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Digital pseudomanifolds, digital weakmanifolds and Jordan–Brouwer separation theorem

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Abstract

In this paper we introduce the new notion of n-pseudomanifold and n-weakmanifold in an $(n+1)$-digital image using $(2(n+1), 3^{(n+1)}-1)$-adjacency. For these classes, we prove the digital version of the Jordan–Brouwer separation theorem. To accomplish this objective, we construct a polyhedral representation of the $(n+1)$-digital image based on a cubical complex decomposition which enables us to translate some results from polyhedral topology into the digital space. Our main result extends the class of "thin" objects that are defined locally and verifying the Jordan–Brouwer separation theorem.

Keywords: Digital topology; Combinatorial topology; Discrete space; Combinatorial manifolds

1. Introduction

Let n be any positive integer. A binary $(n+1)$-digital image is an ordered uplet $\mathscr{P} = (\mathbb{Z}^{n+1}, \mathscr{R}, H)$, where H is a finite subset of $\mathbb{Z}^{n+1}$ and $\mathscr{R}$ represents the adjacency relation in the whole lattice.

Morgenthaler and Rosenfeld [15] introduce for a binary 3-digital image, the concept of a *simple surface point* to characterize a class of objects H that verifies the digital version of the Jordan–Brouwer separation theorem, where the corresponding $\mathscr{R}$ is in $\{(6,26),(26,6)\}$. They define a simple surface point axiomatically by three local

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conditions reflecting basic properties of vertices in a triangulated simple and closed surface of $\mathbb{R}^3$. They show that any finite and connected subset of $\mathbb{Z}^3$ consisting entirely of simple surface points verifies the digital version of the Jordan–Brouwer separation theorem. Thus, the simple surface points criterion determines a class of thin objects called *digital simple surface* of $\mathbb{Z}^3$.

Kong and Roscoe [10] reveal precisely the geometric meaning of the simple surface points and extend its properties to other classes of $\mathscr{R}$-adjacency, i.e., $\mathscr{R} \in \{(18,6),(6,18),(26,18),(18,26)\}$. They introduce the notion of continuous analogs for transferring statements from continuous topology to digital space, and show that a connected finite subset H of $\mathbb{Z}^3$ is a digital simple surface if and only if the continuous analogs of H is a simple and closed polyhedral surface of $\mathbb{R}^3$. Note that [12] extends the concept of continuous analogs to the class of strongly normal 3D digital image.

Bertrand and Malgouyres [5] define strong surfaces and Couprie and Bertrand [7] introduce simplicity surfaces. Those classes of surfaces are richer than Morgenthaler–Rosenfeld surfaces because they contain more local configurations and discrete analogs for these classes of surfaces can be built (see [4] for strong surfaces).

Ayala et al. [3] prove that any digital n-manifold in $\mathbb{Z}^{n+1}$ verifies the Jordan–Brouwer separation theorem (the digital n-manifold is the generalization of the notion of digital simple surface for $n \geqslant 2$).

Malgouyres [13] investigate the reverse problem in $\mathbb{Z}^3$ in the context of the $(26,6)$-adjacency relation. He shows that there is not a local characterization of subset of $\mathbb{Z}^3$ which separates $\mathbb{Z}^3$ in exactly two 6-connected components, see also [6,14].

It is not difficult to realize that there is still a large margin between the local characterization of digital simple surface and the global characterization represented by the digital version of the Jordan–Brouwer separation theorem. Intuitively, the problem is that there exists polyhedra with singularities which verify the Jordan–Brouwer separation theorem, for example the polyhedron obtained by identifying two opposite vertices of a combinatorial 2-sphere admits a singularity at the identified point, however it verifies the Jordan–Brouwer separation theorem. Therefore, the manifold class is too restrictive to represent the class of “thin” objects. In this context, the notion of pseudomanifold (see Definition 5) is more interesting because it allows to formulate weak singularities corresponding to degenerate manifolds.

This work is a first attempt to bring closer local and global properties that define the class of objects verifying the digital version of the Jordan–Brouwer separation theorem. Our approach is based on the combinatorial topology and uses topological properties in addition to local neighborhood structures.

Definitions and statements of n-manifolds and n-pseudomanifolds are given in Section 2. Section 3 is devoted to review basic notions of digital topology in $\mathbb{Z}^{n+1}$. In Section 4, we give a simple method to construct the continuous analogs of a binary $(n+1)$-digital image based on a cubical complex decomposition. This enables us to translate results from polyhedral topology into digital topology. In Section 5, we introduce the notions of a cubical n-weakmanifold and a cubical n-pseudomanifold, then we present some intermediary results. Section 6 is aimed to state our main results, we give the definition of a digital n-weakmanifold, digital n-pseudomanifold and show that they verify the digital version of Jordan–Brouwer separation theorem.

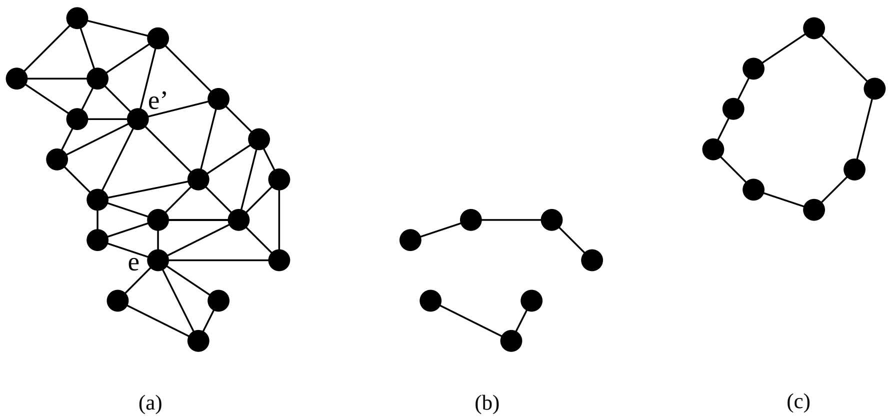


Fig. 1. A simplicial complex K (a), and two links in K: the link of the vertex e (b), and the link of the vertex e' (c).

2. *n*-manifold and *n*-pseudomanifold

In this section, we review some definitions of algebraic topology.

Definition 1 (see Agoston [1, p. 42]). A triangulation of a space X is a pair (K, ϕ), where K is a simplicial complex and $\phi : |K| \to X$ is an homeomorphism. The complex K is said to triangulate X. A polyhedron is any space which admits a triangulation.

Definition 2 (see Agoston [1, p. 187]). A simplicial complex K is said to be homogeneously n-dimensional if every simplex of K is a face of some n-simplex in K.

Definition 3 (see Hudson [8, p. 20]). Let K be an homogeneous n-dimensional simplicial complex. A combinatorial n-sphere is a polyhedron $|K|$ such that certain subdivision of K is simplicially homeomorphic to a subdivision of the boundary of the $(n+1)$-simplex.

Let K be a simplicial complex and $e \in K$. The link of e in K, denoted by $Lk(e;K)$, is a subcomplex of K defined as follows:
$\sigma \in Lk(e;K)$ *if and only if*

1. e is not a face of σ, and
2. $\exists \sigma' \in K$ such that σ is a face of σ', and e is a face of σ'.

Intuitively, $Lk(e;K)$ represents a part of the combinatorial boundary of the smaller neighborhood of e in K. Fig. 1(b) (resp. Fig. 1(c)) illustrates the case where $Lk(e;K)$ is strictly included in (resp. equal to) the combinatorial boundary of the smaller neighborhood of e in K.

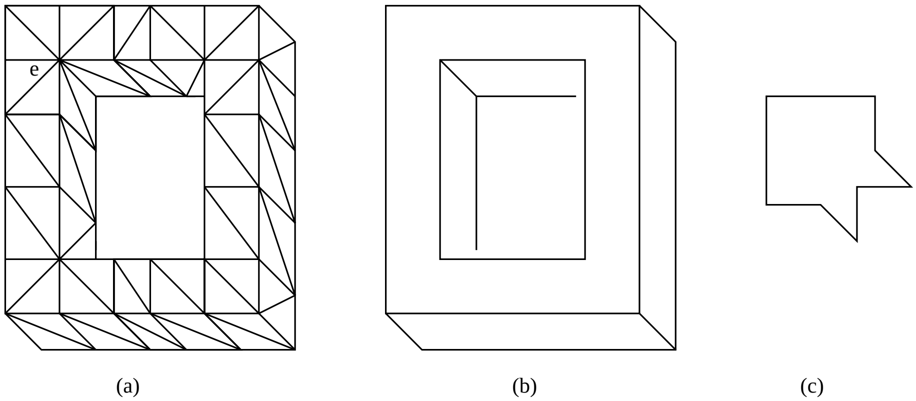


Fig. 2. A combinatorial 2-manifold K (a) representing a torus (b), and the link of the vertex e in K(c).

Definition 4 (see Hudson [8, p. 26]). A connected polyhedron X is said to be an n-manifold without boundary if it admits a triangulation (K,ϕ) satisfying:

1. K is homogeneously n-dimensional.
2. for all $\sigma \in K$, $Lk(\sigma;K)$ is a combinatorial $(n - dim(\sigma) - 1)$-sphere. K is called a combinatorial n-manifold without boundary.

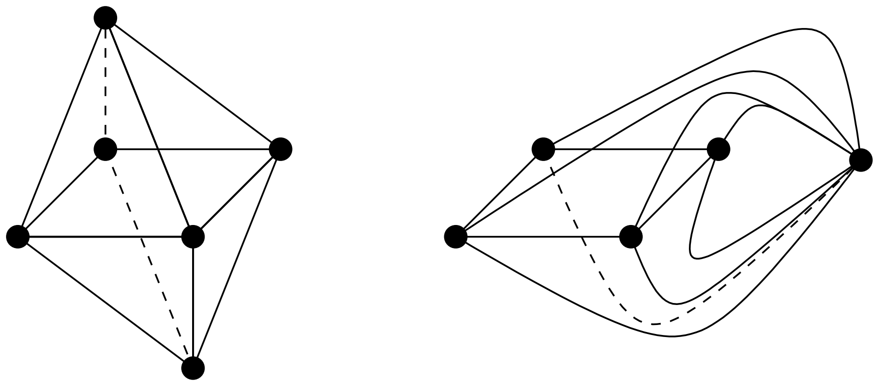
Let us consider the combinatorial torus K of Fig. 2(a), if we look at any vertex e of K, we note that $Lk(e,K)$ is a combinatorial 1-sphere, i.e., combinatorial circle as in Fig. 2(c).

Definition 5 (see Agoston [1, p. 195]). A connected polyhedron X is said to be an n-pseudomanifold without boundary if it admits a triangulation (K,ϕ) such that:

1. K is homogeneously n-dimensional.
2. Every $(n-1)$-simplex of K is a face of precisely two n-simplices of K.
3. If σ and σ' are two distincts n-simplices of K, then there exists a sequence $\sigma_1,\ldots,\sigma_u$ of n-simplices in K such that $\sigma_1 = \sigma$, $\sigma_u = \sigma'$ and σ_i meets σ_{i+1} in an $(n-1)$-dimensional face for $1 \leqslant i < u$.
   K is called a combinatorial n-pseudomanifold without boundary.

It is simple to verify that the combinatorial torus K of Fig. 2(a) is a combinatorial 2-pseudomanifold without boundary.

Now, let us consider the combinatorial octahedron K of Fig. 3(a). We can note that K is a combinatorial 2-manifold without boundary. If we identify two opposite vertices of K as in Fig. 3(b), we will obtain a 2-dimensional curved polyhedron whose triangulations are examples of 2-pseudomanifold without boundary and which are not



(a) (b)

Fig. 3. A combinatorial Octahedron (a), and its transformation by identifying two opposites vertices (b).

2-manifolds without boundary. The example of Fig. 3(b) illustrates the difference between manifold and pseudomanifold.

Proposition 6 (see Agoston [1]).

1. *An n-sphere, denoted by* $\mathbb{S}^n$, *is an n-manifold without boundary.*
2. *Every n-manifold without boundary is an n-pseudomanifold without boundary.*
3. *If* K *and* L *are simplicial complexes and* $|K|$ *is homeomorphic to* $|L|$, *then* K *is homogeneously n-dimensional if and only if* L *is.*
4. *Let* X *be an n-pseudomanifold and let* (L, ψ) *be any triangulation of* X, *then* L *is a combinatorial n-pseudomanifold without boundary.*

Theorem 7 (see Aleksandrov [2, p. 94]). *Every n-pseudomanifold without boundary in* $\mathbb{S}^{n+1}$ *divides* $\mathbb{S}^{n+1}$ *into two domains* [1] *and is the common boundary of both domains.*

This theorem obviously remains true if $\mathbb{R}^{n+1}$ is substituted for $\mathbb{S}^{n+1}$: it is sufficient to map $\mathbb{S}^{n+1}$ onto $\mathbb{R}^{n+1}$ by means of a stereographic projection with center of projection lying outside the given pseudomanifold (see [2, p. 94]).

Theorem 8. *Every n-pseudomanifold without boundary in* $\mathbb{R}^{n+1}$ *divides* $\mathbb{R}^{n+1}$ *into two domains and is the common boundary of both domains.*

Note that the characterisation of a pseudomanifold is simple to implement and less constraining than the characterisation of a manifold.

[1] A domain of $\mathbb{S}^{n+1}$ is an open and connected subset of $\mathbb{S}^{n+1}$.

3. General definitions

In digital image, the study of neighborhood is a very important and significant concept. For a given lattice point, a neighborhood of a point is defined typically using a metric distance.

Let $p, q \in \mathbb{Z}^{n+1}$ with the coordinates $(x_i(p))_{i=1}^{n+1}$ and $(x_i(q))_{i=1}^{n+1}$ respectively. We will consider two types of distance between elements in $\mathbb{Z}^{n+1}$:

$$d_1(p,q) = \sum_{i=1}^{n+1} |x_i(p) - x_i(q)| \quad \text{and} \quad d_\infty(p,q) = \operatorname*{Max}_{i=1,\dots,n+1} |x_i(p) - x_i(q)|.$$

Let $\beta \in \{1, \infty\}$; two points $p, q \in \mathbb{Z}^{n+1}$ are said to be d_β-adjacent if $d_\beta(p,q) = 1$.

We denote by $\mathbb{V}_\beta(p)$ the set of all d_β-neighbors of p. Consequently we have:

(i) $\mathbb{V}_1(p) = \{q \in \mathbb{Z}^{n+1} / d_1(p,q) = 1\}$ and $Card(\mathbb{V}_1(p)) = 2(n+1)$.

(ii) $\mathbb{V}_\infty(p) = \{q \in \mathbb{Z}^{n+1} / d_\infty(p,q) = 1\}$ and $Card(\mathbb{V}_\infty(p)) = 3^{n+1} - 1$.

In the literature ([11,12,16]) the d_1-neighbor is referred as $(2(n+1))$-neighbor and the d_∞-neighbor as $(3^{n+1} - 1)$-neighbor.

Let $T \subset \mathbb{Z}^{n+1}$, we denote by T^c the *complement* of T in $\mathbb{Z}^{n+1}$; $T^c = \mathbb{Z}^{n+1} - T$.

A binary $(n+1)$-digital image is an ordered uplet $\mathscr{P} = (\mathbb{Z}^{n+1}, \mathscr{R}, H)$, where H is a finite subset of $\mathbb{Z}^{n+1}$ and $\mathscr{R}$ represents the adjacency relation in the whole lattice. In this paper, the adjacency relation $\mathscr{R}$ will be taken as follows:

two elements of H are said to *be* $\mathscr{R}$-adjacent *if* they are d_1-adjacent, two elements of H^c are said to be $\mathscr{R}$-adjacent *if* they are d_∞-adjacent, and an element in H is $\mathscr{R}$-adjacent to an element in H^c *if* they are d_∞-adjacent.

Let $T \subset \mathbb{Z}^{n+1}$ and $p, q \in T$. An *$\mathscr{R}$-path* from p to q in T is a sequence of distincts points $(p_1, \dots, p_m)$ in T such that $p_1 = p$, $p_m = q$ and p_i is $\mathscr{R}$-adjacent to p_{i+1}, $1 \leqslant i < m$. T is *$\mathscr{R}$-connected* if given any two elements p and q in T there is an $\mathscr{R}$-path in T from p to q; an *$\mathscr{R}$-component* of T is a maximal $\mathscr{R}$-connected subset of T.

Let $T \subset \mathbb{Z}^{n+1}$ and $p \in \mathbb{Z}^{n+1}$, p is said to be $\mathscr{R}$-adjacent to T if p is $\mathscr{R}$-adjacent to some point in T. T is said to be *strongly thin* if and only if any element of T is $\mathscr{R}$-adjacent to all $\mathscr{R}$-components of T^c. It is said to be *separating* if and only if T^c has exactly two $\mathscr{R}$-components.

4. Polyhedral representation of a binary $(n+1)$-digital image

Let I_i denote the open unit interval $]r_i, r_{i+1}[$ or the single point r_i for some integer r_i. A *k-cube* e_k, $0 \leqslant k \leqslant n+1$, is defined as $e_k = \prod_{i=1}^{n+1} I_i$ where k of the I_i's are intervals and $n+1-k$ are single points. Thus e_k is an open k-cube of $\mathbb{R}^k$ embedded in $\mathbb{R}^{n+1}$. The closure of $e_k = \prod_{i=1}^{n+1} \overline{I_i}$, will be denoted by $\overline{e_k}$. We denote by $Som(e_k)$ the set of all vertices of $\overline{e_k}$. Obviously we have $Som(e_k) \subset \mathbb{Z}^{n+1}$. The subscript of a k-cube will be omitted when irrelevant to the argument at hand. A face of a k-cube e is a k'-cube e' such that $Som(e') \subseteq Som(e)$. We will write $e' \preccurlyeq e$.

A *cubical complex* K is a finite collection of k-cubes in some $\mathbb{R}^{n+1}$, $0 \leqslant k \leqslant n+1$, such that:

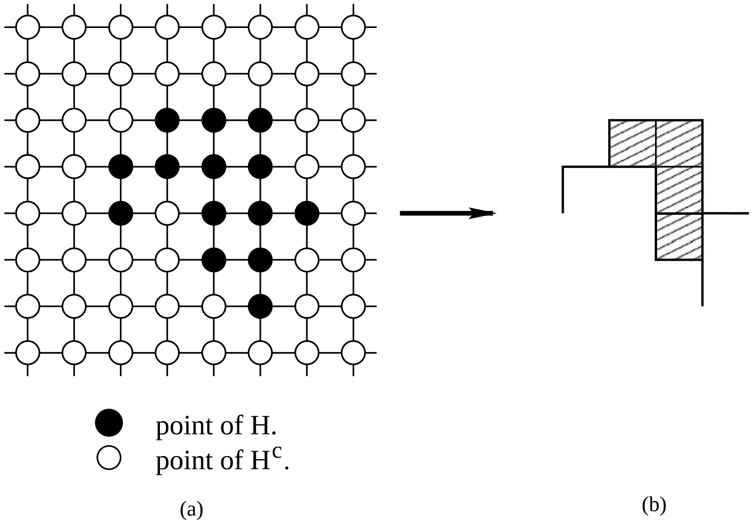


Fig. 4. A binary 2-digital image $\mathscr{P} = (\mathbb{Z}^2, \mathscr{R}, H)$ (a), and its polyhedral representation $|\mathbb{C}(H)|$ (b).

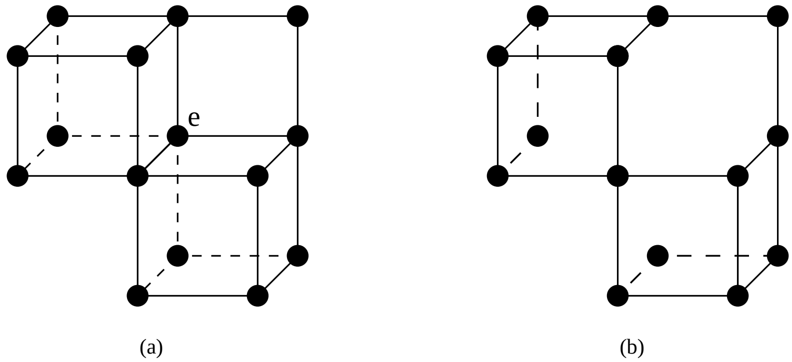


Fig. 5. A cubical complex K (a), and the link of the vertex e in K (b).

- if $e \in K$ then all faces of e belong to K, and
- if $e, e' \in K$ then $e \cap e' = \emptyset$.

Now, we construct the continuous analogs of a binary $(n+1)$-digital image $\mathscr{P} = (\mathbb{Z}^{n+1}, \mathscr{R}, H)$. Intuitively, this construction consists of '*filling in the gaps*' between black points of $\mathscr{P}$, and must be consistent with the $\mathscr{R}$-adjacency relation of $\mathscr{P}$. More precisely, let $\mathbb{C}(H)$ be a collection of k-cubes in $\mathbb{R}^{n+1}$, $0 \leqslant k \leqslant n+1$, defined as follows:

$$\mathbb{C}(H) = \{e\colon k\text{-cube},\ 0 \leqslant k \leqslant n+1/Som(e) \subset H\}.$$

We can note that $\mathbb{C}(H)$ is a cubical complex, the underlying polyhedron $|\mathbb{C}(H)|$ will be called the *polyhedral representation* of $\mathscr{P}$ (see Figs. 4 and 5 for an illustration of this concept).

For each $x \in \mathbb{R}^{n+1}$, $e(x)$ will denote the k-cube, $0 \leqslant k \leqslant n+1$, that contains x.
From the construction of $|\mathbb{C}(H)|$ we can deduce some natural properties:

Remark 9. Let $\mathscr{P} = (\mathbb{Z}^{n+1}, \mathscr{R}, H)$. We have:

1. if $x \in |\mathbb{C}(H)|$ then $e(x) \in \mathbb{C}(H)$.
2. if $x \in \mathbb{R}^{n+1} - |\mathbb{C}(H)|$ then at least one vertex of $e(x)$ belongs to H^c.
3. each component of $|\mathbb{C}(H)|$ or of $\mathbb{R}^{n+1} - |\mathbb{C}(H)|$ meets $\mathbb{Z}^{n+1}$.

The following theorem expresses the fundamental properties that permit us to relate digital topology to Euclidean space topology.

Theorem 10. *Let* $\mathscr{P} = (\mathbb{Z}^{n+1}, \mathscr{R}, H)$.

1. $|\mathbb{C}(H)| \cap \mathbb{Z}^{n+1} = H$ *and* $(\mathbb{R}^{n+1} - |\mathbb{C}(H)|) \cap \mathbb{Z}^{n+1} = H^c$.
2. *Two points in* H *are in the same* $\mathscr{R}$*-component of* H if and only if *they are in the same component of* $|\mathbb{C}(H)|$.
3. *Two points in* H^c *are in the same* $\mathscr{R}$*-component of* H^c if and only if *they are in the same component of* $\mathbb{R}^{n+1} - |\mathbb{C}(H)|$.

The boundary of a component A of $|\mathbb{C}(H)|$ *meets the boundary of a component* B *of* $\mathbb{R}^{n+1} - |\mathbb{C}(H)|$ if and only if *there is a point in* $A \cap \mathbb{Z}^{n+1}$ *that is* $\mathscr{R}$*-adjacent to a point in* $B \cap \mathbb{Z}^{n+1}$.

The proof of the above theorem and other properties related to the concept of continuous analogs in a binary $(n+1)$-digital image are given in [9].

5. Cubical *n*-weakmanifold and cubical *n*-pseudomanifold

We denote by K a cubical complex.

Let $e \in K$. The link of e in K, denoted by $Lk(e;K)$, is the subcomplex of K defined as follows:

$\sigma \in Lk(e;K)$ *if and only if*

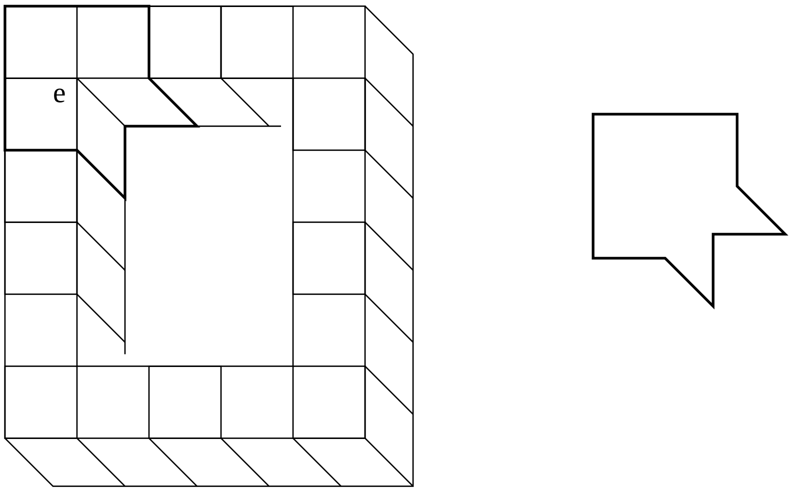
1. $Som(e) \cap Som(\sigma) = \emptyset$, and
2. $\exists \sigma' \in K$ such that σ is a face of σ', and e is a face of σ'.

In the following we suppose that $|K|$ is a connected polyhedron.

Definition 11 (*Cubical n-weakmanifold*). K will be called a cubical n-weakmanifold without boundary if and only if

1. K is homogeneously n-dimensional.
2. for each vertex $p \in K$, $Lk(p;K)$ is a combinatorial $(n-1)$-sphere.

The underlying polyhedron $|K|$ is called an n-weakmanifold without boundary.



(a) (b)

Fig. 6. A cubical 2-weakmanifold without boundary K (a), and the link of the vertex e in K (b).

The above definition is a weak formulation of n-manifold given in Definition 4, it uses uniquely the property of link in vertices of K. For $1 \leqslant n \leqslant 2$ those two notions are equivalent.

The combinatorial torus of Fig. 6(a) is a cubical 2-weakmanifold without boundary. Any vertex of K is a combinatorial 1-sphere, i.e., combinatorial circle.

Definition 12 (Cubical *n-pseudomanifold*). K will be called a cubical n-pseudomanifold without boundary if and only if

1. K is homogeneously n-dimensional.
2. Every $(n-1)$-cube of K is a face of exactly two n-cubes of K.
3. If e and e' are two distincts n-cubes of K, then there exists a sequence $e_1, \ldots, e_u$ of n-cubes in K such that $e_1 = e$, $e_u = e'$ and e_i meets e_{i+1} in an $(n-1)$-cube for $1 \leqslant i < u$.

The underlying polyhedron $|K|$ is called an n-pseudomanifold without boundary.

The combinatorial torus K of Fig. 7(a) is an example of 2-pseudomanifold without boundary. It is easy to see that K is not a cubical 2-weakmanifold without boundary: the condition 2 of the Definition 11 is not verified at the vertex e as illustrated in Fig. 7(b), i.e., $Lk(e,K)$ is not connected, more precisely it is composed by two combinatorial circles.

We can deduce some natural properties:

Proposition 13. *Let K be a cubical complex.*

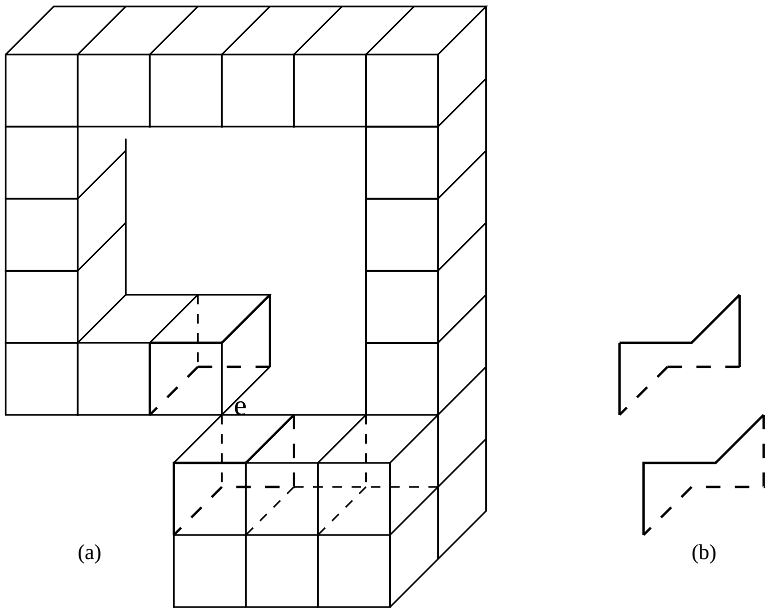


Fig. 7. A cubical 2-pseudomanifold without boundary K (a), and the link of the vertex e in K (b).

1. *If the barycentric subdivision of K is a combinatorial n-pseudomanifold without boundary, then K is a cubical n-pseudomanifold without boundary.*
2. *If K is a cubical n-pseudomanifold without boundary then the barycentric subdivision of K is a combinatorial n-pseudomanifold without boundary.*
3. *Let p be a vertex of K. If $Lk(p,K)$ is a combinatorial $(n-1)$-sphere, then $Lk(p,K)$ is a cubical $(n-1)$-pseudomanifold without boundary.*

Let σ_1 be a k-cube of K and p be a vertex of K such that $p \notin Som(\sigma_1)$. We denote by $p.\sigma_1$ the $(k+1)$-cube (if any) where $(\{p\} \cup Som(\sigma_1)) \subset Som(p.\sigma_1)$.

Remark 14. Let σ and σ_1 be respectively a $(k+1)$-cube and a k-cube of K such that $\sigma_1 \prec \sigma$.

If $p \in Som(\sigma) - Som(\sigma_1)$ then $p.\sigma_1 = \sigma$.

Proposition 15. *Any cubical n-weakmanifold without boundary K is a cubical n-pseudomanifold without boundary.*

Proof. We have to prove properties 2 and 3 in Definition 12.

- Let σ be an $(n-1)$-cube of K, we will show that σ is a face of exactly two n-cubes of K.

Let $p \in Som(\sigma)$, and σ' an $(n-2)$-cube of $Lk(p,K)$ such that $\sigma' \preccurlyeq \sigma$. It is easy to see from Remark 14 that $p.\sigma' = \sigma$. As $Lk(p,K)$ is a combinatorial $(n-1)$-sphere, by using part 1 and 2 of the Proposition 6, we deduce that σ' is a face of exactly two $(n-1)$-cubes $(\sigma_1,\ \sigma_2)$ of $Lk(p,K)$. This implies that $p.\sigma' = \sigma$ is a face of exactly two n-cubes $(p.\sigma_1,\ p.\sigma_2)$ of K, otherwise σ' would be a face for more than two $(n-1)$-cubes of $Lk(p,K)$.

- Let σ and σ' be two n-cubes of K, we will show that there exists a sequence $e_1,\dots,e_u$ of n-cubes in K such that $e_1 = \sigma$, $e_u = \sigma'$ and e_i meets e_{i+1} in an $(n-1)$-cube for $1 \leqslant i < u$.

  Let $p \in Som(\sigma)$ and $p' \in Som(\sigma')$. Since $|K|$ is a connected polyhedron, there is a sequence $(p_0,\dots,p_k)$ of vertices of K that joins $p = p_0$ to $p' = p_k$, i.e., $\bigcup_{i=0}^{k-1} [p_i, p_{i+1}] \subset |K|$. For i, $1 \leqslant i \leqslant k-1$, p_{i-1} and p_{i+1} belong to $Lk(p_i,K)$. Moreover, any two $(n-1)$-cubes e_i and e_i' of $Lk(p_i,K)$ can be joined by a sequence of $(n-1)$-cubes $e_i^0,\dots,e_i^{ki}$ in $Lk(p_i,K)$ such that consecutive $(n-1)$-cubes of this sequence share a common $(n-2)$-cube. So, any two n-cubes of K having p_i as a vertex can be joined by a sequence of n-cubes in K such that consecutive n-cubes of this sequence share a common $(n-1)$-cube.

Let σ_{n,p_i} denotes an n-cube of K such that p_i and p_{i+1} belong to $Som(\sigma_{n,p_i})$. Thus, σ_{n,p_i} and $\sigma_{n,p_{i+1}}$ can be joined by a sequence of n-cubes in K such that consecutive n-cubes of this sequence share a common $(n-1)$-cube, for $0 \leqslant i < k$.

It is the same thing for the pair (σ, σ_{n,p_0}) and $(\sigma_{n,p_k}, \sigma')$. So there exists a sequence of n-cubes in K that joins σ to σ' such that any consecutive n-cubes of this sequence share a common $(n-1)$-cube.

This completes the proof. $\square$

6. Digital n-weakmanifold and digital n-pseudomanifold

Definition 16. Let $\mathscr{P} = (\mathbb{Z}^{n+1}, \mathscr{R}, H)$, and H be $\mathscr{R}$-connected.

1. H will be called *digital n-weakmanifold* if $\mathbb{C}(H)$ is a cubical n-weakmanifold without boundary.
2. H will be called *digital n-pseudomanifold* if $\mathbb{C}(H)$ is a cubical n-pseudomanifold without boundary.

Proposition 17. *Let* $\mathscr{P} = (\mathbb{Z}^{n+1}, \mathscr{R}, H)$.

1. *If H is a digital n-weakmanifold, then H is a digital n-pseudomanifold.*
2. *If H is a digital n-pseudomanifold, then H is a separating set* (*see Section* 3 *for the definition of separating set*).
3. *If H is a digital n-pseudomanifold, then H is strongly thin.*

Proof. 1. Let H be a digital n-weakmanifold.

$\mathbb{C}(H)$ is a cubical n-weakmanifold without boundary. By using Proposition 15, we deduce that $\mathbb{C}(H)$ is a cubical n-pseudomanifold without boundary. This implies that H is a digital n-pseudomanifold.

2. Let H be a digital n-pseudomanifold.

$\mathbb{C}(H)$ is a cubical n-pseudomanifold without boundary. By using the part 2 of the Proposition 13 and part 4 of the Proposition 6, we deduce that $|\mathbb{C}(H)|$ is an n-pseudomanifold without boundary. So, the Theorem 8 allows us to assert that $|\mathbb{C}(H)|$ divides $\mathbb{R}^{n+1}$ in two domains (Int, Ext) and is the common boundary of both domains.

By using Theorem 10, we deduce that H^c has exactly two $\mathscr{R}$-components $Int(H) = Int \cap \mathbb{Z}^{n+1}$ and $Ext(H) = Ext \cap \mathbb{Z}^{n+1}$.

Furthermore, $\partial Int = \partial Ext = |\mathbb{C}(H)|$.

3. Let $p \in H$. Since $\partial Int = |\mathbb{C}(H)|$ and $p \in |\mathbb{C}(H)|$, we can assert that: $\forall \varepsilon > 0$, $\mathbb{B}(p, \varepsilon) \cap Int \neq \emptyset$ where $\mathbb{B}(p, \varepsilon) = \{x \in \mathbb{R}^{n+1} / d_\infty(p, x) \leqslant \varepsilon\}$.

Let $\varepsilon = \frac{1}{3}$. There exists $x \in Int$ such that $d_\infty(p, x) \leqslant \frac{1}{3}$.

Let e be an $(n+1)$-cube such that $p \in Som(e)$ and $x \in \bar{e}$. Note that $e(x) \preccurlyeq e$ (for the definition of $e(x)$ see Section 4).

$e(x) \notin \mathbb{C}(H)$, otherwise $x \in |\mathbb{C}(H)|$. By Property 2 of Remark 9, there exists $p_1 \in Som(e(x))$ such that $p_1 \notin H$. So $[x, p_1] \subset e(x)$.

Since $p_1 \in H^c$ and $x \in Int$, then $p_1 \in Int$ (Int is a connected component of $\mathbb{R}^{n+1} - |\mathbb{C}(H)|$). As $Int(H) = Int \cap \mathbb{Z}^{n+1}$, we deduce that $p_1 \in Int(H)$.

Since $e(x) \preccurlyeq e$, then p and p_1 are vertices of e.

This implies that p is d_∞-adjacent to p_1, i.e., p is $\mathscr{R}$-adjacent to $Int(H)$.

In the same way, we can prove that p is $\mathscr{R}$-adjacent to $Ext(H)$.

This completes the proof. □

Theorem 18. *Let* $\mathscr{P} = (\mathbb{Z}^{n+1}, \mathscr{R}, H)$.

1. *If H is a digital n-pseudomanifold, then H verifies the digital version of the Jordan–Brouwer separation theorem.*
2. *If H is a digital n-weakmanifold, then H verifies the digital version of the Jordan–Brouwer separation theorem.*

Proof. It is easy to deduce this theorem from Proposition 17. □

7. Conclusion

We have defined in a binary $(n+1)$-digital image $\mathscr{P} = (\mathbb{Z}^{n+1}, \mathscr{R}, H)$, where $\mathscr{R} = (2(n+1), 3^{n+1} - 1)$, a new class of objects H (digital n-weakmanifold and digital n-pseudomanifold) that verifies the digital version of the Jordan–Brouwer separation theorem, for $n \geqslant 2$.

References

[1] M.K. Agoston, Algebraic Topology, Marcel Dekker, New York, 1976.

[2] P.S. Aleksandrov, Combinatorial Topology, Vol. 3, Rochester, New York, 1957.

[3] R. Ayala, E. Dominguez, A.R. Francés, A. Quintero, Determining the components of the complement of a digital $(n-1)$-manifold in $\mathbb{Z}^n$, Proceedings of the Sixth International Workshop on Discrete

Geometry for Computer Imagery, DGCI'96, Lectures Notes in Computer Science, Vol. 1176, Springer, Berlin, 1996, pp. 163–176.
[4] R. Ayala, E. Dominguez, A.R. Francés, A. Quintero, A digital lighting function for strong 26-surfaces, Discrete Geometry for Computer Imagery, DGCI'96, Lectures Notes in Computer Science, Vol. 1568, Springer, Berlin, 1999, pp. 91–103.
[5] G. Bertrand, R. Malgouyres, Local property of strong surfaces, SPIE Vision Geometry 3168 (1997) 318–327.
[6] G. Bertrand, R. Malgouyres, Some topological properties of surfaces in $\mathbb{Z}^3$, J. Math. Imaging and Vision 11 (3) (1999) 207–223.
[7] M. Couprie, G. Bertrand, Simplicity surfaces: a new definition of surfaces in $\mathbb{Z}^3$, SPIE Vision Geometry 3454 (1998) 40–51.
[8] J.F.P. Hudson, Piecewise Linear Topology, W.A. Benjamin, New York, 1969.
[9] M. Khachan, P. Chenin, H. Deddi, Polyhedral representation and adjacency graph in *n*-digital image, CVIU (Computer Vision and Image Understanding) 79 (2000) 428–441.
[10] T.Y. Kong, A.W. Roscoe, Continuous analogs of axiomatized digital surfaces, CVGIP (Computer Vision, Graphics, and Image Processing) 29 (1985) 60–86.
[11] T.Y. Kong, A.W. Roscoe, A. Rosenfeld, Concepts of digital topology, Topology Appl. 46 (1992) 219–262.
[12] T.Y. Kong, A. Rosenfeld, Survey digital topology: introduction and survey, CVGIP (Computer Vision, Graphics, and Image Processing) 48 (1989) 357–393.
[13] R. Malgouyres, About surfaces in $\mathbb{Z}^3$, Proceedings of the Fifth International Workshop on Discrete Geometry for Computer Imagery, DGCI'95, 1995, pp. 243–248.
[14] R. Malgouyres, There is no local characterization of separating and thin objects in $\mathbb{Z}^3$, Theoret. Comput. Sci. 163 (1) (1996) 303–309.
[15] D.G. Morgenthaler, A. Rosenfeld, Surfaces in three-dimensional digital images, Inform. and Control 51 (1981) 227–247.
[16] A. Rosenfeld, T.Y. Kong, A.Y. Wu, Digital surfaces, CVGIP 53 (1991) 305–312.