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J. Differential Equations 246 (2009) 845–869

Journal of Differential Equations

www.elsevier.com/locate/jde

Random attractors for stochastic reaction–diffusion equations on unbounded domains

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Received 13 March 2008

Available online 3 July 2008

Abstract

The existence of a pullback attractor is established for a stochastic reaction–diffusion equation on all n-dimensional space. The nonlinearity is dissipative for large values of the state and the stochastic nature of the equation appears as spatially distributed temporal white noise. The reaction–diffusion equation is recast as a random dynamical system and asymptotic compactness for this is demonstrated by using uniform a priori estimates for far-field values of solutions.

MSC: primary 35B40; secondary 35B41, 37L30

Keywords: Stochastic reaction–diffusion equation; Random attractor; Pullback attractor; Asymptotic compactness

1. Introduction

In this paper we investigate the asymptotic behavior of solutions to the following stochastic reaction–diffusion equation with additive noise defined in the entire space $\mathbb{R}^n$:

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[1] Supported in part by NSF grant DMS 0401708.
[2] Supported in part by NSF grant DMS-0703521.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2008.05.017

$$du + (\lambda u - \Delta u)\,dt = \big(f(x,u) + g(x)\big)\,dt + \sum_{j=1}^{m} h_j\,dw_j, \tag{1.1}$$

where λ is a positive constant, g and h_j ($1 \leqslant j \leqslant m$) are given functions defined on $\mathbb{R}^n$, f is a nonlinear function satisfying certain dissipative conditions, and $\{w_j\}_{j=1}^m$ are independent two-sided real-valued Wiener processes on a probability space which will be specified later.

Stochastic differential equations of this type arise from many physical systems when random spatio-temporal forcing is taken into account. These random perturbations are intrinsic effects in a variety of settings and spatial scales. They may be most obviously influential at the microscopic and smaller scales but indirectly they play an important role in macroscopic phenomena. In order to capture the essential dynamics of random systems with wide fluctuations, the concept of pullback random attractor was introduced in [15,17], being an extension to stochastic systems of the theory of attractors for deterministic equations found in [5,20,26,29,32], for instance. The existence of such random attractors has been studied for stochastic PDEs on bounded domains, see, e.g., [10,14,15,17] and the references therein, but little is known for unbounded domains. Here we prove the existence of such a random attractor for the stochastic reaction–diffusion equation (1.1) defined in $\mathbb{R}^n$. It is worth mentioning that, in the case of lattice systems defined on the entire integer set, the existence of a random attractor was proved recently in [8] and the deterministic lattice case was treated in [9].

Notice that the unboundedness of the domain introduces a major difficulty for proving the existence of an attractor because Sobolev embeddings are no longer compact and so the asymptotic compactness of solutions cannot be obtained by the standard method. In the case of deterministic equations, this difficulty can be overcome by the energy equation approach, introduced by Ball in [6,7] and then employed by several authors to prove the asymptotic compactness of deterministic equations in unbounded domains. This and related approaches may be found in, for example, [11,18,19,21–23,28,34] and the references therein. In this paper, we provide uniform estimates on the far-field values of solutions to circumvent the difficulty caused by the unboundedness of the domain. This idea was developed in [33] to prove asymptotic compactness for the deterministic version of (1.1) on $\mathbb{R}^n$, and later used in several other works, see, e.g., [1,2,4,24,25,27,30,31]. The main contribution of this paper is to extend the method of using tail estimates to the case of stochastic dissipative PDEs, and prove the existence of a random attractor for the stochastic reaction–diffusion equation (1.1) in particular, defined on the unbounded domain $\mathbb{R}^n$. It is clear that our method can be used for a variety of other equations, as it was for the deterministic case.

The paper is organized as follows. In the next section, we recall some fundamental results on the existence of a pullback random attractor for random dynamical systems. In Section 3, we transform (1.1) into a continuous random dynamical system. Section 4 is devoted to obtaining uniform estimates of solutions as $t \to \infty$. These estimates are necessary for proving the existence of bounded absorbing sets and the asymptotic compactness of the equation. In the last section, we first establish the asymptotic compactness of the solution operator by giving uniform estimates on the tails of solutions, and then prove the existence of a pullback random attractor.

We denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the norm and the inner product in $L^2(\mathbb{R}^n)$ and use $\|\cdot\|_p$ to denote the norm in $L^p(\mathbb{R}^n)$. Otherwise, the norm of a general Banach space X is written as $\|\cdot\|_X$. The letters c and c_i ($i = 1, 2, \ldots$) are generic positive constants which may change their values from line to line or even in the same line.

2. Preliminaries

We recall some basic concepts related to random attractors for stochastic dynamical systems. The reader is referred to [3,8,13,14,17] for more details.

Let $(X, \|\cdot\|_X)$ be a separable Hilbert space with Borel σ-algebra $\mathcal{B}(X)$, and let $(\Omega, \mathcal{F}, P)$ be a probability space.

Definition 2.1. $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is called a metric dynamical system if $\theta : \mathbb{R} \times \Omega \to \Omega$ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable, θ_0 is the identity on Ω, $\theta_{s+t} = \theta_t \circ \theta_s$ for all $s, t \in \mathbb{R}$ and $\theta_t P = P$ for all $t \in \mathbb{R}$.

Definition 2.2. A continuous random dynamical system (RDS) on X over a metric dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is a mapping

$$\phi : \mathbb{R}^+ \times \Omega \times X \to X, \quad (t, \omega, x) \mapsto \phi(t, \omega, x),$$

which is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable and satisfies, for P-a.e. $\omega \in \Omega$,

(i) $\phi(0, \omega, \cdot)$ is the identity on X;
(ii) $\phi(t + s, \omega, \cdot) = \phi(t, \theta_s\omega, \cdot) \circ \phi(s, \omega, \cdot)$ for all $t, s \in \mathbb{R}^+$;
(iii) $\phi(t, \omega, \cdot) : X \to X$ is continuous for all $t \in \mathbb{R}^+$.

Hereafter, we always assume that ϕ is a continuous RDS on X over $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$.

Definition 2.3. A random bounded set $\{B(\omega)\}_{\omega\in\Omega}$ of X is called tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$ if for P-a.e. $\omega \in \Omega$,

$$\lim_{t\to\infty} e^{-\beta t} d\big(B(\theta_{-t}\omega)\big) = 0 \quad \text{for all } \beta > 0,$$

where $d(B) = \sup_{x\in B} \|x\|_X$.

Definition 2.4. Let $\mathcal{D}$ be a collection of random subsets of X and $\{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. Then $\{K(\omega)\}_{\omega\in\Omega}$ is called a random absorbing set for ϕ in $\mathcal{D}$ if for every $B \in \mathcal{D}$ and P-a.e. $\omega \in \Omega$, there exists $t_B(\omega) > 0$ such that

$$\phi\big(t, \theta_{-t}\omega, B(\theta_{-t}\omega)\big) \subseteq K(\omega) \quad \text{for all } t \geqslant t_B(\omega).$$

Definition 2.5. Let $\mathcal{D}$ be a collection of random subsets of X. Then ϕ is said to be $\mathcal{D}$-pullback asymptotically compact in X if for P-a.e. $\omega \in \Omega$, $\{\phi(t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty}$ has a convergent subsequence in X whenever $t_n \to \infty$, and $x_n \in B(\theta_{-t_n}\omega)$ with $\{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$.

Definition 2.6. Let $\mathcal{D}$ be a collection of random subsets of X. Then a random set $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ of X is called a $\mathcal{D}$-random attractor (or $\mathcal{D}$-pullback attractor) for ϕ if the following conditions are satisfied, for P-a.e. $\omega \in \Omega$,

(i) $\mathcal{A}(\omega)$ is compact, and $\omega \mapsto d(x, \mathcal{A}(\omega))$ is measurable for every $x \in X$;

(ii) $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ is invariant, that is,

$$\phi\big(t,\omega,\mathcal{A}(\omega)\big)=\mathcal{A}(\theta_t\omega) \quad \text{for all } t\geqslant 0;$$

(iii) $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ attracts every set in $\mathcal{D}$, that is, for every $B=\{B(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$,

$$\lim_{t\to\infty} d\big(\phi\big(t,\theta_{-t}\omega,B(\theta_{-t}\omega)\big),\mathcal{A}(\omega)\big)=0,$$

where d is the Hausdorff semi-metric given by $d(Y,Z)=\sup_{y\in Y}\inf_{z\in Z}\|y-z\|_X$ for any $Y\subseteq X$ and $Z\subseteq X$.

The following existence result for a random attractor for a continuous RDS can be found in [8,17]. First, recall that a collection $\mathcal{D}$ of random subsets is called *inclusion closed* if whenever $E(\omega)_{\omega\in\Omega}$ is an arbitrary random set, and $F(\omega)_{\omega\in\Omega}$ is in $\mathcal{D}$ with $E(\omega)\subset F(\omega)$ for all $\omega\in\Omega$, then $E(\omega)_{\omega\in\Omega}$ must belong to $\mathcal{D}$.

Proposition 2.7. *Let $\mathcal{D}$ be an inclusion-closed collection of random subsets of X and ϕ a continuous RDS on X over $(\Omega,\mathcal{F},P,(\theta_t)_{t\in\mathbb{R}})$. Suppose that $\{K(\omega)\}_{\omega\in K}$ is a closed random absorbing set for ϕ in $\mathcal{D}$ and ϕ is $\mathcal{D}$-pullback asymptotically compact in X. Then ϕ has a unique $\mathcal{D}$-random attractor $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ which is given by*

$$\mathcal{A}(\omega)=\bigcap_{\tau\geqslant 0}\overline{\bigcup_{t\geqslant\tau}\phi\big(t,\theta_{-t}\omega,K(\theta_{-t}\omega)\big)}.$$

In this paper, we will take $\mathcal{D}$ as the collection of all tempered random subsets of $L^2(\mathbb{R}^n)$ and prove the stochastic reaction–diffusion equation in $\mathbb{R}^n$ has a $\mathcal{D}$-random attractor.

3. The reaction–diffusion equation on $\mathbb{R}^n$ with additive noise

Here we show that there is a continuous random dynamical system generated by the stochastic reaction–diffusion equation defined on $\mathbb{R}^n$ with additive noise:

$$du+(\lambda u-\Delta u)\,dt=\big(f(x,u)+g(x)\big)\,dt+\sum_{j=1}^{m}h_j\,dw_j, \quad x\in\mathbb{R}^n,\ t>0, \tag{3.1}$$

with the initial condition

$$u(x,0)=u_0(x), \quad x\in\mathbb{R}^n. \tag{3.2}$$

Here λ is a positive constant, g is a given function in $L^2(\mathbb{R}^n)$, for each $j=1,\dots,m$, $h_j\in H^2(\mathbb{R}^n)\cap W^{2,p}(\mathbb{R}^n)$ for some $p\geqslant 2$, $\{w_j\}_{j=1}^m$ are independent two-sided real-valued Wiener processes on a probability space which will be specified below, and f is a nonlinear function satisfying the following conditions: For all $x\in\mathbb{R}^n$ and $s\in\mathbb{R}$,

$$f(x,s)s \leqslant -\alpha_1|s|^p + \psi_1(x), \tag{3.3}$$

$$\left|f(x,s)\right| \leqslant \alpha_2|s|^{p-1} + \psi_2(x), \tag{3.4}$$

$$\frac{\partial f}{\partial s}(x,s) \leqslant \beta, \tag{3.5}$$

$$\left|\frac{\partial f}{\partial x}(x,s)\right| \leqslant \psi_3(x), \tag{3.6}$$

where α_1, α_2 and β are positive constants, $\psi_1 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $\psi_2 \in L^2(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ with $\frac{1}{q} + \frac{1}{p} = 1$, and $\psi_3 \in L^2(\mathbb{R}^n)$.

In the sequel, we consider the probability space $(\Omega, \mathcal{F}, P)$ where

$$\Omega = \left\{\omega = (\omega_1, \omega_2, \dots, \omega_m) \in C\left(\mathbb{R}, \mathbb{R}^m\right):\ \omega(0) = 0\right\},$$

$\mathcal{F}$ is the Borel σ-algebra induced by the compact-open topology of Ω, and P the corresponding Wiener measure on $(\Omega, \mathcal{F})$. Then we will identify ω with

$$W(t) \equiv \left(w_1(t), w_2(t), \dots, w_m(t)\right) = \omega(t) \quad \text{for } t \in \mathbb{R}.$$

Define the time shift by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega,\ t \in \mathbb{R}.$$

Then $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is a metric dynamical system.

We now associate a continuous random dynamical system with the stochastic reaction–diffusion equation over $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$. To this end, we need to convert the stochastic equation with a random additive term into a deterministic equation with a random parameter.

Given $j = 1, \dots, m$, consider the one-dimensional Ornstein–Uhlenbeck equation

$$dz_j + \lambda z_j\, dt = dw_j(t). \tag{3.7}$$

One may easily check that a solution to (3.7) is given by

$$z_j(t) = z_j(\theta_t\omega_j) \equiv -\lambda \int_{-\infty}^{0} e^{\lambda\tau}(\theta_t\omega_j)(\tau)\, d\tau, \quad t \in \mathbb{R}.$$

Note that the random variable $|z_j(\omega_j)|$ is tempered and $z_j(\theta_t\omega_j)$ is P-a.e. continuous. Therefore, it follows from Proposition 4.3.3 in [3] that there exists a tempered function $r(\omega) > 0$ such that

$$\sum_{j=1}^{m}\left(\left|z_j(\omega_j)\right|^2 + \left|z_j(\omega_j)\right|^p\right) \leqslant r(\omega), \tag{3.8}$$

where $r(\omega)$ satisfies, for P-a.e. $\omega \in \Omega$,

$$r(\theta_t\omega) \leqslant e^{\frac{\lambda}{2}|t|}r(\omega), \quad t \in \mathbb{R}. \tag{3.9}$$

Then it follows from (3.8)–(3.9) that, for P-a.e. $\omega \in \Omega$,

$$\sum_{j=1}^{m} \left(\left| z_j(\theta_t \omega_j) \right|^2 + \left| z_j(\theta_t \omega_j) \right|^p \right) \leqslant e^{\frac{\lambda}{2}|t|} r(\omega), \quad t \in \mathbb{R}. \tag{3.10}$$

Putting $z(\theta_t \omega) = \sum_{j=1}^{m} h_j z_j(\theta_t \omega_j)$, by (3.7) we have

$$dz + \lambda z\, dt = \sum_{j=1}^{m} h_j\, dw_j.$$

The existence of a solution to the stochastic partial differential equation (3.1) with initial condition (3.2) follows from [16]. To show that problem (3.1)–(3.2) generates a random dynamical system, we let $v(t) = u(t) - z(\theta_t \omega)$ where u is a solution of problem (3.1)–(3.2). Then v satisfies

$$\frac{\partial v}{\partial t} + \lambda v - \Delta v = f\big(x, v + z(\theta_t \omega)\big) + g + \Delta z(\theta_t \omega). \tag{3.11}$$

By a Galerkin method, one can show that if f satisfies (3.3)–(3.6), then in the case of a bounded domain with Dirichlet boundary conditions, for P-a.e. $\omega \in \Omega$ and for all $v_0 \in L^2$, (3.11) has a unique solution $v(\cdot, \omega, v_0) \in C([0, \infty), L^2) \cap L^2((0, T); H^1)$ with $v(0, \omega, v_0) = v_0$ for every $T > 0$. This was done in [12]. Then, following [24], one may take the domain to be a sequence of balls with radius approaching ∞ to deduce the existence of a weak solution to (3.11) on $\mathbb{R}^n$. Further, one may show that $v(t, \omega, v_0)$ is unique and continuous with respect to v_0 in $L^2(\mathbb{R}^n)$ for all $t \geqslant 0$. Let $u(t, \omega, u_0) = v(t, \omega, u_0 - z(\omega)) + z(\theta_t \omega)$. Then the process u is the solution of problem (3.1)–(3.2). We now define a mapping $\phi : \mathbb{R}^+ \times \Omega \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by

$$\begin{aligned} &\phi(t, \omega, u_0) = u(t, \omega, u_0) = v\big(t, \omega, u_0 - z(\omega)\big) + z(\theta_t \omega) \\ &\quad \text{for all } (t, \omega, u_0) \in \mathbb{R}^+ \times \Omega \times L^2\big(\mathbb{R}^n\big). \end{aligned} \tag{3.12}$$

Then ϕ satisfies conditions (i)–(iii) in Definition 2.2. Therefore, ϕ is a continuous random dynamical system associated with the stochastic reaction–diffusion equation on $\mathbb{R}^n$. In the next two sections, we establish uniform estimates for the solutions of problem (3.1)–(3.2) and prove the existence of a random attractor for ϕ.

4. Uniform estimates of solutions

In this section, we derive uniform estimates on the solutions of (3.1)–(3.2) defined on $\mathbb{R}^n$ when $t \to \infty$ with the purpose of proving the existence of a bounded random absorbing set and the asymptotic compactness of the random dynamical system associated with the equation. In particular, we will show that the tails of the solutions, i.e., solutions evaluated at large values of $|x|$, are uniformly small when time is sufficiently large.

From now on, we always assume that $\mathcal{D}$ is the collection of all tempered subsets of $L^2(\mathbb{R}^n)$ with respect to $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})$. The next lemma shows that ϕ has a random absorbing set in $\mathcal{D}$.

Lemma 4.1. *Assume that $g \in L^2(\mathbb{R}^n)$ and (3.3)–(3.6) hold. Then there exists $\{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ such that $\{K(\omega)\}_{\omega\in\Omega}$ is a random absorbing set for ϕ in $\mathcal{D}$, that is, for any $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and P-a.e. $\omega \in \Omega$, there is $T_B(\omega) > 0$ such that*

$$\phi\big(t, \theta_{-t}\omega, B(\theta_{-t}\omega)\big) \subseteq K(\omega) \quad \text{for all } t \geqslant T_B(\omega).$$

Proof. We first derive uniform estimates on $v(t) = u(t) - z(\theta_t\omega)$ from which the uniform estimates on $u(t)$ follow immediately.

Multiplying (3.11) by v and then integrating over $\mathbb{R}^n$, we find that

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 + \lambda\|v\|^2 + \|\nabla v\|^2 = \int_{\mathbb{R}^n} f\big(x, v + z(\theta_t\omega)\big)v\,dx + (g, v) + \big(\Delta z(\theta_t\omega), v\big). \tag{4.1}$$

For the nonlinear term, by (3.3)–(3.4) we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} f\big(x, v + z(\theta_t\omega)\big)v\,dx \\
&\quad = \int_{\mathbb{R}^n} f\big(x, v + z(\theta_t\omega)\big)\big(v + z(\theta_t\omega)\big)\,dx - \int_{\mathbb{R}^n} f\big(x, v + z(\theta_t\omega)\big)z(\theta_t\omega)\,dx \\
&\quad \leqslant -\alpha_1 \int_{\mathbb{R}^n} |u|^p\,dx + \int_{\mathbb{R}^n} \psi_1(x)\,dx - \int_{\mathbb{R}^n} f(x, u)z(\theta_t\omega)\,dx \\
&\quad \leqslant -\alpha_1 \int_{\mathbb{R}^n} |u|^p\,dx + \int_{\mathbb{R}^n} \psi_1(x)\,dx + \alpha_2 \int_{\mathbb{R}^n} |u|^{p-1}\big|z(\theta_t\omega)\big|\,dx + \int_{\mathbb{R}^n} |\psi_2|\big|z(\theta_t\omega)\big|\,dx \\
&\quad \leqslant -\alpha_1\|u\|_p^p + \|\psi_1\|_1 + \frac{1}{2}\alpha_1\|u\|_p^p + c_1\big\|z(\theta_t\omega)\big\|_p^p + \frac{1}{2}\|\psi_2\|^2 + \frac{1}{2}\big\|z(\theta_t\omega)\big\|^2 \\
&\quad \leqslant -\frac{1}{2}\alpha_1\|u\|_p^p + c_2\big(\big\|z(\theta_t\omega)\big\|_p^p + \big\|z(\theta_t\omega)\big\|^2\big) + c_3.
\end{aligned} \tag{4.2}$$

On the other hand, the last two terms on the right-hand side of (4.1) are bounded by

$$\|g\|\|v\| + \big\|\nabla z(\theta_t\omega)\big\|\|\nabla v\| \leqslant \frac{1}{2}\lambda\|v\|^2 + \frac{1}{2\lambda}\|g\|^2 + \frac{1}{2}\big\|\nabla z(\theta_t\omega)\big\|^2 + \frac{1}{2}\|\nabla v\|^2. \tag{4.3}$$

Then it follows from (4.1)–(4.3) that

$$\frac{d}{dt}\|v\|^2 + \lambda\|v\|^2 + \|\nabla v\|^2 + \alpha_1\|u\|_p^p \leqslant c_4\big(\big\|z(\theta_t\omega)\big\|_p^p + \big\|z(\theta_t\omega)\big\|^2 + \big\|\nabla z(\theta_t\omega)\big\|^2\big) + c_5. \tag{4.4}$$

Note that $z(\theta_t\omega) = \sum_{j=1}^m h_j z_j(\theta_t\omega_j)$ and $h_j \in H^2(\mathbb{R}^n) \cap W^{2,p}(\mathbb{R}^n)$. Therefore, the right-hand side of (4.4) is bounded by

$$c_6 \sum_{j=1}^m \big(\big\|z_j(\theta_t\omega_j)\big\|^p + \big\|z_j(\theta_t\omega_j)\big\|^2\big) + c_7 = p_1(\theta_t\omega) + c_7. \tag{4.5}$$

By (3.10), we find that for P-a.e. $\omega \in \Omega$,

$$p_1(\theta_\tau \omega) \leqslant c_6 e^{\frac{1}{2}\lambda|\tau|} r(\omega) \quad \text{for all } \tau \in \mathbb{R}. \tag{4.6}$$

It follows from (4.4)–(4.5) that, for all $t \geqslant 0$,

$$\frac{d}{dt}\|v\|^2 + \lambda\|v\|^2 + \|\nabla v\|^2 + \alpha_1\|u\|_p^p \leqslant p_1(\theta_t\omega) + c_7, \tag{4.7}$$

which implies that, for all $t \geqslant 0$,

$$\frac{d}{dt}\|v\|^2 + \lambda\|v\|^2 \leqslant p_1(\theta_t\omega) + c_7. \tag{4.8}$$

Applying Gronwall's lemma, we find that, for all $t \geqslant 0$,

$$\big\|v\big(t, \omega, v_0(\omega)\big)\big\|^2 \leqslant e^{-\lambda t}\big\|v_0(\omega)\big\|^2 + \int_0^t e^{\lambda(\tau - t)} p_1(\theta_\tau\omega)\, d\tau + \frac{c_7}{\lambda}. \tag{4.9}$$

By replacing ω by $\theta_{-t}\omega$, we get from (4.9) and (4.6) that, for all $t \geqslant 0$,

$$\begin{aligned}
\big\|v\big(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\big)\big\|^2 &\leqslant e^{-\lambda t}\big\|v_0(\theta_{-t}\omega)\big\|^2 + \int_0^t e^{\lambda(s-t)} p_1(\theta_{s-t}\omega)\, ds + \frac{c_7}{\lambda} \\
&\leqslant e^{-\lambda t}\big\|v_0(\theta_{-t}\omega)\big\|^2 + \int_{-t}^0 e^{\lambda\tau} p_1(\theta_\tau\omega)\, d\tau + \frac{c_7}{\lambda} \\
&\leqslant e^{-\lambda t}\big\|v_0(\theta_{-t}\omega)\big\|^2 + c_6\int_{-t}^0 e^{\frac{1}{2}\lambda\tau} r(\omega)\, d\tau + \frac{c_7}{\lambda} \\
&\leqslant e^{-\lambda t}\big\|v_0(\theta_{-t}\omega)\big\|^2 + \frac{2c_6}{\lambda} r(\omega) + \frac{c_7}{\lambda}.
\end{aligned} \tag{4.10}$$

Note that $\phi(t, \omega, u_0(\omega)) = v(t, \omega, u_0(\omega) - z(\omega)) + z(\theta_t\omega)$. So by (4.10) we get that, for all $t \geqslant 0$,

$$\begin{aligned}
&\big\|\phi\big(t, \theta_{-t}\omega, u_0(\theta_{-t}\omega)\big)\big\|^2 \\
&\quad = \big\|v\big(t, \theta_{-t}\omega, u_0(\theta_{-t}\omega) - z(\theta_{-t}\omega)\big) + z(\omega)\big\|^2 \\
&\quad \leqslant 2\big\|v\big(t, \theta_{-t}\omega, u_0(\theta_{-t}\omega) - z(\theta_{-t}\omega)\big)\big\|^2 + 2\big\|z(\omega)\big\|^2 \\
&\quad \leqslant 2e^{-\lambda t}\big\|u_0(\theta_{-t}\omega) - z(\theta_{-t}\omega)\big\|^2 + c_8 r(\omega) + c_8 + 2\big\|z(\omega)\big\|^2 \\
&\quad \leqslant 4e^{-\lambda t}\big(\big\|u_0(\theta_{-t}\omega)\big\|^2 + \big\|z(\theta_{-t}\omega)\big\|^2\big) + c_8 r(\omega) + c_8 + 2\big\|z(\omega)\big\|^2.
\end{aligned} \tag{4.11}$$

By assumption, $\{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ is tempered. On the other hand, by definition, $\|z(\omega)\|^2$ is also tempered. Therefore, if $u_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$, then there is $T_B(\omega) > 0$ such that for all $t \geqslant T_B(\omega)$,

$$4e^{-\lambda t}\left(\left\|u_0(\theta_{-t}\omega)\right\|^2 + \left\|z(\theta_{-t}\omega)\right\|^2\right) \leqslant c_8 r(\omega) + c_8,$$

which along with (4.11) shows that, for all $t \geqslant T_B(\omega)$,

$$\left\|\phi\left(t, \theta_{-t}\omega, u_0(\theta_{-t}\omega)\right)\right\|^2 \leqslant 2\left(c_8 r(\omega) + c_8 + \left\|z(\omega)\right\|^2\right). \tag{4.12}$$

Given $\omega \in \Omega$, denote by

$$K(\omega) = \left\{u \in L^2\left(\mathbb{R}^n\right):\ \|u\|^2 \leqslant 2\left(c_8 r(\omega) + c_8 + \left\|z(\omega)\right\|^2\right)\right\}.$$

Then $\{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. Further, (4.12) indicates that $\{K(\omega)\}_{\omega\in\Omega}$ is a random absorbing set for ϕ in $\mathcal{D}$, which completes the proof. □

We next derive uniform estimates for v in $H^1(\mathbb{R}^n)$ and for u in $L^p(\mathbb{R}^n)$.

Lemma 4.2. *Assume that $g \in L^2(\mathbb{R}^n)$ and* (3.3)–(3.6) *hold. Let $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $u_0(\omega) \in B(\omega)$. Then for every $T_1 \geqslant 0$ and P-a.e. $\omega \in \Omega$, the solutions $u(t, \omega, u_0(\omega))$ of problem* (3.1)–(3.2) *and $v(t, \omega, v_0(\omega))$ of* (3.11) *with $v_0(\omega) = u_0(\omega) - z(\omega)$ satisfy, for all $t \geqslant T_1$,*

$$\int_{T_1}^{t} e^{\lambda(s-t)}\left\|u\left(s, \theta_{-t}\omega, u_0(\theta_{-t}\omega)\right)\right\|_p^p ds \leqslant e^{-\lambda t}\left\|v_0(\theta_{-t}\omega)\right\|^2 + c\left(1 + r(\omega)\right), \tag{4.13}$$

$$\int_{T_1}^{t} e^{\lambda(s-t)}\left\|\nabla v\left(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\right)\right\|^2 ds \leqslant e^{-\lambda t}\left\|v_0(\theta_{-t}\omega)\right\|^2 + c\left(1 + r(\omega)\right), \tag{4.14}$$

where c is a positive deterministic constant independent of T_1, and $r(\omega)$ is the tempered function in (3.8).

Proof. First, replacing t by T_1 and then replacing ω by $\theta_{-t}\omega$ in (4.9), we find that

$$\left\|v\left(T_1, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\right)\right\|^2 \leqslant e^{-\lambda T_1}\left\|v_0(\theta_{-t}\omega)\right\|^2 + \int_0^{T_1} e^{\lambda(s-T_1)} p_1(\theta_{s-t}\omega)\, ds + c.$$

Multiply the above by $e^{\lambda(T_1-t)}$ and then simplify to get

$$\begin{aligned} &e^{\lambda(T_1-t)}\left\|v\left(T_1, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\right)\right\|^2 \\ &\quad \leqslant e^{-\lambda t}\left\|v_0(\theta_{-t}\omega)\right\|^2 + \int_0^{T_1} e^{\lambda(s-t)} p_1(\theta_{s-t}\omega)\, ds + ce^{\lambda(T_1-t)}. \end{aligned} \tag{4.15}$$

By (4.6), the second term on the right-hand side of (4.15) satisfies

$$\int_0^{T_1} e^{\lambda(s-t)} p_1(\theta_{s-t}\omega)\,ds = \int_{-t}^{T_1-t} e^{\lambda\tau} p_1(\theta_\tau\omega)\,d\tau$$
$$\leqslant c_6 r(\omega) \int_{-t}^{T_1-t} e^{\frac{1}{2}\lambda\tau}\,d\tau \leqslant \frac{2}{\lambda} c_6 r(\omega) e^{\frac{1}{2}\lambda(T_1-t)}. \tag{4.16}$$

From (4.15)–(4.16) it follows that

$$e^{\lambda(T_1-t)} \big\| v\big(T_1, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\big)\big\|^2$$
$$\leqslant e^{-\lambda t} \big\| v_0(\theta_{-t}\omega)\big\|^2 + \frac{2}{\lambda} c_6 r(\omega) e^{\frac{1}{2}\lambda(T_1-t)} + c e^{\lambda(T_1-t)}. \tag{4.17}$$

By (4.7) we find that, for $t \geqslant T_1$,

$$\big\| v\big(t, \omega, v_0(\omega)\big)\big\|^2 + \int_{T_1}^{t} e^{\lambda(s-t)} \big\| \nabla v\big(s, \omega, v_0(\omega)\big)\big\|^2\,ds + \alpha_1 \int_{T_1}^{t} e^{\lambda(s-t)} \big\| u\big(s, \omega, u_0(\omega)\big)\big\|_p^p\,ds$$
$$\leqslant e^{\lambda(T_1-t)} \big\| v\big(T_1, \omega, v_0(\omega)\big)\big\|^2 + \int_{T_1}^{t} e^{\lambda(s-t)} p_1(\theta_s\omega)\,ds + c\int_{T_1}^{t} e^{\lambda(s-t)}\,ds. \tag{4.18}$$

Dropping the first term on the left-hand side of (4.18) and replacing ω by $\theta_{-t}\omega$, we obtain that, for all $t \geqslant T_1$,

$$\int_{T_1}^{t} e^{\lambda(s-t)} \big\| \nabla v\big(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\big)\big\|^2\,ds + \alpha_1 \int_{T_1}^{t} e^{\lambda(s-t)} \big\| u\big(s, \theta_{-t}\omega, u_0(\theta_{-t}\omega)\big)\big\|_p^p\,ds$$
$$\leqslant e^{\lambda(T_1-t)} \big\| v\big(T_1, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\big)\big\|^2 + \int_{T_1}^{t} e^{\lambda(s-t)} p_1(\theta_{s-t}\omega)\,ds + c\int_{T_1}^{t} e^{\lambda(s-t)}\,ds$$
$$\leqslant e^{\lambda(T_1-t)} \big\| v\big(T_1, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\big)\big\|^2 + \int_{T_1-t}^{0} e^{\lambda\tau} p_1(\theta_\tau\omega)\,d\tau + \frac{c}{\lambda}. \tag{4.19}$$

By (4.6), the second term on the right-hand side of (4.19) satisfies, for $t \geqslant T_1$,

$$\int_{T_1-t}^{0} e^{\lambda\tau} p_1(\theta_\tau\omega)\,d\tau \leqslant c_6 r(\omega) \int_{T_1-t}^{0} e^{\frac{1}{2}\lambda\tau}\,d\tau \leqslant \frac{2}{\lambda} c_6 r(\omega). \tag{4.20}$$

Then, using (4.17) and (4.20), it follows from (4.19) that

$$\int_{T_1}^{t} e^{\lambda(s-t)} \left\| \nabla v\big(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\big) \right\|^2 ds + \alpha_1 \int_{T_1}^{t} e^{\lambda(s-t)} \left\| u\big(s, \theta_{-t}\omega, u_0(\theta_{-t}\omega)\big) \right\|_p^p ds$$
$$\leqslant e^{-\lambda t} \left\| v_0(\theta_{-t}\omega) \right\|^2 + c\big(1 + r(\omega)\big).$$

This completes the proof. $\square$

As a special case of Lemma 4.2, we have the following uniform estimates.

Lemma 4.3. *Assume that $g \in L^2(\mathbb{R}^n)$ and* (3.3)–(3.6) *hold. Let $B = \{B(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$ and $u_0(\omega) \in B(\omega)$. Then for P-a.e. $\omega \in \Omega$, there exists $T_B(\omega) > 0$ such that the solutions $u(t, \omega, u_0(\omega))$ of problem* (3.1)–(3.2) *and $v(t, \omega, v_0(\omega))$ of* (3.11) *with $v_0(\omega) = u_0(\omega) - z(\omega)$ satisfy, for all $t \geqslant T_B(\omega)$,*

$$\int_{t}^{t+1} \left\| u\big(s, \theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega)\big) \right\|_p^p ds \leqslant c\big(1 + r(\omega)\big),$$

$$\int_{t}^{t+1} \left\| \nabla v\big(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)\big) \right\|^2 ds \leqslant c\big(1 + r(\omega)\big),$$

where c is a positive deterministic constant and $r(\omega)$ is the tempered function in (3.8).

Proof. First replacing t by $t + 1$ and then replacing T_1 by t in (4.14), we find that

$$\int_{t}^{t+1} e^{\lambda(s-t-1)} \left\| \nabla v\big(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)\big) \right\|^2 ds$$
$$\leqslant e^{-\lambda(t+1)} \left\| v_0(\theta_{-t-1}\omega) \right\|^2 + c\big(1 + r(\omega)\big). \tag{4.21}$$

Note that $e^{\lambda(s-t-1)} \geqslant e^{-\lambda}$ for $s \in [t, t + 1]$. Hence, from (4.21) we get that

$$e^{-\lambda} \int_{t}^{t+1} \left\| \nabla v\big(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)\big) \right\|^2 ds$$
$$\leqslant e^{-\lambda(t+1)} \left\| v_0(\theta_{-t-1}\omega) \right\|^2 + c\big(1 + r(\omega)\big)$$
$$\leqslant 2e^{-\lambda(t+1)} \big(\left\| u_0(\theta_{-t-1}\omega) \right\|^2 + \left\| z(\theta_{-t-1}\omega) \right\|^2 \big) + c\big(1 + r(\omega)\big). \tag{4.22}$$

Since $\|u_0(\omega)\|^2$ and $\|z(\omega)\|^2$ are tempered, there is $T_B(\omega) > 0$ such that for all $t \geqslant T_B(\omega)$,

$$2e^{-\lambda(t+1)} \big(\left\| u_0(\theta_{-t-1}\omega) \right\|^2 + \left\| z(\theta_{-t-1}\omega) \right\|^2 \big) \leqslant c\big(1 + r(\omega)\big),$$

which along with (4.22) shows that, for all $t \geqslant T_B(\omega)$,

$$\int_t^{t+1} \left\| \nabla v\big(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)\big)\right\|^2 ds \leqslant 2e^{\lambda}\big(1 + r(\omega)\big). \tag{4.23}$$

Using (4.13) and repeating the above process, we also find that, for $t \geqslant T_B(\omega)$,

$$\int_t^{t+1} \left\| u\big(s, \theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega)\big)\right\|_p^p ds \leqslant 2e^{\lambda}\big(1 + r(\omega)\big). \tag{4.24}$$

Then the lemma follows from (4.23)–(4.24). □

Lemma 4.4. *Assume that $g \in L^2(\mathbb{R}^n)$ and* (3.3)–(3.6) *hold. Let $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $u_0(\omega) \in B(\omega)$. Then for P-a.e. $\omega \in \Omega$, there exists $T_B(\omega) > 0$ such that the solution $u(t, \omega, u_0(\omega))$ of* (3.1)–(3.2) *satisfies, for all $t \geqslant T_B(\omega)$,*

$$\int_t^{t+1} \left\| \nabla u\big(s, \theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega)\big)\right\|^2 ds \leqslant c\big(1 + r(\omega)\big),$$

where c is a positive deterministic constant and $r(\omega)$ is the tempered function in (3.8).

Proof. Let $T_B(\omega)$ be the positive constant in Lemma 4.3, take $t \geqslant T_B(\omega)$ and $s \in (t, t+1)$. By (3.12) we find that

$$\begin{aligned}
&\left\| \nabla u\big(s, \theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega)\big)\right\|^2 \\
&\quad = \left\| \nabla v\big(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)\big) + \nabla z(\theta_{s-t-1}\omega)\right\|^2 \\
&\quad \leqslant 2\left\| \nabla v\big(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)\big)\right\|^2 + 2\left\| \nabla z(\theta_{s-t-1}\omega)\right\|^2.
\end{aligned} \tag{4.25}$$

By (3.10) we have

$$2\left\| \nabla z(\theta_{s-t-1}\omega)\right\|^2 \leqslant c\sum_{j=1}^m \big|z_j(\theta_{s-t-1}\omega_j)\big|^2 \leqslant ce^{\frac{\lambda}{2}(t+1-s)} r(\omega) \leqslant ce^{\frac{\lambda}{2}} r(\omega). \tag{4.26}$$

Now integrating (4.25) with respect to s over $(t, t+1)$, by Lemma 4.3 and inequality (4.26), we get that

$$\int_t^{t+1} \left\| \nabla u\big(s, \theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega)\big)\right\|^2 ds \leqslant c_1 + c_2 r(\omega). \tag{4.27}$$

Then the lemma follows from (4.27). □

Lemma 4.5. *Assume that $g \in L^2(\mathbb{R}^n)$ and (3.3)–(3.6) hold. Let $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $u_0(\omega) \in B(\omega)$. Then for P-a.e. $\omega \in \Omega$, there exists $T_B(\omega) > 0$ such that for all $t \geqslant T_B(\omega)$,*

$$\big\|\nabla u\big(t, \theta_{-t}\omega, u_0(\theta_{-t}\omega)\big)\big\|^2 \leqslant c\big(1 + r(\omega)\big),$$

where c is a positive deterministic constant and $r(\omega)$ is the tempered function in (3.8).

Proof. Taking the inner product of (3.11) with Δv in $L^2(\mathbb{R}^n)$, we get that

$$\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2 + \lambda\|\nabla v\|^2 + \|\Delta v\|^2 = -\int_{\mathbb{R}^n} f(x,u)\Delta v\,dx - \big(g + \Delta z(\theta_t\omega), \Delta v\big). \tag{4.28}$$

We first estimate the nonlinear term in (4.28) for which, by (3.4)–(3.6), we have

$$\begin{aligned}
&-\int_{\mathbb{R}^n} f(x,u)\Delta v\,dx \\
&\quad= -\int_{\mathbb{R}^n} f(x,u)\Delta u\,dx + \int_{\mathbb{R}^n} f(x,u)\Delta z(\theta_t\omega)\,dx \\
&\quad= \int_{\mathbb{R}^n} \frac{\partial f}{\partial x}(x,u)\nabla u\,dx + \int_{\mathbb{R}^n} \frac{\partial f}{\partial u}(x,u)|\nabla u|^2\,dx + \int_{\mathbb{R}^n} f(x,u)\Delta z(\theta_t\omega)\,dx \\
&\quad\leqslant \|\psi_3\|\|\nabla u\| + \beta\|\nabla u\|^2 + \int_{\mathbb{R}^n} \big|f(x,u)\big|\big|\Delta z(\theta_t\omega)\big|\,dx \\
&\quad\leqslant \|\psi_3\|\|\nabla u\| + \beta\|\nabla u\|^2 + \alpha_2\int_{\mathbb{R}^n} |u|^{p-1}\big|\Delta z(\theta_t\omega)\big|\,dx + \int_{\mathbb{R}^n} \big|\psi_2(x)\big|\big|\Delta z(\theta_t\omega)\big|\,dx \\
&\quad\leqslant c\|\nabla u\|^2 + \frac{\alpha_2}{q}\int_{\mathbb{R}^n} |u|^p\,dx + \frac{\alpha_2}{p}\int_{\mathbb{R}^n} \big|\Delta z(\theta_t\omega)\big|^p\,dx + c\big(\|\psi_2\|^2 + \|\psi_3\|^2\big) + c\big\|\Delta z(\theta_t\omega)\big\|^2 \\
&\quad\leqslant c\big(\|\nabla u\|^2 + \|u\|_p^p\big) + c\big(\big\|\Delta z(\theta_t\omega)\big\|^2 + \big\|\Delta z(\theta_t\omega)\big\|_p^p + 1\big).
\end{aligned} \tag{4.29}$$

On the other hand, the last term on the right-hand side of (4.28) is bounded by

$$\big|(g, \Delta v)\big| + \big|\big(\Delta z(\theta_t\omega), \Delta v\big)\big| \leqslant \frac{1}{2}\|\Delta v\|^2 + \|g\|^2 + \big\|\Delta z(\theta_t\omega)\big\|^2. \tag{4.30}$$

By (4.28)–(4.30) we see that

$$\begin{aligned}
&\frac{d}{dt}\|\nabla v\|^2 + 2\lambda\|\nabla v\|^2 + \|\Delta v\|^2 \\
&\qquad\leqslant c\big(\|\nabla u\|^2 + \|u\|_p^p\big) + c\big(\big\|\Delta z(\theta_t\omega)\big\|^2 + \big\|\Delta z(\theta_t\omega)\big\|_p^p + 1\big).
\end{aligned} \tag{4.31}$$

Let

$$p_2(\theta_t\omega) = c\big(\big\|\Delta z(\theta_t\omega)\big\|^2 + \big\|\Delta z(\theta_t\omega)\big\|_p^p + 1\big). \tag{4.32}$$

Since $z(\theta_t\omega)=\sum_{j=1}^m h_j z_j(\theta_t\omega_j)$ and $h_j\in H^2(\mathbb{R}^n)\cap W^{2,p}(\mathbb{R}^n)$, there are positive constants c_1 and c_2 such that

$$p_2(\theta_t\omega)\leqslant c_1\sum_{j=1}^m\left(\left|z_j(\theta_t\omega_j)\right|^2+\left|z_j(\theta_t\omega_j)\right|_p^p\right)+c_2,$$

which along with (3.10) shows that

$$p_2(\theta_t\omega)\leqslant c_1 e^{\frac{\lambda}{2}|t|}r(\omega)+c_2\quad\text{for all } t\in\mathbb{R}. \tag{4.33}$$

By (4.31)–(4.32), we find that

$$\frac{d}{dt}\|\nabla v\|^2\leqslant c\left(\|\nabla u\|^2+\|u\|_p^p\right)+p_2(\theta_t\omega). \tag{4.34}$$

Let $T_B(\omega)$ be the positive constant in Lemma 4.3, take $t\geqslant T_B(\omega)$ and $s\in(t,t+1)$. Then integrate (4.34) over $(s,t+1)$ to get

$$\begin{aligned}\left\|\nabla v\left(t+1,\omega,v_0(\omega)\right)\right\|^2&\leqslant\left\|\nabla v\left(s,\omega,v_0(\omega)\right)\right\|^2+\int_s^{t+1}p_2(\theta_\tau\omega)\,d\tau\\&\quad+c\int_s^{t+1}\left(\left\|\nabla u\left(\tau,\omega,u_0(\omega)\right)\right\|^2+\left\|u\left(\tau,\omega,u_0(\omega)\right)\right\|_p^p\right)d\tau\\&\leqslant\left\|\nabla v\left(s,\omega,v_0(\omega)\right)\right\|^2+\int_t^{t+1}p_2(\theta_\tau\omega)\,d\tau\\&\quad+c\int_t^{t+1}\left(\left\|\nabla u\left(\tau,\omega,u_0(\omega)\right)\right\|^2+\left\|u\left(\tau,\omega,u_0(\omega)\right)\right\|_p^p\right)d\tau.\end{aligned}$$

Now integrating the above with respect to s over $(t,t+1)$, we find that

$$\begin{aligned}\left\|\nabla v\left(t+1,\omega,v_0(\omega)\right)\right\|^2&\leqslant\int_t^{t+1}\left\|\nabla v\left(s,\omega,v_0(\omega)\right)\right\|^2ds+\int_t^{t+1}p_2(\theta_\tau\omega)\,d\tau\\&\quad+c\int_t^{t+1}\left(\left\|\nabla u\left(\tau,\omega,u_0(\omega)\right)\right\|^2+\left\|u\left(\tau,\omega,u_0(\omega)\right)\right\|_p^p\right)d\tau.\end{aligned}$$

Replacing ω by $\theta_{-t-1}\omega$, we obtain that

$$\begin{aligned}
&\big\|\nabla v\big(t+1,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)\big)\big\|^2\\
&\quad\leqslant \int_t^{t+1}\big\|\nabla v\big(s,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)\big)\big\|^2\,ds + \int_t^{t+1} p_2(\theta_{\tau-t-1}\omega)\,d\tau\\
&\qquad + c\int_t^{t+1}\Big(\big\|\nabla u\big(\tau,\theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega)\big)\big\|^2\\
&\qquad + \big\|u\big(\tau,\theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega)\big)\big\|_p^p\Big)\,d\tau.
\end{aligned}\tag{4.35}$$

By Lemmas 4.3 and 4.4, it follows from (4.35) and (4.33) that, for all $t \geqslant T_B(\omega)$,

$$\begin{aligned}
\big\|\nabla v\big(t+1,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)\big)\big\|^2 &\leqslant c_3 + c_4 r(\omega) + \int_{-1}^{0} p_2(\theta_s\omega)\,ds\\
&\leqslant c_3 + c_4 r(\omega) + \int_{-1}^{0}\big(c_1 e^{-\frac{\lambda}{2}s} r(\omega) + c_2\big)\,ds\\
&\leqslant c_5 + c_6 r(\omega).
\end{aligned}\tag{4.36}$$

Then by (4.36) and (3.10), we have, for all $t \geqslant T_B(\omega)$,

$$\begin{aligned}
\big\|\nabla u\big(t+1,\theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega)\big)\big\|^2 &= \big\|\nabla v\big(t+1,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)\big) + \nabla z(\omega)\big\|^2\\
&\leqslant 2\big\|\nabla v\big(t+1,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)\big)\big\|^2 + 2\big\|\nabla z(\omega)\big\|^2\\
&\leqslant c_7 + c_8 r(\omega),
\end{aligned}$$

which completes the proof. □

Lemma 4.6. *Assume that $g \in L^2(\mathbb{R}^n)$ and (3.3)–(3.6) hold. Let $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $u_0(\omega) \in B(\omega)$. Then for every $\epsilon > 0$ and P-a.e. $\omega \in \Omega$, there exist $T^* = T_B^*(\omega,\epsilon) > 0$ and $R^* = R^*(\omega,\epsilon) > 0$ such that the solution $v(t,\omega,v_0(\omega))$ of (3.11) with $v_0(\omega) = u_0(\omega) - z(\omega)$ satisfies, for all $t \geqslant T^*$,*

$$\int_{|x|\geqslant R^*}\big|v\big(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega)\big)(x)\big|^2\,dx \leqslant \epsilon.$$

Proof. Let ρ be a smooth function defined on $\mathbb{R}^+$ such that $0 \leqslant \rho(s) \leqslant 1$ for all $s \in \mathbb{R}^+$, and

$$\rho(s) = \begin{cases} 0 & \text{for } 0 \leqslant s \leqslant 1,\\ 1 & \text{for } s \geqslant 2.\end{cases}$$

Then there exists a positive constant c such that $|\rho'(s)| \leqslant c$ for all $s \in \mathbb{R}^+$. Taking the inner product of (3.11) with $\rho(\frac{|x|^2}{k^2})v$ in $L^2(\mathbb{R}^n)$, we get that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)|v|^2\,dx + \lambda\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)|v|^2\,dx - \int_{\mathbb{R}^n} (\Delta v)\rho\left(\frac{|x|^2}{k^2}\right)v\,dx$$
$$= \int_{\mathbb{R}^n} f(x,u)\rho\left(\frac{|x|^2}{k^2}\right)v\,dx + \int_{\mathbb{R}^n} \big(g + \Delta z(\theta_t\omega)\big)\rho\left(\frac{|x|^2}{k^2}\right)v\,dx. \tag{4.37}$$

We now estimate the terms in (4.37) as follows. First we have

$$\begin{aligned}
&-\int_{\mathbb{R}^n} (\Delta v)\rho\left(\frac{|x|^2}{k^2}\right)v\,dx \\
&\quad= \int_{\mathbb{R}^n} |\nabla v|^2\rho\left(\frac{|x|^2}{k^2}\right)dx + \int_{\mathbb{R}^n} v\rho'\left(\frac{|x|^2}{k^2}\right)\frac{2x}{k^2}\cdot\nabla v\,dx \\
&\quad= \int_{\mathbb{R}^n} |\nabla v|^2\rho\left(\frac{|x|^2}{k^2}\right)dx + \int_{k\leqslant|x|\leqslant\sqrt{2}k} v\rho'\left(\frac{|x|^2}{k^2}\right)\frac{2x}{k^2}\cdot\nabla v\,dx.
\end{aligned} \tag{4.38}$$

Note that the second term on the right-hand side of (4.38) is bounded by

$$\begin{aligned}
\left|\int_{k\leqslant|x|\leqslant\sqrt{2}k} v\rho'\left(\frac{|x|^2}{k^2}\right)\frac{2x}{k^2}\cdot\nabla v\,dx\right| &\leqslant \frac{2\sqrt{2}}{k}\int_{k\leqslant|x|\leqslant\sqrt{2}k} |v|\left|\rho'\left(\frac{|x|^2}{k^2}\right)\right||\nabla v|\,dx \\
&\leqslant \frac{c}{k}\int_{\mathbb{R}^n} |v||\nabla v|\,dx \leqslant \frac{c}{k}\big(\|v\|^2 + \|\nabla v\|^2\big).
\end{aligned} \tag{4.39}$$

By (4.38)–(4.39), we find that

$$-\int_{\mathbb{R}^n} (\Delta v)\rho\left(\frac{|x|^2}{k^2}\right)v\,dx \geqslant \int_{\mathbb{R}^n} |\nabla v|^2\rho\left(\frac{|x|^2}{k^2}\right)dx - \frac{c}{k}\big(\|v\|^2 + \|\nabla v\|^2\big). \tag{4.40}$$

For the nonlinear term, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} f(x,u)\rho\left(\frac{|x|^2}{k^2}\right)v\,dx &= \int_{\mathbb{R}^n} f(x,u)\rho\left(\frac{|x|^2}{k^2}\right)u\,dx \\
&\quad - \int_{\mathbb{R}^n} f(x,u)\rho\left(\frac{|x|^2}{k^2}\right)z(\theta_t\omega)\,dx.
\end{aligned} \tag{4.41}$$

By (3.3), the first term on the right-hand side of (4.41) is bounded by

$$\int_{\mathbb{R}^n} f(x,u)\rho\left(\frac{|x|^2}{k^2}\right)u\,dx \leqslant -\alpha_1\int_{\mathbb{R}^n}|u|^p\rho\left(\frac{|x|^2}{k^2}\right)dx+\int_{\mathbb{R}^n}\psi_1\rho\left(\frac{|x|^2}{k^2}\right)dx. \tag{4.42}$$

By (3.4), the second term on the right-hand side of (4.42) is bounded by

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} f(x,u)\rho\left(\frac{|x|^2}{k^2}\right)z(\theta_t\omega)\,dx\right| \\
&\quad\leqslant \alpha_2\int_{\mathbb{R}^n}|u|^{p-1}\rho\left(\frac{|x|^2}{k^2}\right)\left|z(\theta_t\omega)\right|dx+\int_{\mathbb{R}^n}|\psi_2|\rho\left(\frac{|x|^2}{k^2}\right)\left|z(\theta_t\omega)\right|dx \\
&\quad\leqslant \frac{1}{2}\alpha_1\int_{\mathbb{R}^n}|u|^p\rho\left(\frac{|x|^2}{k^2}\right)dx+c\int_{\mathbb{R}^n}\left|z(\theta_t\omega)\right|^p\rho\left(\frac{|x|^2}{k^2}\right)dx \\
&\qquad+\frac{1}{2}\int_{\mathbb{R}^n}\left|z(\theta_t\omega)\right|^2\rho\left(\frac{|x|^2}{k^2}\right)dx+\frac{1}{2}\int_{\mathbb{R}^n}\psi_2^2\rho\left(\frac{|x|^2}{k^2}\right)dx.
\end{aligned} \tag{4.43}$$

Then it follows from (4.41)–(4.43) that

$$\begin{aligned}
&\int_{\mathbb{R}^n} f(x,u)\rho\left(\frac{|x|^2}{k^2}\right)v\,dx \\
&\quad\leqslant -\frac{1}{2}\alpha_1\int_{\mathbb{R}^n}|u|^p\rho\left(\frac{|x|^2}{k^2}\right)dx+\int_{\mathbb{R}^n}\psi_1\rho\left(\frac{|x|^2}{k^2}\right)dx \\
&\qquad+\frac{1}{2}\int_{\mathbb{R}^n}\psi_2^2\rho\left(\frac{|x|^2}{k^2}\right)dx+c\int_{\mathbb{R}^n}\left(\left|z(\theta_t\omega)\right|^p+\left|z(\theta_t\omega)\right|^2\right)\rho\left(\frac{|x|^2}{k^2}\right)dx.
\end{aligned} \tag{4.44}$$

For the last term on the right-hand side of (4.37), we have that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\left(g+\Delta z(\theta_t\omega)\right)\rho\left(\frac{|x|^2}{k^2}\right)v\,dx\right| \\
&\quad\leqslant \frac{1}{2}\lambda\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|v|^2\,dx+\frac{1}{\lambda}\int_{\mathbb{R}^n}\left(g^2+\left|\Delta z(\theta_t\omega)\right|^2\right)\rho\left(\frac{|x|^2}{k^2}\right)dx.
\end{aligned} \tag{4.45}$$

Finally, by (4.37), (4.40) and (4.44)–(4.45), we obtain that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|v|^2\,dx+\frac{1}{2}\lambda\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|v|^2\,dx$$

$$
\begin{aligned}
&+\frac{1}{2}\alpha_1\int\limits_{\mathbb{R}^n}|u|^p\rho\left(\frac{|x|^2}{k^2}\right)dx+\int\limits_{\mathbb{R}^n}|\nabla v|^2\rho\left(\frac{|x|^2}{k^2}\right)dx\\
&\leqslant\frac{c}{k}\left(\|\nabla v\|^2+\|v\|^2\right)+\int\limits_{\mathbb{R}^n}\left(|\psi_1|+\frac{1}{2}|\psi_2|^2+\frac{1}{\lambda}g^2\right)\rho\left(\frac{|x|^2}{k^2}\right)dx\\
&+c\int\limits_{\mathbb{R}^n}\left(\left|\Delta z(\theta_t\omega)\right|^2+\left|z(\theta_t\omega)\right|^2+\left|z(\theta_t\omega)\right|^p\right)\rho\left(\frac{|x|^2}{k^2}\right)dx.
\end{aligned}\tag{4.46}
$$

Note that (4.46) implies that

$$
\begin{aligned}
&\frac{d}{dt}\int\limits_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|v|^2\,dx+\lambda\int\limits_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|v|^2\,dx\\
&\quad\leqslant\frac{c}{k}\left(\|\nabla v\|^2+\|v\|^2\right)+\int\limits_{\mathbb{R}^n}\left(2|\psi_1|+|\psi_2|^2+\frac{2}{\lambda}g^2\right)\rho\left(\frac{|x|^2}{k^2}\right)dx\\
&\quad+c\int\limits_{\mathbb{R}^n}\left(\left|\Delta z(\theta_t\omega)\right|^2+\left|z(\theta_t\omega)\right|^2+\left|z(\theta_t\omega)\right|^p\right)\rho\left(\frac{|x|^2}{k^2}\right)dx.
\end{aligned}\tag{4.47}
$$

By Lemmas 4.1 and 4.5, there is $T_1=T_1(B,\omega)>0$ such that for all $t\geqslant T_1$,

$$
\left\|v\left(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\right)\right\|^2_{H^1(\mathbb{R}^n)}\leqslant c\left(1+r(\omega)\right).\tag{4.48}
$$

Now integrating (4.47) over (T_1,t), we get that, for all $t\geqslant T_1$,

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left|v\left(t,\omega,v_0(\omega)\right)\right|^2dx\\
&\quad\leqslant e^{\lambda(T_1-t)}\int\limits_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left|v\left(T_1,\omega,v_0(\omega)\right)\right|^2dx\\
&\qquad+\frac{c}{k}\int\limits_{T_1}^{t}e^{\lambda(s-t)}\left(\left\|\nabla v\left(s,\omega,v_0(\omega)\right)\right\|^2+\left\|v\left(s,\omega,v_0(\omega)\right)\right\|^2\right)ds\\
&\qquad+\int\limits_{T_1}^{t}e^{\lambda(s-t)}\int\limits_{\mathbb{R}^n}\left(2|\psi_1|+|\psi_2|^2+\frac{2}{\lambda}g^2\right)\rho\left(\frac{|x|^2}{k^2}\right)dx\,ds\\
&\qquad+c\int\limits_{T_1}^{t}e^{\lambda(s-t)}\int\limits_{\mathbb{R}^n}\left(\left|\Delta z(\theta_s\omega)\right|^2+\left|z(\theta_s\omega)\right|^2+\left|z(\theta_s\omega)\right|^p\right)\rho\left(\frac{|x|^2}{k^2}\right)dx\,ds.
\end{aligned}\tag{4.49}
$$

Replacing ω by $\theta_{-t}\omega$, we obtain from (4.49) that, for all $t\geqslant T_1$,

$$\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)\left|v\left(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\right)\right|^2 dx$$

$$\leqslant e^{\lambda(T_1-t)} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)\left|v\left(T_1, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\right)\right|^2 dx$$

$$+ \frac{c}{k}\int_{T_1}^{t} e^{\lambda(s-t)}\left\|\nabla v\left(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\right)\right\|^2 ds + \frac{c}{k}\int_{T_1}^{t} e^{\lambda(s-t)}\left\|v\left(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\right)\right\|^2 ds$$

$$+ \int_{T_1}^{t} e^{\lambda(s-t)} \int_{\mathbb{R}^n} \left(2|\psi_1| + |\psi_2|^2 + \frac{2}{\lambda} g^2\right)\rho\left(\frac{|x|^2}{k^2}\right) dx\, ds$$

$$+ \tilde{c} \int_{T_1}^{t} e^{\lambda(s-t)} \int_{\mathbb{R}^n} \left(\left|\Delta z(\theta_{s-t}\omega)\right|^2 + \left|z(\theta_{s-t}\omega)\right|^2 + \left|z(\theta_{s-t}\omega)\right|^p\right)\rho\left(\frac{|x|^2}{k^2}\right) dx\, ds. \tag{4.50}$$

In what follows, we estimate the terms in (4.50). First replacing t by T_1 and then replacing ω by $\theta_{-t}\omega$ in (4.9), we have the following bounds for the first term on the right-hand side of (4.50):

$$e^{\lambda(T_1-t)} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)\left|v\left(T_1, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\right)\right|^2 dx$$

$$\leqslant e^{\lambda(T_1-t)}\left(e^{-\lambda T_1}\left\|v_0(\theta_{-t}\omega)\right\|^2 + \int_0^{T_1} e^{\lambda(s-T_1)} p_1(\theta_{s-t}\omega)\, ds + c\right)$$

$$\leqslant e^{-\lambda t}\left\|v_0(\theta_{-t}\omega)\right\|^2 + ce^{\lambda(T_1-t)} + \int_{-t}^{T_1-t} e^{\lambda\tau} p_1(\theta_\tau\omega)\, d\tau$$

$$\leqslant e^{-\lambda t}\left\|v_0(\theta_{-t}\omega)\right\|^2 + ce^{\lambda(T_1-t)} + \int_{-t}^{T_1-t} e^{\frac{1}{2}\lambda\tau} c_6 r(\omega)\, d\tau$$

$$\leqslant e^{-\lambda t}\left\|v_0(\theta_{-t}\omega)\right\|^2 + ce^{\lambda(T_1-t)} + \frac{2}{\lambda} c_6 r(\omega) e^{\frac{1}{2}\lambda(T_1-t)}, \tag{4.51}$$

where we have used (4.6). By (4.51), we find that, given $\epsilon > 0$, there is $T_2 = (B, \omega, \epsilon) > T_1$ such that for all $t \geqslant T_2$,

$$e^{\lambda(T_1-t)} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)\left|v\left(T_1, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\right)\right|^2 dx \leqslant \epsilon. \tag{4.52}$$

By Lemma 4.2, there is $T_3 = T_3(B, \omega) > T_1$ such that the second term on the right-hand side of (4.50) satisfies

$$\frac{c}{k}\int_{T_1}^{t} e^{\lambda(s-t)} \left\| \nabla v\big(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\big)\right\|^2 ds \leqslant \frac{c}{k}\big(1 + r(\omega)\big).$$

And hence, there is $R_1 = R_1(\omega, \epsilon) > 0$ such that for all $t \geqslant T_3$ and $k \geqslant R_1$,

$$\frac{c}{k}\int_{T_1}^{t} e^{\lambda(s-t)} \left\| \nabla v\big(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\big)\right\|^2 ds \leqslant \epsilon. \tag{4.53}$$

First replacing t by s and then replacing ω by $\theta_{-t}\omega$ in (4.9), we find that the third term on the right-hand side of (4.50) satisfies

$$\begin{aligned}
&\frac{c}{k}\int_{T_1}^{t} e^{\lambda(s-t)} \left\| v\big(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\big)\right\|^2 ds \\
&\quad \leqslant \frac{c}{k}\int_{T_1}^{t} e^{-\lambda t}\left\| v_0(\theta_{-t}\omega)\right\|^2 ds + \frac{c}{k}\int_{T_1}^{t} e^{\lambda(s-t)}\int_0^s e^{\lambda(\tau-s)} p_1(\theta_{\tau-t}\omega)\, d\tau\, ds + \frac{c}{k}\int_{T_1}^{t} e^{\lambda(s-t)}\, ds \\
&\quad \leqslant \frac{c}{k} e^{-\lambda t}(t - T_1)\left\| v_0(\theta_{-t}\omega)\right\|^2 + \frac{c}{k} + \frac{c}{k}\int_{T_1}^{t}\int_0^s e^{\lambda(\tau-t)} p_1(\theta_{\tau-t}\omega)\, d\tau\, ds \\
&\quad \leqslant \frac{c}{k} e^{-\lambda t}(t - T_1)\left\| v_0(\theta_{-t}\omega)\right\|^2 + \frac{c}{k} + \frac{c}{k}\int_{T_1}^{t}\int_{-t}^{s-t} e^{\lambda\tau} p_1(\theta_{\tau}\omega)\, d\tau\, ds \\
&\quad \leqslant \frac{c}{k} e^{-\lambda t}(t - T_1)\left\| v_0(\theta_{-t}\omega)\right\|^2 + \frac{c}{k} + \frac{c}{k} c_6 r(\omega)\int_{T_1}^{t}\int_{-t}^{s-t} e^{\frac{1}{2}\lambda\tau}\, d\tau\, ds \\
&\quad \leqslant \frac{c}{k} e^{-\lambda t}(t - T_1)\left\| v_0(\theta_{-t}\omega)\right\|^2 + \frac{c}{k} + \frac{4c}{\lambda^2 k} c_6 r(\omega).
\end{aligned}$$

This implies that there exist $T_4 = T_4(B, \omega, \epsilon) > T_1$ and $R_2 = R_2(\omega, \epsilon)$ such that for all $t \geqslant T_4$ and $k \geqslant R_2$,

$$\frac{c}{k}\int_{T_1}^{t} e^{\lambda(s-t)} \left\| v\big(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\big)\right\|^2 ds \leqslant \epsilon. \tag{4.54}$$

Note that $\psi_1 \in L^1(\mathbb{R}^n)$ and $\psi_2, g \in L^2(\mathbb{R}^n)$. Therefore, there is $R_3 = R_3(\epsilon)$ such that for all $k \geqslant R_3$,

$$\int_{|x|\geqslant k} \left(2|\psi_1| + |\psi_2|^2 + \frac{2}{\lambda} g^2\right) dx \leqslant \lambda\epsilon.$$

Then for the fourth term on the right-hand side of (4.50), we have

$$\int_{T_1}^{t} e^{\lambda(s-t)} \int_{\mathbb{R}^n} \left(2|\psi_1| + |\psi_2|^2 + \frac{2}{\lambda} g^2\right) \rho\left(\frac{|x|^2}{k^2}\right) dx\, ds$$

$$\leqslant \int_{T_1}^{t} e^{\lambda(s-t)} \int_{|x|\geqslant k} \left(2|\psi_1| + |\psi_2|^2 + \frac{2}{\lambda} g^2\right) dx\, ds \leqslant \lambda\epsilon \int_{T_1}^{t} e^{\lambda(s-t)}\, ds \leqslant \epsilon. \tag{4.55}$$

Note that $z(\theta_t\omega) = \sum_{j=1}^m h_j z_j(\theta_t\omega_j)$ and $h_j \in H^2(\mathbb{R}^n) \cap W^{2,p}(\mathbb{R}^n)$. Hence there is $R_4 = R_4(\omega, \epsilon)$ such that for all $k \geqslant R_4$ and $j = 1, 2, \ldots, m$,

$$\int_{|x|\geqslant k} \left(\left|h_j(x)\right|^2 + \left|h_j(x)\right|^p + \left|\Delta h_j(x)\right|^2\right) dx \leqslant \min\left\{\frac{\lambda\epsilon}{4m^p \tilde{c} r(\omega)}, \frac{\epsilon}{2m^2 r(\omega)}\right\}, \tag{4.56}$$

where $r(\omega)$ is the tempered function in (3.8) and $\tilde{c}$ is the positive constant in the last term on the right-hand side of (4.50). By (4.56) and (3.8)–(3.9), we have the following bounds for the last term on the right-hand side of (4.50):

$$\tilde{c} \int_{T_1}^{t} e^{\lambda(s-t)} \int_{\mathbb{R}^n} \left(\left|\Delta z(\theta_{s-t}\omega)\right|^2 + \left|z(\theta_{s-t}\omega)\right|^2 + \left|z(\theta_{s-t}\omega)\right|^p\right) \rho\left(\frac{|x|^2}{k^2}\right) dx\, ds$$

$$\leqslant \tilde{c} \int_{T_1}^{t} e^{\lambda(s-t)} \int_{|x|\geqslant k} \left(\left|\Delta z(\theta_{s-t}\omega)\right|^2 + \left|z(\theta_{s-t}\omega)\right|^2 + \left|z(\theta_{s-t}\omega)\right|^p\right) dx\, ds$$

$$\leqslant m^p \tilde{c} \int_{T_1}^{t} e^{\lambda(s-t)} \sum_{j=1}^{m} \int_{|x|\geqslant k} \left(|\Delta h_j|^2 \left|z_j(\theta_{s-t}\omega_j)\right|^2 + |h_j|^2 \left|z_j(\theta_{s-t}\omega_j)\right|^2 \right.$$

$$\left. + |h_j|^p \left|z_j(\theta_{s-t}\omega_j)\right|^p\right) dx\, ds$$

$$\leqslant \frac{\lambda\epsilon}{2r(\omega)} \int_{T_1}^{t} e^{\lambda(s-t)} \sum_{j=1}^{m} \left(\left|z_j(\theta_{s-t}\omega_j)\right|^2 + \left|z_j(\theta_{s-t}\omega_j)\right|^p\right) ds$$

$$\leqslant \frac{\lambda\epsilon}{2r(\omega)} \int_{T_1}^{t} e^{\lambda(s-t)} r(\theta_{s-t}\omega)\, ds \leqslant \frac{\lambda\epsilon}{2r(\omega)} \int_{T_1-t}^{0} e^{\lambda\tau} r(\theta_\tau\omega)\, d\tau$$

$$\leqslant \frac{\lambda\epsilon}{2r(\omega)} \int_{T_1-t}^{0} e^{\frac{1}{2}\lambda\tau} r(\omega)\, d\tau \leqslant \epsilon. \tag{4.57}$$

Let $T_5 = T_5(B, \omega, \epsilon) = \max\{T_1, T_2, T_3, T_4\}$ and $R_5 = R_5(\omega, \epsilon) = \max\{R_1, R_2, R_3, R_4\}$. Then it follows from (4.50), (4.52)–(4.57) that, for all $t \geqslant T_5$ and $k \geqslant R_5$, one has

$$\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)\left|v\left(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\right)\right|^2 dx \leqslant 5\epsilon,$$

which shows that for all $t \geqslant T_5$ and $k \geqslant R_5$,

$$\int_{|x| \geqslant \sqrt{2}k} \left|v\left(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\right)\right|^2 dx \leqslant \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)\left|v\left(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\right)\right|^2 dx \leqslant 5\epsilon.$$

This completes the proof. $\square$

Lemma 4.7. *Assume that $g \in L^2(\mathbb{R}^n)$ and (3.3)–(3.6) hold. Let $B = \{B(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$ and $u_0(\omega) \in B(\omega)$. Then for every $\epsilon > 0$ and P-a.e. $\omega \in \Omega$, there exist $T^* = T_B^*(\omega, \epsilon) > 0$ and $R^* = R^*(\omega, \epsilon) > 0$ such that, for all $t \geqslant T^*$,*

$$\int_{|x| \geqslant R^*} \left|u\left(t, \theta_{-t}\omega, u_0(\theta_{-t}\omega)\right)(x)\right|^2 dx \leqslant \epsilon.$$

Proof. Let T^* and R^* be the constants in Lemma 4.6. By (4.56) and (3.8) we have, for all $t \geqslant T^*$ and $k \geqslant R^*$,

$$\begin{aligned}\int_{|x| \geqslant R^*} \left|z(\omega)\right|^2 dx &= \int_{|x| \geqslant R^*} \left|\sum_{j=1}^m h_j z_j(\omega_j)\right|^2 dx \\ &\leqslant m^2 \int_{|x| \geqslant R^*} \sum_{j=1}^m |h_j|^2 \left|z_j(\omega_j)\right|^2 dx \leqslant \frac{\epsilon}{2r(\omega)} \sum_{j=1}^m \left|z_j(\omega_j)\right|^2 \leqslant \frac{\epsilon}{2}. \qquad (4.58)\end{aligned}$$

Then by (4.58) and Lemma 4.6, we get that, for all $t \geqslant T^*$ and $k \geqslant R^*$,

$$\begin{aligned}&\int_{|x| \geqslant R^*} \left|u\left(t, \theta_{-t}\omega, u_0(\theta_{-t}\omega)\right)\right|^2 dx \\ &\quad = \int_{|x| \geqslant R^*} \left|v\left(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\right) + z(\omega)\right|^2 dx \\ &\quad \leqslant 2 \int_{|x| \geqslant R^*} v\left(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega)\right)^2 dx + 2 \int_{|x| \geqslant R^*} z(\omega)^2 dx \\ &\quad \leqslant 3\epsilon,\end{aligned}$$

which completes the proof. $\square$

5. Random attractors

In this section, we prove the existence of a $\mathcal{D}$-random attractor for the random dynamical system ϕ associated with the stochastic reaction–diffusion equation (3.1)–(3.2) on $\mathbb{R}^n$. It follows from Lemma 4.1 that ϕ has a closed random absorbing set in $\mathcal{D}$, which along with the $\mathcal{D}$-pullback asymptotic compactness will imply the existence of a unique $\mathcal{D}$-random attractor. The $\mathcal{D}$-pullback asymptotic compactness of ϕ is given below and will be proved by using the uniform estimates on the tails of solutions.

Lemma 5.1. *Assume that $g \in L^2(\mathbb{R}^n)$ and (3.3)–(3.6) hold. Then the random dynamical system ϕ is $\mathcal{D}$-pullback asymptotically compact in $L^2(\mathbb{R}^n)$; that is, for P-a.e. $\omega \in \Omega$, the sequence $\{\phi(t_n, \theta_{-t_n}\omega, u_{0,n}(\theta_{-t_n}\omega))\}$ has a convergent subsequence in $L^2(\mathbb{R}^n)$ provided $t_n \to \infty$, $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $u_{0,n}(\theta_{-t_n}\omega) \in B(\theta_{-t_n}\omega)$.*

Proof. Let $t_n \to \infty$, $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $u_{0,n}(\theta_{-t_n}\omega) \in B(\theta_{-t_n}\omega)$. Then by Lemma 4.1, for P-a.e. $\omega \in \Omega$, we have that

$$\left\{\phi\big(t_n, \theta_{-t_n}\omega, u_{0,n}(\theta_{-t_n}\omega)\big)\right\}_{n=1}^{\infty} \quad \text{is bounded in } L^2\big(\mathbb{R}^n\big).$$

Hence, there is $\xi \in L^2(\mathbb{R}^n)$ such that, up to a subsequence,

$$\phi\big(t_n, \theta_{-t_n}\omega, u_{0,n}(\theta_{-t_n}\omega)\big) \to \xi \quad \text{weakly in } L^2\big(\mathbb{R}^n\big). \tag{5.1}$$

Next, we prove the weak convergence of (5.1) is actually strong convergence. Given $\epsilon > 0$, by Lemma 4.7, there is $T_1 = T_1(B, \omega, \epsilon)$ and $R_1 = R_1(\omega, \epsilon)$ such that for all $t \geqslant T_1$,

$$\int\limits_{|x|\geqslant R_1} \left|\phi\big(t, \theta_{-t}\omega, u_0(\theta_{-t}\omega)\big)\right|^2 dx \leqslant \epsilon. \tag{5.2}$$

Since $t_n \to \infty$, there is $N_1 = N_1(B, \omega, \epsilon)$ such that $t_n \geqslant T_1$ for every $n \geqslant N_1$. Hence, it follows from (5.2) that for all $n \geqslant N_1$,

$$\int\limits_{|x|\geqslant R_1} \left|\phi\big(t_n, \theta_{-t_n}\omega, u_{0,n}(\theta_{-t_n}\omega)\big)\right|^2 dx \leqslant \epsilon. \tag{5.3}$$

On the other hand, by Lemmas 4.1 and 4.5, there is $T_2 = T_2(B, \omega)$ such that for all $t \geqslant T_2$,

$$\left\|\phi\big(t, \theta_{-t}\omega, u_0(\theta_{-t}\omega)\big)\right\|^2_{H^1(\mathbb{R}^n)} \leqslant c\big(1 + r(\omega)\big). \tag{5.4}$$

Let $N_2 = N_2(B, \omega)$ be large enough such that $t_n \geqslant T_2$ for $n \geqslant N_2$. Then by (5.4) we find that, for all $n \geqslant N_2$,

$$\left\|\phi\big(t_n, \theta_{-t_n}\omega, u_{0,n}(\theta_{-t_n}\omega)\big)\right\|^2_{H^1(\mathbb{R}^n)} \leqslant c\big(1 + r(\omega)\big). \tag{5.5}$$

Denote by Q_{R_1} the set $\{x \in \mathbb{R}^n\colon\ |x| \leqslant R_1\}$. By the compactness of embedding $H^1(Q_{R_1}) \hookrightarrow L^2(Q_{R_1})$, it follows from (5.5) that, up to a subsequence,

$$\phi\big(t_n, \theta_{-t_n}\omega, u_{0,n}(\theta_{-t_n}\omega)\big) \to \xi \quad \text{strongly in } L^2(Q_{R_1}),$$

which shows that for the given $\epsilon > 0$, there exists $N_3 = N_3(B, \omega, \epsilon)$ such that for all $n \geqslant N_3$,

$$\left\| \phi\big(t_n, \theta_{-t_n}\omega, u_{0,n}(\theta_{-t_n}\omega)\big) - \xi \right\|^2_{L^2(Q_{R_1})} \leqslant \epsilon. \tag{5.6}$$

Note that $\xi \in L^2(\mathbb{R}^n)$. Therefore there exists $R_2 = R_2(\epsilon)$ such that

$$\int\limits_{|x| \geqslant R_2} \left|\xi(x)\right|^2 dx \leqslant \epsilon. \tag{5.7}$$

Let $R_3 = \max\{R_1, R_2\}$ and $N_4 = \max\{N_1, N_3\}$. By (5.3), (5.6), and (5.7), we find that for all $n \geqslant N_4$,

$$\begin{aligned} \left\| \phi\big(t_n, \theta_{-t_n}\omega, u_{0,n}(\theta_{-t_n}\omega)\big) - \xi \right\|^2_{L^2(\mathbb{R}^n)} &\leqslant \int\limits_{|x| \leqslant R_3} \left| \phi\big(t_n, \theta_{-t_n}\omega, u_{0,n}(\theta_{-t_n}\omega)\big) - \xi \right|^2 dx \\ &\quad + \int\limits_{|x| \geqslant R_3} \left| \phi\big(t_n, \theta_{-t_n}\omega, u_{0,n}(\theta_{-t_n}\omega)\big) - \xi \right|^2 dx \leqslant 5\epsilon, \end{aligned}$$

which shows that

$$\phi\big(t_n, \theta_{-t_n}\omega, u_{0,n}(\theta_{-t_n}\omega)\big) \to \xi \quad \text{strongly in } L^2\big(\mathbb{R}^n\big),$$

as desired. □

We are now in a position to present our main result: the existence of a $\mathcal{D}$-random attractor for ϕ in $L^2(\mathbb{R}^n)$.

Theorem 5.2. *Assume that $g \in L^2(\mathbb{R}^n)$ and* (3.3)–(3.6) *hold. Then the random dynamical system ϕ has a unique $\mathcal{D}$-random attractor in $L^2(\mathbb{R}^n)$.*

Proof. Notice that ϕ has a closed random absorbing set $\{K(\omega)\}_{\omega \in \Omega}$ in $\mathcal{D}$ by Lemma 4.1, and is $\mathcal{D}$-pullback asymptotically compact in $L^2(\mathbb{R}^n)$ by Lemma 5.1. Hence the existence of a unique $\mathcal{D}$-random attractor for ϕ follows from Proposition 2.7 immediately. □

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