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Solving systems of fractional differential equations using differential transform method

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Abstract

This paper presents approximate analytical solutions for systems of fractional differential equations using the differential transform method. The fractional derivatives are described in the Caputo sense. The application of differential transform method, developed for differential equations of integer order, is extended to derive approximate analytical solutions of systems of fractional differential equations. The solutions of our model equations are calculated in the form of convergent series with easily computable components. Some examples are solved as illustrations, using symbolic computation. The numerical results show that the approach is easy to implement and accurate when applied to systems of fractional differential equations. The method introduces a promising tool for solving many linear and nonlinear fractional differential equations.

MSC: 26A33; 65L05; 65D15

Keywords: Differential transform method; Fractional differential equation; Caputo fractional derivative; Numerical solutions

1. Introduction

In this study, we consider the system of fractional differential equations :

$$\begin{aligned}
D_*^{\alpha_1} x_1(t) &= f_1(t, x_1, x_2, \ldots, x_n),\\
D_*^{\alpha_2} x_2(t) &= f_2(t, x_1, x_2, \ldots, x_n),\\
&\vdots\\
D_*^{\alpha_n} x_n(t) &= f_n(t, x_1, x_2, \ldots, x_n),
\end{aligned} \tag{1.1}$$

where $D_*^{\alpha_i}$ is the derivative of x_i of order α_i in the sense of Caputo and $0 < \alpha_i \leqslant 1$, subject to the initial conditions

$$x_1(0) = c_1, \quad x_2(0) = c_2, \ \ldots, \ x_n(0) = c_n. \tag{1.2}$$

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Differential equations of fractional order have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering. Recently, a large amount of literatures developed concerning the application of fractional differential equations in nonlinear dynamics [5–9,19]. Consequently, considerable attention has been given to the solution of fractional ordinary differential equations, integral equations and fractional partial differential equations of physical interest. Since most fractional differential equations do not have exact analytic solutions, approximation and numerical techniques, therefore, they are used extensively. Recently, the Adomian decomposition method and variational iteration method have been used for solving a wide range of problems [3,6,10–13,15–18,20]. The two methods were used in a direct way without using linearization, perturbation or restrictive assumptions.

The differential transform method was first applied in the engineering domain in [21]. In general, the differential transform method is applied to the solution of electric circuit problems. The differential transform method is a numerical method based on the Taylor series expansion which constructs an analytical solution in the form of a polynomial. The traditional high order Taylor series method requires symbolic computation. However, the differential transform method obtains a polynomial series solution by means of an iterative procedure. Recently, the application of differential transform method is successfully extended to obtain analytical approximate solutions to linear and nonlinear ordinary differential equations of fractional order [1]. A comparison between the differential transform method and Adomian decomposition method for solving fractional differential equations is given in [1]. The fact that the differential transform method solves nonlinear equations without using Adomian polynomials can be considered as an advantage of this method over the Adomian decomposition method.

In this paper, we introduce a new application of the differential transform method [1] to provide approximate solutions for the system of fractional differential equations (1.1). There are several definitions of a fractional derivative of order $\alpha > 0$ [2,19]. The two most commonly used definitions are the Riemann–Liouville and Caputo. Each definition uses Riemann–Liouville fractional integration and derivatives of whole order. The difference between the two definitions is in the order of evaluation. Riemann–Liouville fractional integration of order α is defined as

$$J_{x_0}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} f(t)\, \mathrm{d}t, \quad \alpha > 0, \quad x > 0. \tag{1.3}$$

The next two equations define Riemann–Liouville and Caputo fractional derivatives of order α, respectively,

$$D_{x_0}^{\alpha} f(x) = \frac{\mathrm{d}^m}{\mathrm{d}x^m} [J^{m-\alpha} f(x)], \tag{1.4}$$

$$D_{*x_0}^{\alpha} f(x) = J^{m-\alpha} \left[\frac{\mathrm{d}^m}{\mathrm{d}x^m} f(x) \right], \tag{1.5}$$

where $m-1 < \alpha \leqslant m$ and $m \in N$. For now, the Caputo fractional derivative will be denoted by D_*^{α} to maintain a clear distinction with the Riemann–Liouville fractional derivative. The Caputo fractional derivative first computes an ordinary derivative followed by a fractional integral to achieve the desired order of fractional derivative. The Riemann–Liouville fractional derivative is computed in the reverse order. We have chosen to use the Caputo fractional derivative because it allows traditional initial and boundary conditions to be included in the formulation of the problem, but for homogeneous initial condition assumption, these two operators coincide. For more details on the geometric and physical interpretation for fractional derivatives of both the Riemann–Liouville and Caputo types see [19].

2. Fractional differential transform method

In this section, we introduce the fractional differential transform method used in this paper to obtain approximate analytical solutions for the system of fractional differential equations (1.1). This method has been developed in [1] as follows:

The fractional differentiation in Riemann–Liouville sense is defined by

$$D_{x_0}^{q} f(x) = \frac{1}{\Gamma(m-q)} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \left[\int_{x_0}^{x} \frac{f(t)}{(x-t)^{1+q-m}}\, \mathrm{d}t \right], \tag{2.1}$$

for $m-1 \leqslant q < m, m \in Z^+, x > x_0$. Let us expand the analytical and continuous function $f(x)$ in terms of a fractional power series as follows:

$$f(x) = \sum_{k=0}^{\infty} F(k)(x-x_0)^{k/\alpha}, \tag{2.2}$$

where α is the order of fraction and $F(k)$ is the fractional differential transform of $f(x)$.

In order to avoid fractional initial and boundary conditions, we define the fractional derivative in the Caputo sense. The relation between the Riemann–Liouville operator and Caputo operator is given by

$$D_{*x_0}^q f(x) = D_{x_0}^q \left[f(x) - \sum_{k=0}^{m-1} \frac{1}{k!}(x-x_0)^k f^{(k)}(x_0) \right]. \tag{2.3}$$

Setting $f(x) = f(x) - \sum_{k=0}^{m-1} \frac{1}{k!}(x-x_0)^k f^{\ (k)}(x_0)$ in Eq. (2.1) and using Eq. (2.3), we obtain fractional derivative in the Caputo sense [2] as follows:

$$D_{*x_0}^q f(x) = \frac{1}{\Gamma(m-q)} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \left\{ \int_{x_0}^{x} \left[\frac{f(t) - \sum_{k=0}^{m-1}(1/k!)(t-x_0)^k f^{\ (k)}(x_0)}{(x-t)^{1+q-m}} \right] \mathrm{d}t \right\}. \tag{2.4}$$

Since the initial conditions are implemented to the integer order derivatives, the transformation of the initial conditions are defined as follows:

$$F(k) = \begin{cases} \text{If } k/\alpha \in Z^+, \ \dfrac{1}{(k/\alpha)!}\left[\dfrac{\mathrm{d}^{k/\alpha} f(x)}{\mathrm{d}x^{k/\alpha}}\right]_{x=x_0} & \text{for } k = 0, 1, 2, \ldots, (q\alpha - 1) \\ \text{If } k/\alpha \notin Z^+ & 0, \end{cases} \tag{2.5}$$

where, q is the order of fractional differential equation considered. The following theorems that can be deduced from Eqs. (2.1) and (2.2) are given below, for proofs and details see [1]:

Theorem 1. *If $f(x) = g(x) \pm h(x)$, then $F(k) = G(k) \pm H(k)$.*

Theorem 2. *If $f(x) = g(x)h(x)$, then $F(k) = \sum_{l=0}^{k} G(l)H(k-l)$.*

Theorem 3. *If $f(x) = g_1(x)g_2(x)\ldots g_{n-1}(x)g_n(x)$, then*

$$F(k) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} G_1(k_1)G_2(k_2-k_1)\ldots G_{n-1}(k_{n-1}-k_{n-2})G_n(k-k_{n-1}).$$

Theorem 4. *If $f(x) = (x-x_0)^p$, then $F(k) = \delta(k - \alpha p)$ where,*

$$\delta(k) = \begin{cases} 1 & \textit{if } k = 0, \\ 0 & \textit{if } k \neq 0. \end{cases}$$

Theorem 5. *If $f(x) = D_{x_0}^q[g(x)]$, then $F(k) = \frac{\Gamma(q+1+k/\alpha)}{\Gamma(1+k/\alpha)} G(k + \alpha q)$.*

Proof. Utilizing Eqs. (2.2), (2.4) and (2.5) we get

$$
\begin{aligned}
D_{x_0}^{q}[g(x)] &= \frac{1}{\Gamma(m-q)}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left\{\int_{x_0}^{x}\left[\frac{\sum_{k=0}^{\infty}G(k)(t-x_0)^{k/\alpha}-\sum_{k=0}^{q\alpha-1}G(k)(t-x_0)^{k/\alpha}}{(x-t)^{1+q-m}}\right]\mathrm{d}t\right\}\\
&= \frac{1}{\Gamma(m-q)}\sum_{k=\alpha q}^{\infty}G(k)\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left[\int_{x_0}^{x}\frac{(t-x_0)^{k/\alpha}}{(x-t)^{1+q-m}}\mathrm{d}t\right]\\
&= \sum_{k=\alpha q}^{\infty}\frac{\Gamma(1+k/\alpha)}{\Gamma(1-q+k/\alpha)}G(k)(x-x_0)^{k/\alpha-q}.
\end{aligned}
$$

Starting the index of this series from $k=0$, we have

$$
f(x)=\sum_{k=0}^{\infty}\frac{\Gamma(q+1+k/\alpha)}{\Gamma(1+k/\alpha)}G(k+\alpha q)(x-x_0)^{k/\alpha}.
$$

From the definition of transform in Eq. (2.2), the following expression is obtained:

$$
F(k)=\frac{\Gamma(q+1+k/\alpha)}{\Gamma(1+k/\alpha)}G(k+\alpha q). \qquad \square
$$

Theorem 6. *For the production of fractional derivatives in the most general form* $f(x)=\frac{\mathrm{d}^{q_1}}{\mathrm{d}x^{q_1}}[g_1(x)]\frac{\mathrm{d}^{q_2}}{\mathrm{d}x^{q_2}}[g_2(x)]\cdots\frac{\mathrm{d}^{q_{n-1}}}{\mathrm{d}x^{q_{n-1}}}[g_{n-1}(x)]\frac{\mathrm{d}^{q_n}}{\mathrm{d}x^{q_n}}[g_n(x)]$, *then*

$$
\begin{aligned}
F(k) &= \sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}\frac{\Gamma(q_1+1+k_1/\alpha)}{\Gamma(1+k_1/\alpha)}\frac{\Gamma[q_2+1+(k_2-k_1)/\alpha]}{\Gamma[1+(k_2-k_1)/\alpha]}\\
&\quad\cdots\frac{\Gamma[q_{n-1}+1+(k_{n-1}-k_{n-2})/\alpha]}{\Gamma[1+(k_{n-1}-k_{n-2})/\alpha]}\frac{\Gamma[q_n+1+(k-k_{n-1})/\alpha]}{\Gamma[1+(k-k_{n-1})/\alpha]}G_1(k_1+\alpha q_1)\\
&\quad\times G_2(k_2-k_1+\alpha q_2)\cdots G_{n-1}(k_{n-1}-k_{n-2}+\alpha q_{n-1})\\
&\quad\times G_n(k-k_{n-1}+\alpha q_n)\ \textit{where}\ \alpha q_i\in Z^{+}\ \textit{for}\ i=1,2,3,\ldots,n.
\end{aligned}
$$

Proof. Let the differential transform of $\frac{\mathrm{d}^{q_i}}{\mathrm{d}x^{q_i}}[g_i(x)]$ be $C_i(k)$ at $x=x_0$ for $i=1,2,3,\ldots,n$. Then by using Theorem 3, we have the fractional differential transform of $f(x)$ as

$$
F(k)=\sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}C_1(k_1)C_2(k_2-k_1)\ldots C_{n-1}(k_{n-1}-k_{n-2})C_n(k-k_{n-1}),
$$

and using Theorem 5 one can deduce that

$$
\begin{aligned}
C_1(k_1) &= \frac{\Gamma(q_1+1+k_1/\alpha)}{\Gamma(1+k_1/\alpha)}G_1(k_1+\alpha q_1),\\
C_2(k_2-k_1) &= \frac{\Gamma[q_2+1+(k_2-k_1)/\alpha]}{\Gamma[1+(k_2-k_1)/\alpha)}G_2(k_2-k_1+\alpha q_2),\ldots,\\
C_{n-1}(k_{n-1}-k_{n-2}) &= \frac{\Gamma[q_{n-1}+1+(k_{n-1}-k_{n-2})/\alpha]}{\Gamma[1+(k_{n-1}-k_{n-2})/\alpha)}G_{n-1}(k_{n-1}-k_{n-2}+\alpha q_{n-1}),\\
C_n(k-k_{n-1}) &= \frac{\Gamma[q_n+1+(k-k_{n-1})/\alpha]}{\Gamma[1+(k-k_{n-1})/\alpha)}G_n(k-k_{n-1}+\alpha q_n).
\end{aligned}
$$

By utilizing these values we have

$$\begin{aligned}F(k) = {} & \sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}\frac{\Gamma(q_1+1+k_1/\alpha)}{\Gamma(1+k_1/\alpha)}\frac{\Gamma[q_2+1+(k_2-k_1)/\alpha]}{\Gamma[1+(k_2-k_1)/\alpha]}\\ & \cdots\frac{\Gamma[q_{n-1}+1+(k_{n-1}-k_{n-2})/\alpha]}{\Gamma[1+(k_{n-1}-k_{n-2})/\alpha]}\frac{\Gamma[q_n+1+(k-k_{n-1})/\alpha]}{\Gamma[1+(k-k_{n-1})/\alpha]}\\ & \times G_1(k_1+\alpha q_1)G_2(k_2-k_1+\alpha q_2)\cdots G_{n-1}(k_{n-1}-k_{n-2}+\alpha q_{n-1})\\ & \times G_n(k-k_{n-1}+\alpha q_n),\end{aligned}$$

where $\alpha q_i \in Z^+$ for $i = 1, 2, 3, \ldots, n$. $\square$

3. Numerical examples

To demonstrate the effectiveness of the proposed algorithm, three special cases of the system of fractional differential equations (1.1) will be studied. All the results are calculated by using the symbolic calculus software Mathematica.

Example 1. Let us consider the following system of two linear fractional differential equations

$$\begin{aligned}&D_*^{\beta}x(t) = x(t) + y(t),\\ &D_*^{\gamma}y(t) = -x(t) + y(t),\end{aligned}\tag{3.1}$$

subject to the initial conditions

$$x(0) = 0, \quad y(0) = 1.\tag{3.2}$$

System (3.1) is transformed by using Theorems 1 and 5 as follows:

$$\begin{aligned}X(k+\beta\alpha_1) &= \frac{\Gamma(1+k/\alpha_1)}{\Gamma(\beta+1+k/\alpha_1)}[X(k)+Y(k)],\\ Y(k+\gamma\alpha_2) &= \frac{\Gamma(1+k/\alpha_2)}{\Gamma(\gamma+1+k/\alpha_2)}[-X(k)+Y(k)],\end{aligned}\tag{3.3}$$

where α_1 and α_2 are the unknown values of the fractions of β and γ, respectively.

Initial conditions in Eq. (3.2) are transformed by using Eq. (2.5) as follows:

$$\begin{aligned}&X(k) = 0 \quad \text{for } k = 0, 1, \ldots, \beta\alpha_1 - 1,\\ &Y(k) = 0 \quad \text{for } k = 1, \ldots, \gamma\alpha_2 - 1,\\ &Y(0) = 1.\end{aligned}\tag{3.4}$$

Using Eqs. (3.3) and (3.4), $X(k)$ and $Y(k)$ are obtained up to $k = 10$ for the values of $\beta = 1$ and $\gamma = 1$, and then using Eq. (2.2), $x(t)$ and $y(t)$ are obtained as follows:

$$\begin{aligned}x(t) &= t + t^2 + \frac{t^3}{3} - \frac{t^5}{30} - \frac{t^6}{90} - \frac{t^7}{630} + \frac{t^9}{22680} + \frac{t^{10}}{113400} + \cdots,\\ y(t) &= 1 + t - \frac{t^3}{3} - \frac{t^4}{6} - \frac{t^5}{30} + \frac{t^7}{630} + \frac{t^8}{2520} + \frac{t^9}{22680} + \cdots.\end{aligned}$$

Fig. 1 shows the approximate solutions for system (3.1) obtained for the values of $\beta = \gamma = 1$. This is the only case for which we know the exact solution ($x(t) = e^t \sin t$, $y(t) = e^t \cos t$) and our approximate solutions using the method are in good agreement with the exact solution. Fig. 2 shows the approximate solutions for system (3.1) obtained for

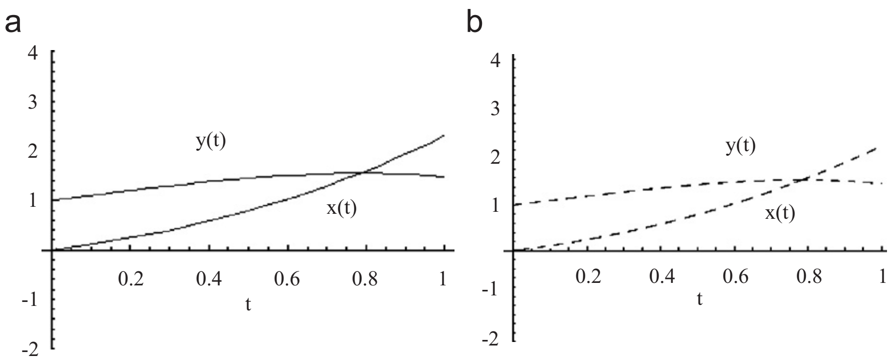

Fig. 1. Plots of system (3.1) when $\beta = 1$ and $\gamma = 1$: (a) the present method; (b) ADM [14].

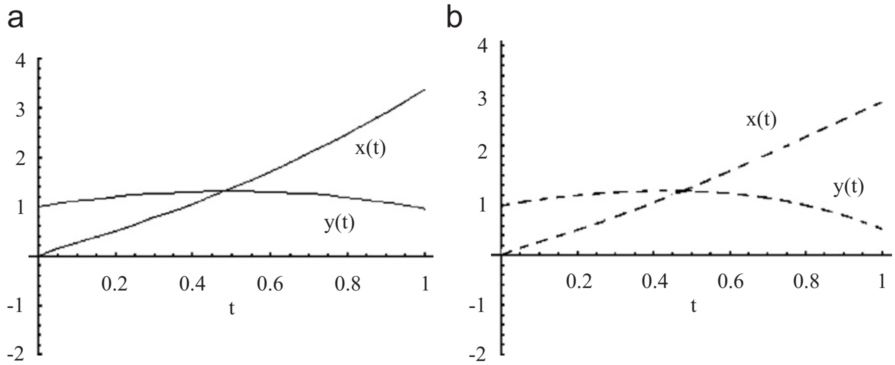

Fig. 2. Plots of system (3.1) when $\beta = 0.7$ and $\gamma = 0.9$: (a) the present method; (b) ADM [14].

the values of $\beta = 0.7$ and $\gamma = 0.9$. It is to be noted that the following twenty five terms were used in evaluating the approximate solutions for Fig. 2:

$$x(t) = \frac{8t^{5/2}}{15\Gamma(\frac{1}{2})} + \frac{t^{7/10}}{\Gamma(\frac{17}{10})} + \frac{t^{7/5}}{\Gamma(\frac{12}{5})} + \frac{t^{8/5}}{\Gamma(\frac{13}{5})} + \frac{t^{21/10}}{\Gamma(\frac{31}{10})} + \cdots$$

and

$$y(t) = 1 - \frac{16t^{5/2}}{15\Gamma(\frac{1}{2})} + \frac{t^{9/10}}{\Gamma(\frac{19}{10})} - \frac{t^{8/5}}{\Gamma(\frac{13}{5})} + \frac{t^{9/5}}{\Gamma(\frac{14}{5})} - \frac{t^{23/10}}{\Gamma(\frac{33}{10})} + \cdots.$$

It is evident that the efficiency of this approach can be dramatically enhanced by computing further terms of $x(t)$, $y(t)$ when the differential transform method is used. The results in Figs. 1 and 2 are in full agreement with the results obtained in [14] using the Adomian decomposition method.

Example 2. Following [4], we consider the following system of two nonlinear fractional differential equations:

$$\begin{aligned} D_*^{1.3} y_1 &= y_1 + y_2^2, \\ D_*^{2.4} y_2 &= y_1 + 5y_2 \end{aligned} \tag{3.5}$$

with the initial conditions

$$y_1(0) = 0, \quad y_1'(0) = 1, \quad y_2(0) = 0, \quad y_2'(0) = 1, \quad y_2''(0) = 1. \tag{3.6}$$

By using Theorems 1, 2 and 5, system (3.5) transforms to

$$Y_1(k+13) = \frac{\Gamma(1+k/10)}{\Gamma(1.3+1+k/10)}\left[Y_1(k) + \sum_{k_1=0}^{k} Y_2(k_1)Y_2(k-k_1)\right],$$

$$Y_2(k+24) = \frac{\Gamma(1+k/10)}{\Gamma(2.4+1+k/10)}[Y_1(k) + 5Y_2(k)]. \tag{3.7}$$

Using Eq. (2.5) the initial conditions in Eq. (3.6) are transformed to

$$Y_1(k) = 0 \quad \text{for } k = 0, 1, \ldots, 9, 11, 12,$$

$$Y_1(10) = 1,$$

$$Y_2(k) = 0 \quad \text{for } k = 0, \ldots, 9, 11, \ldots, 19, 21, 22, 23,$$

$$Y_2(10) = 1 \quad Y_2(20) = \tfrac{1}{2}. \tag{3.8}$$

Using Eqs. (3.7) and (3.8), $Y_1(k)$ and $Y_2(k)$ are obtained up to $k = 50$ and then using Eq. (2.2), the following series solutions are obtained for $y_1(t)$ and $y_2(t)$, respectively:

$$y_1(t) = t + \frac{t^{23/10}}{\Gamma(\frac{33}{10})} + \frac{2t^{33/10}}{\Gamma(\frac{43}{10})} + \frac{t^{18/5}}{\Gamma(\frac{23}{5})} + \frac{6t^{43/10}}{\Gamma(\frac{53}{10})} + \frac{2t^{23/5}}{\Gamma(\frac{28}{5})} + \frac{t^{49/10}}{\Gamma(\frac{59}{10})} + \cdots, \tag{3.9}$$

$$y_2(t) = t + \frac{t^2}{2} + \frac{6t^{17/15}}{\Gamma(\frac{22}{5})} + \frac{5t^{22/5}}{\Gamma(\frac{27}{5})} + \frac{t^{47/10}}{\Gamma(\frac{57}{10})} + \cdots. \tag{3.10}$$

In Fig. 3, we draw the curves $y_1(t)$ and $y_2(t)$. The graphical results are in very good agreement with the results in [4].

Example 3. Lastly we consider the following system of three nonlinear fractional differential equations:

$$D_*^{\alpha} x = 2y^2, \quad 0 < \alpha \leqslant 1,$$

$$D_*^{\beta} y = tx, \quad 0 < \beta \leqslant 1,$$

$$D_*^{\gamma} z = yz, \quad 0 < \gamma \leqslant 1, \tag{3.11}$$

subject to the initial conditions

$$x(0) = 0, \quad y(0) = 1, \quad z(0) = 1, \tag{3.12}$$

which is studied in [7].

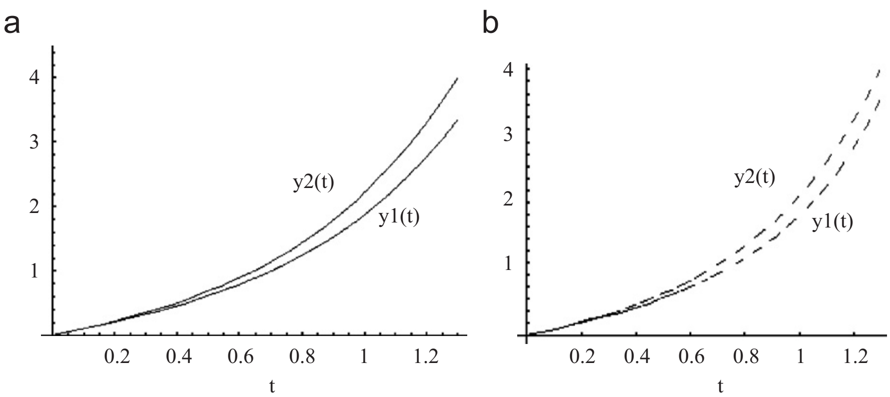

Fig. 3. Plots of system (3.5): (a) the present method; (b) the method [4].

Using Theorems 2 and 5, system of equations (3.11) can be transformed as follows:

$$X(k+\alpha\alpha_1)=\frac{\Gamma(1+k/\alpha_1)}{\Gamma(\alpha+1+k/\alpha_1)}\left[2\sum_{l=0}^{k}Y(l)Y(k-l)\right],$$

$$Y(k+\beta\alpha_2)=\frac{\Gamma(1+k/\alpha_2)}{\Gamma(\beta+1+k/\alpha_2)}\left[\sum_{l=0}^{k}\delta(l-\alpha_2)X(k-l)\right],$$

$$Z(k+\gamma\alpha_3)=\frac{\Gamma(1+k/\alpha_3)}{\Gamma(\gamma+1+k/\alpha_3)}\left[\sum_{l=0}^{k}Y(l)Z(k-l)\right], \tag{3.13}$$

where α_1, α_2 and α_3 are the unknown values of the fractions of α, β and γ, respectively.

The initial conditions in Eq. (3.12) can be transformed by using Eq. (2.5) as follows:

$$X(k)=0 \quad \text{for } k=0,1,\ldots,\alpha.\alpha_1-1,$$

$$Y(k)=0 \quad \text{for } k=1,\ldots,\beta\alpha_2-1,$$

$$Z(k)=0 \quad \text{for } k=1,\ldots,\gamma\alpha_3-1,$$

$$Y(0)=1, \quad Z(0)=1. \tag{3.14}$$

Using Eqs. (3.13) and (3.14), $X(k)$ for $k=\alpha.\alpha_1, \alpha.\alpha_1+1,\ldots,n$, $Y(k)$ for $k=\beta.\alpha_2, \beta.\alpha_2+1,\ldots,n$ and $Z(k)$ for $k=\gamma.\alpha_3, \gamma.\alpha_3+1,\ldots,n$ are calculated and using the inverse transformation rule in Eq. (2.2), $x(t)$, $y(t)$ and $z(t)$ are calculated for different values of α, β and γ. Using Eqs. (3.13) and (3.14), $x(t)$, $y(t)$ and $z(t)$ are obtained up to $k=10$ and then using the inverse transformation rule in Eq. (2.2) the following series solutions are obtained for the values of $\alpha=\beta=\gamma=1$:

$$x(t)=2t+\frac{2t^4}{3}+\frac{4t^7}{21}+\frac{4t^{10}}{105}+\cdots,$$

$$y(t)=1+\frac{2t^3}{3}+\frac{t^6}{9}+\frac{4t^9}{189}+\cdots,$$

$$z(t)=1+t+\frac{t^2}{2}+\frac{t^3}{6}+\frac{5t^4}{24}+\frac{7t^5}{40}+\frac{61t^6}{720}+\frac{221t^7}{5040}+\frac{1481t^8}{40320}+\frac{1685t^9}{75576}+\frac{43321t^{10}}{3628800}+\cdots.$$

Fig. 4 shows the approximate solutions for system (3.11) obtained for the values of $\alpha=\beta=\gamma=1$.

By taking $\alpha=0.5$, $\beta=0.4$ and $\gamma=0.3$, the solutions $x(t)$, $y(t)$ and $z(t)$ are evaluated up to $k=70$. The first few terms of the series solutions are given by

$$x(t)=\frac{4\sqrt{t}}{\sqrt{\pi}}+\frac{12t^{12/5}}{\Gamma(\frac{17}{15})}+\cdots,$$

$$y(t)=1+\frac{3t^{19/10}}{\Gamma(\frac{29}{10})}+\cdots,$$

$$z(t)=1+\frac{4t^{3/2}}{3\sqrt{\pi}}+\frac{t^{3/10}}{\Gamma(\frac{13}{10})}+\frac{t^{3/5}}{\Gamma(\frac{8}{5})}+\frac{t^{9/10}}{\Gamma(\frac{19}{10})}+\frac{t^{6/5}}{\Gamma(\frac{11}{5})}+\frac{t^{9/5}}{\Gamma(\frac{14}{5})}+\frac{t^{21/10}}{\Gamma(\frac{31}{10})}+\frac{3t^{11/5}}{\Gamma(\frac{16}{5})}+\cdots.$$

Fig. 5 shows the approximate solutions for system (3.11) obtained for the values of $\alpha=0.5$, $\beta=0.4$ and $\gamma=0.3$.

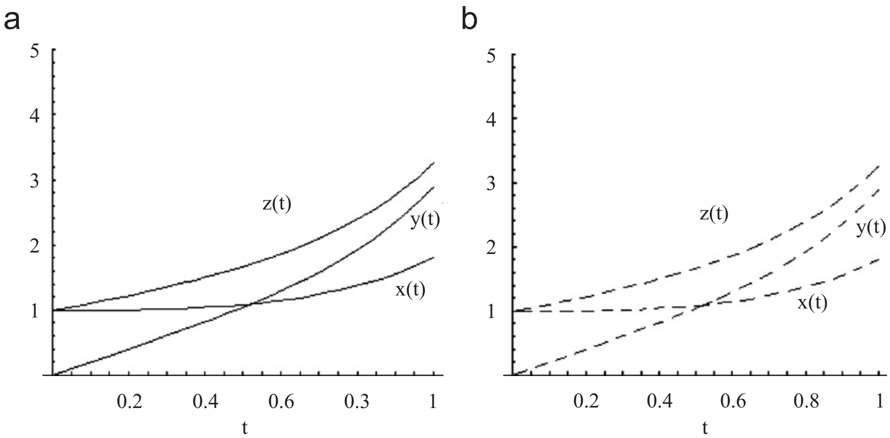

Fig. 4. Plots of system (3.11) when $\alpha = 1$, $\beta = 1$ and $\gamma = 1$ (a) the present method; (b) ADM [7].

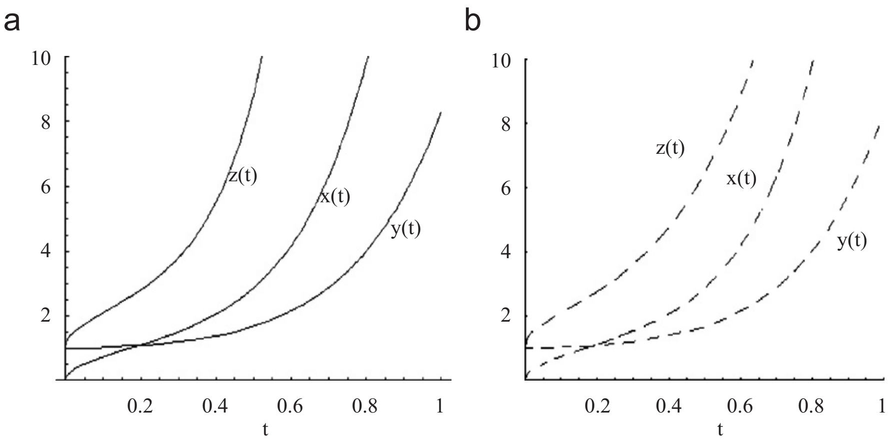

Fig. 5. Plots of system (3.11) when $\alpha = 0.5$, $\beta = 0.4$ and $\gamma = 0.3$: (a) the present method; (b) ADM [7].

Graphical results in Figs. 4 and 5 are in very good agreement with the results obtained in [7] using the Adomian decomposition method.

4. Conclusions

This present analysis exhibits the applicability of the differential transform method to solve systems of differential equations of fractional order. The work emphasized our belief that the method is a reliable technique to handle linear and nonlinear fractional differential equations. It provides the solutions in terms of convergent series with easily computable components in a direct way without using linearization, perturbation or restrictive assumptions. The results of this method are in good agreement with those obtained by using the variational iteration method and the Adomian decomposition method. As an advantage of this method over the Adomian decomposition method, in this method we do not need to do the difficult computation for finding the Adomian polynomials. Generally speaking, the proposed method is promising and applicable to a broad class of linear and nonlinear problems in the theory of fractional calculus.

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