\pi x) - 1, \\ h_n(x) &= 2\phi_1(\sqrt{\lambda_{l,n,0}x}) - 1.
\end{align*}
\end{equation}
We know that $\{1, f_n, g_n\}$ is complete, and we wish to deduce the same thing about $\{1, f_n, h_n\}$ (recall that $\lambda_{0,n,0} = n\pi$). According to the Paley–Wiener Theorem ([Y, p. 38]), the answer is affirmative provided that there exists $L \in (0, 1)$ such that
\begin{equation}
\left\| \sum_{n=1}^{N} c_n (g_n - h_n) \right\|_{L^2(0,1)} \leq L \left\| d_0 + \sum_{n=1}^{N} d_n f_n + c_n g_n \right\|_{L^2(0,1)}
\end{equation}
for any finite choice of scalars $c_1, \ldots, c_n$. Equivalently, we can show that the sequence $\{\tilde{h}_n\}$ spans the odd subspace of $L^2(0,1)$ where
\( \tilde{h}_n = \mathbf{P}_n h_n \) is the odd projection of \( h_n \), and for this it suffices to show that
\[ \left\| \sum_{n=1}^N c_n (\tilde{g}_n - \tilde{h}_n) \right\|_{L^2(0,1)} \leq L \left\| \sum_{n=1}^N c_n \tilde{g}_n \right\|_{L^2(0,1)} \] for some \( L \in (0, 1) \), any scalars \( c_1, \ldots, c_N \), and \( \tilde{g}_n = \mathbf{P}_n \tilde{g}_n \).

Now for any fixed \( N \) it is not hard to check that the maximum value of the quotient
\[ \frac{\left\| \sum_{n=1}^N c_n (\tilde{g}_n - \tilde{h}_n) \right\|_{L^2(0,1)}}{\left\| \sum_{n=1}^N c_n \tilde{g}_n \right\|_{L^2(0,1)}} \] is the magnitude of the largest eigenvalue of \( B^{-1}A \) (or of the generalized eigenvalue of \( AC = \lambda BC \)), where \( A, B \) are the Gram matrices with entries \( a_{nm} = \int_0^1 (\tilde{g}_n - \tilde{h}_n) \tilde{g}_m - \tilde{h}_m \) \( dx \), \( b_{nm} = \int_0^1 \tilde{g}_n \tilde{g}_m \) \( dx \). These are completely explicit integrals which may be computed with high accuracy. If we denote by \( L_N \) this maximum value for various values of \( N \), then we found \( L_N \approx 0.1963, 0.2052, 0.2094, 0.2114 \) for \( N = 5, 10, 20, 40 \), and extrapolating \( N \to \infty \) gives \( L \approx 0.213 \). Thus, although we cannot quite regard it as a theorem, it seems quite clear that for \( l = 0, 1 \), \( \{1, \phi_l(\sqrt{\mu_{l,n}})\}_{n \geq 1} \) is complete for the choice \( \mu_{l,n} = \lambda_{l,n,0} \), and in fact a much larger variation of \( \mu_{l,n} \) from the base value \((n + \frac{1}{2})^2 \pi^2 \) can be allowed without losing completeness.

6. UNIQUENESS FOR THE CASE \( l = 0, 2 \)

Next we consider the case when the two indices are \( l_1 = 0, l_2 = 2 \).

PROPOSITION 6.1. The set \( \Phi_{0,2} \) is complete in \( L^2(0,1) \). In particular, the operator (2.15) with \( A = A_{0,2} \) and \( \Lambda_{l,n,0} \) replaced with \((n + \frac{1}{2})^2 \pi^2 \) is one to one on \( \{\xi \in L^2(0,1); \int_0^1 \xi = 0\} \).

Proof. Suppose that \( \xi \in L^2(0,1) \) is orthogonal to \( \Phi_{0,2} := \{1, \phi_0(n \pi x), \phi_2(n + \frac{1}{2}) \pi x\}, n = 1, 2, \ldots \). Then, as before, \( \xi \) must be odd and, by Lemma 3.2, \( T_2[\xi] \) must be orthogonal to \( \cos 2(n + 1) \pi x \) for \( n = 1, 2, \ldots \). Thus \( T_2[\xi] \) must be a linear combination of an odd function, a constant, and \( \cos 2 \pi x \); that is,
\[ \zeta(x) + 12x \int_x^1 \frac{\xi(t)}{t^2} dt - 24x^3 \int_x^1 \frac{\xi(t)}{t^3} dt = \chi(x) + \epsilon_0 + \epsilon_2 \cos 2 \pi x, \] where \( \chi(x) = -\chi(1 - x) \) and \( \epsilon_0 \) and \( \epsilon_2 \) are real constants.

From (6.1) and the fact that \( T_2^* [1] = -1 \), it immediately follows that \( \epsilon_0 = \langle T_2[\zeta], 1 \rangle = \langle \zeta, T_2^* [1] \rangle = -\langle \zeta, 1 \rangle = 0 \). Taking limits in (6.1) gives
\[ 5\zeta(0) = \chi(0) + \epsilon_2, \quad \zeta(1) = \chi(1) + \epsilon_2, \quad \zeta'(0) = \zeta'(1) = 0, \] from which we conclude that \( \zeta(0) = -\zeta(1) = \epsilon_2 / 2 \) and consequently that \( \int_0^1 \zeta(x) dx = -\epsilon_2 \).
Suppose that a nontrivial solution triple \((\zeta, \chi, \epsilon_2)\) of (6.1) exists. If we differentiate four times, we obtain
\[
\zeta'''' + \frac{12\zeta'''}{x} + \frac{24\zeta''}{x^2} - \frac{24\zeta'}{x^3} = \chi'''' + (2\pi)^4\epsilon_2 \cos(2\pi x). \tag{6.3}
\]
Since \(\zeta''''\) and \(\chi''''\) are also odd, we may project this equation onto the subspace of even functions to eliminate \(\chi\). If we again set \(y(x) = \zeta'(x)\) then the result is
\[
(x^2(1-x)^2 )y' + (6x - 6x^2 - 2)y = \frac{4}{3}\epsilon_2 \pi^4 x^3(1-x)^3 \cos(2\pi x), \tag{6.4}
\]
with
\[
y(0) = y(1) = 0, \quad \int_0^1 y(x) dx = -\epsilon_2. \tag{6.5}
\]
Again, since (6.4) is a linear equation, we may assume that \(\epsilon_2 > 0\). Equation (6.5) shows that \(y\) must have a negative minimum at some \(x_0\) in \([\frac{1}{2}, 1]\). But at such a point (6.4) implies that
\[
(6x - 6x^2 - 2)y(x) - \frac{4}{3}\pi^4 x^3 (1-x)^3 \cos(2\pi x) \leq 0,
\]
so that
\[
y(x) \geq \frac{4\epsilon_2 \pi^4 x^3 (1-x)^3 \cos(2\pi x)}{3(6x - 6x^2 - 2)} \tag{6.6}
\]
at \(x = x_0\). But minimizing the right-hand side of this over \([\frac{1}{2}, 1]\) gives \(y(x) \geq c_0\) where \(c_0 \approx -0.18\epsilon_2\), which contradicts (6.5). This shows that \(y(x) = 0\) and so \(\zeta\) is a constant. The condition \(\int_0^1 \zeta(x) dx = 0\) then shows that \(\zeta = 0\).

7. UNIQUENESS FOR OTHER \(l\) VALUES

We will obtain a completeness result in several more cases, but first we derive some general formulas. First consider the case where \(l_1 = l\) and \(l_2 = l + 1\). The parity of \(l\) makes some difference, so consider first the case when it is odd. The equations coming from (4.7) are
\[
T_l[\zeta](x) = f(x) := \chi_e(x) + \sum_{k=1}^l \epsilon_k \cos k \pi x, \tag{7.1}
\]
\[
T_{l+1}[\zeta](x) = g(x) := \chi_o(x) + \sum_{k=0}^{l+1} \epsilon_k \cos k \pi x,
\]
where \(\epsilon_k \in \mathbb{R}\) for \(k = 0, 1, \ldots, l + 1\), \(\chi_e\) is even, and \(\chi_o\) is odd. On account of Lemma 3.4 it is sufficient to show that \(\epsilon_k = 0\) for \(k = 0, 1, \ldots, l + 1\).
Our primary tools in this task are Eq. (3.4) and the information on the nullspaces of $T_j$ as provided in Lemma 3.2.

Since $T^*_{i+1}[1] = -1$ and $\zeta$ has mean value zero, $\epsilon_0 = -\langle T_{i+1}[\zeta], 1 \rangle = -\langle \zeta, T^*_{i+1}[1] \rangle = \langle \zeta, 1 \rangle = 0$, and similarly $\langle f, 1 \rangle = \langle g, 1 \rangle = 0$. Clearly $S_{i+1} \{ f(x) = g(x) \}$; by Lemma 3.2 we have the orthogonality conditions

$$\langle f, x^{2n} \rangle = 0, \ n = 0, \ldots, l, \quad \langle g, x^{2n} \rangle = 0, \ n = 0, \ldots, l + 1, \quad (7.2)$$

and from (34) we have the coupling equations between $f$ and $g$,

$$\langle x^{2n-1}, g \rangle = \langle x^{2n-1}, S_{i+1}[f] \rangle = \frac{2(n-l) - 3}{2(n+l) + 1} \langle x^{2n-1}, f \rangle. \quad (7.3)$$

The decompositions

$$x^{2n-1} = \sum_{j=1}^{n} a_{nj} \left( x - \frac{1}{2} \right)^{2j} + \sum_{j=0}^{n} b_{nj} x^{2j}$$

$$= \sum_{j=1}^{n} c_{nj} \left( x - \frac{1}{2} \right)^{2j-1} + \sum_{j=0}^{n-1} d_{nj} x^{2j}, \quad (7.4)$$

for values of $a_{nj}$, $b_{nj}$, $c_{nj}$, and $d_{nj}$ that can be recursively computed, are easily verified. Define $I_{n,k} := \int_0^1 (x - \frac{1}{2})^n \cos k \pi x \, dx$. For each value of $n$, $1 \leq n \leq l + 1$, we have

$$\int_0^1 x^{2n-1} f(x) \, dx = \sum_{j=1}^{n} c_{nj} \int_0^1 \left( x - \frac{1}{2} \right)^{2j-1} f(x) \, dx + \sum_{j=0}^{n-1} d_{nj} \int_0^1 x^{2j} f(x) \, dx$$

$$= \sum_{j=1}^{n} c_{nj} \int_0^1 \left( x - \frac{1}{2} \right)^{2j-1} \chi_e(x) \, dx + \sum_{k=1 \ odd}^{l} \epsilon_k \sum_{j=1}^{n} c_{nj} I_{2j-1, k}, \quad (7.5)$$

$$\int_0^1 x^{2n-1} g(x) \, dx = \sum_{j=1}^{n} a_{nj} \int_0^1 \left( x - \frac{1}{2} \right)^{2j} g(x) \, dx + \sum_{j=0}^{n} b_{nj} \int_0^1 x^{2j} g(x) \, dx$$

$$= \sum_{j=1}^{n} a_{nj} \int_0^1 \left( x - \frac{1}{2} \right)^{2j} \chi_o(x) \, dx + \sum_{k=2 \ even}^{l+1} \epsilon_k \sum_{j=1}^{n} a_{nj} I_{2j, k}$$

$$= \sum_{k=2 \ even}^{l+1} \epsilon_k \sum_{j=1}^{n} a_{nj} I_{2j, k}. \quad (7.6)$$
Set $A_{n,k} := \sum_{j=1}^{n} a_{nj} I_{2j,k}$ and $C_{n,k} := \sum_{j=1}^{n} c_{nj} I_{2j-1,k}$ so that from (7.3) we have the $(l + 1) \times (l + 1)$ system for the $\epsilon_{k}^{l+1}$.

\[
\sum_{k \text{ even}}^{l+1} A_{n,k} \epsilon_{k} - 2(n-l-2) \frac{1}{2(n+l)+1} \sum_{k \text{ odd}}^{l} C_{n,k} \epsilon_{k} = 0, \quad n = 1, \ldots, l + 1. \quad (7.7)
\]

We have not been able to show that the system of equations (7.7) is uniquely invertible for all $l$, although one can check this for any specific value.

For $l = 1$ we obtain the equation $\epsilon_{0} = 0$ and the pair $12 \epsilon_{1} + 5 \epsilon_{2} = 0$ and $16 \epsilon_{1} + 7 \epsilon_{2} = 0$, from which it readily follows that $\epsilon_{1} = \epsilon_{2} = 0$. For the case $l = 3$, then in (7.7) we obtain a $4 \times 4$ homogeneous system for $\epsilon_{1}, \ldots, \epsilon_{4}$, and one can verify that the determinant of the coefficient matrix is nonzero. (After a calculation using symbolic manipulation we found that the determinant of the coefficient matrix is $78365/13343616 \pi^{30}$.) Thus uniqueness also holds in this case.

The case when $l$ is even is slightly more difficult for although we still have the conditions (7.2) we must exchange the roles of $f$ and $g$ in (7.5) and (7.6), and then (7.6) is only valid as stated for $n \leq l$. This means that the corresponding system (7.7) is underdetermined, providing only $l$ equations for the $l + 1$ unknowns $\epsilon_{1}, \ldots, \epsilon_{l+1}$ ($\epsilon_{0} = 0$ as always). We can, however, attempt to obtain one further relation by finding an element $v$ in the nullspace of $P_{0}S_{l+1}P_{o}$ as we did in (5.6). If such a $v$ can be found then it follows from (7.1) with odd and even reversed that it must be orthogonal to

\[
\psi(x) = \sum_{k \text{ odd}}^{l+1} \epsilon_{k} \cos(k \pi x) + \sum_{k \text{ even}}^{l} \epsilon_{k} P_{0} S_{l+1} [\cos(k \pi x)], \quad (7.8)
\]

providing an additional homogeneous linear equation for $\epsilon_{1}, \ldots, \epsilon_{l+1}$. Again, we have no general argument that this linear system is nonsingular.

In the case $l = 2$, for example, the two equations we get from (7.7) are

\[
252 \epsilon_{1} + 45 \epsilon_{2} + 28 \epsilon_{3} = 0, \quad 12 \epsilon_{1} - \frac{1}{4} \epsilon_{2} + \frac{4}{27} \epsilon_{3} = 0. \quad (7.9)
\]

By using the form of the solution in (5.6) as a guide, we can verify that the function

\[
u(x) = \left( z^3 - \frac{9}{22} z^2 + \frac{1}{22} z - \frac{1}{924} \right) \log \left( \frac{x}{1-x} \right) + \left( x - \frac{1}{2} \right) \left( z^2 - \frac{8}{33} z + \frac{7}{660} \right), \quad z = x(1-x), \quad (7.10)\]

is an element in the null space of $P_{0} S_{l+1} P_{o}$. The condition $\int_{0}^{1} \nu(x) \psi(x) \, dx = 0$ is, by direct calculation, equivalent to

\[
I_{1} \epsilon_{1} + I_{2} \epsilon_{2} + I_{3} \epsilon_{3} = 0, \quad (7.11)
\]

where $I_{1} \approx 0.047356$, $I_{2} \approx -0.000689$, and $I_{3} \approx -0.073139$. 

---

Rundell and Sacks, 370
It is easily verified that \( \epsilon_1 = \epsilon_2 = \epsilon_3 = 0 \) is the only solution to (7.9) and (7.11).

We can summarize our findings on the conjecture in

**Proposition 7.1.** For the case \((l, l + 1)\) for \(l = 0, 1, 2, 3\), the sets \(\Phi_{l, l+1}\) are complete in \(L^2(0, 1)\). In particular, the operator (2.15) with \(\Lambda = \Lambda_{l, l+1}\) and \(\lambda_{l, n, 0}\) replaced with \((n + \frac{1}{2})^2 \pi^2\) is one to one on \(\{\xi \in L^2(0, 1): \int_0^1 \xi = 0\}\).

We now briefly consider the case \(l_2 - l_1 = 2\) for there is considerable overlap with the previous situation. If we assume \(l\) is odd then (7.1) becomes

\[
T_l[\xi](x) = f(x) := \chi_e(x) + \sum_{k=1 \text{ even}}^l \epsilon_k \cos k \pi x,
\]

\[
T_{l+2}[\xi](x) = g(x) := \tilde{\chi}_e(x) + \sum_{k=1 \text{ odd}}^{l+2} \tilde{\epsilon}_k \cos k \pi x,
\]

where \(\chi_e\) and \(\tilde{\chi}_e\) are even. There is an obvious reversal of parities in these quantities if \(l\) is even. There are a total of \(l + 2\) constants \(\epsilon_k, \tilde{\epsilon}_k\) in these equations and thus the maximum dimension of the nullspace is \(l + 2\). Of course there are constraints similar to (7.2) and (7.3) from which we can obtain relations between the values of \(\epsilon_k\) and \(\tilde{\epsilon}_k\) and thus reduce the dimension of this null space.

We considered the \((0, 2)\) case in the previous section and for the \((1, 3)\) case we can argue as follows. The relevant equations are

\[
S_1S_2[f] = g, \quad f(x) = \chi_e(x) + \epsilon_1 \cos \pi x,
\]

\[
g(x) = \tilde{\chi}_e(x) + \eta_1 \cos \pi x + \eta_3 \cos 3 \pi x.
\]

The orthogonality relations (7.2) are

\[
\langle f, 1 \rangle = \langle f, x^2 \rangle = 0, \quad \langle g, 1 \rangle = \langle g, x^2 \rangle = \langle g, x^4 \rangle = \langle g, x^6 \rangle = 0.
\]

If we use \(\chi\) to denote either \(\chi_e\) or \(\tilde{\chi}_e\) then \(\int_0^1 \chi(x) \, dx = 0\) and since \(\chi\) is even, \(\int_0^1 x \chi(x) \, dx = \int_0^1 (x - \frac{1}{2}) \chi(x) \, dx = 0\). Thus (3.4) with \(n = 1\) gives

\[
\langle x, \cos \pi x \rangle \eta_1 + \langle x, \cos 3 \pi x \rangle \eta_3 = \frac{1}{2} \langle x, \cos \pi x \rangle \epsilon_1
\]

or

\[
\eta_1 + \frac{1}{6} \eta_3 - \frac{3}{7} \epsilon_1 = 0.
\]

From (7.5) we obtain \(\langle x^3, \chi \rangle = \frac{3}{2} \langle x^2, \chi \rangle\) which, in view of the orthogonality relations, is equal to \(-\frac{3}{2} \langle x^2, \cos \pi x \rangle \epsilon_1\) in the case of \(\chi_e\) and \(-\frac{3}{2} \langle x^2, \cos 3 \pi x \rangle \eta_1 + \langle x^2, \cos 3 \pi x \rangle \eta_3\) in the case of \(\tilde{\chi}_e\). Using (3.4) with \(n = 3\) gives \(\langle g, x^3 \rangle = \frac{1}{21} \langle f, x^3 \rangle\) and combining these relations we obtain

\[
\eta_1 + \frac{1}{81} \eta_3 - \frac{1}{21} \epsilon_1 = 0.
\]
The homogeneous equations (7.15) and (7.16) are of course insufficient to show that $\epsilon_1$, $\eta_1$, and $\eta_3$ are all zero. A third equation would have to be provided by one of the approaches used for the $(0, 1)$ and $(0, 2)$ cases.

8. NUMERICAL EXAMPLES

We now describe a computational method for the inverse spectral problem, based on the analysis of the earlier sections. The specific approach is very much analogous to that used for the regular inverse Sturm–Liouville problem in [LPR]. We do not claim that it is a practical method in the simple form presented here (among other things, a practical method should use more than two $l$ values), but it is a reasonable indication that it may be possible to develop a constructive method based on properties of the linearized mappings we have been studying. Furthermore, we can easily illustrate some expected ill-posedness in the numerical examples.

Let us suppose that data consisting of a finite set of eigenvalues $\{\lambda_{l,n}\}$ are given for $(l, n) \in \Lambda = \{(l, n): l = l_1, l_2, 1 \leq n \leq N\}$. As discussed in Section 2, we may then seek to solve the inverse spectral problem by obtaining an approximate solution of

$$F_{\Lambda}(q, \lambda_{l,n}) = 0,$$

(8.1)

where $F_{\Lambda}$ is defined in (2.13).

Of course the solution will no longer be unique, so we will seek the unknown $q$ in a finite-dimensional set, assuming in particular that it is representable by the basis expansion

$$q(x) = \sum_{m=1}^{M} c_m v_m(x)$$

(8.2)

for some set of basis functions $\{v_m(x)\}_{m=1}^{M} \in L^2[0, 1]$. Since by preliminary calculation we reduce to the case that $\int_0^1 q(x) \, dx = 0$, it is natural to choose the basis functions with this property also. Clearly we should take $M \leq 2N$ and we may regard $F_{\Lambda}$ as a function of $c_1, \ldots, c_M$ when convenient to do so. Let $\delta$ be a $2N \times M$ matrix approximating $D_q F_{\Lambda}(q_k, \lambda_{l,n})$. We seek to solve the equation (8.1) by some form of Newton iteration,

$$\delta \delta_q = F_{\Lambda}(q_k, \lambda_{l,n}), \quad q_{k+1} = q_k - \delta_q.$$  

(8.3)

The specific approximation we will use is

$$\delta = D_q F_{\Lambda}(0, \lambda_{l,n,0})$$

(8.4)

for which an explicit expression is given in (2.15). We might also replace $\lambda_{l,n,0}$ in (2.15) with either the exact eigenvalue $\lambda_{l,n}$ or by the leading term
in the eigenvalue asymptotics, namely \((n + \frac{1}{2})^2 \pi^2\), and the end result would be virtually the same. If \(M < 2N\), \(\delta_q\) is understood to be the solution of the above system in a suitable least-square sense. As in the regular Sturm–Liouville inverse problem, there is little computational advantage to using a full Newton scheme ([RS]).

Evaluation of the nonlinear forward map requires the computation of the solution \(\Psi(x, \lambda, q)\) of (1.3) as described in Section 2. The most convenient way to do this is by using the transformation \(\psi(x) = x^{l+1}y(x)\) leading to the integral equation

\[
y(x) = 1 + \frac{1}{2l + 1} \int_0^x t(1 - (t/x)^{2l+1})(q(t) - \lambda)y(t) \, dt, \quad (8.5)
\]

on which is easily solved by standard numerical schemes for \(y(x)\). We use this form to evaluate the map \(F\) in our numerical reconstructions. In the examples below, a numerical solution of (8.5) was accomplished by using Simpson’s rule with the 3/8 rule for the endpoint panels.

Computing the increment \(\delta q\) then requires the solution of a linear system with coefficient matrix \(\mathcal{A}\). Although the analysis of the preceding sections proves or suggests that \(\mathcal{A}\) will be nonsingular, in practice it is always rather ill-conditioned, so that some regularization will always be necessary.

Assuming the eigenvalue data is of high accuracy, we can invert (8.3) using a singular value decomposition provided that \(2N \geq M\). If \(M\) is sufficiently small then we are effectively regularizing by spectral cutoff and no further regularization is required. With increasing error in the data we can simply delete all singular values of \(\mathcal{A}\) less than a certain amount (depending on the noise level) and this in effect automatically reduces the basis size \(N\) to accommodate the additional error. An alternative is Tikhonov regularization in which \(\delta q\) is obtained by solving \((\mathcal{A}^*\mathcal{A} + \alpha \mathcal{E})\delta q = \mathcal{A}^*F(\lambda_k, \lambda_{l,n})\).

Here \(\alpha\) is a regularization parameter to be chosen with respect to the estimated noise level (more noise requires a larger \(\alpha\)) and \(\mathcal{E}\) is a matrix that depends on the structure of the function \(q\) to be recovered. Typically one takes \(\mathcal{E}\) to be the identity, although other choices are possible [K].

As usual in methods relying on a basis representation for the unknown, there are some ad hoc elements about the choice of the basis and some expectation that the condition numbers of the resulting matrix \(\mathcal{A}\) will depend on the choice. When we used either a trigonometric basis set or a spline basis we noted that the singular values of the matrix \(\mathcal{A}\) decreased more rapidly with larger values of \(l\). This is to be expected since from (2.12) the kernel depends on the term \(x^l\), and indeed the decay of the singular values of \(\mathcal{A}\) is much more rapid than in the regular inverse Sturm–Liouville problem. The usual “folk theorem” here is that the optimal choice from the point of view of best conditioning of the inversion process is to use eigenfunctions of the underlying differential operator. For the regular
inverse Sturm–Liouville problem this amounts to choosing a trigonometric basis set (solutions of the base equation \(-u'' = \lambda u\)) ([RS]). In the singular case this would involve using a basis set consisting of spherical Bessel functions with the order dictated by the values of \(l\) being used. The gain here is in general illusionary; a basis set consisting of the constant function and spherical Bessel functions of order \(l\) (where \(l\) is the highest of the orders used) will give a poor representation for any function with features near the origin. This just illustrates the point that in computing an “effective potential” \(l(l + 1)/x^2 + q(x)\) from (1.3), the known singular term dominates the unknown part near the origin.

The figures show numerical reconstructions using the above techniques. In each case synthetic numerically-generated data \(\lambda_{n,l}\) were obtained using an eigenvalue solver adapted for the singular case [BEZ], and a certain level of uniformly random noise was added. Just how this noise was added makes a considerable difference to the inversion process.

In reconstructions in Fig. 1 we subtracted the leading terms of the asymptotic expansion \((n + \frac{1}{2})^2 \pi^2 - l(l + 1)\) as indicated by (2.11). To the remainder we added 5% random error, effectively, to the sequence \(r_{l,n}\). It could
be argued that the “true data” are the eigenvalues themselves and that this
construct is artificial. Indeed, we are in actuality removing a substantial part
of the ill-conditioning in the problem by ensuring that the dominant term,
as distinct from the term holding information about the unknown potential,
is error free. A similar situation holds in the non singular case; see [RS].

A cosine basis, \( v_n(x) = \cos(n-1)\pi x \), was used and we took \( N = M = 10 \). Regularization was by the Tikhonov method and the figures show the
reconstructions using four pairs of \( l \) values. With this noise level, effective
numerical convergence was obtained with about six iterations in each case.
The general features we point out are that in general smaller \( l \) values give
better reconstructions and this is particularly noticeable near the endpoint
\( x = 0 \); this is exactly as should be expected from the above discussion.
The figures shown were taken from the median values (as measured by the
difference in the \( L^2 \) norm of \( q \)) of the results of 10 experiments each with
a different random vector added to the residual part \( r_{l,n} \) of the eigenvalue
sequences.

The two reconstructions in Fig. 2 consider the \( l = 0, 1 \) case except that
the noise is now added directly to the eigenvalues themselves. The level
of error was 0.5%, this being the maximum level for which we could reli-
ably obtain reconstructions. The figure on the left uses data consisting of
only the first 5 eigenvalues in each sequence; that on the right uses the
first 10 eigenvalues, as in Fig. 1. Fewer data here give a superior recon-
struction since an error \( \epsilon \) in \( \lambda_{5,l} \) is approximately \((5 + l/2)^2 \pi^2 \epsilon \) and this
is in effect the error transferred to the information-holding term \( r_{5,l} \). For
the same percentage error in \( \lambda_{10,l} \), considerably more error is transferred
to the corresponding term \( r_{10,l} \). Clearly, as more eigenvalues are included
in this process the farther removed from an \( l^2 \) sequence the information-
holding term \( r_{n,l} \) will appear. This effect will become more marked with
higher values of \( l \).
APPENDIX A: PROOF OF PROPOSITION 2.1

First note that the general solution of (1.3) when \( q \equiv 0 \) is
\[
\psi(x) = x(C_1 j_l(\sqrt{\lambda}x) + C_2 y_l(\sqrt{\lambda}x)).
\]  
(A.1)

The spherical Bessel functions have small \( x \) behavior,
\[
j_l(x) = \frac{x^l}{(2l + 1)!!} + O(x^{l+2}),
\]
\[
y_l(x) = -((2l - 1)!!x^{-l-1} + O(x^{-l+1})) \quad x \to 0,
\]

(A.2)

where \((2n + 1)!! = 1 \cdot 3 \cdot 5 \cdots (2n + 1)\). To satisfy (1.5) we must have \( C_2 = 0 \), and so by Eq. (1.4) the eigenvalues are characterized as the positive solutions of \( j_l(\sqrt{\lambda}) = 0 \); i.e., \( j_l(\sqrt{\lambda_{l,n,0}}) = 0 \).

Now for any \( q \) we can choose a fundamental set of solutions \( \Psi_1 = \Psi_1(x, \lambda, q), \Psi_2 = \Psi_2(x, \lambda, q) \) of (1.3) with the normalizations
\[
\lim_{x \to 0} \frac{\Psi_1(x, \lambda, q)}{x^{l+1}} = 1 \quad \lim_{x \to 0} \frac{\Psi_2(x, \lambda, q)}{x^{l+1}} = 1.
\]  
(A.3)

In particular, we then have
\[
\Psi_1(x, \lambda, 0) = \frac{(2l + 1)!!x_j(\sqrt{\lambda}x)}{\lambda^l} \quad \Psi_2(x, \lambda, 0) = -\frac{\lambda^{l+1/2}x_l(\sqrt{\lambda}x)}{(2l - 1)!!}
\]  
(A.4)

and
\[
\Psi_1(1, \lambda_{l,n}, q) = 0 \quad l = 0, 1, 2, \ldots, n = 1, 2, \ldots.
\]  
(A.5)

Define \( w(x, \lambda, q) = (D_x \Psi_1)(\zeta)(x, \lambda, q) = \frac{d}{dx} \Psi_1(x, \lambda, q + t\zeta)|_{t=0} \), so that
\[
w'' + \left( \lambda - \frac{l(l + 1)}{x^2} - q \right) w = \Psi_1\zeta, \quad 0 < x < 1, \quad \text{so that}
\]
\[
\lim_{x \to 0} \frac{w(x, \lambda, q)}{x^{l+1}} = 0.
\]  
(A.6)

(A.7)

Specializing to the case \( q = 0 \) and using (A.4) give
\[
w'' + \left( \lambda - \frac{l(l + 1)}{x^2} \right) w = \frac{(2l + 1)!!x_j(\sqrt{\lambda}x)\zeta(x)}{\lambda^l}.
\]  
(A.8)

We claim that
\[
w(1, \lambda, 0) = \frac{(2l + 1)!!}{\lambda^{l+1/2}} \int_0^1 x^l j_l(\sqrt{\lambda}x) y_l(\sqrt{\lambda}x) \left[ \sqrt{\lambda} j_l(\sqrt{\lambda}x) \right] \zeta(x) \, dx.
\]  
(A.9)
To see this, we make use of the fundamental set \( \{ \Psi_1(x, \lambda, 0), \Psi_2(x, \lambda, 0) \} \) to represent the solution \( w \). The identity \( j_1(x)y_1(x) - j_0(x)y_1(x) = x^{-2} \) ([AS]) shows that the Wronskian is \( W(\Psi_1, \Psi_2)(x) = -(2l + 1) \) for \( q = 0 \) and any \( \lambda \). Substituting \( u(x) = a_1(x)\Psi_1(x, \lambda, 0) + a_2(x)\Psi_2(x, \lambda, 0) \), the usual variation-of-parameters technique leads to

\[
a'_1(x) = -\frac{\Psi_1(x, \lambda, 0)\Psi_2(x, \lambda, 0)\xi(x)}{W(\Psi_1, \Psi_2)(x)}
= -\sqrt{x^2}j_1(\sqrt{x})y_1(\sqrt{x})\xi(x)
\tag{A.10}
\]

and

\[
a'_2(x) = \frac{\Psi_1^2(x, \lambda, 0)\xi(x)}{W(\Psi_1, \Psi_2)(x)} = -\frac{(2l + 1)!!x^2j_1^2(\sqrt{x})\xi(x)}{(2l + 1)\lambda}.
\tag{A.11}
\]

The asymptotic behaviors \( a'_1(x) = O(x) \) and \( a'_2(x) = O(x^{2l+2}) \) as \( x \to 0 \) show that \( a_1, a_2 \) are continuous on \([0, 1]\). The condition (A.7) can then only be satisfied if \( a_1(0) = a_2(0) = 0 \). Thus we obtain \( a_1(1), a_2(1) \) by integration of (A.10) and (A.11). Then \( w(1, \lambda, 0) = a_1(1)\Psi_1(1, \lambda, 0) + a_2(1)\Psi_2(1, \lambda, 0) \) yields (A.9) after substitution of (A.4) and some algebraic simplification. Finally, substitute \( \lambda = \lambda_{l, n, 0} \) to get (2.15) with

\[
c_{l, n} = \frac{(2l + 1)!!y_1(\sqrt{\lambda_{l, n, 0}})}{\lambda^{(l-n)/2}_{l, n, 0}},
\tag{A.12}
\]

which must be nonzero since \( j_1(\sqrt{\lambda_{l, n, 0}}) = 0 \) and \( j_1, y_1 \) cannot simultaneously vanish.

**APPENDIX B: PROOFS OF LEMMAS 3.1–3.4**

**Proof of Lemma 3.1.** Parts (i) and (ii) are easily verified by direct computation. For part (iii), if we define the function \( h \) and the operator \( \mathcal{B} \) by \( h = \mathcal{B}g := x^{-2l}f_0^x s^{2l-1}g(s) \, ds \), then since \( x^{-2l}f_0^x s^{2l-1}g(s) \, ds \leq x^{-1}f_0^x |g(s)| \, ds \), it follows that \( \|\mathcal{B}g\| \leq \|x^{-1}f_0^x |g(s)| \, ds \| \|x\|_2 \) and the Hardy inequality applied to the last term gives \( \|\mathcal{B}g\|_2 \leq 2\|g\|_2 \). This is sufficient for our present purposes, although the estimate is far from the best possible. It can be improved to \( \|\mathcal{B}\| \leq 2/(4l - 1) \). For \( \alpha > -\frac{1}{2}, x^\alpha \) is an eigenfunction of \( B \) with eigenvalue \( 1/(2l + \alpha) \). Therefore we must have \( \|\mathcal{B}\| = 2/(4l - 1) \), showing that this estimate on \( \|\mathcal{B}\| \) is sharp. It follows that \( S^*_i = I - 4lB \) is a bounded operator and hence \( S^*_i \) is also bounded.

To prove (iv), suppose that \( S^*_i[f] = h \). Setting \( \tilde{f}(x) = x^{-2l}f(x) \) and \( \tilde{h}(x) = x^{-2l}h(x) \) we obtain \( \tilde{f}(1) = \tilde{h}(1) \) and the equation \( x\tilde{h} - \tilde{f} + 4l \int_x^1 \tilde{f} \, ds = 0 \). Differentiating gives \( xf'(x) + (4l + 1)f(x) = x\tilde{h}(x) + \tilde{h}(x) \) or \( f(1) - x^{4l+1}f = \int_x^1 s^{4l+1}\tilde{h}(s) \, ds + \int_x^1 s^{4l}\tilde{h}(s) \, ds \). Integrating by parts
gives \( x^{4l+1} f(x) - x^{4l+1} h(x) = 4l \int_s^1 s^{4l} h(s) \, ds \) or by converting to the original functions \( x^{2l+1} f(x) = x^{2l+1} h(x) + 4l \int_s^1 s^{2l} h(s) \, ds \). From this it follows that if \( h = 0 \) then \( f = 0 \). This shows that \( S_l \) is one to one and also verifies (3.3).

For part (v), direct computation shows that \( S_l^*[x^n] = (n-2l)/(n+2l)x^n \) for each \( n > -\frac{1}{2} \). This verifies (3.4) and also shows that \( x^{2l} \) is in the null space of \( S_l^* \). Suppose that \( g(x) \) is also in the null space so that \( h(x) := x^{2l-1} g(x) \) satisfies \( h(x) = (4l/x) \int_0^1 h(s) \, ds \). From this it follows that \( h'(x) = \frac{4l-1}{2} h(x) \). The general solution of this equation is \( h(x) = C x^{4l-1} \). Thus \( g(x) = C x^{2l} \). This shows the null space of \( S_l^* \) consists of exactly the function \( x^{2l} \).

Next, the spherical Bessel functions obey the recursion relations ([AS, 10.1])

\[
x_j = (l-1)j_{l-1} - x_{j_{l-1}}, \quad x_{j_{l-1}} = (l+1)j_l + x_j.
\] (B.1)

Inserting these two relations into the identity \( x_j j_{l-1} = x_j j_{l-1} \) gives

\[
j_{l-1}((l-1)j_{l-1} - x_{j_{l-1}}) = j_l((l+1)j_l + x_j),
\] (B.2)

from which we obtain \( x(j_j + j_{l-1} j_{l-1}) + (l+1)j_l^2 = (l-1)j_{l-1}^2 \). This last equation can be written in the form

\[
x^{2l+2}(2j_j + 2j_{l-1}j_{l-1}) + (2l + 2)x^{2l+1}(j_l^2 + j_{l-1}^2) = 4l x^{2l+1}j_{l-1}^2
\] (B.3)

or equivalently as \( (x^{2l+1}(j_l^2 + j_{l-1}^2))' = 4l x^{2l+1}j_{l-1}^2 \). We integrate this to obtain

\[
x^{2l+2}(j_l^2 + j_{l-1}^2) = 4l \int x^{2l+1}j_{l-1}^2(s) \, ds.
\] (B.4)

From the definition of \( S_l^* \) and \( \phi_l = x^2 j_l^2 \) the above equation is equivalent to

\[
x^{2l}(\phi_l + \phi_{l-1}) = 4l \int x^{2l-1} \phi_{l-1}(s) \, ds.
\] (B.5)

This is precisely the statement that \( \phi_l = -S_l^*[\phi_{l-1}] \), which shows (vi).

To show the final part we differentiate \( S_l[f] = g \) once to obtain

\[
f'(x) + \frac{4l}{x} f(x) - 4l(2l-1)x^{2l-2} \int_x^1 \frac{f(s)}{s^{2l}} \, ds = g'(x).
\] (B.6)

Eliminating the integral term between this and the original equation gives the relation

\[
x f'(x) + (2l+1)f(x) = x g'(x) - (2l-1)g(x),
\] (B.7)

and after differentiating \( k - 1 \) times we obtain

\[
x f^{(k)}(x) + (2l+k)f^{(k-1)}(x) = x g^{(k)}(x) - (2l-k)g^{(k-1)}(x).
\] (B.8)

In particular, taking \( k = 2l \) gives (3.6).
Proof of Lemma 3.2. For \( f \in L^2[0, 1] \) and \( \mu > 0 \) we define \( f_\mu(x) = f(\mu x) \). Then by a simple change of variables in (3.2) we see that \( S^\mu_i[f] = f_\mu(x) \). This shows that (3.5) can be written as \( \phi_L(\sqrt{\lambda}x) = -S^\mu_i[f]\phi_{i-1}(\sqrt{\lambda}x) \). Then from (3.7) we get \( \phi_L(\sqrt{\lambda}x) = -T^\mu_i[f]\phi_{i-1}(\sqrt{\lambda}x) \), and by putting this together with the fact that \( S^\mu_i[1] = -1 \) we get (3.8). The boundedness and one-to-one property of \( T^\mu_i \) follows directly from (3.7) and Lemma 3.1. Since \( T^\mu_i \) is the product of \( S^\mu_1 \cdots S^\mu_n \), we also note that the null space of \( T^\mu_i \) is of dimension at most \( l \). Since each monomial \( x^n \) is an eigenfunction of every operator \( S^\mu_i \) we see that this is also true of \( T^\mu_i \) and indeed \( T^\mu_i \) will annihilate every function that each of \( S^\mu_1, S^\mu_2, \ldots, S^\mu_n \) does. This is precisely the set \( \{x^2, x^4, \ldots, x^{2l}\} \).

Proof of Lemma 3.3. We write \( g_0 = f, g_1 = S^\mu_1[f], g_{j+1} = S^\mu_j[g_j] \), and so \( g = g_0 = S^\mu_{l+m}[g_{m-1}] \). From \( g_1 = S^\mu_1[g_0] \) it follows that \( g_1 \) and hence by extension \( g_j \in C^\infty(0, 1] \). The continuity of the derivatives at \( x = 0 \) will in fact follow from the results below.

From (B.7) with \( l \) replaced with \( l + j \) we obtain

\[
xg_j(x) + (2l + 2j + 1)g_j(x) = xg_{j+1}(x) - (2l + 2j - 1)g_{j+1}(x).
\]

and after differentiating \( k - 1 \) times we obtain for each \( j \) the formula

\[
xg_j^{(k)}(x) + (2l + 2j + k)g_j^{(k-1)}(x) = xg_{j+1}^{(k)}(x) - (2l + 2j - k)g_{j+1}^{(k-1)}(x).
\]

First note that \( g_1(1) = f(1) \), and by taking limits as \( x \to 0 \) in \( g_1 = S^\mu_1[f] \) we see that \( \lim_{x \to 0} g_1(x) = -\frac{2l+1}{2}f(0) \), and so \( g_1 \) extends to be continuous on \([0, 1]\). Continuing in this manner we have \( g(1) = f(1) \) and \( g(0) = (1)^m \frac{2l+2m+1}{2l-1}f(0) \). The only circumstances in which these pair of equations can hold under the assumption of parity on \( f \) and \( g \) are for \( f(0) = g(0) = f(1) = g(1) = 0 \). In addition, each intermediate function extends to be continuous on \([0, 1]\) and must also satisfy \( g_j(0) = g_j(1) = 0 \).

Suppose that for some \( k, g_j^{(k-1)}(0) = g_j^{(k-1)}(1) = 0 \) for \( j = 0, \ldots, m \). From repeatedly differentiating \( g_{j+1} = S^\mu_{j+1}[g_j] \) it follows that for each \( j, g_j^{(k)}(1) = g_j^{(k)}(1), \) and taking limits in (B.10) we must have \( g_j^{(k)}(0) = \frac{2l+2j+k+1}{2l+2j-k-1}g_j^{(k)}(0) \). Together these last two equations imply that \( g^{(k)}(1) = f^{(k)}(1) \) and \( g^{(k)}(0) = (1)^m \frac{2l+2m+k+1}{2l-2k+1}f^{(k)}(0) \). The parity conditions on \( f \) and \( g \) now imply that \( g^{(k)}(0) = f^{(k)}(0) = 0 \) and \( g^{(k)}(1) = f^{(k)}(1) = 0 \). The conclusion now follows by induction.

Proof of Lemma 3.4. For the case \( m = 0 \) if \( f \) and \( g \) have the same parity then it follows by projecting (3.6) onto the subspace of opposite parity that \( f^{(2l-1)}(x) = 0 \). In this situation \( f \) is a polynomial and the facts that \( f \)}
hence $g$ equal 0 follow from Lemma 3.3. If $f$ and $g$ have opposite parity then we set $Y(x) = f^{(2l-1)}(x)$, and so from (3.6) we get

$$Y'(x) + \frac{4l}{x}Y(x) = g^{(2l)}(x). \quad \text{(B.11)}$$

Projecting onto the subspace of $L^2(0, 1)$ with opposite parity to $g$ then yields

$$Y'(x) + \left(\frac{2l}{x} - \frac{2l}{1-x}\right)Y(x) = 0. \quad \text{(B.12)}$$

The general solution of (B.12) can be computed; it is $Y(x) = C[x(1-x)]^{-2l}$ for some constant $C$. In order for $f \in L^2$, $C$ must be zero and thus $f$ must be a polynomial of degree $p \leq 2l-2$, and the conclusion again follows from Lemma 3.3.

For the case $m = 1$, by applying (3.6) successively with $l$ and $l+1$ we obtain

$$g^{(2l+2)} = f^{(2l+2)} + 4(2l + 1)\left(\frac{1}{x}f^{(2l+1)} + \frac{2l}{x^2}f^{(2l)} - \frac{2l}{x^3}f^{(2l-1)}\right). \quad \text{(B.13)}$$

The case of interest is when $f$ and $g$ share the same parity, and if we project this equation onto the subspace of $L^2(0, 1)$ of opposite parity to $f$ and set $Y(x) = f^{(2l-1)}(x)$ we get

$$Y''(x) + 2l \frac{1 - 2x}{x(1-x)}Y'(x) - 2l \frac{3x^2 - 3x + 1}{x^2(1-x)^2}Y(x) = 0. \quad \text{(B.14)}$$

Equation (B.14) has $x = 0, 1$ as regular singular points, and we may compute that the roots of the corresponding indicial equation are $-2l$ and 1. Since these differ by an integer, there exists a fundamental set $\{Y_1, Y_2\}$ in which $Y_1(x) = O(x)$ and $Y_2(x) = CY_1(x) \log x + O(x^{-2l})$ as $x \to 0$. The solution $Y_2$ cannot correspond to $f \in L^2(0, 1)$ and so must be excluded. It follows immediately that $Y(x) \equiv 0$ since otherwise $Y$ must have a positive maximum or negative minimum at a point $x_0 \in (0, 1)$ and (B.14) evaluated at $x = x_0$ would give a contradiction, since $3x^2 - 3x + 1 > 0$ on $(0, 1)$. Thus $f, g$ are polynomials, with all derivatives vanishing at $x = 0, 1$ by Lemma 3.3, and so $f, g \equiv 0$. (One could also argue directly that $f, g$ must have analytic extensions to an open interval containing $[0, 1]$ so that the conclusion would again follow from Lemma 3.3.)

The case when $m = 1$ and $f, g$ have opposite parity is not needed in what follows, so we just sketch the argument. Projecting (B.13) onto the subspace of opposite parity from $g$, we get a third-order ODE for $Y = f^{(2l-1)}$, which is again regular singular at $x = 0, 1$. Checking the indicial equation for possible behaviors at the endpoints we find the roots $1, -4l, -4l+2$. The latter
two possibilities must be excluded as too singular, so that the only admissi-
ble solution of $Y(x) = O(x)$ at $x = 0$, and similarly at $x = 1$. Thus $f, g$
must have analytic extensions to an open interval containing $[0, 1]$ and so
are zero by Lemma 3.3.

Remark. We can extend part of this argument to higher values of $m$.
In general we obtain $Y(x) = f^{(2l-1)}(x)$ as the solution of an equation with
regular singular points at $x = 0, 1$ and we can compute the roots of the
appropriate indicial equation. The roots turn out to be positive odd integer
powers plus complex roots with real part $-2l$. Specifically, there are $m$
nroots with real part $-2l$ plus $m - 1$ roots $1, 3, 5, \ldots$ in the case that $m$
is odd and $m - 1$ roots with real part $-2l$ plus $m - 1$ roots $1, 3, 5, \ldots$ in the
case that $m$ is even. However, at present we are unable to show that all
solutions which are not regular at $x = 0, 1$ must be excluded.

REFERENCES


[BEZ] P. B. Bailey, W. N. Everitt, A. Zettl, Computing eigenvalues of singular
Sturm-Liouville problems. Results Math. 20, Nos. 1–2 (1991), 391–423 [see also
www.math.niu.edu/~zettl/SL2/].

78 (1946), 1–96.

[C] R. Carlson, A Borg–Levinson theorem for Bessel operators, Pacific J. Math. 177

[CS] R. Carlson and C. Shubin, Spectral rigidity for radial Schrödinger operators,

[G] D. Gough, Comments on helioseismic inference, in “Progress of Seismology of the
Sun and Stars,” Lecture Notes in Physics, Vol. 367, pp. 283–318, Springer-Verlag,


[GR] J. C. Guillot and J. Ralston, Inverse spectral theory for a singular Sturm–Liouville


Springer-Verlag, New York, 1996.

[LPR] B. Lowe, M. S. Pilant, and W. Rundell, The recovery of potentials from finite spectral


1987.

[RS] W. Rundell and P. E. Sacks, Reconstruction techniques for classical inverse Sturm–