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Generalized comaximal factorization of ideals

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ABSTRACT

We generalize the notion of comaximal factorization of ring ideals to the language of weak ideal systems on monoids and prove several results generalizing and extending previous work. We also develop some topological methods for dealing with comaximal factorization and some related finitary weak ideal system problems.

1. Introduction

Let R be a commutative ring with $1 \neq 0$. A *comaximal factorization* of a proper ideal I of R is a product $I = I_1 \cdots I_n$ of proper ideals with $I_i + I_j = R$ for $i \neq j$, a proper ideal is called *pseudo-irreducible* if it has no comaximal factorizations other than the trivial one $I = I$, and a *complete comaximal factorization* is a comaximal factorization into pseudo-irreducibles. Some of the early work on complete comaximal factorizations was done by Noether, who proved in her monumental paper [21] that if R is Noetherian, then every proper ideal has a unique (up to order) complete comaximal factorization. Much more recently, McAdam and Swan [19] laid the foundations for a systematic study of comaximal factorization of ideals in arbitrary commutative rings with $1 \neq 0$. They proved that complete comaximal factorizations are always unique when they exist, generalized Noether's existence theorem with a weaker sufficient criterion for the existence of complete comaximal factorizations, and showed that the comaximal factorizations of an ideal are in a natural one-to-one correspondence with the comaximal factorizations of its radical.

A natural generalization of comaximal factorizations of ideals is the notion of $\star$-comaximal factorizations of $\star$-ideals, where $\star$ is a star operation. (See Section 3 for definitions concerning star operations.) A *$\star$-comaximal factorization* of a proper $\star$-ideal I of R is a $\star$-product $I = (I_1 \cdots I_n)^\star$ of proper $\star$-ideals with $(I_i + I_j)^\star = R$ for $i \neq j$, a proper $\star$-ideal is called *$\star$-pseudo-irreducible* if it has

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no nontrivial $\star$-comaximal factorizations, and a *complete $\star$-comaximal factorization* is a $\star$-comaximal factorization into $\star$-pseudo-irreducibles. Taking $\star = d$, we reduce to the definitions of comaximal factorization, pseudo-irreducible, and complete comaximal factorization, respectively. Baghdadi, Gabelli, and Zafrullah [7] did some work on complete $\star$-comaximal factorizations, but they focused on the special case of (necessarily complete) $\star$-comaximal factorizations into $\star$-ideals with prime radical, where $\star$ is a finitary star operation on an integral domain. A domain where every principal ideal (equivalently, every $\star$-finitely generated $\star$-ideal) has such a $\star$-comaximal factorization is called a *$\star$-unique representation domain* (*$\star$-URD*), and those authors proved that, if $\star$ is a finitary star operation on a domain D, then D is a $\star$-URD if and only if every principal ideal has only finitely many minimal primes and any two incomparable prime $\star$-ideals are $\star$-comaximal.

In this paper we will lay the foundations for the most general theory of comaximal factorization, one using the language of weak ideal systems on monoids. By a *monoid*, we mean a commutative multiplicative semigroup with $1 \neq 0$. If r is a weak ideal system on a monoid H, then an *r-comaximal factorization* of a proper r-ideal I is an r-product $I = (I_1 \cdots I_n)_r$ of proper r-ideals with $(I_i \cup I_j)_r = H$ for $i \neq j$, a proper r-ideal is called *r-pseudo-irreducible* if it has no nontrivial r-comaximal factorizations, and a *complete r-comaximal factorization* is an r-comaximal factorization into r-pseudo-irreducibles. (See Section 3 for definitions concerning weak ideal systems.) If H is a ring and r is a star operation, then we reduce to the definitions of the previous paragraph. We will show that a majority of the theorems of the aforementioned authors are special cases of analogous theorems in this more general setting. Although we work in this setup in order to preserve maximum generality, that is not to say that specializing our theorems to ring-theoretic language will not give new results. The following are a sample of some of the noteworthy developments from a purely ring-theoretic point of view.

It is well known that if $\star$ is a finitary star operation on an integral domain and $I_1, \ldots, I_n$ are pairwise $\star$-comaximal $\star$-ideals, then $I_1 \cap \cdots \cap I_n = (I_1 \cdots I_n)^\star$. One proof, which requires some familiarity with the $\star_w$ operation introduced by Anderson and Cook [3], is given in the introduction to [7]. Corollary 4.3 gives a more elementary proof that does not rely on a finitary assumption or on the ring being an integral domain.

In [20], the authors proved that every comaximal factorization of a principal ideal has factors of the form (a, c), where there is a $b \in R$ with $a \mid bc$ and $(b, c) = R$. This is a special case of Corollary 4.14, which shows that every comaximal factorization of a proper ideal I has factors of the form (I, a) with $a(a-1) \in I$.

In Theorem 4.15, we will prove (in a more general setting) necessary and sufficient conditions for a proper ideal to have a complete comaximal factorization. In short, the set of pseudo-irreducible ideals containing a given proper ideal I has minimal elements, called *minimal pseudo-irreducibles* of I, and I has a complete comaximal factorization if and only if it has only finitely many minimal pseudo-irreducibles. Equivalently, a proper ideal I has a complete comaximal factorization if and only if there are pseudo-irreducible ideals $P_1, \ldots, P_m$ containing I such that for every minimal prime P of I we have some $P + P_k \neq R$. Replacing "pseudo-irreducible" with the stronger "prime", we get the sufficient condition proven by McAdam and Swan.

Our main theorem is a generalization of McAdam and Swan's aforementioned theorem that the comaximal factorizations of an ideal are in a natural one-to-one correspondence with those of its radical. We generalize this in the furthest possible way, but stating Theorem 4.18 in terms of star operations on rings, we have the following: If $\star$ is a finitary star operation and I and J are $\star$-ideals with $I \subseteq J \subseteq \operatorname{rad}(I)$, then the map $I = (I_1 \cdots I_n)^\star \to J = ((I_1 + J)^\star \cdots (I_n + J)^\star)^\star$ is a bijection between the $\star$-comaximal factorizations of I and J. The original proof of McAdam and Swan makes heavy use of the connection between comaximal factorizations and direct product decompositions. As this method does not generalize to star operations, we instead turn to a method using a generalized version of semistar operations. As an application of the theorem, we show how it can be used to give an alternate proof of one of the main results of [7]. The definition of $\star$-URD generalizes to a *$\star$-unique representation ring* (*$\star$-URR*) in an obvious way, and our new proof, which does not rely on the ring being an integral domain, shows that the aforementioned characterization of $\star$-URD's works just as well for $\star$-URR's.

Finally, in the last section we will show how our main theorem translates statements about $\star$-comaximal factorizations into equivalent topological statements about the Zariski topology on the $\star$-spectrum. This makes a topological approach to $\star$-comaximal factorizations possible, and we demonstrate this and a few other applications with examples. We summarize some of the theory here. Let $\star$ be a finitary star operation on a ring R. (We note that the following statements are also true in the more general situation of finitary weak ideal systems on monoids.) The $\star$-spectrum $\mathrm{Spec}_\star(R)$ is given the *Zariski topology* by declaring the closed subsets to be those of the form $V_\star(I) = \{P \in \mathrm{Spec}_\star(R) \mid P \supseteq I\}$. We will show that the $\star$-comaximal factorizations of a proper $\star$-ideal I are in one-to-one correspondence with the separations of the subspace $V_\star(I)$, so I is connected (resp., has a complete $\star$-comaximal factorization) if and only if $V_\star(I)$ is connected (resp., has a separation into connected subspaces). We also show that the $\star$-spectrum is a spectral space, and hence homeomorphic to the d-spectrum of some reduced ring. This observation allows one to reduce the proof of certain statements about finitary star operations to the d-operation case. Similarly, we will show that the collection of strongly $\star$-irreducible $\star$-ideals (see Section 4) forms a spectral space with its Zariski topology (see Section 5).

2. Preliminaries

Throughout this paper, all monoids will be multiplicative and commutative with $1 \neq 0$ unless stated otherwise. Similarly, all rings will be commutative with $1 \neq 0$. For a set X, we denote the set of (finite) subsets of X by $\mathcal{P}(X)$ ($\mathcal{P}_f(X)$).

Let H be a monoid. We denote the units of H by $H^\times$ and we set $H^* = H \setminus \{0\}$, $H^\# = H^* \setminus H^\times$, and $H_0^\# = H \setminus H^\times$. We call $a \in H^*$ *cancellative* or *regular* if $ax = ay \Rightarrow x = y$, and we denote the regular elements of H by $\mathrm{Reg}(H)$. We say H is *cancellative* if $\mathrm{Reg}(H) = H^*$. If H is a ring, then its regular elements are precisely the elements that are not zero divisors, so a cancellative ring is simply an integral domain. We outline some of the theory of standard factorizations. Our main references for this are [2,4,1]. Those papers deal with rings, but most of the theory translates to the more general language of monoids with trivial changes.

A *factorization* of $b \in H_0^\#$ of *length* n is a product $b = b_1 \cdots b_n$ with each $b_i \in H^\#$. We call a length 1 factorization *trivial*. We will find it convenient to also regard $0 = 0$ as a trivial factorization. A *refinement* of a factorization $b = b_1 \cdots b_n$ is a factorization $b = b_{1,1} \cdots b_{1,m_1} \cdots b_{n,1} \cdots b_{n,m_n}$ with each $b_i = b_{i,1} \cdots b_{i,m_i}$; we say the refinement is *proper* if some $m_i > 1$. If one factorization is a (proper) refinement of another, then we call the latter factorization a (*proper*) *partition* of the former.

Let $\equiv$ be a relation on H. We say that factorizations $a_1 \cdots a_m$ and $b_1 \cdots b_n$ are $\equiv$*-homomorphic* if for each $i \in \{1, \dots, m\}$ there is a $j \in \{1, \dots, n\}$ with $a_i \equiv b_j$, and for each $j \in \{1, \dots, n\}$ there is an $i \in \{1, \dots, m\}$ with $b_j \equiv a_i$. If $\equiv$ is symmetric, we say that factorizations $a_1 \cdots a_m$ and $b_1 \cdots b_n$ are $\equiv$*-isomorphic* if $m = n$ and each $a_i \equiv b_i$ after a suitable reordering. If $\equiv$ is reflexive and transitive, then "$\equiv$-homomorphic" is an equivalence relation. If $\equiv$ is an equivalence relation, then so is "$\equiv$-isomorphic".

Theorem 2.1. *Let τ be a transitive relation on H and define a relation $\equiv$ on H by $a \equiv b \Leftrightarrow a\tau b$ and $b\tau a$. Consider two factorizations $a_1 \cdots a_m$ and $b_1 \cdots b_n$. If they are $\equiv$-isomorphic, then they are τ-homomorphic. If $m \geqslant n$ and $a_i \not\tau a_j$ for $i \neq j$, then the converse is true.*

Proof. ($\Rightarrow$): Clear. ($\Leftarrow$): Assume that $a_1 \cdots a_m$ and $b_1 \cdots b_n$ are τ-homomorphic, $m \geqslant n$, and $a_i \not\tau a_j$ for $i \neq j$. Reorder if necessary so that $a_1 \equiv b_1, \dots, a_k \equiv b_k$ (where $0 \leqslant k \leqslant n$) and no $a_i \equiv b_j$ for $i, j > k$. Suppose $k < m$. Then $a_{k+1} \tau b_j \tau a_i$ for some i, j, and hence $i = k+1$ and $b_j \equiv a_{k+1}$. Therefore $j \leqslant k$ and $a_j \equiv b_j \equiv a_{k+1}$, a contradiction. Therefore $k = m = n$. $\square$

We will note one special case of the above theorem. Two factorizations are called *isomorphic* if they are $\sim$-isomorphic, and *weakly homomorphic* if they are $|$-homomorphic. We say that a factorization $a_1 \cdots a_n$ has *incomparable factors* if $a_i \nmid a_j$ for $i \neq j$, or, equivalently, if $a_1 H, \dots, a_n H$ are incomparable. Applying the above theorem with $\tau = |$ shows that two weakly homomorphic factorizations with incomparable factors are isomorphic. We invite the reader to formulate other results using different transitive relations for τ.

3. Review of module systems and ideal systems

The theory of (weak) ideal systems is carefully developed in our main ideal systems reference [13], and later many of the results were generalized to (weak) module systems in [14] and [15]. We briefly give a summary of some of the most important facts and definitions. All the definitions and results in this section will come from these sources unless stated otherwise.

Let H be a monoid. For $X \subseteq H$, we define $X_s = X_{s_H}$ to be the monoid ideal generated by X, and, if H is a ring, then we define $X_d = X_{d_H}$ to be the ring ideal generated by X. A *weak module system* on H is an operation r on $\mathcal{P}(H)$ satisfying the following for all $X, Y \subseteq H$ and $a \in H$:

(M1) $X \cup \{0\} \subseteq X_r$;
(M2) $X \subseteq Y_r \Rightarrow X_r \subseteq Y_r$; and
(M3) $aX_r \subseteq (aX)_r$.

In view of (M1), we could equivalently replace (M2) in the above with the following two axioms:

(M2a) $(X_r)_r = X_r$; and
(M2b) $X \subseteq Y \Rightarrow X_r \subseteq Y_r$.

A *module system* on H is an operation r on $\mathcal{P}(H)$ satisfying (M1), (M2), and the following stronger version of (M3):

(M3$^+$) $aX_r = (aX)_r$.

A (*weak*) *ideal system* is a (weak) module system with the following stronger axiom in place of (M1):

(I1) $X_s \subseteq X_r$.

(For consistency, we relabel (M2)–(M3) as (I2)–(I3) in this case.) Equivalently, one could define "(weak) ideal system" by leaving (M1) unchanged and adding on this additional axiom:

(IS) $\{1\}_r = H$.

For any weak module system r we have $H_r = H$ and $\{0\} \subseteq \emptyset_r = \{0\}_r$, with equality holding if r is a module system. Given a (weak) module system r on a monoid H, we define another (weak) module system r_f by $X_{r_f} = \bigcup_{Y \in \mathcal{P}_f(X)} Y_r$. We say a weak module system r is *finitary* if $r = r_f$. It is easily seen that s and d are finitary ideal systems. If we need to use one of the nice properties of finitary weak ideal systems, we have generally just stated the theorems with an assumption that r is finitary for the sake of simplicity. Isolating exactly which properties are used in a proof should not be overly difficult for one familiar enough with weak ideal systems to care about such a thing. Defining $X_x = H$ for all $X \subseteq H$ yields the *trivial weak ideal system* x. A more interesting weak ideal system a is obtained by defining $X_a = \operatorname{ann}(\operatorname{ann}(X))$ for $X \subseteq H$, where $\operatorname{ann}(X) = \{c \in H \mid cX = \{0\}\}$.

Let r be a weak module system on a monoid H. We call $I \subseteq H$ an r-*module* of H if $I_r = I$. For $n \geqslant 0$ we call an r-module n-*generated* if it is of the form X_r for some $X \in \mathcal{P}_f(H)$ of cardinality n. We call an r-module r-*finitely generated* if it is n-generated for some $n \geqslant 0$. We denote the set of r-modules (resp., r-finitely generated r-modules, n-generated r-modules) of H by $\mathcal{M}_r(H)$ (resp., $\mathcal{M}_{r,f}(H)$, $\mathcal{M}_{r,n}(H)$), and the set of principal ideals of H by $\operatorname{Princ}(H)$, where a *principal ideal* of H is a subset of the form aH for some $a \in H$. Any intersection of r-modules is an r-module, and if r is a module system, then $\operatorname{Princ}(H) \subseteq \mathcal{M}_{r,1}(H)$. If r is a weak ideal system, then we modify our terminology as follows: an r-*ideal* is an r-module and the set of r-ideals (resp., r-finitely generated r-ideals, n-generated r-ideals) is denoted $\mathcal{I}_r(H)$ (resp., $\mathcal{I}_{r,f}(H)$, $\mathcal{I}_{r,n}(H)$). For example, the d-ideals of a ring are the usual ring ideals, and the s-ideals of a monoid are the usual monoid ideals. We can partially order the weak module systems on H by defining $r \leqslant r'$ if $X_r \subseteq X_{r'}$ for every $X \subseteq H$, or, equivalently, if $\mathcal{M}_{r'}(H) \subseteq \mathcal{M}_r(H)$. With this ordering, the module system $X \to X \cup \{0\}$ is the smallest weak module system on H and the ideal system s is the smallest weak ideal system on H. If H is a ring, then a *star operation* on H is an ideal system $\star \geqslant d$. For subsets X and Y of the total quotient monoid of H, the *residuation* of X with respect to Y is $[X : Y] = [X :_H Y] = \{a \in H \mid aY \subseteq X\}$. For $y \in H$, we abbreviate $[X : \{y\}] = [X : y]$. If $I \in \mathcal{M}_r(H)$ and $X \subseteq H$, then $[I : X] = [I : X_r] \in \mathcal{M}_r(H)$. For $X, Y \subseteq H$, we set $XY = \{xy \mid x \in X,\ y \in Y\}$, and we have $(XY)_r = (X_rY)_r = (X_rY_r)_r$, and it follows that $\mathcal{M}_r(H)$ is a commutative monoid under the *r-multiplication* $I \cdot_r J = (IJ)_r$. The zero is $\{0\}_r$ and the identity is $\{1\}_r$. If r is a weak ideal system, then $\mathcal{I}_r(H)^\times = \{H\}$. We have $\mathcal{M}_r(H) = \{H\}$ if and only if r is the trivial weak ideal system. (Note that, even in the case when H is a ring and I and J are ring ideals, we set $IJ = \{ab \mid a \in I,\ b \in J\}$. The usual product of ring ideals is then $(IJ)_d$. Conveniently, if $\star$ is a star operation, it doesn't matter which way we interpret the "IJ" in "$(IJ)^\star$". This note allows one to directly translate our results to ring-theoretic terminology if desired.)

Let r be a weak ideal system on a monoid H. We call $I \in \mathcal{I}_r(H)$ *proper* if $I \neq H$. Thus $\mathcal{I}_r(H)_0^\#$ is the set of proper r-ideals. (We are tacitly ignoring the trivial weak ideal system.) We say that a proper r-ideal P is *prime* if $H \setminus P$ is multiplicatively closed (or, equivalently, if $(I_1I_2)_r \subseteq P \Rightarrow$ some $I_i \subseteq P$ for $I_1, I_2 \in \mathcal{I}_r(H)$), and *maximal* if H is the only r-ideal properly containing it. We denote the sets of prime (resp., maximal) r-ideals by $\mathrm{Spec}_r(H)$ (resp., $\mathrm{Max}_r(H)$). We will sometimes call $\mathrm{Spec}_r(H)$ the *prime r-spectrum* or simply *r-spectrum* of H. Each maximal r-ideal is prime. If r is finitary then every proper r-ideal is contained in a maximal r-ideal. The *r-Krull dimension* or simply *r-dimension* $\dim_r(H)$ of H is the supremum of the lengths of the properly descending chains $P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq P_n$ of prime r-ideals. For $X \subseteq H$ we denote the set of prime (resp., maximal) r-ideals containing X by $V_r(X)$ (resp., $W_r(X)$), and the *minimal primes* or *prime divisors* of X are the minimal elements of $V_s(X)$. Any prime s-ideal containing $X \subseteq H$ can be shrunk to a minimal prime of X, so every proper s-ideal has a minimal prime. The *minimal primes* of H are the minimal primes of $\{0\}$, or, equivalently, the minimal elements of $\mathrm{Spec}_s(H)$. (Different sources vary slightly on this definition. Some, like [13], specify that a minimal prime of H is a prime s-ideal minimal among the nonzero prime s-ideals.) If r is finitary, then every minimal prime of an r-ideal is an r-ideal. If $I_1, \ldots, I_n \in \mathcal{I}_r(H)$ and $I = (I_1 \cdots I_n)_r$, then (1) $V_r(I) = \bigcup_{i=1}^n V_r(I_i)$, (2) $W_r(I) = \bigcup_{i=1}^n W_r(I_i)$, and (3) a minimal prime of I is a minimal prime of some I_i. If additionally $(I_i \cup I_j)_r = H$ for $i \neq j$, then (1) the unions are disjoint, and (2) a minimal prime of any I_i is a minimal prime of I.

Let r be a weak ideal system on a monoid H. For $I \in \mathcal{I}_s(H)$, the *radical* of I is $\mathrm{rad}(I) = \{x \in H \mid x^n \in I \text{ for some } n \geqslant 1\} = \bigcap_{I \subseteq P \in \mathrm{Spec}_s(H)} P = \bigcap\{\text{minimal primes of } I\} \in \mathcal{I}_s(H)$. We will find it convenient to introduce a related definition: for $I \in \mathcal{I}_r(H)$, the *r-radical* of I is $\mathrm{rad}_r(I) = \bigcap_{I \subseteq P \in \mathrm{Spec}_r(H)} P \in \mathcal{I}_r(H)$. If r is finitary, then the notions of radical and r-radical coincide. An r-ideal is called *(r-)radical* if it is equal to its (r-)radical. For $I \in \mathcal{I}_r(H)$, we define the *r-Jacobson radical* of I to be $\mathcal{J}_r(I) = \bigcap_{I \subseteq M \in \mathrm{Max}_r(H)} M$, and we have $I \subseteq \mathrm{rad}(I) \subseteq \mathrm{rad}_r(I) \subseteq \mathcal{J}_r(I)$. For $I, J \in \mathcal{I}_r(H)$ we have $\mathrm{rad}_r(I) \subseteq \mathrm{rad}_r(J) \Leftrightarrow V_r(I) \supseteq V_r(J)$, and if r is finitary then these are equivalent to every minimal prime of J containing I. For $I, J \in \mathcal{I}_r(H)$ we have $\mathcal{J}_r(I) \subseteq \mathcal{J}_r(J) \Leftrightarrow W_r(I) \supseteq W_r(J)$. Thus $\mathrm{rad}_r(I) \subseteq \mathrm{rad}_r(J) \Rightarrow \mathcal{J}_r(I) \subseteq \mathcal{J}_r(J)$ for $I, J \in \mathcal{I}_r(H)$. Many of the nice properties of radicals of r-ideals follow from the following fact: if r is finitary, then we have a finitary weak ideal system $\mathfrak{r}(r) \geqslant r$ given by $X_{\mathfrak{r}(r)} = \mathrm{rad}(X_r)$. (See [13, Corollary 6.7].)

Let r be a weak module system on a monoid H. A submonoid of H that is also an r-module is called an *r-submonoid*. If D is a submonoid of H, then D_r is an r-submonoid. For each submonoid D of H, we define a weak module system $r[D] \geqslant r$ on H by $X_{r[D]} = (XD)_r$ for $X \subseteq H$, and we note that $r[D]$ is a module system if r is. If D is an r-submonoid of H, then the restriction of $r[D]$ to $\mathcal{P}(D)$ yields a weak ideal system on D; this weak ideal system is finitary if r is, and it is an ideal system if r is a module system. Let S be a nonempty multiplicatively closed subset of H. We use $H_S = \{\frac{a}{b} \mid a \in H,\ b \in S\}$ to denote the *localization* of H with respect to S, where multiplication is defined in the obvious way and $\frac{a}{b} = \frac{c}{d}$ if and only if there is an $s \in S$ with $sad = sbc$. We have a natural monoid homomorphism $j_S : H \to H_S : a \to \frac{a}{1}$. If $S \subseteq \mathrm{Reg}(H)$, then j_S embeds H in H_S. For $P \in \mathrm{Spec}_s(H)$, we write j_P for $j_{H \setminus P}$ and H_P for $H_{H \setminus P}$. If r is a finitary weak module system and $aX_r = (aX)_r$ for $a \in S$ and $X \subseteq H$, then r uniquely extends to a finitary weak module system on H_S by defining

$Y_r = c^{-1}(cY)_r$ for $Y \in \mathcal{P}_f(H_S)$ and $c \in S$ with $cY \subseteq H$, and defining $X_r = \bigcup_{Y \in \mathcal{P}_f(X)} Y_r$ for $X \subseteq H_S$; if r is a module system on H, then its extension is a module system on H_S.

Let q be a finitary weak ideal system on a monoid H, and let S be a nonempty multiplicatively closed subset of $\mathrm{Reg}(H)$ such that $aX_r = (aX)_r$ for $a \in S$ and $X \subseteq H$. (For example, this last condition automatically holds if $S \subseteq H^\times$, in which case $H_S = H$.) We call a nonempty set $\mathcal{L}$ of q-ideals a *q-localizing system* if it satisfies the following properties:

(L1) If $I \in \mathcal{L}$ and $I \subseteq J \in \mathcal{I}_q(H)$, then $J \in \mathcal{L}$.
(L2) If $I \in \mathcal{L}$, $J \in \mathcal{I}_q(H)$, and $[J :_H a] \in \mathcal{L}$ for every $a \in I$, then $J \in \mathcal{L}$.

We say that a q-localizing system $\mathcal{L}$ is *finitary* if it additionally satisfies:

(L3) For every $I \in \mathcal{L}$ there is a q-finitely generated $J \in \mathcal{L}$ with $J \subseteq I$.

Let $\mathcal{L}$ be a q-localizing system. By a small adjustment to the proof of [14, Theorem 4.3], the localizing system $\mathcal{L}$ induces a weak module system $\rho_{\mathcal{L},S} = \rho_{\mathcal{L}}$ on H_S by defining

$$X_{\rho_{\mathcal{L},S}} = X_{\mathcal{L},S} = X_{\mathcal{L}} = \{x \in H_S \mid [X_q :_H x] \in \mathcal{L}\} = \bigcup_{J \in \mathcal{L}} [X_q :_{H_S} J] \in \mathcal{M}_q(H_S)$$

for $X \subseteq H_S$. If H is cancellative and q is an ideal system, then $\rho_{\mathcal{L}}$ is a module system. Minor adaptations of arguments used in [14] yield the following facts: (1) $q \leqslant q[H_{\mathcal{L}}] \leqslant \rho_{\mathcal{L}} = \rho_{\mathcal{L}}[H_{\mathcal{L}}]$, (2) $\rho_{\mathcal{L}}$ is finitary if $\mathcal{L}$ is, (3) the q-ideals of H that $\rho_{\mathcal{L}}$ maps to $H_{\mathcal{L}}$ are precisely those in $\mathcal{L}$, (4) the map $\rho_{\mathcal{L}} : \mathcal{I}_q(H) \to \mathcal{I}_{\rho_{\mathcal{L}}}(H_{\mathcal{L}})$ is an inclusion-preserving monoid epimorphism with the map $\pi_{\mathcal{L}} : \mathcal{I}_{\rho_{\mathcal{L}}}(H_{\mathcal{L}}) \to \mathcal{I}_q(H) : J \to J \cap H$ a right inverse, and (5) the map $\rho_{\mathcal{L}} : \mathrm{Spec}_q(H) \setminus \mathcal{L} \to \mathrm{Spec}_{\rho_{\mathcal{L}}}(H_{\mathcal{L}})$ is a bijection. In the important special case $S \subseteq H^\times$, the map $\rho_{\mathcal{L}}$ is a weak ideal system on $H = H_{\mathcal{L}}$ and fixes the prime q-ideals not in $\mathcal{L}$. The theory of localizing systems generalizes localization in the following sense. If S is a nonempty multiplicatively closed subset of $\mathrm{Reg}(H)$, then $\mathcal{L}_S = \{I \in \mathcal{I}_q(H) \mid I \cap S \neq \emptyset\}$ is a finitary q-localizing system, $I_{\mathcal{L}_S} = I_S$ for $I \in \mathcal{I}_q(H)$, and $\rho_{\mathcal{L}_S}$ is the weak ideal system q_S on H_S induced by q. (See [13, Theorem 4.4] for a definition of q_S and some of its properties. Of particular interest are the facts that $s_S = s_{H_S}$ and that $d_S = d_{H_S}$ if H is a ring, showing that this gives an appropriate generalization of the standard theory.) The topics of this paragraph are discussed in a ring-theoretic context in [11].

4. *r*-Comaximal factorizations of *r*-ideals

Throughout this section, we will use r to denote a weak ideal system on a monoid H unless noted otherwise.

We say two subsets X and Y of H are *r-comaximal* if $(X \cup Y)_r = H$, and we say $x, y \in H$ are *r-comaximal* if $\{x, y\}_r = H$. (Note: If H is a ring, $\star$ is a star operation on H, and $I, J \in I_d(H)$, then $(I \cup J)_d = I + J$ and $(I \cup J)^\star = (I + J)^\star$, so this definition generalizes the familiar ring-theoretic definitions.) An *r-comaximal factorization* of a proper r-ideal is a factorization with the factors pairwise r-comaximal. The r-ideals that appear in r-comaximal factorizations of a proper r-ideal are called its *r-comaximal factors*. An *r-comaximal refinement* is a refinement where each factor is replaced with an r-comaximal factorization. If one factorization is an r-comaximal refinement of another, then we call the latter factorization an *r-comaximal partition* of the former. A proper r-ideal is *r-pseudo-irreducible* if it has no nontrivial r-comaximal factorizations. A *complete r-comaximal factorization* is an r-comaximal factorization with no proper r-comaximal refinements, or, equivalently, an r-comaximal factorization whose factors are r-pseudo-irreducible. For $X, Y_1, Y_2 \subseteq H$ with each $(X \cup Y_i)_r = H$, we have $(X \cup Y_1Y_2)_r = (X_s \cup Y_1Y_2)_r \supseteq ((X \cup Y_1)(X \cup Y_2))_r = H$. It follows that any nontrivial r-comaximal factorization is an r-comaximal refinement of a length 2 r-comaximal factorization.

The following lemma is one of the most useful computational tools for dealing with r-comaximal factorizations.

Lemma 4.1. *Assume $I, I_1, \dots, I_n \in \mathcal{I}_r(H)$ satisfy $(I_1 \cdots I_n)_r \subseteq I$ and $(I \cup I_i \cup I_j)_r = H$ for $i \neq j$. Then $I = ((I \cup I_1)_r \cdots (I \cup I_n)_r)_r$.*

Proof. The case $n = 1$ is trivial, so assume $n \geqslant 2$. Let $J = (I_2 \cdots I_n)_r$. Then $(I \cup I_1 \cup J)_r = H$, so

$$I = \big(I(I \cup I_1 \cup J)\big)_r \subseteq \big((I \cup I_1)(I \cup J)\big)_r \subseteq (I \cup I_1 \cdots I_n)_r = I.$$

Now, $J \subseteq (I \cup J)_r$ and $(I \cup J \cup I_i \cup I_j)_r = H$ for distinct $i, j \geqslant 2$, so by induction we have

$$(I \cup J)_r = \big((I \cup J \cup I_2)_r \cdots (I \cup J \cup I_n)_r\big)_r = \big((I \cup I_2)_r \cdots (I \cup I_n)_r\big)_r,$$

and thus $I = ((I \cup I_1)_r (I \cup J)_r)_r = ((I \cup I_1)_r \cdots (I \cup I_n)_r)_r$. $\square$

Corollary 4.2. *Let I be an r-pseudo-irreducible and $m \geqslant 1$. Then there is a length-preserving map from the r-comaximal factorizations of $(I^m)_r$ to those of I, namely $(I^m)_r = (I_1 \cdots I_n)_r \to I = ((I \cup I_1)_r \cdots (I \cup I_n)_r)_r$. In particular, a power of an r-pseudo-irreducible is r-pseudo-irreducible.*

Proof. Let $(I^m)_r = (I_1 \cdots I_n)_r$ be an r-comaximal factorization. Because each $(I^m \cup I_i)_r = I_i \neq H$, each $(I \cup I_i)_r \neq H$, and the rest follows from Lemma 4.1. $\square$

Corollary 4.3. *If $I_1, \dots, I_n \in \mathcal{I}_r(H)$ are pairwise r-comaximal, then $(I_1 \cdots I_n)_r = I_1 \cap \cdots \cap I_n$.*

Proof. Apply Lemma 4.1 to $(I_1 \cdots I_n)_r \subseteq I_1 \cap \cdots \cap I_n$. $\square$

We now collect some alternate equivalent definitions of r-pseudo-irreducible, which we will use freely in the rest of the paper.

Theorem 4.4. *The following are equivalent for a proper r-ideal I.*

(1) *I is r-pseudo-irreducible.*
(2) *For every r-comaximal $I_1, I_2 \in \mathcal{I}_r(H)$ with $(I_1 I_2)_r = I$, we have some $I_i = I$.*
(3) *For every r-comaximal $I_1, I_2 \in \mathcal{I}_r(H)$ with $(I_1 I_2)_r \subseteq I$, we have some $I_i \subseteq I$.*

Proof. (1) $\Rightarrow$ (2): Clear. (2) $\Rightarrow$ (3): Assume (2) and that $(I_1 I_2)_r \subseteq I$ for some r-comaximal $I_1, I_2 \in \mathcal{I}_r(H)$. Lemma 4.1 yields $I = ((I \cup I_1)_r (I \cup I_2)_r)_r$. By (2) we obtain $I = (I \cup I_i)_r \supseteq I_i$ for some i. (3) $\Rightarrow$ (1): Assume (3) and take any r-comaximal $I_1, I_2 \in \mathcal{I}_r(H)$ with $I = (I_1 I_2)_r$. Without loss of generality, we have $I_1 \subseteq I$, so $I_2 = (I \cup I_2)_r \supseteq (I_1 \cup I_2)_r = H$, showing I to be r-pseudo-irreducible. $\square$

Note that as a consequence of (3) above, if $q \leqslant r$ are weak ideal systems on H, then prime r-ideal $\Rightarrow$ r-pseudo-irreducible $\Rightarrow$ q-pseudo-irreducible. It also implies that any two complete r-comaximal factorizations of the same element are $\supseteq$-homomorphic, so we could apply Theorem 2.1 with $\tau = \supseteq$ to show that complete r-comaximal factorizations are always unique (up to order) when they exist. We will find it illuminating to give an additional proof in Theorem 4.11 below that uses the idea of common refinements.

The paper [19] showed that the following statements are equivalent to the corresponding versions of (1)–(3) above.

(4) The ring R/I is connected.
(5) For every d-comaximal $x_1, x_2 \in R$ with $x_1 x_2 \in I$, we have some $x_i \in I$.

(Recall that a ring is called *connected* or *indecomposable* if it has no idempotents other than the trivial ones 0 and 1, or, equivalently, if it has no nontrivial direct product decompositions, or, equivalently, if $\mathrm{Spec}_d(R)$ is connected in the Zariski topology. We will explore the Zariski topology in the next section. In particular, we will show that if r is a finitary weak ideal system on a monoid H, then a proper r-ideal J is r-pseudo-irreducible if and only if $V_r(J)$ is a connected subspace of $\mathrm{Spec}_r(H)$ with the Zariski topology. As one would expect, the topological spaces $\mathrm{Spec}_d(R/I)$ and $V_d(I)$ are homeomorphic, so in some sense our result generalizes that of [19].) Elaborating on (4), the authors note that there is a bijection between the d-comaximal factorizations of I and the nontrivial direct product decompositions of R/I, taking $I = (I_1 \cdots I_n)_d$ to $R/I \cong R/I_1 \times \cdots \times R/I_n$. So I has a complete d-comaximal factorization if and only if R/I is a finite direct product of connected rings. In particular, the d-ideal $\{0\}$ is d-pseudo-irreducible (resp., has a complete d-comaximal factorization) if and only if R is connected (resp., is a finite direct product of connected rings). So the uniqueness of complete d-comaximal factorizations noted above gives one way to see that finite direct product decompositions into connected rings are unique when they exist.

The most convenient thing about studying factorizations in the setting of a cancellative monoid or an integral domain is undoubtedly the ability to cancel. Of course, the monoid $\mathcal{I}_r(H)$ is usually not cancellative, but it turns out that we always can cancel when dealing with r-comaximal factorizations. In the following theorem we collect some simple "cancelation", divisibility, and uniqueness results about r-comaximal factorizations. For example, if r is finitary, then we have uniqueness of r-comaximal factorizations with equal r-Jacobson radicals. We note that part (1) is a weak ideal systems version of [7, Lemma 2.2].

Theorem 4.5. *Let $I_1, I_2, J_1, J_2 \in \mathcal{I}_r(H)$.*

(1) *If $I_1 \supseteq (J_1 J_2)_r$ and $(I_1 \cup J_1)_r = H$, then $I_1 \supseteq J_2$.*
(2) *If $(I_1 I_2)_r = (J_1 J_2)_r$ and $(I_1 \cup J_2)_r = (J_1 \cup I_2)_r = H$, then $I_1 = J_1$ and $I_2 = J_2$.*
(3) *Assume $(I_1 I_2)_r = (J_1 J_2)_r$, $(I_1 \cup I_2)_r = (J_1 \cup J_2)_r = H$, and $I_1 \supseteq J_1$. Then $I_2 \subseteq J_2$. If $I_1 = J_1$, then $I_2 = J_1$.*
(4) *Assume that r is finitary, $(I_1 I_2)_r = (J_1 J_2)_r$, $(I_1 \cup I_2)_r = (J_1 \cup J_2)_r = H$, and $\mathcal{J}_r(I_1) \supseteq J_1$. Then $I_1 \supseteq J_1$ and $I_2 \subseteq J_2$. If $\mathcal{J}_r(I_1) = \mathcal{J}_r(J_1)$, then $I_1 = J_1$ and $I_2 = J_2$.*

Proof. (1) If $I_1 \supseteq (J_1 J_2)_r$ and $(I_1 \cup J_1)_r = H$, then $J_2 = (J_2(I_1 \cup J_1))_r \subseteq (I_1 \cup J_1 J_2)_r = I_1$.

(2) Follows from (1).

(3) Similar to the proof of (4) below.

(4) It will suffice to show the first part. For that, we observe that $(I_1 \cup J_2)_r = H$ and apply part (1) to $I_1 \supseteq (J_1 J_2)_r$ and $J_2 \supseteq (I_1 I_2)_r$. □

Theorem 4.6. *Given a factorization and an r-comaximal factorization of the same proper r-ideal, the former has an r-comaximal refinement that is a refinement of the latter (up to order).*

Proof. Let $(I_1 \cdots I_m)_r = (J_1 \cdots J_n)_r$ be factorizations with the latter an r-comaximal factorization. By Lemma 4.1, each $I_i = ((I_i \cup J_1)_r \cdots (I_i \cup J_n)_r)_r$, so we have r-comaximal factorizations

$$\Big(\big((I_1 \cup J_1)_r \cdots (I_m \cup J_1)_r\big)_r \cdots \big((I_1 \cup J_n)_r \cdots (I_m \cup J_n)_r\big)_r\Big)_r = (J_1 \cdots J_n)_r.$$

By Theorem 4.5 part (2), each $J_i = ((I_1 \cup J_i)_r \cdots (I_m \cup J_i)_r)_r$, so $((I_1 \cup J_1)_r \cdots (I_m \cup J_n)_r)_r$ is an r-comaximal refinement of $(I_1 \cdots I_m)_r$ that is (up to order) a refinement of $(J_1 \cdots J_n)_r$. □

Corollary 4.7. *Any two r-comaximal factorizations of a proper r-ideal have a common r-comaximal refinement (up to order).*

Corollary 4.8. *Any two distinct r-pseudo-irreducible r-comaximal factors of a proper r-ideal are r-comaximal.*

Proof. Let $(PI)_r = (QJ)_r$ be r-comaximal factorizations with $P \neq Q$ r-pseudo-irreducibles. By Corollary 4.7, the two factorizations have a common r-comaximal refinement, so $Q \supseteq I$ and hence $(P \cup Q)_r \supseteq (P \cup I)_r = H$. □

Corollary 4.9. *Any r-product of r-pseudo-irreducibles is a refinement of a complete r-comaximal factorization. So, if a proper r-ideal is an r-product of r-pseudo-irreducibles, then a minimum length such product is its unique complete r-comaximal factorization.*

Proof. Assume $I_1, \dots, I_m$ are r-pseudo-irreducibles. Theorem 4.6 implies that $(I_1 \cdots I_m)_r$ is a refinement of each of its r-comaximal factorizations, and thus each r-comaximal factorization is of length at most m, and an r-comaximal factorization of maximum length is necessarily complete. □

Corollary 4.10. *Let I and J be r-pseudo-irreducibles. Then $(I \cup J)_r = H$ or $(IJ)_r$ is r-pseudo-irreducible.*

Theorem 4.11. *Let r be a weak ideal system on a monoid H. The following are equivalent for a proper r-ideal I.*

(1) *I is a product of r-pseudo-irreducibles.*
(2) *I has a complete r-comaximal factorization.*
(3) *I has a unique complete r-comaximal factorization (up to order).*
(4) *I has only finitely many r-comaximal factorizations.*
(5) *I has only finitely many r-comaximal factors.*
(6) *There is an upper bound on the lengths of the r-comaximal factorizations of I.*

Proof. (1) ⇔ (2): Corollary 4.9. (2) ⇒ (3): Follows from Corollary 4.7 and the fact that a complete r-comaximal factorization has no proper r-comaximal refinements, or by applying Theorem 2.1 as noted above. (3) ⇒ (4): If I has a unique complete r-comaximal factorization (up to order), then each of its r-comaximal factorizations is a reordering of a partition of this complete r-comaximal factorization, so it has only finitely many r-comaximal factorizations. (4) ⇒ (5) ⇒ (6): Clear. (6) ⇒ (2): An r-comaximal factorization of maximum length is necessarily complete. □

We've been giving $\mathcal{I}_r(H)$ a monoid structure with the operation $\cdot_r$, but we could have also done it with $\cap$. By Corollary 4.3, we can think of r-comaximal factorizations as a special kind of factorization in either $(\mathcal{I}_r(H), \cdot_r)$ or $(\mathcal{I}_r(H), \cap)$. A simple consequence of Theorem 4.4 is that an arbitrary nonempty intersection of pairwise non-r-comaximal r-pseudo-irreducibles is r-pseudo-irreducible. It is then easy to see that any finite intersection of r-pseudo-irreducibles is a refinement of a *complete r-comaximal intersection*, i.e., a nonempty finite intersection of pairwise r-comaximal r-pseudo-irreducibles. So, if a proper r-ideal is a finite intersection of r-pseudo-irreducibles, then a minimum length such intersection is its unique complete r-comaximal one. Alternately, we could prove this fact using the following more general theorem.

Theorem 4.12. *Let r be a weak ideal system on a monoid H. Given any finite intersections $\bigcap_{i=1}^m I_i = \bigcap_{j=1}^n J_j$ of proper r-ideals with the J_i's pairwise r-comaximal, the former has an r-comaximal refinement that is (up to order) a refinement of the latter.*

Proof. Replace r-products with intersections in the proof of Theorem 4.6 as appropriate. □

We are now at a point where we can compare r-pseudo-irreducible r-ideals with other types of "r-irreducible" r-ideals. In [13], a proper r-ideal I is said to be *r-irreducible* if whenever $I_1, I_2 \in \mathcal{I}_r(H)$ and $I = I_1 \cap I_2$, then some $I_i = I$. In analogy with [16], we define a proper r-ideal I to be *strongly r-irreducible* if whenever $I_1, I_2 \in \mathcal{I}_r(H)$ and $I \supseteq I_1 \cap I_2$, we have some $I_i \subseteq I$. Finally, analogously to [5], we call a proper r-ideal I *r-nonfactorable* if whenever $I_1, I_2 \in \mathcal{I}_r(H)$ and $I = (I_1 I_2)_r$, then some $I_i = H$ or $\{0\}_r$. (For those familiar with the factorization terminology of [4], we note that the r-irreducibles and strong r-irreducibles are precisely the atoms and primes, respectively, in the monoid $(\mathcal{I}_r(H), \cap)$, and the r-nonfactorables are precisely the very strong atoms in the monoid $(\mathcal{I}_r(H), \cdot_r)$.) With these definitions, we note the following implications: prime r-ideal ⇒ strongly r-irreducible ⇒ r-irreducible ⇒ r-pseudo-irreducible ⇐ r-nonfactorable ⇐ prime r-ideal. None of the implications reverse, even in the case $r = d$. As the other sorts of "d-irreducibles" have been studied extensively, we content ourselves with showing that the d-pseudo-irreducible property is strictly weaker than any of the others. For this, we note that we will see in Theorem 4.15 that a d-ideal with prime radical is d-pseudo-irreducible, while it has been shown in the papers referenced that there are no implications between prime radical and the other kinds of "d-irreducibles".

For $I \in \mathcal{I}_r(H)$, we call an r-pseudo-irreducible P containing I a *minimal r-pseudo-irreducible* of I if it is minimal among the r-pseudo-irreducibles containing I. We record some facts about minimal r-pseudo-irreducibles in the following theorem. The d-operation case for part (5) is [12, Proposition 7.36].

Theorem 4.13. *Let $I \in \mathcal{I}_r(H)$.*

(1) *Any r-pseudo-irreducible containing I can be shrunk to a minimal r-pseudo-irreducible of I. Thus minimal r-pseudo-irreducibles of I exist if r is finitary and $I \neq H$.*
(2) *The minimal r-pseudo-irreducibles of I are pairwise r-comaximal. Hence the number of minimal r-pseudo-irreducibles of I is bounded above by the size of any collection of proper r-ideals Ω with each minimal r-pseudo-irreducible contained in some element of Ω.*
(3) *Let Ω be a set of r-pseudo-irreducible r-ideals with the following two properties*: (1) *the r-ideal I and each of its minimal r-pseudo-irreducibles are contained in an element of Ω, and* (2) *each element of Ω containing a given proper r-ideal can be shrunk to a minimal such element. Then every minimal r-pseudo-irreducible of I is contained in an element of Ω minimal over I, and the number of minimal r-pseudo-irreducibles of I is at most the number of elements of Ω minimal over I.*
(4) *If $I = (I_1 \cdots I_n)_r$ is an r-comaximal factorization, then the set of r-pseudo-irreducibles containing I is the disjoint union of the sets of r-pseudo-irreducibles containing the I_i's, and an r-pseudo-irreducible is minimal over I if and only if it is minimal over some I_i. In particular, if I has a complete r-comaximal factorization, then the factors in that factorization are the minimal r-pseudo-irreducibles of I.*
(5) *If r is finitary, then I is the intersection of the r-pseudo-irreducibles containing it, or, equivalently, the intersection of its minimal r-pseudo-irreducibles. In fact, in this case I is the intersection of the r-irreducibles containing it.*
(6) *Every minimal r-pseudo-irreducible of $\{0\}_r$ is idempotent.*
(7) *Every r-comaximal factor of $\{0\}_r$ is idempotent.*
(8) *If r is finitary, then each r-comaximal factor of I is r-generated by I and a finite number of elements of H.*

Proof. (1) By Zorn's Lemma it suffices to show that the intersection of any nonempty chain of r-pseudo-irreducibles is r-pseudo-irreducible, and this easily follows from Theorem 4.4.

(2) If P and Q are distinct minimal r-pseudo-irreducibles of I, then $I \subseteq P \cap Q \subsetneq P$, so $P \cap Q$ is not r-pseudo-irreducible and hence $(P \cup Q)_r = H$.

(3) Let P be a minimal r-pseudo-irreducible of I and Q be an element of Ω minimal over P. Shrink Q to $Q_0 \in \Omega$ minimal over I. By part (1), the r-pseudo-irreducible Q_0 contains a minimal r-pseudo-irreducible of I, which must be P by part (2) and the fact that Q_0 and P are not r-comaximal. Therefore $P \subseteq Q_0 \subseteq Q$, and $Q = Q_0$ by minimality.

(4) Can be proven in the same way as the analogous facts about prime r-ideals.

(5) It will suffice to prove the "in fact" part. Assume r is finitary and take any $x \in H \setminus I$. By the fact that r is finitary, the union of any chain of r-ideals not containing x is an r-ideal not containing x. (See [13, Proposition 3.1(v)].) By Zorn's Lemma, we may then enlarge I to an r-ideal J maximal with respect to the exclusion of x. If $J = J_1 \cap J_2$ for some $J_1, J_2 \supsetneq J$, then by maximality $x \in J_1 \cap J_2 = J$, a contradiction. Therefore J is an r-irreducible containing I but not x.

(6) Follows from Corollary 4.2.

(7) If $(IJ)_r = \{0\}_r$ is an r-comaximal factorization, then we have r-comaximal factorizations $((I^2)_r J)_r = (IJ)_r$ and Theorem 4.5 part (2) gives us $I = (I^2)_r$.

(8) Assume r is finitary and $I = (I_1 I_2)_r$ is a nontrivial r-comaximal factorization. By the fact that r is finitary, there is an r-finitely generated r-ideal $J_1 \subseteq I_1$ such that $(J_1 \cup I_2)_r = H$. Then $I \subseteq (I \cup J_1)_r \cap I_2 \subseteq I_1 \cap I_2 = I$, so $(I_1 I_2)_r = ((I \cup J_1)_r I_2)_r$ are r-comaximal factorizations. By Theorem 4.5 part (2) we obtain $I_1 = (I \cup J_1)_r$. □

In the course of proving (5), we proved a fact worth noting in its own right: An r-ideal maximal with respect to the exclusion of an element of H is r-irreducible. More generally, an r-ideal maximal with respect to not containing some nonempty $X \subseteq H$ is r-irreducible.

For examples of sets Ω we can use in (3) when r is finitary, we have $\mathrm{Spec}_r(H)$, $\mathrm{Jac}_r(H)$, $\mathrm{Max}_r(H)$, or $\mathrm{SSpec}_r(H)$. Here $\mathrm{SSpec}_r(H)$ (resp., $\mathrm{Jac}_r(H)$) denotes the *strongly r-irreducible spectrum* (resp., *r-Jacobson subspace of* $\mathrm{Spec}_r(H)$), which consists of the strongly r-irreducible r-ideals (resp., r-Jacobson r-ideals), where an *r-Jacobson r-ideal* (*r-J-ideal*) is a prime r-ideal that is an intersection of maximal r-ideals. We will discuss r-J-ideals and some of their properties in the next section. Of course, we could also take Ω to be the set of r-pseudo-irreducibles itself, and we note that this choice of Ω gives the smallest number of minimal elements over I.

Corollary 4.14. *Let R be a ring and I be a proper d-ideal. Every d-comaximal factorization of I is of the form* $I = ((I + a_1R)\cdots(I + a_nR))_d$, *where each* $a_i(a_i - 1) \in I$.

Proof. Passing to R/I, we reduce the problem to showing the following: If $\{0\} = (I_1 \cdots I_n)_d$ is a d-comaximal factorization, then each I_i is a principal ideal generated by an idempotent. But this now follows directly from Theorem 4.13 and the fact that a d-finitely generated idempotent d-ideal is a principal ideal generated by an idempotent. □

Theorem 4.15. *Assume r is finitary. The following are equivalent for a proper r-ideal I.*

(1) *I has a complete r-comaximal factorization.*
(2) *I has only finitely many minimal r-pseudo-irreducibles.*
(3) *There are r-pseudo-irreducibles* $P_1, \ldots, P_m$ *containing I such that every r-pseudo-irreducible* $P \supseteq I$ *contains some* P_k.
(4) *For some* (*resp., each*) Ω *as in Theorem* 4.13 *part* (3) *above, there are r-pseudo-irreducibles* $P_1, \ldots, P_m$ *containing I such that for any* $P \in \Omega$ *minimal over I we have some* $(P \cup P_k)_r \neq H$.

In this case, the unique complete r-comaximal factorization of I is $I = (I_1 \cdots I_n)_r$, *where* $n \leqslant m$ *and* $I_1, \ldots, I_n \in \mathcal{I}_r(H)$ *are the minimal r-pseudo-irreducibles of I.*

Proof. (1) $\Leftrightarrow$ (2): Theorem 4.13. (2) $\Rightarrow$ (3) $\Rightarrow$ (4): Clear. (4) $\Rightarrow$ (2): Assume (4). We may assume that each P_i is a distinct minimal r-pseudo-irreducible of I. If I has some minimal r-pseudo-irreducible Q other than $P_1, \ldots, P_m$, then Q may be enlarged to an element of Ω minimal over I that is r-comaximal with each of the P_i's, a contradiction. Therefore I has exactly m minimal r-pseudo-irreducibles.

The part about the factors of the complete r-comaximal factorization of I being the minimal r-pseudo-irreducibles of I follows from Theorem 4.13, and the part about $n \leqslant m$ follows from the proof of "(4) $\Rightarrow$ (2)". □

In [19, Theorem 5.4] it was shown that in the case where H is a ring and $r = d$, statement (4) with the P_i's prime d-ideals and $\Omega = \mathrm{Spec}_r(H)$ is sufficient for I to have a complete d-comaximal factorization. By slightly weakening the hypothesis, we have managed to give necessary and sufficient conditions for an r-ideal to have a complete r-comaximal factorization.

Let r be a finitary weak ideal system on a monoid H. Theorems 4.13 and 4.15 give us one of our main methods for deducing that a proper r-ideal has a complete r-comaximal factorization and estimating its length. For each Ω as in Theorems 4.13 and 4.15, a proper r-ideal with only a finite number n of elements of Ω minimal over it has a complete r-comaximal factorization of length at most n. (Also, we can always get n to be equal to the length of the complete r-comaximal factorization by taking Ω to be the set of r-pseudo-irreducibles, but of course this would be rather circular. Also, given Ω, we can always shrink it to an Ω that is "optimal" in this way by picking out one element minimal over each of the minimal r-pseudo-irreducibles of I, but again this is somewhat circular.) In particular, if a proper r-ideal has a unique element of Ω minimal over it, then it is r-pseudo-irreducible. Writing out the most important special cases $\Omega = \mathrm{Spec}_r(H)$ and $\Omega = \mathrm{Max}_r(H)$ explicitly, we have the following facts.

(a) A proper r-ideal with only finitely many minimal primes has a complete r-comaximal factorization.
(b) An r-ideal with prime radical is r-pseudo-irreducible.
(c) A proper r-ideal contained in only finitely many maximal r-ideals has a complete r-comaximal factorization.
(d) An r-ideal contained in a unique maximal r-ideal is r-pseudo-irreducible.

As a special case of (1), if H is r-Noetherian, or more generally is $\mathfrak{r}(r)$-Noetherian (i.e., it satisfies the ascending chain condition on radical r-ideals), then every proper r-ideal has a complete r-comaximal factorization. (Recall from [13] that if r is a (not necessarily finitary) weak ideal system on H, then H is called *r-Noetherian* if it satisfies the ascending chain condition on r-ideals, or, equivalently, if r is finitary and every r-ideal is r-finitely generated. By [13, Lemma 7.8.1], if r is finitary, then H is $\mathfrak{r}(r)$-Noetherian if and only if it satisfies the ascending chain condition on prime r-ideals and every r-ideal has only finitely many minimal primes.)

We now give two examples, one illustrating a method to show that a complete r-comaximal factorization does not exist, and one showing that there are times when d-pseudo-irreducibility cannot always be verified by counting minimal primes or maximal d-ideals.

Example 4.16. (An example of a 1-dimensional Bézout domain in which every nonzero proper principal ideal has arbitrarily long d-comaximal factorizations into principal ideals. Hence no proper principal ideal has a complete d-comaximal factorization.) Let K be an algebraically closed field, D be the monoid domain $K[x;\mathbb{Q}^+] = K[\{x^r \mid r \in \mathbb{Q}^+\}]$, S be the set of nonzero monomials in D, and $R = D_S$. The proof of [18, Corollary 3.3] shows that R is our desired example. (If K has characteristic $p \geqslant 0$, then we can replace $\mathbb{Q}^+$ with any p-pure submonoid, where a submonoid of $\mathbb{Q}^+$ is *p-pure* if it is locally cyclic and has the property that for every $t \in T$ there is a natural number $n \geqslant 2$ with $t/n \in T$ and n not a power of p.) Note that we localized to remove the d-pseudo-irreducible principal ideals in D, namely the ones generated by non-constant monomials. To see that these principal ideals are d-pseudo-irreducible, it suffices by Corollary 4.14 and the fact that D is Bézout to show that they have no nontrivial d-comaximal factorizations into principal ideals, which follows from the observations that the nonzero monomials are a saturated subset of D and that no two non-constant monomials are relatively prime. Alternatively, we could simply note that each non-constant monomial is contained in a unique maximal d-ideal, namely the one consisting of the elements with zero constant term.

Example 4.17. Example 5.23 from the next section gives an example of a d-pseudo-irreducible d-ideal with infinitely many minimal primes and which is contained in infinitely many maximal d-ideals. More specifically, it gives a connected ring with infinitely many minimal primes and infinitely many maximal d-ideals. The ring in the example has d-dimension 1, but one might ask if it is possible to construct such a ring of d-dimension 0. In the next section we will use topological methods to show that 1 is in fact the minimum d-dimension of such a ring.

In [19, Lemma 5.5] it is shown that if I and J are d-ideals of a ring R with $I \subseteq J \subseteq \mathrm{rad}(I)$, then there is a bijection between the d-comaximal factorizations of I and those of J, mapping $I = (I_1 \cdots I_n)_d$ to $J = ((J + I_1) \cdots (J + I_n))_d$. Unfortunately, the technique used to prove the above in [19] does not seem to generalize to weak ideal systems. Assume r is finitary and let $I, J \in \mathcal{I}_r(H)$ be such that $I \subseteq J \subseteq \mathrm{rad}(I)$. The corresponding version of the above map, taking $I = (I_1 \cdots I_n)_r$ to $J = ((J \cup I_1)_r \cdots (J \cup I_n)_r)_r$ still takes an r-comaximal factorization to an r-comaximal factorization by Lemma 4.1 (and we can relax our condition on I and J to $I \subseteq J \subseteq \mathcal{J}_r(I)$ here), so it's not hard to apply Theorem 4.11 to obtain: If J has a complete r-comaximal factorization of length n, then I has a complete r-comaximal factorization of length at most n. However, we wish to go a step further and obtain that the above map is in fact a bijection, and for this we need considerably more work.

Theorem 4.18. *Let r be a finitary weak ideal system on a monoid H. Let I and J be proper r-ideals with $I \subseteq J \subseteq \mathcal{J}_r(I)$.*

(1) *There is an injection ϕ from the r-comaximal factorizations of I into the r-comaximal factorizations of J, given by*

$$I = (I_1 \cdots I_n)_r \to J = \big((J \cup I_1)_r \cdots (J \cup I_n)_r\big)_r.$$

(2) *The map ϕ takes r-comaximal refinements to r-comaximal refinements.*
(3) *If J has a complete r-comaximal factorization, then I has one that is no longer.*
(4) *If $I \subseteq J \subseteq \operatorname{rad}(I)$, then ϕ is a bijection.*

Proof. (1) If $I = (I_1 \cdots I_n)_r$ is an r-comaximal factorization, then so is $J = ((J \cup I_1)_r \cdots (J \cup I_n)_r)_r$ by Lemma 4.1 and the fact that each $(J \cup I_i)_r \subseteq \mathcal{J}_r(I_i) \subsetneq H$. So ϕ does indeed take r-comaximal factorizations to r-comaximal factorizations. If $I = (I_1 \cdots I_n)_r = (I'_1 \cdots I'_n)_r$ are two r-comaximal factorizations with each $(J \cup I_i)_r = (J \cup I'_i)_r$, then each $\mathcal{J}_r(I_i) = \mathcal{J}_r((J \cup I_i)_r) = \mathcal{J}_r((J \cup I'_i)_r) = \mathcal{J}_r(I'_i)$, and hence each $I_i = I'_i$ by Theorem 4.5. This shows that ϕ is injective.

(2) This is a simple consequence of the definitions and Lemma 4.1.

(3) Assume J has a complete r-comaximal factorization, and consider the pre-image of a maximum length r-comaximal partition of that factorization that is in the image of ϕ. By part (2), such a factorization must be the unique complete r-comaximal factorization of I.

(4) (For this part, we use some of the ideas seen in the proof of [7, Proposition 2.4].) Assume $I \subseteq J \subseteq \operatorname{rad}(I)$. It only remains to show that our map is surjective, so let $J = (J_1 \cdots J_n)_r$ be any r-comaximal factorization of length $n \geqslant 2$. For each $i = 1, \ldots, n$ we have $(J_i \cup \prod_{j \neq i} J_j)_r = H$, and by the fact that r is finitary there is a finite $E_i \subseteq \prod_{j \neq i} J_j$ such that $(J_i \cup E_i)_r = H$.

For $i = 1, \ldots, n$, let $\mathcal{L}_i$ be the set of $J \in \mathcal{I}_r(H)$ with $E_i^k \subseteq J$ for some $k \geqslant 1$. Each $\mathcal{L}_i$ is finitary and closed under r-products, and hence a finitary r-localizing system by [14, Proposition 4.2(2)]. For each $i = 1, \ldots, n$ let $\rho_{\mathcal{L}_i} = \rho_{\mathcal{L}_i, H^\times}$ and $I_i = I_{\mathcal{L}_i} \in \mathcal{I}_{\rho_{L_i}}(H) \subseteq \mathcal{I}_r(H)$. If I is in some $\mathcal{L}_i$, then there is a $k \geqslant 1$ with $E_i^k \subseteq I \subseteq J \subseteq J_i$, contradicting the fact that J_i and E_i are r-comaximal. Therefore for each i we have $I \subseteq I_i \subsetneq H$.

Take any $i \in \{1, \ldots, n\}$ and let P be a minimal prime of I_i. Because $(J_1 \cdots J_n)_r \subseteq \operatorname{rad}(I) \subseteq \operatorname{rad}(I_i) \subseteq P$, we have some $J_j \subseteq P$. If $j \neq i$, then $E_i \subseteq J_j \subseteq P$, so $P \in \mathcal{L}_i$ and $P = P_{\mathcal{L}_i} = H$, a contradiction. Therefore J_i is the unique J_j contained in P. We've now shown that any minimal prime of I_i contains J_i, so $V_r(I_i) \subseteq V_r(J_i)$ and $\operatorname{rad}(I_i) \supseteq \operatorname{rad}(J_i)$.

We claim that $I_1 \cap \cdots \cap I_n \subseteq j_M^{-1}(I_M)$ for any $M \in \operatorname{Max}_r(H)$. The only nontrivial case is $I \subseteq M \in \operatorname{Max}_r(H)$. In this case, $M \supseteq \operatorname{rad}(I) \supseteq (J_1 \cdots J_n)_r$, so some $J_i \subseteq M$. But $(M \cup E_i)_r \supseteq (J_i \cup E_i)_r = H$, so there is a $c \in E_i \setminus M$. For any $a \in I_i$, there is a $k \geqslant 1$ with $ac^k \in I$, so $\frac{a}{1} = \frac{ac^k}{c^k} \in I_M$, and thus $a \in j_M^{-1}(I_M)$. Therefore $I_1 \cap \cdots \cap I_n \subseteq I_i \subseteq j_M^{-1}(I_M)$, as desired.

By the above, $I \subseteq I_1 \cap \cdots \cap I_n \subseteq \bigcap_{M \in \operatorname{Max}_r(H)} j_M^{-1}(I_M) = I$, where the last equality is [13, Theorem 7.4]. For each $i \neq j$ we have $(\operatorname{rad}(I_i) \cup \operatorname{rad}(I_j))_r \supseteq (\operatorname{rad}(J_i) \cup \operatorname{rad}(J_j))_r = H$, showing that $I = I_1 \cap \cdots \cap I_n = (I_1 \cdots I_n)_r$ is an r-comaximal factorization. We have

$$\bigsqcup_{i=1}^{n} V_r(J_i) = V_r(J) = V_r(I) = \bigsqcup_{i=1}^{n} V_r(I_i),$$

where the unions are disjoint by r-comaximality. Because each $V_r(I_i) \subseteq V_r(J_i)$, we are in fact led to the conclusion that each $V_r(I_i) = V_r(J_i)$. We have $((J \cup I_1)_r \cdots (J \cup I_n)_r)_r = J = (J_1 \cdots J_n)_r$ and each $\operatorname{rad}(J_i) = \operatorname{rad}((J \cup I_i)_r)$, and Theorem 4.5 implies that each $J_i = (J \cup I_i)_r$, showing ϕ to be surjective. □

We note that in the case $J = \operatorname{rad}(I)$, the bijection ϕ takes $I = (I_1 \cdots I_n)_r$ to $\operatorname{rad}(I) = (\operatorname{rad}(I_1) \cdots \operatorname{rad}(I_n))_r$. To see this, we note that

$$\big(\mathrm{rad}(I)\cup I_1\big)_r\cap\cdots\cap\big(\mathrm{rad}(I)\cup I_n\big)_r=\mathrm{rad}(I)=\mathrm{rad}\big((I_1\cdots I_n)_r\big)=\mathrm{rad}(I_1)\cap\cdots\cap\mathrm{rad}(I_n),$$

and that each $\mathrm{rad}(I_i)=(\mathrm{rad}(I)\cup I_i)_r$ by Theorem 4.5. In particular, every r-comaximal factor of a radical r-ideal is radical.

Example 4.19. The map ϕ need not be a bijection if we only assume $I\subseteq J\subseteq\mathcal{J}_r(I)$. Indeed, if R is any connected ring with a finite number n of maximal d-ideals, then $\{0\}$ is d-pseudo-irreducible and $\mathcal{J}_d(\{0\})$ has a complete d-comaximal factorization of length n. We can construct such an R with various nice properties. For example, choosing K to be a field containing at least n distinct elements $a_1,\dots,a_n$, $M_i=(x-a_i)K[x]$ for $i=1,\dots,n$, $S=K[x]\setminus(M_1\cup\cdots\cup M_n)$, and $R=K[x]_S$, we get such an example where R is a PID.

Corollary 4.20. *Assume r is finitary and I is a proper r-ideal. There is a bijection χ between the minimal r-pseudo-irreducibles of I and those of* $\mathrm{rad}(I)$*, taking P to* $(\mathrm{rad}(I)\cup P)_r$.

Proof. First we show that χ does indeed take minimal r-pseudo-irreducibles of I to minimal r-pseudo-irreducibles of $\mathrm{rad}(I)$. Let P be any minimal r-pseudo-irreducible of I. We have $P\subseteq(\mathrm{rad}(I)\cup P)_r\subseteq\mathrm{rad}(P)$, so $(\mathrm{rad}(I)\cup P)_r$ is r-pseudo-irreducible by Theorem 4.18. Now we show minimality over $\mathrm{rad}(I)$. If Q is any r-pseudo-irreducible with $\mathrm{rad}(I)\subseteq Q\subseteq(\mathrm{rad}(I)\cup P)_r$, then we can shrink Q to a minimal r-pseudo-irreducible of I, which must be P by the fact that P and Q are not r-comaximal, showing that $Q=(\mathrm{rad}(I)\cup P)_r$.

If $(\mathrm{rad}(I)\cup P)_r=(\mathrm{rad}(I)\cup Q)_r$ for minimal r-pseudo-irreducibles P and Q of I, then $(P\cup Q)_r\subseteq(\mathrm{rad}(I)\cup P)_r\subsetneq H$, so $P=Q$. Therefore χ is injective.

For surjectivity, let Q be any minimal r-pseudo-irreducible of $\mathrm{rad}(I)$, and shrink Q to a minimal r-pseudo-irreducible P of I. Then $\mathrm{rad}(I)\subseteq(\mathrm{rad}(I)\cup P)_r\subseteq Q$ and $(\mathrm{rad}(I)\cup P)_r$ is r-pseudo-irreducible, so by minimality we have $Q=(\mathrm{rad}(I)\cup P)_r$. □

Let r be a weak ideal system on a monoid H. We call H an *r-unique representation monoid* (*r-URM*) if every r-finitely generated proper r-ideal has a unique r-comaximal factorization into r-ideals with prime radical. (We note that [7] defines a *$\star$-unique representation domain* ($\star$*-URD*) in a somewhat different but equivalent way.) If r is finitary, then Theorem 4.15 shows that an r-ideal with prime radical is r-pseudo-irreducible, so Theorem 4.11 tells us that r-comaximal factorizations into r-ideals with prime radical are unique whenever they exist. (This gives an alternate proof of [7, Theorem 2.3].) The paper [7] extensively studied these concepts in the context of star operations on integral domains. As an application of Theorem 4.18, we give a different proof of a slightly generalized version of [7, Proposition 2.4].

Corollary 4.21. *Let r be a finitary weak ideal system on a monoid H. The following are equivalent for a proper r-ideal I.*

(1) *I has only finitely many minimal primes, and these minimal primes are pairwise r-comaximal.*
(2) *I has a unique r-comaximal factorization into r-ideals with prime radical.*
(3) *I has a factorization into r-ideals with only finitely many minimal primes, and the minimal primes of I are pairwise r-comaximal.*

In this case, the following observations hold:

(a) *The length of the factorization in* (2) *is the number n of minimal primes of I.*
(b) *Any factorization of I into r-pseudo-irreducibles has length at least n.*
(c) *Any length n factorization of I into r-pseudo-irreducibles is its unique r-comaximal factorization.*

Proof. (1) $\Rightarrow$ (2): Using Theorem 4.18, we can reduce the problem to showing that $\mathrm{rad}(I)$ has an r-comaximal factorization into prime r-ideals, which is obvious. (2) $\Rightarrow$ (1), (3): Clear. (3) $\Rightarrow$ (2):

Assume (3). We can partition the given factorization to obtain an r-comaximal factorization $I = (I_1 \cdots I_n)_r$ with each I_i having only finitely many minimal primes. Each minimal prime of an I_i is a minimal prime of I, so the minimal primes of any given I_i are pairwise r-comaximal. Using "(1) $\Rightarrow$ (2)", we get the desired conclusion.

Observation (a) follows from the observation that the length of the factorization in (2) is the length of the complete r-comaximal factorization of $\operatorname{rad}(I)$. Observations (b) and (c) follow from Corollary 4.9. □

We can now formulate some characterizations of r-URM's analogous to those found in [7] for star operations on integral domains. We will only do the following one. The interested reader is encouraged to read [7] for more information. One definition we will need for the following is that $\operatorname{Spec}_r(H)$ is called *treed* if any two incomparable prime r-ideals are r-comaximal. If r is finitary, then this is equivalent to saying that the prime r-ideals that are contained in a given maximal r-ideal are linearly ordered.

Corollary 4.22. *Let r be a finitary weak ideal system on a monoid H. The following are equivalent.*

(1) *H is an r-URM.*
(2) *Every (nonzero) 1-generated r-ideal has only finitely many minimal primes, and these minimal primes are pairwise r-comaximal.*
(3) $\operatorname{Spec}_r(H)$ *is treed and every r-finitely generated r-ideal has only finitely many minimal primes.*
(4) *There is a collection $\mathcal{C}$ of $\mathfrak{r}(r)$-ideals such that (1) $\mathcal{C}$ is closed under finite intersections, (2) every $\mathfrak{r}(r)$-ideal is $\mathfrak{r}(r)$-generated by a collection of elements of $\mathcal{C}$, (3) every element of $\mathcal{C}$ has only finitely many minimal primes, and (4) the minimal primes of an element of $\mathcal{C}$ are pairwise r-comaximal.*

We have added characterization (4) to capture the essential property of the 1-generated r-ideals that allows for the key implication (2) $\Rightarrow$ (3). For the topological approach that we will take in the next section, it will be useful to have such a characterization that does not refer to the number of generators.

Proof of Corollary 4.22. (We note that this proof is essentially given in [7], but our generalized Corollary 4.21 lets us get by without the cancellative assumption present in that paper.) (2) $\Rightarrow$ (4): Take $\mathcal{C}$ to be the collection of radicals of 1-generated r-ideals. (4) $\Rightarrow$ (3): Assume (4). Now let $P_1, P_2 \in \operatorname{Spec}_r(H)$ be any incomparable prime r-ideals. We can observe that P_1 and P_2 are $\mathfrak{r}(r)$-generated by the elements of $\mathcal{C}$ they contain, so there are $J_1, J_2 \in \mathcal{C}$ with $J_1 \nsubseteq P_2$, $J_2 \nsubseteq P_1$, and each $J_i \subseteq P_i$. Corollary 4.21 gives us $J_1 \cap J_2 = (I_1 \cdots I_n)_r$, where the I_i's are pairwise r-comaximal and have prime radical. Then $(I_1 \cdots I_n)_r \subseteq P_1 \cap P_2$, so some $I_i \subseteq P_1$ and $I_j \subseteq P_2$. If $i = j$, then $J_1 \cap J_2$ is contained in the prime r-ideal $\operatorname{rad}(I_i)$, so some $J_k \subseteq \operatorname{rad}(I_i) \subseteq P_1 \cap P_2$, a contradiction. Therefore $i \neq j$, so $(P_1 \cup P_2)_r \supseteq (I_i \cup I_j)_r = H$, showing that $\operatorname{Spec}_r(H)$ is treed. The statement that every r-finitely generated r-ideal has only finitely many minimal primes is equivalent to the statement that every $\mathfrak{r}(r)$-finitely generated $\mathfrak{r}(r)$-ideal has only finitely many minimal primes. Any r-ideal I of the latter type can be written in the form $I = (J_1 \cup \cdots \cup J_n)_{\mathfrak{r}(r)}$, where each $J_i \in \mathcal{C}$. To show that I has only finitely many minimal primes, it will suffice to show that a minimal prime P of I is minimal over some J_i. For each $i = 1, \ldots, n$, shrink P to a minimal prime P_i of J_i. Because $\operatorname{Spec}_r(H)$ is treed, we can reorder so that $P_1 \subseteq \cdots \subseteq P_n \subseteq P$. So P_n contains every J_i, and by the minimality of P we get $P = P_n$. (3) $\Rightarrow$ (1) $\Rightarrow$ (2): Corollary 4.21. □

5. Topological methods

Throughout this section, we will use r for a finitary weak ideal system on a monoid H unless noted otherwise.

The subsets of $\operatorname{Spec}_r(H)$ of the form $V_r(I)$ for $I \in \mathcal{I}_r(H)$ form the closed sets of a topology on $\operatorname{Spec}_r(H)$, called the *Zariski topology*. (This follows from observing that $V_r(H) = \emptyset$, $V_r(\{0\}_r) = \operatorname{Spec}_r(H)$, $V_r(I) \cup V_r(J) = V_r(I \cap J) = V_r((IJ)_r)$ for $I, J \in \mathcal{I}_r(H)$, and that $\bigcap_{\lambda \in \Lambda} V_r(I_\lambda) = V_r((\bigcup_{\lambda \in \Lambda} I_\lambda)_r)$ for any family $\{I_\lambda\}_{\lambda\in\Lambda}$ of r-ideals.) The map V_r is a bijection between the radical r-ideals and the closed subsets of $\mathrm{Spec}_r(H)$, so we could equivalently restrict our attention to the radical r-ideals when defining the Zariski topology. Whenever we talk about a subset of $\mathrm{Spec}_r(H)$, we are always giving it the subspace topology. Perhaps the most fundamental special case are the subspaces of the form $V_r(I)$, whose closed subsets are simply those of the form $V_r(J)$ for some r-ideal $J \supseteq I$. The Zariski topology is of considerable interest to algebraic geometers, so in this section we will translate some of our results into topological terms, as well as exploring how topological methods can help us better understand the r-comaximal factorizations we have been exploring throughout this paper, in addition to a multitude of other topics about rings and monoids. We will just touch the surface of the many possibilities.

First we review some standard topological terminology and results. If X is a topological space, we will say that a *separation* of X of *length* $n < \infty$ is an expression of X as a disjoint union of n nonempty disjoint open (equivalently, closed) subsets. Calling a separation of length 1 *trivial*, a topological space is *connected* if has no nontrivial separations, or, equivalently, if it has no nonempty proper subsets that are both open and closed. A topological space is called *irreducible* if it cannot be written as the union of two proper closed subsets. Any irreducible topological space is connected. A subspace Y of a topological space X is irreducible if and only if whenever $Y \subseteq Z_1 \cup \cdots \cup Z_n$ and each Z_i is a closed subset of X, then some $Z_i \supseteq Y$. A subspace Y of a topological space X is connected if and only if whenever $Y \subseteq Z_1 \sqcup \cdots \sqcup Z_n$ and the Z_i's are pairwise disjoint open (equivalently, closed) subsets of X, we have some $Z_i \supseteq Y$. The *irreducible* (resp., *connected*) *components* of a topological space are those subspaces that are maximal with respect to being irreducible (resp., connected). The union of any chain of irreducible (resp., connected) subspaces is irreducible (resp., connected), so by Zorn's Lemma any irreducible (resp., connected) subspace can be enlarged to an irreducible (resp., connected) component. Clearly, subspaces containing exactly one element are irreducible, so each element of a topological space is contained in an irreducible (resp., connected) component, and the irreducible (resp., connected) components are nonempty except in the trivial case where the space is empty. Because the closure of an irreducible (resp., connected) subspace is irreducible (resp., connected), the irreducible (resp., connected) components of a topological space are closed subsets. Because the union of two non-disjoint connected subspaces is connected, the distinct connected components are disjoint. Thus the connected components are simultaneously open and closed if there are only finitely many of them. A topological space is *disconnected* if it is not connected, and *totally disconnected* if its connected components are of size at most 1. (The way that we have stated the definitions, the empty space is both connected and totally disconnected, but there is no general consensus on this. For our purposes, it will not make a difference.) We will shortly see that for a proper r-ideal I we have $V_r(I)$ irreducible (resp., connected) if and only if I has prime radical (resp., is r-pseudo-irreducible).

A topological space is called *Kolmogorov* or T_0 if for each pair of distinct points there is an open (equivalently, a closed) subset that contains exactly one of the points, *accessible* or T_1 if every point is closed, *Hausdorff* or T_2 if any two distinct points have disjoint open neighborhoods, and *quasicompact* if each of its open coverings has a finite subcovering. A subset of a topological space is called *locally closed* if it is the intersection of a closed and an open subset, a subset of a topological space is *dense* (resp., *strongly dense*) if it intersects every nonempty open (resp., locally closed) subset, and a topological space is called T_D if every point is locally closed. Another characterization of strongly dense subsets is given by the G-topology, which was defined by Picavet [22] on the d-spectrum and later generalized to an arbitrary topological space in [9]. By [9, Proposition 4.1], if X is a topological space, then the operation $A \to \overline{A}^G = \{x \in X \mid \bar{x} = \overline{A \cap \bar{x}}\}$ on the subsets of X yields a new topology called the *G-topology*, where the *G-closed subsets* are those fixed by this operation. Then by [9, Corollary 4.3] the notions of strongly dense and G-dense are equivalent. An element of a topological space is called a *generic point* if its closure is the whole space, and a topological space is called *sober* if every nonempty irreducible closed subspace has a unique generic point. Equivalently, a sober topological space is a T_0 space in which every nonempty irreducible closed subspace has a generic point. As we will soon see, the r-spectrum is a sober space, with a nonempty closed subset $V_r(I)$ irreducible if and only if $\mathrm{rad}(I)$ is prime, in which case $\mathrm{rad}(I)$ is the unique generic point of $V_r(I)$. Of course, the only T_1 space with a generic point is the space with one element, so one would expect the r-spectrum to typically be somewhat strange from the point of one who is accustomed to Hausdorff spaces. Indeed, since the closed points of the r-spectrum correspond to maximal r-ideals, the r-spectrum is T_1 if and only if $\dim_r(H) = 0$. (We will later see that V_r is an inclusion-reversing bijection between the prime r-ideals and the nonempty irreducible closed subspaces of $\mathrm{Spec}_r(H)$, so $\dim_r(H)$ is equivalent to the combinatorial dimension of $\mathrm{Spec}_r(H)$, where the (*combinatorial*) *dimension* of a topological space is the supremum of the lengths of properly descending chains $X_0 \supsetneq X_1 \supsetneq \cdots \supsetneq X_n$ of nonempty irreducible closed subspaces.) It is easy to see that two prime r-ideals are incomparable if and only if they have disjoint open neighborhoods, so in fact $\mathrm{Spec}_r(H)$ being T_1 is equivalent to it being a *Boolean space* (also called a *Stone space*), i.e., quasicompact, Hausdorff, and totally disconnected. To see that it is totally disconnected, we note that we will later see that $\mathrm{Spec}_r(H)$ always has a basis of open quasicompact subspaces, and it is well known that a quasicompact subspace of a Hausdorff space is closed, and any T_0 space with a basis of open subsets that are also closed is totally disconnected. To summarize, we have $T_2 \Rightarrow T_1 \Rightarrow T_D \Rightarrow T_0 \Leftarrow$ sober, and no other implications exist.

Several distinguished subspaces of the d-spectrum of a ring have been studied, and we have already defined the natural weak ideal system generalizations of two of the most important ones in the previous section: the r-Jacobson subspace $\mathrm{Jac}_r(H)$ and the maximal r-ideal space $\mathrm{Max}_r(H)$. Recall that the *r-Jacobson subspace* consists of the *r-Jacobson r-ideals* (*r-J-ideals*), which are the prime r-ideals that are intersections of maximal r-ideals. Another very important subspace whose d-operation counterpart has been studied intensively is the *r-goldspectrum* $\mathrm{Gold}_r(H)$ whose elements are the *r-Goldman r-ideals* (*r-G-ideals*), which are the prime r-ideals that are not equal to the intersection of the prime r-ideals properly containing them. We could go about generalizing several different results about the d-Jacobson subspace and the d-goldspectrum to the r-Jacobson subspace and the r-goldspectrum, but we will instead do all of this at once by proving that the r-spectrum is always homeomorphic to the d-spectrum of some reduced ring, which means that any sort of topological property that the d-spectrum possesses is automatically inherited by the r-spectrum. As it turns out, the r-goldspectrum and the r-Jacobson subspace can be recovered from the r-spectrum in the exact same purely topological way that their counterparts can from the d-spectrum, so these subspaces will also automatically inherit all the topological properties that their d-operation counterparts possess.

Following [9], we define the *Goldman subspace* of a topological space X to be the set $\mathrm{Gold}(X)$ of locally closed points. This definition was motivated by a result of Picavet [22] that showed that $\mathrm{Gold}(\mathrm{Spec}_d(R)) = \mathrm{Gold}_d(R)$ for a ring R. We generalize this to the r-spectrum.

Theorem 5.1. $\mathrm{Gold}(\mathrm{Spec}_r(H)) = \mathrm{Gold}_r(H)$.

Proof. First assume $P \in \mathrm{Spec}_r(H)$ is a locally closed point. Then $\{P\}$ is the intersection of a closed and an open subset of $\mathrm{Spec}_r(H)$. The former we can take to be $V_r(P)$, and the latter we can take to have its complement properly contained in $V_r(P)$, so $\{P\} = V_r(P) \setminus V_r(I)$ for some r-ideal I properly containing P. Thus the intersection of the prime r-ideals properly containing P contains I and hence is not equal to P.

Conversely, for any $P \in \mathrm{Gold}_r(H)$ we have $\{P\} = V_r(P) \setminus V_r(\bigcap_{P \subsetneq Q \in \mathrm{Spec}_r(H)} Q)$. $\square$

Echi [10] defined the *Jacobson subspace* of a topological space X to be $\mathrm{Jac}(X) = \overline{\mathfrak{C}(X)}^G$, where $\mathfrak{C}(X)$ denotes the closed points of X. A topological space is called a *Jacobson space* if its set of closed points is a strongly dense subset, or, equivalently, if $\mathrm{Jac}(X) = X$. As expected, the Jacobson subspace of a topological space is always a Jacobson space since $\mathrm{Jac}(\mathrm{Jac}(X)) = \mathrm{Jac}(X)$. Picavet [22] proved that $\mathrm{Jac}(\mathrm{Spec}_d(R)) = \mathrm{Jac}_d(R)$ for a ring R. Again this generalizes to the r-spectrum.

Theorem 5.2. $\mathrm{Jac}(\mathrm{Spec}_r(H)) = \mathrm{Jac}_r(H)$.

Proof. Straight from the definitions we compute

$$\begin{aligned}\mathrm{Jac}\big(\mathrm{Spec}_r(H)\big) &= \overline{\mathfrak{C}\big(\mathrm{Spec}_r(H)\big)}^G \\ &= \overline{\mathrm{Max}_r(H)}^G\end{aligned}$$

$$
\begin{aligned}
&= \{P \in \operatorname{Spec}_r(H) \mid \overline{P} = \overline{\operatorname{Max}_r(H) \cap \overline{P}}\} \\
&= \Big\{P \in \operatorname{Spec}_r(H) \;\Big|\; V_r(P) = \bigcap \{V_r(I) \mid I \in \mathcal{I}_r(H),\ V_r(I) \supseteq W_r(P)\}\Big\} \\
&= \Big\{P \in \operatorname{Spec}_r(H) \;\Big|\; V_r(P) = \bigcap \{V_r(I) \mid I \in \mathcal{I}_r(H),\ \mathcal{J}_r(I) \subseteq \mathcal{J}_r(P)\}\Big\} \\
&= \{P \in \operatorname{Spec}_r(H) \mid V_r(P) = V_r(\mathcal{J}_r(P))\} \\
&= \{P \in \operatorname{Spec}_r(H) \mid P = \mathcal{J}_r(P)\} \\
&= \operatorname{Jac}_r(P). \qquad \square
\end{aligned}
$$

A topological space is called *spectral* if it is homeomorphic to the d-spectrum of some (reduced) ring, or, equivalently [17, Proposition 4], is a nonempty quasicompact sober space with a basis of open quasicompact subspaces that is closed under finite intersections, or, equivalently [17, Proposition 10], is an inverse limit of finite T_0 spaces. If R is a ring and $I \in \mathcal{I}_d(R)$, then $V_d(I)$ is homeomorphic to $\operatorname{Spec}_d(R/I)$. In particular, any closed subspace of a spectral space is spectral. As we will soon see, the space $\operatorname{Spec}_r(H)$ is spectral, and hence homeomorphic to the d-spectrum of some reduced ring. This fact lets us sometimes prove results about weak ideal systems using classical ring theory. Roughly speaking, we call a statement about monoids and weak ideal systems *topologizable* if it is equivalent to some purely topological statement about the corresponding Zariski topology. For example, the statement "$\dim_r(H) = 0$" is topologizable because it is equivalent to the purely topological statement "$\operatorname{Spec}_r(H)$ is T_1". Sometimes we are not concerned about whether a certain statement is topologizable for all monoids and weak ideal systems, but only whether it is topologizable on some proper subclass. For example, the statement "H has no zero divisors" is not topologizable in general (or even on the subclass consisting of d-operations on rings) because a ring with a unique prime d-ideal need not be a field. However, on the proper subclass consisting of pairs (H, r) with $\{0\} \in \mathcal{I}_{\mathfrak{r}(r)}(H)$ (in particular, this class includes pairs (R, d) for R a reduced ring), this statement is equivalent to "H has a unique minimal prime", which can be topologized as "$\operatorname{Spec}_r(H)$ is irreducible". Using the previously-mentioned homeomorphism, we see that to prove any topologizable statement about monoids and weak ideal systems, it is sufficient to prove it for reduced rings and the d-operation. As it will turn out, many properties dealing with prime r-ideals, r-pseudo-irreducibles, r-comaximal factorization, etc. are topologizable, so there are many different situation where this technique will apply. We will make our comments on this technique somewhat more precise later and give a few examples.

The fact that $\operatorname{Spec}_r(H)$ is spectral is perhaps in the folklore, and in any case is certainly not surprising, but in the interests of completeness and keeping this paper somewhat self-contained, we will include proofs of this and a few other basic facts. We start by proving some theorems telling us how we can topologize some ideal-theoretic concepts.

Theorem 5.3.

(1) *The closed points of* $\operatorname{Spec}_r(H)$ *are the maximal r-ideals.*
(2) *A nonempty closed subspace of* $\operatorname{Spec}_r(H)$ *is irreducible* $\Leftrightarrow$ *it is of the form* $V_r(I)$ *for some* $I \in \mathcal{I}_r(H)$ *with prime radical* $\Leftrightarrow$ *it is of the form* $V_r(P)$ *for some* $P \in \operatorname{Spec}_r(H)$.
(3) *For* $I \in \mathcal{I}_r(H)$, *the map* V_r *is an inclusion-reversing bijection between the prime r-ideals containing I and the nonempty closed irreducible subspaces of* $V_r(I)$; *it takes minimal primes to irreducible components.*
(4) *Two closed subspaces of* $\operatorname{Spec}_r(H)$ *are disjoint if and only if they are of the form* $V_r(I)$ *and* $V_r(J)$ *for* $I, J \in \mathcal{I}_r(H)$ *with* $(I \cup J)_r = H$.

Proof. (1) This has already been noted.

(2) Let $P \in \mathcal{I}_r(H)$ be radical. From the equivalences $P \supseteq (IJ)_r \Leftrightarrow P \supseteq \operatorname{rad}((IJ)_r) \Leftrightarrow V_r(P) \subseteq V_r((IJ)_r) = V_r(I) \cup V_r(J)$ and $P \supseteq I \Leftrightarrow P \supseteq \operatorname{rad}(I) \Leftrightarrow V_r(P) \subseteq V_r(I)$, we easily see that $V_r(P)$ is irreducible if and only if P is prime.

(3) Follows from (2).

(4) This follows from the equivalence $(I \cup J)_r = H \Leftrightarrow V_r(I) \cap V_r(J) = V_r((I \cup J)_r) = V_r(H) = \emptyset$. $\square$

Our chief tool for translating results about r-comaximal factorizations into topological terms will be the following theorem.

Theorem 5.4. *Let I be a proper r-ideal. There is a bijection ψ from the r-comaximal factorizations of I onto the separations of $V_r(I)$, taking $I = (I_1 \cdots I_n)_r$ to $V_r(I) = V_r(I_1) \sqcup \cdots \sqcup V_r(I_n)$. If I is radical, then ψ^{-1} takes $V_r(I) = V_r(I_1) \sqcup \cdots \sqcup V_r(I_n)$ to $I = (\mathrm{rad}(I_1) \cdots \mathrm{rad}(I_n))_r$.*

Proof. We have already noted that if $I = (I_1 \cdots I_n)_r$ is a comaximal factorization, then $V_r(I) = V_r(I_1) \sqcup \cdots \sqcup V_r(I_n)$, and each of these sets are nonempty by the fact that r is finitary and each I_i is proper. Therefore the map is well defined, and Theorem 4.5 part (4) gives us injectivity.

For surjectivity, let $V_r(I) = V_r(J_1) \sqcup \cdots \sqcup V_r(J_n)$ be any separation of $V_r(I)$, where each J_i is an r-ideal containing I. Then the J_i's are pairwise r-comaximal, and $I \subseteq J_1 \cap \cdots \cap J_n = (J_1 \cdots J_n)_r \subseteq \mathrm{rad}((J_1 \cdots J_n)_r) = \mathrm{rad}(I)$, so by Theorem 4.18 there is an r-comaximal factorization $I = (I_1 \cdots I_n)_r$ with each $J_i = (I \cup I_i)_r$, hence each $V_r(J_i) = V_r((I \cup I_i)_r) = V_r(I_i)$.

Finally, assume I is radical and $V_r(I) = V_r(I_1) \sqcup \cdots \sqcup V_r(I_n)$ is a separation of $V_r(I)$. Then each $V_r(I_i) = V_r(\mathrm{rad}(I_i))$ and $V_r(I) = V_r(\mathrm{rad}(I_1)) \sqcup \cdots \sqcup V_r(\mathrm{rad}(I_n)) = V_r(\mathrm{rad}(I_1) \cap \cdots \cap \mathrm{rad}(I_n))$. Because V_r is one-to-one on radical r-ideals we obtain $I = \mathrm{rad}(I_1) \cap \cdots \cap \mathrm{rad}(I_n) = (\mathrm{rad}(I_1) \cdots \mathrm{rad}(I_n))_r$, and of course this is the inverse image of the original separation. $\square$

Corollary 5.5. *Let I be a proper r-ideal.*

(1) *In the following statements we have* (a) $\Leftrightarrow$ (b) $\Leftarrow$ (c).
   (a) *I is r-pseudo-irreducible.*
   (b) *$V_r(I)$ is connected.*
   (c) *$W_r(I)$ is connected.*

(2) *$V_r(I)$ has at most $|W_r(I)|$ connected components.*

(3) *V_r is an inclusion-reversing bijection between the radical r-pseudo-irreducibles containing I and the nonempty connected closed subspaces of $V_r(I)$; it takes the* (*radicals of*) *minimal r-pseudo-irreducibles to connected components.*

Proof. (1) (a) $\Leftrightarrow$ (b): Theorem 5.4. (c) $\Rightarrow$ (b): Because $W_r(I)$ intersects every closed subset of $V_r(I)$, a nontrivial separation of $V_r(I)$ gives rise to a nontrivial separation of $W_r(I)$.

(2) Each connected component of $V_r(I)$ intersects $W_r(I)$, and the connected components are disjoint.

(3) Follows from (1) and the fact that a proper r-ideal is r-pseudo-irreducible if and only if its radical is. $\square$

Example 5.6. Given $n \geqslant 1$, Example 4.19 constructs a ring R with $V_d(\{0\})$ connected and $W_d(\{0\})$ having n connected components.

Corollary 5.7. *The following are equivalent for a proper r-ideal I.*

(1) *I has a complete r-comaximal factorization.*
(2) *$V_r(I)$ has a separation into connected spaces.*
(3) *$V_r(I)$ has only finitely many connected components.*

In this case, the separation of $V_r(I)$ into connected spaces is the disjoint union of its connected components.

Proof. We have (1) $\Leftrightarrow$ (2) by Theorem 5.4. Elementary point-set topology shows that (2) $\Leftrightarrow$ (3) is true with any topological space in place of $V_r(I)$, and the same goes for the last statement. $\square$

Using Corollary 5.5 to translate, it is obvious that there is a strong analogy between Theorem 4.15 and Corollary 5.7, and to some extent one can prove one from the other.

Theorem 5.8. *The following are equivalent for $I \in \mathcal{I}_r(H)$.*

(1) $\operatorname{rad}(I) \in \mathcal{I}_{\mathfrak{r}(r),f}(H)$.
(2) *Every collection of closed subsets of* $\operatorname{Spec}_r(H)$ *with intersection contained in $V_r(I)$ has a finite subcollection whose intersection is contained in $V_r(I)$.*
(3) *Every collection of closed subsets of* $\operatorname{Spec}_r(H)$ *whose intersection equals $V_r(I)$ has a finite subcollection whose intersection equals I.*
(4) $\operatorname{Spec}_r(H) \setminus V_r(I)$ *is quasicompact.*

Proof. It is simple to see that (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) holds with any topological space in place of $\operatorname{Spec}_r(H)$ and any closed subset in place of $V_r(I)$. (1) $\Rightarrow$ (2): Assume $\operatorname{rad}(I) \in \mathcal{I}_{\mathfrak{r}(r),f}(H)$ and let $\{I_\lambda\}_{\lambda\in\Lambda}$ be any collection of r-ideals with $V_r(I) \supseteq \bigcap_{\lambda\in\Lambda} V_r(I_\lambda) = V_r((\bigcup_{\lambda\in\Lambda} I_\lambda)_r)$. Then $\operatorname{rad}(I) \subseteq (\bigcup_{\lambda\in\Lambda} I_\lambda)_{\mathfrak{r}(r)}$, so by the fact that $\operatorname{rad}(I)$ is $\mathfrak{r}(r)$-finitely generated and $\mathfrak{r}(r)$ is finitary, there is some finite subcollection $\Lambda_0 \subseteq \Lambda$ with $\operatorname{rad}(I) \subseteq (\bigcup_{\lambda\in\Lambda_0} I_\lambda)_{\mathfrak{r}(r)}$. Thus $V_r(I) \supseteq V_r((\bigcup_{\lambda\in\Lambda_0} I_\lambda)_r) = \bigcap_{\lambda\in\Lambda_0} V_r(I_\lambda)$, as desired. (3) $\Rightarrow$ (1): We have $\bigcap_{a\in I} V_r(\{a\}_r) = V_r(\bigcup_{a\in I}\{a\}_r) = V_r(I)$, so if (3) holds then there are $a_1,\dots,a_m \in I$ with $V_r(I) = \bigcap_{i=1}^m V_r(\{a_i\}_r) = V_r(\{a_1,\dots,a_m\}_r)$ and hence $\operatorname{rad}(I) = \{a_1,\dots,a_m\}_{\mathfrak{r}(r)}$. □

Theorem 5.9. *The r-spectrum is a spectral topological space, hence homeomorphic to the d-spectrum of some reduced ring.*

Proof. Recall that we need to prove that $\operatorname{Spec}_r(H)$ is T_0 and quasicompact, each of its nonempty irreducible subspaces contains a generic point, and the collection of its open quasicompact subspaces is closed under finite intersections and forms a basis of open subsets.

T_0: Let $P, Q \in \operatorname{Spec}_r(H)$ be distinct, say $Q \nsubseteq P$. Then $V_r(Q)$ is a closed subset containing Q but not P.

Quasicompact: [13, Exercise 6.7].

Generic points: For $P \in \operatorname{Spec}_r(H)$, every closed subset containing P contains $V_r(P)$, and hence P is a generic point of $V_r(P)$.

Open quasicompact subspaces: By Theorem 5.8, the open quasicompact subspaces of $\operatorname{Spec}_r(H)$ are the complements of the closed subsets of the form $V_r(I)$ with $\operatorname{rad}(I) \in \mathcal{I}_{\mathfrak{r}(r),f}(H)$. Therefore it will suffice to show that the collection of closed subsets of this form is closed under finite unions and forms a basis of closed subsets. The closure under finite unions follows from the equality $V_r(I) \cup V_r(J) = V_r((IJ)_{\mathfrak{r}(r)})$. To show that this collection forms a basis of closed subsets of $\operatorname{Spec}_r(H)$, we need to show that for each $I \in \mathcal{I}_r(H)$ and $P \notin V_r(I)$, there is a $J \in \mathcal{I}_{\mathfrak{r}(r),f}(H)$ with $P \notin V_r(J) \supseteq V_r(I)$. In this setup, there is some $a \in I \setminus P$, so $P \notin V_r(\{a\}_{\mathfrak{r}(r)}) \supseteq V_r(I)$, as desired. □

Theorem 5.10. *Let $\mathcal{C}$ be a class of pairs (H,r) such that H is a monoid and r is a finitary weak ideal system on H. Assume that $(R,d) \in \mathcal{C}$ for each reduced ring R. Let $S : \mathcal{C} \to \{true, false\}$ be some statement about pairs in $\mathcal{C}$, and assume that it is* topologizable. *More precisely, assume that there is some property T of topological spaces such that for each $(H,r) \in \mathcal{C}$ we have $S(H,r)$ if and only if* $\operatorname{Spec}_r(H)$ *has property T. Then $S(H,r)$ is true for all $(H,r) \in \mathcal{C}$ if and only if $S(R,d)$ is true for all reduced rings R.*

Proof. Each $\operatorname{Spec}_r(H)$ is spectral and thus homeomorphic to the d-spectrum of some reduced ring. Thus $S(H,r)$ holds for each $(H,r) \in \mathcal{C}$ $\Leftrightarrow$ T holds for $\operatorname{Spec}_r(H)$ for each $(H,r) \in \mathcal{C}$ $\Leftrightarrow$ T holds for the d-spectrum of every reduced ring $\Leftrightarrow$ $S(R,d)$ holds for all reduced rings R. □

In [8] it is noted that the Zariski topology can be given to the strongly d-irreducible spectrum of a ring. This remark can be extended to the strongly r-irreducible spectrum. More specifically, for $X \subseteq H$ we define $SV_r(X) = \{P \in \operatorname{SSpec}_r(H) \mid P \supseteq X\}$, and the *Zariski topology* on $\operatorname{SSpec}_r(H)$ is obtained by declaring the subsets of the form $SV_r(I)$ for $I \in \mathcal{I}_r(H)$ to be closed. (As was the case for the r-spectrum, the observation that these subsets do in fact form the closed sets of a topology follows from observing that $SV_r(H) = \emptyset$, $SV_r(\{0\}_r) = \operatorname{SSpec}_r(H)$, $SV_r(I) \cup SV_r(J) = SV_r(I \cap J)$ for $I, J \in \mathcal{I}_r(H)$, and that $\bigcap_{\lambda\in\Lambda} SV_r(I_\lambda) = SV_r((\bigcup_{\lambda\in\Lambda} I_\lambda)_r)$ for any family $\{I_\lambda\}_{\lambda\in\Lambda}$ of r-ideals.) The topology that $\mathrm{Spec}_r(H)$ inherits as a subspace of $\mathrm{SSpec}_r(H)$ is its usual Zariski topology. For $I \in \mathcal{I}_r(H)$, we define the *weak r-radical* of I to be $\mathrm{wrad}_r(I) = \bigcap_{I \subseteq P \in \mathrm{SSpec}_r(H)} P \in \mathcal{I}_r(H)$, and we say I is *weakly r-radical* if $I = \mathrm{wrad}_r(I)$. In the following theorem we give some elementary properties of strongly r-irreducible r-ideals and weak r-radicals. This theorem extends and generalizes some work previously done for the d-operation in [6, Theorems 2.1, 4.1, and 4.2].

Theorem 5.11. *Let r be a (not necessarily finitary) weak ideal system on a monoid H, and let* $I, J \in \mathcal{I}_r(H)$*.*

(1) $I \subseteq \mathrm{wrad}_r(I) = \mathrm{wrad}_r(\mathrm{wrad}_r(I)) \subseteq \mathrm{rad}_r(I)$.
(2) $\mathrm{wrad}_r(I \cap J) = \mathrm{wrad}_r(I) \cap \mathrm{wrad}_r(J)$.
(3) $\mathrm{wrad}_r(I) \subseteq \mathrm{wrad}_r(J) \Leftrightarrow SV_r(J) \subseteq SV_r(I)$.
(4) *If* $I \neq H$*, then the following are equivalent.*
   (a) $\mathrm{wrad}_r(I)$ *is r-irreducible.*
   (b) $\mathrm{wrad}_r(I)$ *is strongly r-irreducible.*
   (c) $SV_r(I)$ *is irreducible.*
   In particular, the notions of r-irreducible and strongly r-irreducible are equivalent for weakly r-radical proper r-ideals.
(5) *If* $I \neq H$*, then the following are equivalent.*
   (a) $\mathrm{rad}_r(I)$ *is prime.*
   (b) $\mathrm{rad}_r(I)$ *is r-irreducible.*
   (c) $\mathrm{rad}_r(I)$ *is strongly r-irreducible.*
   (d) $V_r(I)$ *is irreducible.*
   In particular, the notions of prime and (strongly) r-irreducible are equivalent for r-radical r-ideals.
(6) *Every strongly r-irreducible r-ideal containing an r-ideal I can be shrunk to a minimal such strongly r-irreducible r-ideal, called a* minimal strong r-irreducible *of I.*
(7) *The map* $Q \to SV_r(Q)$ *is a one-to-one correspondence between the minimal strong r-irreducibles of I and the irreducible components of* $SV_r(I)$*.*
(8) *Assume r is finitary and* $I \neq H$*. Then there is a bijection* θ *from the r-comaximal factorizations of I onto the separations of* $SV_r(I)$*, taking* $I = (I_1 \cdots I_n)_r$ *to* $SV_r(I) = SV_r(I_1) \sqcup \cdots \sqcup SV_r(I_n)$*. Therefore the separations of* $SV_r(I)$ *and* $V_r(I)$ *are in one-to-one correspondence. If I is weakly r-radical, then* θ^{-1} *takes* $SV_r(I) = SV_r(I_1) \sqcup \cdots \sqcup SV_r(I_n)$ *to* $I = (\mathrm{wrad}_r(I_1) \cdots \mathrm{wrad}_r(I_n))_r$*.*
(9) *If r is finitary and* $I \neq H$*, then the following are equivalent.*
   (a) *I is r-pseudo-irreducible.*
   (b) $SV_r(I)$ *is connected.*
   (c) $V_r(I)$ *is connected.*

Proof. (1)–(3) Simple consequences of the definitions.

(4) Assume $I \neq H$. (b) $\Rightarrow$ (a): Clear. (a) $\Rightarrow$ (c): Assume $\mathrm{wrad}_r(I)$ is r-irreducible and $SV_r(I) = SV_r(I_1) \cup SV_r(I_2) = SV_r(I_1 \cap I_2)$ for some $I_1, I_2 \in \mathcal{I}_r(H)$. We have $\mathrm{wrad}_r(I) = \mathrm{wrad}_r(I_1 \cap I_2) = \mathrm{wrad}_r(I_1) \cap \mathrm{wrad}_r(I_2)$, so some $\mathrm{wrad}_r(I_i) = \mathrm{wrad}_r(I)$ and hence $SV_r(I_i) = SV_r(I)$. Therefore $SV_r(I)$ is irreducible. (c) $\Rightarrow$ (b): Assume $SV_r(I)$ is irreducible and $\mathrm{wrad}_r(I) \supseteq J_1 \cap J_2$ for some $J_1, J_2 \in \mathcal{I}_r(H)$. Then $\mathrm{wrad}_r(I) \supseteq \mathrm{wrad}_r(J_1 \cap J_2)$, so $SV_r(I) \subseteq SV_r(J_1 \cap J_2) = SV_r(J_1) \cup SV_r(J_2)$. By the irreducibility of $SV_r(I)$, some $SV_r(J_i) \supseteq SV_r(I)$, and hence $\mathrm{wrad}_r(I) \supseteq \mathrm{wrad}_r(J_i) \supseteq J_i$, showing that $\mathrm{wrad}_r(I)$ is strongly r-irreducible.

(5) Assume $I \neq H$. We have (a) $\Leftrightarrow$ (d) by Theorem 5.3 part (2) (the proof does not involve the finitary property), the implication (b) $\Leftrightarrow$ (c) follows from part (4), and (a) $\Rightarrow$ (c) is clear, so all that remains is (c) $\Rightarrow$ (a). Assume $\mathrm{rad}_r(I)$ is strongly r-irreducible and $\mathrm{rad}_r(I) \supseteq (I_1 I_2)_r$. Then $\mathrm{rad}_r(I) \supseteq \mathrm{rad}_r((I_1 I_2)_r) = \mathrm{rad}_r(I_1) \cap \mathrm{rad}_r(I_2)$, so some $I_i \subseteq \mathrm{rad}_r(I_i) \subseteq \mathrm{rad}_r(I)$ by strong r-irreducibility, showing that $\mathrm{rad}_r(I)$ is prime.

(6) It is clear from the definitions that any intersection of a chain of strongly r-irreducible r-ideals is strongly r-irreducible, so the statement follows by Zorn's Lemma.

(7) Follows from parts (4) and (6).

(8) Similar to the proof of Theorem 5.4.

(9) Follows from part (8). $\square$

The inclusions in part (1) can be proper. For example, any d-irreducible d-ideal that is not strongly d-irreducible is not weakly d-radical by part (4), and any strongly d-irreducible d-ideal that is not prime is weakly d-radical but not radical by part (5).

Theorem 5.12. *The space* $\mathrm{SSpec}_r(H)$ *is a spectral topological space, hence homeomorphic to the d-spectrum of some reduced ring.*

Proof. Similar to the proof of Theorem 5.9. □

The above theorem gives rise to the following variant of Theorem 5.10.

Theorem 5.13. *Let* $\mathcal{C}$ *be a class of pairs* (H, r) *such that* H *is a monoid and* r *is a finitary weak ideal system on* H*, let* $\mathcal{R}$ *be the class of pairs* (R, d) *with* R *a reduced ring, and assume* $\mathcal{R} \subseteq \mathcal{C}$*. Let* $S : \mathcal{C} \to \{true, false\}$ *be some statement about pairs in* $\mathcal{C}$*, and assume that there is some property* T *of topological spaces such that for each* $(H, r) \in \mathcal{C}$ *we have* $S(H, r)$ *if and only if* $\mathrm{SSpec}_r(H)$ *has property* T*. Let* $S' : \mathcal{R} \to \{true, false\}$ *be a statement about pairs in* $\mathcal{R}$ *such that for reduced rings* R *we have* $S'(R, d)$ *if and only if* $\mathrm{Spec}_d(R)$ *has property* T*. Then* $S(H, r)$ *is true for all* $(H, r) \in \mathcal{C}$ *if and only if* $S'(R, d)$ *is true for all reduced rings* R*.*

Proof. Similar to the proof of Theorem 5.10. □

In the remainder of the section we will give a (far from exhaustive) collection of several examples of applications of the topological approach to finitary weak ideal systems. Sometimes translating a statement into a topological form allows one to instantly prove it by reducing to the d-operation case as in Theorem 5.10, sometimes this translation suggests a topological generalization, and sometimes this translation allows one to apply topological techniques to more easily construct rings with a desired property.

Example 5.14. Consider Corollary 4.7: "Any two r-comaximal factorizations of the same proper r-ideal have a common refinement (up to order)". By Theorem 5.4, this statement is topologizable as "Any two separations of a nonempty closed subspace of $\mathrm{Spec}_r(H)$ have a common refinement (up to order)". Hence we can obtain an alternate proof by reducing to the d-operation case, which is covered by the proof of [19, Theorem 5.1]. Alternatively, it is very easy to prove the topological statement directly. Indeed, if $X_1 \sqcup \cdots \sqcup X_m = Y_1 \sqcup \cdots \sqcup Y_n$ are separations of a topological space, then so is $(X_1 \cap Y_1) \sqcup \cdots \sqcup (X_m \cap Y_n)$ (ignoring any empty sets in this disjoint union), and each $X_i = (X_i \cap Y_1) \sqcup \cdots \sqcup (X_i \cap Y_n)$ and each $Y_j = (X_1 \cap Y_j) \sqcup \cdots \sqcup (X_m \cap Y_j)$.

Example 5.15. Consider the statement "Complete r-comaximal factorizations are unique (up to order) whenever they exist". (See Theorem 4.11.) By Theorem 5.4, it is topologizable as "Separations of closed subspaces of $\mathrm{Spec}_r(H)$ into connected subspaces are unique whenever they exist". Thus an alternate proof can be obtained by reducing to the d-operation case, which is covered in [19, Theorem 5.1]. Of course, a different more direct topological proof is available by noting that it is standard that separations of topological spaces into connected subspaces are unique whenever they exist. (For those who are keeping track, we have now seen four completely different proofs of this uniqueness fact. The first used Theorems 2.1 and 4.4, the second used Corollary 4.7, and the last two were given in this example.)

Example 5.16. We give some alternate topological proofs of some consequences of Theorem 4.13. It is an elementary topological fact that every connected component of a topological space contains an irreducible component, so this provides an alternate topological proof of the fact that every minimal r-pseudo-irreducible of a proper r-ideal is contained in a minimal prime. Taking this further, the number of connected components of a topological space is at most its number of irreducible components, so the number of minimal r-pseudo-irreducibles of an r-ideal is at most its number of minimal primes. Corollary 5.5 part (2) shows that the number of minimal r-pseudo-irreducibles of an r-ideal is at most the number of maximal r-ideals containing it.

Example 5.17. Let I be a proper r-ideal. Passing to $\mathrm{rad}(I)$ and translating into Zariski topology terminology, we see that I has an r-comaximal factorization into r-ideals with prime radical $\Leftrightarrow$ $\mathrm{rad}(I)$ has an r-comaximal factorization into prime r-ideals $\Leftrightarrow$ $V_r(I) = V_r(\mathrm{rad}(I))$ has a separation into irreducible subspaces. Using topology, it is easy to see that this last statement is equivalent to saying that (1) $V_r(I)$ has only finitely many irreducible components, and (2) the irreducible components of $V_r(I)$ are disjoint. Translating this back into ideal-theoretic terms, we get that I has an r-comaximal factorization into r-ideals with prime radical if and only if (1) it has only finitely many minimal primes, and (2) its minimal primes are pairwise r-comaximal, recovering Corollary 4.21.

Example 5.18. On a similar note to the previous example, the characterization of r-URM's in Corollary 4.22 is topologizable. Consider the following statements. (We know these are equivalent by Corollary 4.22.)

(1) H is an r-URM.
(2) $\mathrm{Spec}_r(H)$ is treed and every r-finitely generated r-ideal has only finitely many minimal primes.
(3) There is a collection $\mathcal{C}$ of $\mathfrak{r}(r)$-ideals such that (1) $\mathcal{C}$ is closed under finite intersections, (2) every $\mathfrak{r}(r)$-ideal is $\mathfrak{r}(r)$-generated by a collection of elements of $\mathcal{C}$, (3) every element of $\mathcal{C}$ has only finitely many minimal primes, and (4) the minimal primes of an element of $\mathcal{C}$ are pairwise r-comaximal.

These are respectively topologizable as follows. (We have omitted the equivalent condition about 1-generated r-ideals because it does not translate into topological terms and is obviously equivalent to (2) and (3) if those two statements are equivalent.)

(T1) Every closed subspace of $\mathrm{Spec}_r(H)$ whose complement is quasicompact has a (unique) separation into irreducible subspaces.
(T2) Any two incomparable irreducible closed subspaces of $\mathrm{Spec}_r(H)$ are disjoint, and every closed subspace of $\mathrm{Spec}_r(H)$ whose complement is quasicompact has only finitely many irreducible components.
(T3) The topological space $\mathrm{Spec}_r(H)$ has a basis $\mathcal{C}$ of closed subsets such that (1) $\mathcal{C}$ is closed under finite unions, (2) every element of $\mathcal{C}$ has only finitely many irreducible components, and (3) the irreducible components of an element of $\mathcal{C}$ are disjoint.

Hence we may reduce proving Corollary 4.22 to the d-operation case. Alternatively, it is possible to directly prove the following generalization of the topological version of the theorem.

Theorem 5.19. *Let X be a topological space whose collection of quasicompact open subspaces forms a basis and is closed under finite intersections. The following are equivalent.*

(1) *Every closed subspace of X whose complement is quasicompact has a* (*unique*) *separation into irreducible subspaces.*
(2) *Any two incomparable irreducible closed subspaces of X are disjoint, and every closed subspace of X whose complement is quasicompact has only finitely many irreducible components.*
(3) *The topological space X has a basis $\mathcal{C}$ of closed subsets such that* (1) *$\mathcal{C}$ is closed under finite unions,* (2) *every element of $\mathcal{C}$ has only finitely many irreducible components, and* (3) *the irreducible components of an element of $\mathcal{C}$ are disjoint.*

Proof. First we observe that a nonempty topological space has a (unique) separation into irreducible subspaces if and only if it has only finitely many irreducible components and its irreducible components are disjoint. From this observation we obtain (2) $\Rightarrow$ (1), and we also note that we can get (1) $\Rightarrow$ (3) by taking $\mathcal{C}$ to be the collection of nonempty closed subspaces of X with quasicompact complement. (3) $\Rightarrow$ (2): Assume (3). Now let Y_1 and Y_2 be any incomparable irreducible closed subspaces of X. We can observe that Y_1 and Y_2 are the intersections of the elements of $\mathcal{C}$

that contain them, so there are $Z_1, Z_2 \in \mathcal{C}$ with $Z_1 \not\supseteq Y_2$, $Z_2 \not\supseteq Y_1$, and each $Z_i \supseteq Y_i$. The subspace $Z_1 \cup Z_2 \in \mathcal{C}$ has a unique separation into irreducible closed subspaces, say $Z_1 \cup Z_2 = W_1 \sqcup \cdots \sqcup W_n$. Then $Y_1 \cup Y_2 \subseteq Z_1 \cup Z_2 = W_1 \sqcup \cdots \sqcup W_n$, so some $W_i \supseteq Y_1$ and $W_j \supseteq Y_2$. If $i = j$, then $Z_1 \cup Z_2 \supseteq W_i$, so some $Z_k \supseteq W_i \supseteq Y_1 \cup Y_2$, a contradiction. Therefore $i \neq j$, so $Y_1 \cap Y_2 \subseteq W_i \cap W_j = \emptyset$. Now that we have shown that any two incomparable irreducible closed subspaces of X are disjoint, we show that every nonempty closed subspace of X whose complement is quasicompact has only finitely many irreducible components. Any subspace Y of the latter type can be written in the form $Y = Z_1 \cap \cdots \cap Z_n$, where each $Z_i \in \mathcal{C}$. To show that Y has only finitely many irreducible components, it will suffice to show that an irreducible component W of Y is an irreducible component of some Z_i. For each $i = 1, \ldots, n$, enlarge W to an irreducible component W_i of Z_i. Because any two incomparable irreducible closed subspaces of X are disjoint, we may reorder so that $W_1 \supseteq \cdots \supseteq W_n \supseteq W$. Thus $W \subseteq W_n \subseteq Y$, and by the maximality of W we get $W = W_n$. □

Example 5.20. A particularly interesting example of the topological approach concerns the relationship between the ascending chain condition on radical r-ideals and each r-ideal having only finitely many minimal primes. In [13, Lemma 7.8.1] it is shown that H is $\mathfrak{r}(r)$-Noetherian if and only if (1) it satisfies the ascending chain condition on prime r-ideals, and (2) every r-ideal has only finitely many minimal primes.

Using the idea of Noetherian topological spaces, this theorem is topologizable. Recall that a *Noetherian topological space* is defined by one of the following equivalent statements: (1) the open subsets satisfy the ascending chain condition, (2) the closed subsets satisfy the descending chain condition, (3) every nonempty collection of open subsets has a maximal element, (4) every nonempty collection of closed subsets has a minimal element, or (5) every subspace is quasicompact. By (5) we see that every subspace of a Noetherian topological space is Noetherian. Using equivalent condition (2), we see that $\mathrm{Spec}_r(H)$ is Noetherian if and only if H is $\mathfrak{r}(r)$-Noetherian.

Thus the above theorem is topologizable as "$\mathrm{Spec}_r(H)$ is Noetherian if and only if (1) it satisfies the descending chain condition on irreducible closed subspaces, and (2) every closed subspace has only finitely many irreducible components", and therefore may be immediately deduced from the well-known d-operation special case. Alternatively, there is a direct proof of a more general topological statement.

Theorem 5.21. *A topological space is Noetherian if and only if* (1) *it satisfies the descending chain condition on irreducible closed subspaces, and* (2) *every* (*closed*) *subspace has only finitely many irreducible components. In particular, a finite-dimensional topological space is Noetherian if and only if every* (*closed*) *subspace has only finitely many irreducible components.*

Proof. ($\Rightarrow$): (This direction is well known.) Because every subspace of a Noetherian space is Noetherian, it will suffice to show that every Noetherian space has only finitely many irreducible components. Suppose to the contrary that there is a Noetherian space X with infinitely many irreducible components. Using the Noetherian property, the nonempty collection of closed subspaces of X with infinitely many irreducible components has a minimal element Y. Since Y is not irreducible, it may be written as a union of two proper closed subspaces. However, by minimality each of these closed subspaces must have only finitely many irreducible components, so we may write Y as a union of finitely many irreducible subspaces, and it follows that Y has only finitely many irreducible components, a contradiction. ($\Leftarrow$): By contradiction. Suppose that there is a topological space satisfying (1) and the weaker version of (2), and that there is a properly descending chain $X_1 \supsetneq X_2 \supsetneq \cdots$ of closed subsets. For each i let $Y_{i,1}, \ldots, Y_{i,n_i}$ be the irreducible components of X_i that are not contained in the closed subspace $X = \bigcap_{n=1}^{\infty} X_n$, noting that each $n_i \geqslant 1$ since no $X_i = X$. Let $Y_1, \ldots, Y_n$ be the irreducible components of X, and note that $X = Y_1 \cup \cdots \cup Y_n$ and each $X_i = X \cup Y_{i,1} \cup \cdots \cup Y_{i,n_i}$. Take any $i \geqslant 2$ and any $k_i \in \{1, \ldots, n_i\}$. Then $Y_1 \cup \cdots \cup Y_n \cup Y_{i-1,1} \cup \cdots \cup Y_{i-1,n_{i-1}} = X_{i-1} \supsetneq X_i \supseteq Y_{i,k_i}$. In this case, each $Y_j \not\supseteq Y_{i,k_i}$ since $X \not\supseteq Y_{i,k_i}$, so by the irreducibility of Y_{i,k_i} we have some $Y_{i-1,k_{i-1}} \supseteq Y_{i,k_i}$. Repeating this process, we obtain a finite sequence $\{k_j\}_{j=1}^{i}$ of positive integers so that $Y_{1,k_1} \supseteq \cdots \supseteq Y_{i,k_i}$. We wish to construct a sequence $\{v_j\}_{j=1}^{\infty}$ of positive integers with each $v_j \in \{1, \ldots, n_j\}$ and $Y_{1,v_1} \supseteq Y_{2,v_2} \supseteq \cdots$.

By the above argument, we can construct such sequences of arbitrarily long finite length. Given $n \geqslant 0$ and $v_1, \ldots, v_n$ such that there are arbitrarily long finite sequences of the desired form starting with $v_1, \ldots, v_n$, there are only a finite number of choices for the next entry in the sequence, so we can choose some v_{n+1} so that there are arbitrarily long finite sequences of the desired type starting with $v_1, \ldots, v_{n+1}$. This recursively constructs an infinite sequence of the desired form, and we have a descending chain $Y_{1,v_1} \supseteq Y_{2,v_2} \supseteq \cdots$, which by hypothesis stabilizes. Thus $X = \bigcap_{n=1}^{\infty} X_n \supseteq \bigcap_{n=1}^{\infty} Y_{n,v_n} = Y_{N,v_N}$ for some N, a contradiction. □

Example 5.22. We give one example of an application of Theorem 5.13. We return to the well-known theorem that a ring is $\mathfrak{r}(d)$-Noetherian if and only if it satisfies the ascending chain condition on prime d-ideals and every d-ideal has only finitely many minimal primes. Translating this into an equivalent purely topological statement T about the d-spectrum, then finding a statement equivalent to T holding for $\mathrm{SSpec}_r(H)$ yields the following theorem: H satisfies the ascending chain condition on weakly r-radical r-ideals if and only if it satisfies the ascending chain condition on strongly r-irreducible r-ideals and every r-ideal has only finitely many minimal strong r-irreducibles. We can similarly apply this method to any other topologizable theorem to get a corresponding theorem relating to strongly r-irreducible notions.

We end this paper by filling in the details for the facts alluded to in Example 4.17 in the last section. For any $I \in \mathcal{I}_r(H)$ whose minimal primes are maximal r-ideals, we have $V_r(I)$ a Boolean space, so such an r-ideal has a complete r-comaximal factorization if and only if it is contained in only finitely many prime r-ideals. In particular, a ring of d-dimension 0 is connected if and only if it has a unique prime d-ideal. However, we give an example of a connected ring of d-dimension 1 with infinitely many minimal primes and infinitely many maximal d-ideals.

Example 5.23. (There is a connected ring of d-dimension 1 with infinitely many minimal primes and infinitely many maximal d-ideals.) There are undoubtedly many different ways to construct such a ring, but we find it interesting to proceed by topological methods. That is, we show that there is a 1-dimensional connected spectral space with infinitely many irreducible components and infinitely many closed points, and we do this by a very general construction where we "glue together" spectral spaces. Let Y and Z be disjoint spectral spaces and $X = Y \sqcup Z$. We can topologize X so that the given topologies of Y and Z correspond with their topologies as subspaces of X by declaring the closed subsets of X to be the closed subsets of Z and the subsets of the form $C \sqcup Z$ for closed $C \subseteq Y$. The nonempty irreducible closed subspaces of X are precisely those of the forms $C \sqcup Z = \overline{\{y\}}$ and $D = \overline{\{z\}}$ for nonempty closed irreducible subspaces C of Y and D of Z with unique generic points y and z, respectively. Since X is clearly T_0, this proves that it is sober. The quasicompactness of X follows easily from that of Z. Let $\mathcal{C}_1$ (resp., $\mathcal{C}_2$, $\mathcal{C}_3$) denote the collection of closed subsets of X (resp., Y, Z) with quasicompact complement in X (resp., Y, Z). Then $\mathcal{C}_1 = \{C \sqcup Z \mid C \in \mathcal{C}_2\} \cup \mathcal{C}_3$. The fact that $\mathcal{C}_1$ forms a basis of closed subsets of X and is closed under finite unions follows from the corresponding properties of $\mathcal{C}_2$ and $\mathcal{C}_3$. We have now shown that X is a spectral space. Since any two closed subsets of X whose union is X must contain Z, we see that X is connected. Furthermore, any maximal chain of nonempty irreducible closed subspaces of X must be of the form $C_0 \sqcup Z \supsetneq \cdots \supsetneq C_m \sqcup Z \supsetneq D_0 \supsetneq \cdots \supsetneq D_n$, where $C_0 \supsetneq \cdots \supsetneq C_m$ and $D_0 \supsetneq \cdots \supsetneq D_n$ are chains of nonempty irreducible closed subspaces of Y and Z, respectively. Thus $\dim(X) = \dim(Y) + \dim(Z) + 1$. Taking Y and Z to be infinite 0-dimensional spectral spaces, we get X to be a 1-dimensional connected spectral space with infinitely many irreducible components (namely $\{y\} \sqcup Z$ for $y \in Y$) and infinitely many closed points (namely the points of Z).

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