Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Linear Algebra and its Applications 418 (2006) 394–415
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Minimum rank and maximum eigenvalue multiplicity of symmetric tree sign patterns

Luz Maria DeAlba [a,1], Timothy L. Hardy [b], Irvin Roy Hentzel [c], Leslie Hogben [c,*], Amy Wangsness [c]

[a] *Department of Mathematics and Computer Science, Drake University, Des Moines, IA 50311, USA*
[b] *Department of Mathematics, Chadron State College, Chadron, NE 69337, USA*
[c] *Department of Mathematics, Iowa State University, Ames, IA 50011, USA*

Received 24 May 2005; accepted 19 February 2006
Available online 18 April 2006
Submitted by S. Fallat

Abstract

The set of real matrices described by a sign pattern (a matrix whose entries are elements of $\{+, -, 0\}$) has been studied extensively but only loose bounds were available for the minimum rank of a tree sign pattern. A simple graph has been associated with the set of symmetric matrices having a zero–nonzero pattern of off-diagonal entries described by the graph, and the minimum rank/maximum eigenvalue multiplicity among matrices in this set is readily computable for a tree. In this paper, we extend techniques for trees to tree sign patterns and trees allowing loops (with the presence or absence of loops describing the zero–nonzero pattern of the diagonal), allowing precise computation of the minimum rank of a tree sign pattern and a tree allowing loops. For a symmetric tree sign pattern or a tree that allows loops, we provide an algorithm that allows exact computation of maximum multiplicity and minimum rank, and can be used to obtain a symmetric integer matrix realizing minimum rank.

AMS classification: 05C50; 15A18; 05C05; 05C85

Keywords: Sign pattern matrix; Symmetric tree sign pattern; Minimum rank; Maximum multiplicity; Tree; Graph

* Corresponding author. Tel.: +1 515 294 8168; fax: +1 515 294 5454.
E-mail addresses: luz.dealba@drake.edu (L.M. DeAlba), thardy@csc.edu (T.L. Hardy), hentzel@iastate.edu (I.R. Hentzel), lhogben@iastate.edu (L. Hogben), awangsne@fsc.edu (A. Wangsness).
[1] Partially supported by a Drake University Research Grant.

0024-3795/$ - see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2006.02.018

1. Introduction

Much of the material we present is inspired by previous work in two somewhat different but related areas: sign patterns of matrices, and graphs of matrices.

Sign pattern matrices have many important applications; in fact, the study of sign patterns arose more than fifty years ago in economics. Brualdi and Shader [4] provide a thorough mathematical treatment of sign patterns through 1995. For a current survey with extensive bibliography, see [7].

Recently there has been substantial interest in minimum rank and the related question of the maximal multiplicity of an eigenvalue for sign patterns, e.g., [5,6]. In addition, many other papers concerning related parameters of sign patterns, such as inertia [8], rank [9], diagonalizability [14], etc. have appeared. In the last ten years there have been numerous papers on minimum rank and multiplicities of eigenvalues for symmetric matrices associated with a graph, e.g., [11,3,1,2]. There are similarities in techniques and results in the study of sign patterns and matrices of graphs, but also important differences, caused by the issue of what set of matrices is associated with a graph or a sign pattern. For sign patterns, the matrices have traditionally not been required to be symmetric and the diagonal has been constrained by the pattern; for graphs, traditionally symmetric matrices have been required and the diagonal has been unconstrained. Algorithms are known [13] for the exact computation of the minimum rank among the family of symmetric matrices associated with a tree (with no restriction on diagonal entries of the matrices), whereas only loose bounds have been given for the minimum rank of tree sign patterns [5]. We generalize the techniques used for trees (with unconstrained diagonal) to obtain an algorithm for an exact computation of minimum rank of a tree sign pattern (Section 2), and also present an algorithm to realize minimum rank with a symmetric integer matrix in Section 3.

1.1. Matrices and sign patterns

We begin by introducing some terminology. Let $\mathcal{N} = \{1, \dots, n\}$. An $n \times n$ matrix $B = [b_{ij}]$, $i, j \in \mathcal{N}$ can be described in a natural way as being indexed by $\mathcal{N}$. Every matrix discussed in this paper is real and square. Because we will be extracting submatrices of submatrices, and because we will be associating principal submatrices with induced subgraphs, we will need to retain information about the original row and column indices. Thus, we explicitly attach the index set to the matrix.

An *index set* is a finite set of positive integers. We require every matrix B to have an index set, denoted $\iota(B)$; the *order*, denoted $o(B)$, is the cardinality $|\iota(B)|$ of its index set. Thus B is an $o(B) \times o(B)$ matrix with entries b_{ij}, $i, j \in \iota(B)$; B is written as a square array using the natural order of the indices. The standard index set for an $n \times n$ matrix is $\mathcal{N}$, and this is used for an ordinary matrix (that does not arise from a graph or as a principal submatrix).

Matrix functions, such as the rank and the spectrum of B, are computed ignoring the index set (here the spectrum $\sigma(B)$ is the multiset of roots of the characteristic polynomial). We will use the definition of the determinant in terms of permutations, with the permutations acting on the index set; this results in the same value of the determinant as obtained by ignoring the index set and evaluating as usual.

If B is a matrix and $R \subseteq \iota(B)$, define the *principal submatrix* $B[R]$ to be the submatrix of B lying in rows and columns that have indices in R, together with the index set R. This definition has the desirable feature that if $R \subseteq Q \subseteq \iota(B)$, $B[Q][R] = B[R]$. We also define $B(R)$ to be the principal submatrix obtained from B by deleting from B all rows and columns with indices in R, with $\iota(B(R)) = \overline{R}$, where $\overline{R} = \iota(B) - R$. Equivalently, $B(R) = B[\overline{R}]$. If R and Q are disjoint subsets of $\iota(B)$, then $B(R)(Q) = B(R \cup Q)$. When $\{k\}$ is a singleton set, we use $B(k)$ to denote $B(\{k\})$.

A *sign pattern matrix* (sign pattern for short) is a square matrix $Z = [z_{ij}]$ whose entries z_{ij} are elements of $\{+, -, 0\}$, with index set $\iota(Z)$ and order $o(Z) = |\iota(Z)|$. For Z a sign pattern and $R \subseteq \iota(Z)$, define the *principal subpattern* $Z[R]$ to be the subpattern of Z lying in rows and columns that have indices in R, together with the index set R. Define $Z(R) = Z[\overline{R}]$; when $\{k\}$ is a singleton set, $Z(\{k\})$ is denoted $Z(k)$. The determinant of an order n sign pattern Z is evaluated as a formal sum of $n!$ terms that are products of $+, -, 0$, where each product is evaluated in the obvious way to be one of $+, -, 0$.

For a real number b, the *sign* of b, denoted $\text{sgn}(b)$, is $+, -, 0$ according as $b > 0$, $b < 0$, $b = 0$. For B a matrix, define $\mathscr{Z}(B)$ to be the sign pattern matrix with $(\mathscr{Z}(B))_{ij} = \text{sgn}(b_{ij})$ and $\iota(\mathscr{Z}(B)) = \iota(B)$. The *qualitative class* of sign pattern Z is

$$\mathscr{Q}(Z) = \{B : \mathscr{Z}(B) = Z\}.$$

The qualitative class of a sign pattern has been studied extensively, cf. [4]. For the sign patterns of primary interest to us (tree sign patterns defined in the next subsection) we will be able to reduce the class of matrices studied to symmetric matrices, and thus for a symmetric sign pattern Z we define

$$\mathscr{S}(Z) = \{A : A \text{ is a symmetric matrix and } \mathscr{Z}(A) = Z\}.$$

1.2. Graphs

For our purposes, a *graph* G allows loops but does not allow multiple edges. A *simple graph* is a graph that does not have loops. The set of vertices $V(G)$ of G is a finite set of positive integers. An *edge* of G is an unordered multiset of two vertices of G, denoted vw or vv, and the set of edges of G is denoted $E(G)$. The *simple graph associated with* G, $\widehat{G}$, is obtained from G by suppressing all loops. We will also use $\widehat{G}$ to denote an arbitrary simple graph. If $Q \subseteq V(G)$, $G - Q$ is the graph obtained from G by deleting all vertices in Q and all edges incident with a vertex in Q. An *induced subgraph* of G is a graph of the form $G - Q$, also denoted $\langle R \rangle$ where $R = V(G) - Q$.

A matrix or sign pattern Z is *symmetric* if for all $i, j \in \iota(Z)$, $z_{ij} = z_{ji}$, and Z is *combinatorially symmetric* if for all $i, j \in \iota(Z)$, either z_{ij} and z_{ji} are both nonzero, or they are both 0. Let Z be a combinatorially symmetric sign pattern or matrix. Then we define

- $\mathscr{G}(Z)$ to be the graph with vertices $\iota(Z)$ such that ij is an edge of $\mathscr{G}(Z)$ if and only if $z_{ij} \neq 0$.
- $\widehat{\mathscr{G}}(Z)$ to be the simple graph with vertices $\iota(Z)$ such that ij is an edge of $\widehat{\mathscr{G}}(Z)$ if and only if $i \neq j$ and $z_{ij} \neq 0$. Note the diagonal is ignored.

Let G be a graph, and $\widehat{G}$ a simple graph. Then we define

- $\mathscr{S}(G) = \{A : A \text{ is a symmetric matrix and } \mathscr{G}(A) = G\}$.
- $\widehat{\mathscr{S}}(\widehat{G}) = \{A : A \text{ is a symmetric matrix and } \widehat{\mathscr{G}}(A) = \widehat{G}\}$.

$\widehat{\mathscr{S}}(\widehat{G})$ is the traditional class of symmetric matrices associated with a simple graph, e.g., [1,3,11].

Let Z be a symmetric sign pattern and let G be a graph. For $S \in \{\mathscr{S}(Z), \mathscr{S}(G)\}$, every matrix A in S has the same index set and graph, so we can extend various definitions to S, i.e., for $S \in \{\mathscr{S}(Z), \mathscr{S}(G)\}$,

$$\mathscr{G}(S) = \mathscr{G}(A), \quad \widehat{\mathscr{G}}(S) = \widehat{\mathscr{G}}(A), \quad \iota(S) = \iota(A), \quad o(S) = o(A) \quad \text{for } A \in S.$$

A *component* of a graph G is a maximal connected induced subgraph of G. If $S \in \{\mathscr{S}(Z), \mathscr{S}(G)\}$ and $\langle R \rangle$ is a component of $\mathscr{G}(S)$, then we call the family of principal submatrices $S[R]$ a *component* of S.

1.3. Trees and tree sign patterns

The standard terms tree and forest are customarily defined for simple graphs. To distinguish between graphs and simple graphs, we will preface these terms with the word "simple" when referring to a simple graph. We extend these terms to graphs by ignoring loops. Thus, a graph T is a *tree* if its associated simple graph $\widehat{T}$ is a simple tree and is a *forest* if $\widehat{T}$ is a simple forest (where a simple forest is a disjoint union of one or more simple trees).

A combinatorially symmetric sign pattern Z is a *tree sign pattern* (*forest sign pattern*) if $\mathscr{G}(Z)$ is a tree (forest); equivalently, Z is a tree sign pattern (forest sign pattern) if $\widehat{\mathscr{G}}(Z)$ is a simple tree (simple forest).

The results in Lemmas 1.1–1.4 are generally known; as indicated, several were stated in [5].

Lemma 1.1 [5]. *Let Z be a symmetric tree sign pattern and $B \in \mathscr{Q}(Z)$. Then there exists a positive real diagonal matrix D such that $A = DBD^{-1}$ is symmetric and has the same sign pattern as B, i.e., $A \in \mathscr{S}(Z)$.*

Lemma 1.2. *Let Z be a symmetric tree sign pattern. There exists a nonsingular diagonal sign pattern D such that all nonzero off-diagonal entries of DZD are $+$.*

Proof. There exists exactly one path between any two vertices of $\mathscr{G}(Z)$. Let D be the diagonal matrix with index set $\iota(Z)$ defined by $D = \operatorname{diag}\big(d_{\iota(1)}, d_{\iota(2)}, \ldots, d_{\iota(n)}\big)$. Set $d_{\iota(1)} = +$. Let $P(v_0 = \iota(1), v_1, \ldots, v_{k-1}, v_k = v)$ be the path from $\iota(1)$ to v, and set $d_v = \prod_{i=1}^{k} z_{v_{i-1}v_i}$. Then DZD has all off-diagonal entries equal to $+$. □

Lemma 1.3 [5]. *If Z is a symmetric forest sign pattern such that all nonzero off-diagonal entries of Z are + and $B \in \mathscr{Q}(Z)$, then there exist positive diagonal matrices D_1, D_2 such that all the nonzero off-diagonal entries of $A = D_1 B D_2$ are one, and $A \in \mathscr{S}(Z)$.*

Lemma 1.4 [5]. *Let Z be a forest sign pattern. There exists a nonsingular diagonal sign pattern D and symmetric forest sign pattern Z_1 such that $Z = Z_1 D$.*

1.4. Minimum rank and maximum eigenvalue multiplicity

The multiplicity of eigenvalue λ for the symmetric matrix A will be denoted by $m_A(\lambda)$.

For a real number λ and a symmetric sign pattern Z, the *maximum multiplicity of λ for Z* is

$$M_\lambda(Z) = \max\{m_A(\lambda) : A \in \mathscr{S}(Z)\},$$

and for a graph G, the *maximum multiplicity of* λ *for* G is

$$M_\lambda(G) = \max\{m_A(\lambda) : A \in \mathscr{S}(G)\}.$$

For convenience, we extend this notation to sets of matrices: for $S \in \{\mathscr{S}(Z), \mathscr{S}(G)\}$, where G is a graph and Z is a symmetric sign pattern,

$$M_\lambda(S) = \max\{m_A(\lambda) : A \in S\}.$$

Note that if G is a graph, $M_\lambda(G) = M_\lambda(\mathscr{S}(G))$, and if Z is a symmetric sign pattern, $M_\lambda(Z) = M_\lambda(\mathscr{S}(Z))$.

For a simple graph $\widehat{G}$, where no restriction is placed on the diagonal of associated matrices, the *maximum multiplicity for* $\widehat{G}$ is

$$\widehat{M}(\widehat{G}) = \max\{m_A(\lambda) : A \in \widehat{\mathscr{S}}(\widehat{G})\}.$$

The subscript λ is omitted since translation by a scalar multiple of the identity matrix makes λ irrelevant. What we are denoting by $\widehat{M}(\widehat{G})$ is often denoted by $M(\widehat{G})$ in the literature.

We now state a standard result, which applies to all real symmetric matrices, and a more general version that we will need.

Theorem 1.5 (Interlacing Theorem [10, p. 185]). *If the eigenvalues of a symmetric matrix* A *are* $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$, $k \in \iota(A)$, *and the eigenvalues of* $A(k)$ *are* $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_{n-1}$, *then* $\lambda_1 \leqslant \mu_1 \leqslant \lambda_2 \leqslant \mu_2 \leqslant \cdots \leqslant \lambda_{n-1} \leqslant \mu_{n-1} \leqslant \lambda_n$.

Corollary 1.6 (Interlacing Corollary). *Let* G *be a graph, let* Z *be a symmetric sign pattern, let* $S \in \{\mathscr{S}(Z), \mathscr{S}(G)\}$, *and let* $R \subseteq \iota(S)$. *Then*

$$M_\lambda(S) - |R| \leqslant M_\lambda(S(R)) \leqslant M_\lambda(S) + |R|.$$

Proof. We prove that for $k \in \iota(S)$, $M_\lambda(S) - 1 \leqslant M_\lambda(S(k)) \leqslant M_\lambda(S) + 1$, and the more general result follows by repeated application.

Choose $A \in S$ such that $m_A(\lambda) = M_\lambda(S)$. Then $M_\lambda(S(k)) \geqslant m_{A(k)}(\lambda) \geqslant m_A(\lambda) - 1 = M_\lambda(S) - 1$. Choose $A' \in S$ such that $m_{A'(k)}(\lambda) = M_\lambda(S(k))$. Then $M_\lambda(S(k)) = m_{A'(k)}(\lambda) \leqslant m_{A'}(\lambda) + 1 \leqslant M_\lambda(S) + 1$. □

One of the parameters of primary interest in this work is the minimum rank of the family of matrices associated with a tree or tree sign pattern. Although it is possible to give the following definitions more generally for sign patterns and graphs, we restrict our attention to trees and tree sign patterns to avoid having to distinguish minimum rank from symmetric minimum rank.

Let Z be a symmetric tree sign pattern. The minimum rank is

$$\operatorname{mr}(Z) = \min\{\operatorname{rank} A : A \in \mathscr{S}(Z)\}.$$

Note that by Lemma 1.1, $\operatorname{mr}(Z) = \min\{\operatorname{rank} B : B \in \mathscr{Q}(Z)\}$. For a tree T, the *minimum rank of* T is

$$\operatorname{mr}(T) = \min\{\operatorname{rank} A : A \in \mathscr{S}(T)\}.$$

For a simple tree $\widehat{T}$, the minimum rank is

$$\widehat{\operatorname{mr}}(\widehat{T}) = \min\{\operatorname{rank} A : A \in \widehat{\mathscr{S}}(\widehat{T})\}.$$

What we are denoting by $\widehat{\operatorname{mr}}(\widehat{T})$ is often denoted by $\operatorname{mr}(\widehat{T})$ in the literature.

Observation 1.7. *For a simple tree* $\widehat{T}$ *and for* $S \in \{\mathscr{S}(Z), \mathscr{S}(T)\}$ *where* T *is a tree and* Z *is a symmetric tree sign pattern*,

$$M_0(S) + \text{mr}(S) = o(S) \quad \text{and} \quad \widehat{M}(\widehat{T}) + \widehat{\text{mr}}(\widehat{T}) = o(S).$$

1.5. Allowing a given eigenvalue

Let Z be a symmetric pattern. If there exists a matrix $A \in \mathscr{S}(Z)$ such that $\lambda \in \sigma(A)$, then we say $\mathscr{S}(Z)$ (or Z) *allows eigenvalue* λ. Equivalently, $\mathscr{S}(Z)$ allows eigenvalue λ if $M_\lambda(\mathscr{S}(Z)) \geqslant 1$. Analogously, for a graph G, $\mathscr{S}(G)$ (or G) *allows eigenvalue* λ if there exists a matrix $A \in \mathscr{S}(G)$ such that $\lambda \in \sigma(A)$.

Lemma 1.8. *Let* G *be a graph and let* Z *be a symmetric sign pattern. Let* $S \in \{\mathscr{S}(G), \mathscr{S}(Z)\}$.

1. *If* $\widehat{\mathscr{G}}(S)$ *has an edge*, *then* S *allows any nonzero eigenvalue.*
2. *If* G *has a loop*, *then* $\mathscr{S}(G)$ *allows any nonzero eigenvalue.*
3. *If* Z *has a positive* (*negative*) *diagonal entry*, *then* $\mathscr{S}(Z)$ *allows any positive* (*negative*) *eigenvalue.*

Proof. Suppose $\widehat{\mathscr{G}}(S)$ has edge kj with $k \neq j$. Choose $A \in S$ with $a_{kj} = a_{jk} = 1$ (or $a_{kj} = a_{jk} = -1$) and $a_{kk}, a_{jj} \in \{0, 0.1, -0.1\}$, depending on whether the loop is present (or $z_{kk}, z_{jj} \in \{0, +, -\}$). Then $\det(A[\{k, j\}]) \leqslant -0.99$, so $A[\{k, j\}]$ must have both a positive and a negative eigenvalue. Then, by the Interlacing Theorem 1.5, A has both a positive and a negative eigenvalue. Now scale A.

For the second and third statements, apply the Interlacing Theorem 1.5 to the 1×1 matrix associated with the loop or the correctly signed diagonal entry and scale. □

It is traditional (cf. [4]) in the study of sign patterns to say that a sign pattern Z *requires* property P if every matrix in $\mathscr{Q}(Z)$ has property P and to say that Z *allows* property P if there exists a matrix in $\mathscr{Q}(Z)$ that has property P.

In our study of minimum rank, we are interested in sign patterns that allow singularity, or equivalently, that do not require nonsingularity. Let Z be a symmetric tree sign pattern and let T be a tree. If $S \in \{\mathscr{S}(Z), \mathscr{S}(T)\}$ allows eigenvalue zero, then we say $S \in \{\mathscr{S}(Z), \mathscr{S}(T)\}$ *allows singularity*. With this definition, $\mathscr{S}(Z)$ allows singularity if and only if Z allows singularity (with the standard definition of "allows" based on $\mathscr{Q}(Z)$) for a symmetric tree sign pattern Z, by Lemma 1.1. It is worth noting that the analogous result is not true for symmetric sign patterns that are not tree sign patterns, where allowing symmetric singularity must be distinguished from allowing singularity, and minimum rank must be distinguished from symmetric minimum rank, cf. [5]. However, our interest here is restricted to tree sign patterns.

Using the well-known result that a sign pattern Z of order n requires nonsingularity if and only if at least one of the $n!$ terms in the standard expansion of the determinant as a sum of products is nonzero and all nonzero terms have the same sign [4], we have the following criterion for $\mathscr{S}(Z)$ to allow singularity.

Observation 1.9. *Let* Z *be a symmetric tree sign pattern and let* X_Z *be the* $o(Z) \times o(Z)$ *matrix defined as follows*: *For* $i \leqslant j$, $i, j \in \iota(Z)$, *let* x_{ij} *be independent indeterminates and define* $(X_Z)_{ij} = z_{ij}x_{ij}$ *and* $(X_Z)_{ji} = z_{ij}x_{ij}$. *Then* $\mathscr{S}(Z)$ *requires singularity if and only if* $\det X_Z$ *is*

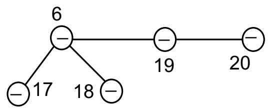

Fig. 1.1. The tree $\widehat{\mathscr{G}}(Z)$ of Z in Example 1.10 (showing signs of diagonal entries).

identically zero and $\mathscr{S}(Z)$ allows but does not require singularity if and only if $\det X_Z$ *is not identically zero and not all the nonzero terms have the same sign. Thus one can determine whether Z allows singularity by evaluating the determinant of X_Z.*

Example 1.10. Let Z be the symmetric tree sign pattern having $\iota(Z) = \{6, 17, 18, 19, 20\}$, every diagonal entry equal to $-$, every nonzero off-diagonal entry equal to $+$, and graph shown in Fig. 1.1 (the index set was chosen for use in Example 2.5).

Then

$$X_Z = \begin{bmatrix} -x_{6,6} & x_{6,17} & x_{6,18} & x_{6,19} & 0 \\ x_{6,17} & -x_{17,17} & 0 & 0 & 0 \\ x_{6,18} & 0 & -x_{18,18} & 0 & 0 \\ x_{6,19} & 0 & 0 & -x_{19,19} & x_{19,20} \\ 0 & 0 & 0 & x_{19,20} & -x_{20,20} \end{bmatrix}$$

and

$$\begin{aligned} \det X_Z = & -x_{18,18}x_{6,17}^2x_{19,20}^2 - x_{17,17}x_{6,18}^2x_{19,20}^2 + x_{17,17}x_{18,18}x_{66}x_{19,20}^2 \\ & + x_{18,18}x_{19,19}x_{20,20}x_{6,17}^2 + x_{17,17}x_{19,19}x_{20,20}x_{6,18}^2 \\ & + x_{17,17}x_{18,18}x_{20,20}x_{6,19}^2 - x_{17,17}x_{18,18}x_{19,19}x_{20,20}x_{6,6}. \end{aligned}$$

Since $\det X_Z$ has terms of both signs, Z allows singularity (a singular integer matrix having this sign pattern is actually constructed in Example 3.3).

For a tree T, we could determine whether $\mathscr{S}(T)$ allows singularity by evaluation of the determinants of all possible sign patterns having the graph T; however, this would be extraordinarily inefficient. Fortunately, it is unnecessary. For a tree T, define its *matrix of indeterminates* X_T as follows: For $i \leqslant j$, $i, j \in \iota(T)$, let x_{ij} be independent indeterminates and $(X_T)_{ij} = x_{ij}$ and $(X_T)_{ji} = x_{ij}$. We can use X_T to determine whether T allows singularity.

Lemma 1.11. *Let $\widehat{T}$ be a simple forest. If the order of $\widehat{T}$ is odd, then* $\det X_{\widehat{T}} = 0$. *If the order of $\widehat{T}$ is even, then* $\det X_{\widehat{T}}$ *has at most one nonzero term.*

Proof. Since $\widehat{T}$ has no loops, all the diagonal entries of $X_{\widehat{T}}$ are zero, so every term in $\det X_{\widehat{T}}$ must be a product of distinct squares x_{ij}^2 with $i \neq j$. Thus if $|\widehat{T}| = n$ is odd, then $\det X_{\widehat{T}} = 0$. Suppose the order $n = 2k$ of $\widehat{T}$ is even. We show by induction on k that there is at most one nonzero term of $\det X_{\widehat{T}}$. The result is clear for $k = 1$, i.e., $n = 2$. Assume true for k. Let the order of $\widehat{T}$ be $2(k+1) = 2k+2$. If $\widehat{T}$ has an isolated vertex, $\det X_{\widehat{T}} = 0$; otherwise, let v be a vertex of degree 1, and let u be the unique neighbor of v. In any nonzero term in $\det X_{\widehat{T}}$, x_{uv}^2 must appear since there is no other way to cover v. So delete u and v from T to obtain simple forest $T - \{u, v\}$ of order $2k$, which by the induction hypothesis has at most one nonzero term in its determinant. □

Lemma 1.12. *Let T be a forest that has at least two nonzero terms in its determinant. Then T has a loop ii such that there is a nonzero term of* $\det X_T$ *that includes x_{ii} and another nonzero term of* $\det X_T$ *that does not include x_{ii}.*

Proof. Let t_1 and t_2 be nonzero terms of $\det X_T$. If t_1 and t_2 do not have identical sets of diagonal elements x_{jj}, then one has a diagonal element x_{ii} that is not in the other. If they have identical sets of diagonal elements, then let L be the set of indices j such that the diagonal x_{jj} appears in t_1 and t_2. Dividing t_1 and t_2 by $\prod_{j\in L} x_{jj}$ gives two nonzero terms in the determinant of the simple graph $\widehat{T - L}$, contradicting Lemma 1.11. □

Theorem 1.13. *Let T be a forest. Then T requires singularity if and only if* $\det X_T$ *is identically zero and T allows but does not require singularity if and only if* $\det X_T$ *has at least two nonzero terms.*

Proof. If $\det X_T = 0$ then T requires singularity. If $\det X_T$ has one term then T requires nonsingularity.

Let T be a forest such that $\det X_T$ has at least two nonzero terms. We show there is a symmetric sign pattern Z with $\mathscr{G}(Z) = T$ such that Z allows singularity.

Choose any symmetric sign pattern Z such that $\mathscr{G}(Z) = T$. Compute $\det X_Z$, which has at least two nonzero terms. If there are terms of opposite sign, then Z allows singularity. Now suppose all nonzero terms in $\det X_Z$ have the same sign. By Lemma 1.12, there is a loop ii of T and two terms in $\det X_T$ such that one includes x_{ii} and another does not include x_{ii}. Reverse the sign of the ith diagonal element in Z to obtain a new sign pattern Z'. The determinant of $X_{Z'}$ is obtained from the determinant of X_Z by reversing the signs of exactly those terms containing x_{ii}. Thus, at least one term changes sign and at least one does not. Thus Z' allows singularity, and hence T allows singularity. □

1.6. Generalized Parter–Wiener Theorem

Let A be a symmetric matrix. Index $k \in \iota(A)$ is a *Parter–Wiener vertex of A for eigenvalue* λ if $m_{A(k)}(\lambda) = m_A(\lambda) + 1$. Furthermore, k is a *strong Parter–Wiener vertex of A for* λ if λ is an eigenvalue of at least three of the principal submatrices of A corresponding to components of $\widehat{\mathscr{G}}(A) - k$ and k is a Parter–Wiener vertex of A for λ.

Theorem 1.14 (Parter–Wiener Theorem [15,16,12]). *If A is a symmetric matrix, $\widehat{\mathscr{G}}(A)$ is a simple tree, and $m_A(\lambda) \geqslant 2$, then there is a strong Parter–Wiener vertex of A for λ.*

Theorem 1.15 (Generalized Parter–Wiener Theorem). *Let $S \in \{\mathscr{S}(Z), \mathscr{S}(T)\}$, where T is a tree and Z is a symmetric tree sign pattern. If $M_\lambda(S) \geqslant 2$, then there exists $k \in \iota(S)$ such that $M_\lambda(S(k)) = M_\lambda(S) + 1$ and $S(k)$ has at least three components that allow eigenvalue λ.*

Proof. If $M_\lambda(S) \geqslant 2$, then there exists $A \in S$ such that $m_A(\lambda) = M_\lambda(S) \geqslant 2$. So by the Parter–Wiener Theorem, there exists $k \in \iota(A) = \iota(S)$ such that k is a strong Parter–Wiener vertex of A for

λ. That is, λ is an eigenvalue of the principal submatrices $A[R_i]$ of A corresponding to at least three of the components $\langle R_i\rangle$ of $\widehat{\mathscr{G}}(A(k)) = \widehat{\mathscr{G}}(S(k))$ and $m_{A(k)}(\lambda) = m_A(\lambda) + 1 = M_\lambda(S) + 1$. Thus, $S(k)$ must have at least three components that allow eigenvalue λ and $M_\lambda(S(k)) \geqslant M_\lambda(S) + 1$. But $M_\lambda(S(k)) \leqslant M_\lambda(S) + 1$ by the Interlacing Corollary 1.6. $\square$

An index k with the properties in Theorem 1.15 is called a *strong Parter–Wiener vertex* for S.

In [11], the parameter Δ is defined for simple trees as $\Delta(\widehat{T}) = \max\{p_Q - |Q| : Q \subseteq V(\widehat{T})$ and $\widehat{T} - Q$ consists of p_Q disjoint paths$\}$.

One of the main results of [11] is that for $\widehat{T}$ a simple tree, $\Delta(\widehat{T}) = \widehat{M}(\widehat{T})$; this is useful because there are algorithms for the computation of Δ, e.g., [13], which render the otherwise challenging computation of M straightforward. In the next section we introduce a new parameter that generalizes Δ and is readily computable.

2. Algorithm for determination of minimum rank and maximum multiplicity for trees and tree sign patterns

Chen et al. [5] give a variety of lower bounds for the minimum rank of a tree sign pattern. Specifically, both the diameter and half the number of loops of $\mathscr{G}(Z)$ are lower bounds for the minimum rank of tree sign pattern Z. Those authors also provide a means of computing the exact value of minimum rank for certain tree sign patterns having "star-like" graphs. In this section, we introduce a parameter $\mathscr{C}_\lambda$, show $\mathscr{C}_\lambda = M_\lambda$, and give an algorithm for the computation of $\mathscr{C}_\lambda$ that allows exact calculation of the minimum rank of *any* tree sign pattern. Throughout this section, Z will denote a symmetric tree sign pattern and T will denote a tree.

For $S \in \{\mathscr{S}(Z), \mathscr{S}(T)\}$ and $Q \subseteq \iota(S)$, define $c_\lambda(Q)$ to be the number of components of $S(Q)$ that allow eigenvalue λ. Then our readily computable new parameter is

$$\mathscr{C}_\lambda(S) = \max\{c_\lambda(Q) - |Q| : Q \subseteq \iota(S)\},$$

and we define

$$\mathscr{C}_\lambda(T) = \mathscr{C}_\lambda(\mathscr{S}(T)) \quad \text{and} \quad \mathscr{C}_\lambda(Z) = \mathscr{C}_\lambda(\mathscr{S}(Z)).$$

Theorem 2.1. *$\mathscr{C}_\lambda(S) = M_\lambda(S)$ for $S \in \{\mathscr{S}(Z), \mathscr{S}(T)\}$ where T is a tree and Z is a symmetric tree sign pattern.*

Proof. Let $S \in \{\mathscr{S}(Z), \mathscr{S}(T)\}$. Let Q be a subset of vertices such that $c_\lambda(Q) = \mathscr{C}_\lambda(S) + |Q|$. Let $S[R_1], \dots, S[R_{c_\lambda(Q)}]$ be the components of $S(Q)$ that allow eigenvalue λ. Since $S[R_i]$ allows eigenvalue λ, there exists a matrix $A_i \in S[R_i]$ such that $\lambda \in \sigma(A_i)$. Construct a matrix $A \in S$ such that $A[R_i] = A_i$ for $i = 1, \dots, c_\lambda(Q)$, so $\lambda \in \sigma(A[R_i])$. Thus $m_{A(Q)}(\lambda) \geqslant c_\lambda(Q)$ and $M_\lambda(S(Q)) \geqslant c_\lambda(Q)$. Then by the Interlacing Corollary 1.6, $M_\lambda(S) \geqslant c_\lambda(Q) - |Q| = \mathscr{C}_\lambda(S)$.

We show by induction on the order of S that $\mathscr{C}_\lambda(S) \geqslant M_\lambda(S)$. Note first that if $M_\lambda(S) = 1$, $\mathscr{C}_\lambda(S) \geqslant M_\lambda(S)$ by choosing $R = \emptyset$; this includes the base case where $o(S) = 1$. Now assume the theorem is true for every $S' \in \{\mathscr{S}(Z'), \mathscr{S}(G')\}$ such that $o(S') < o(S)$ (where Z' denotes a symmetric tree sign pattern and G' denotes a tree). The case $M_\lambda(S) = 1$ is done; if $M_\lambda(S) > 1$, then by the Generalized Parter–Wiener Theorem 1.15, there exists an index k such that $M_\lambda(S(k)) = M_\lambda(S) + 1$. Each component $S[R_i]$ of $S(k)$ is in $\{\mathscr{S}(Z[R_i]), \mathscr{S}(\langle R_i\rangle)\}$ and $o(S[R_i]) < o(S)$, so by the induction hypothesis, $\mathscr{C}_\lambda(S[R_i]) = M_\lambda(S[R_i])$. Thus there exists a subset $Q_i \subseteq R_i$ such that there are at least $M_\lambda(S[R_i]) + |Q_i|$ components of $S[R_i](Q_i)$ that allow eigenvalue λ.

Let $Q = (\cup Q_i) \cup \{k\}$. Then $S(Q)$ has at least $\sum M_\lambda(S[R_i]) + \sum |Q_i| = M_\lambda(S(k)) + \sum |Q_i| = M_\lambda(S) + 1 + |Q| - 1$ components that allow eigenvalue λ, so $\mathscr{C}_\lambda(S) \geqslant M_\lambda(S)$. □

Observation 2.2. *Let T be a tree. When computing $\mathscr{C}_0(T)$, by Theorem 1.13, $c_0(Q)$ is the number of components $\langle R_i \rangle$ of $T - Q$ such that* det $X_{\langle R_i \rangle}$ *is identically zero or has at least two nonzero terms. For $\lambda \neq 0$, by Lemma 1.8, $c_\lambda(Q)$ is the number of components of $T - Q$ that have an edge* (*with a loop considered to be an edge*).

Observation 2.3. *Let Z be a symmetric tree sign pattern. When computing $\mathscr{C}_0(Z)$, by Observation 1.9, $c_0(Q)$ is the number of components $Z[R_i]$ of $Z(Q)$ such that* det $X_{Z[R_i]}$ *is identically zero or has nonzero terms of opposite sign. For $\lambda \neq 0$, by Lemma 1.8, $c_\lambda(Q)$ is the number of components of $Z(Q)$ that have a nonzero off-diagonal entry or a diagonal entry whose sign matches the sign of λ.*

For a simple tree $\widehat{T}$ and subset $R \subseteq V(\widehat{T})$, we say $\widehat{T}$ is *R-free* if $R \cap V(\widehat{T}) = \emptyset$. A *high degree vertex* in a simple forest $\widehat{T}$ is a vertex whose degree is at least three.

Algorithm 2.4. Let $S \in \{\mathscr{S}(Z), \mathscr{S}(T)\}$ with T a tree and Z a symmetric tree sign pattern.
Initialize: $\widehat{T} = \widehat{\mathscr{G}}(S)$, H is the set of all high degree vertices of $\widehat{T}$, $Q = \emptyset$, and $i = 1$.
While $H \neq \emptyset$:

1. Set $\widehat{T_i}$ = the unique component of $\widehat{T} - Q$ that contains an H-vertex.
2. Set $S_i = S[V(\widehat{T_i})]$, the associated component of $S(Q)$.
3. Set $Q_i = \emptyset$.
4. Set $W_i = \{w \in H$: all but possibly one component of $\widehat{T_i} - w$ is H-free$\}$.
5. For each vertex $w \in W_i$,
   if there are at least two H-free components of $S_i(w)$ that allow eigenvalue λ, then $Q_i = Q_i \cup \{w\}$.
6. $Q = Q \cup Q_i$.
7. Remove all the vertices of W_i from H.
8. For each $v \in H$,
   if $\deg_{\widehat{T}-Q} v \leqslant 2$, remove v from H.
9. $i = i + 1$.

In Theorem 2.8 we will show that for the set Q produced by Algorithm 2.4,

$$c_\lambda(Q) - |Q| = \mathscr{C}_\lambda(S).$$

Before doing so, we illustrate how the algorithm is used in several examples. As noted in Observation 2.3, it is easy to determine whether a component allows a positive or allows a negative eigenvalue for a sign pattern or a graph (cf. Example 2.6). However, the case of $\lambda = 0$ is of more interest, because of the connection to minimum rank, so we begin with that example, even though it is more difficult.

Example 2.5. We compute the minimum rank of the tree sign pattern Z shown in Fig. 2.1, by computing $M_0(Z)$. The sign of each diagonal entry is shown on the vertex, with the absence of a sign indicating 0; the signs of the nonzero off-diagonal entries can be assumed to be + by Lemmas 1.4 and 1.2. Initially, $Q = \emptyset$, $i = 1$, and $H = \{1, 2, 3, 4, 5, 6\}$ is the set of high degree vertices.

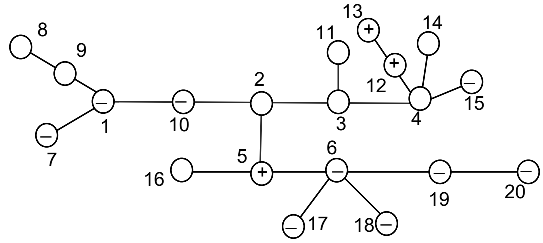

Fig. 2.1. The tree $\widehat{\mathscr{G}}(Z)$ for Z in Example 2.5, showing signs of diagonal entries.

For the first iteration of Algorithm 2.4, $\widehat{T}_1 = \widehat{\mathscr{G}}(Z)$, and $W_1 = \{1, 4, 6\}$.

Deletion of vertex 1 leaves two H-free components of $\mathscr{S}(Z)$, but neither allows singularity, since in each case the determinant of $X_{Z[R]}$ has a single nonzero term. Thus $1 \notin Q_1$.

Deletion of vertex 4 leaves three H-free components, two of which allow singularity (since $z_{14,14} = 0$ and $\det X_{Z[\{12,13\}]} = x_{12,12}x_{13,13} - x_{12,13}^2$). Thus $4 \in Q_1$.

Deletion of vertex 6 leaves three H-free components, but only one allows singularity, so $6 \notin Q_1$.

Vertex 3 is no longer high degree, and so is removed from H also.

Now $Q = Q_1 = \{4\}$, $H = \{2, 5\}$, and the signed forest $\widehat{\mathscr{G}}(Z) - Q_1$ is shown in Fig. 2.2 (the only labels now shown are for vertices currently in H).

For the second iteration of Algorithm 2.4, $\widehat{T}_2$ is the component that contains 2 and 5, and $W_2 = \{2, 5\}$.

$\widehat{T}_2 - 2$ has two H-free components. Vertex 2 is not an element of Q_2 because $Z[\{3, 11\}]$ does not allow singularity (use Fig. 2.1 to see the vertex numbers). It is unnecessary to verify that $Z[\{1, 7, 8, 9, 10\}]$ allows singularity.

$\widehat{T}_2 - 5$ has two H-free components. The component $Z[\{16\}]$ requires singularity because $z_{16,16} = 0$. The fact that the component $Z[\{6, 17, 18, 19, 20\}]$ allows singularity was established by evaluation of the determinant in Example 1.10. Thus $5 \in Q_2$.

Thus $Q = \{4, 5\}$. The signed forest $\widehat{\mathscr{G}}(Z) - Q$ is shown in Fig. 2.3. Note that H is now empty. From previous remarks, the components $Z[\{14\}]$, $Z[\{16\}]$, $Z[\{12, 13\}]$, and $Z[\{6, 17, 18, 19, 20\}]$ allow singularity. By evaluating the determinant we can also see that $Z[\{1, 2, 3, 7, 8, 9, 10, 11\}]$ allows singularity, so five components allow singularity. Since $|Q| = 2$, by Theorems 2.1 and 2.8, $M_0(Z) = \mathscr{C}_0(Z) = 5 - 2 = 3$. Thus $\mathrm{mr}(Z) = 20 - 3 = 17$. Note that the lower bound for minimum rank given by Corollary 2.9 of [5] is 6, since $\mathscr{G}(Z)$ has 12 loops, and the lower bound given by the diameter (Corollary 2.3 of [5]) is 8. A specific symmetric integer matrix $A \in \mathscr{S}(Z)$ of rank 17 is constructed in Example 3.3.

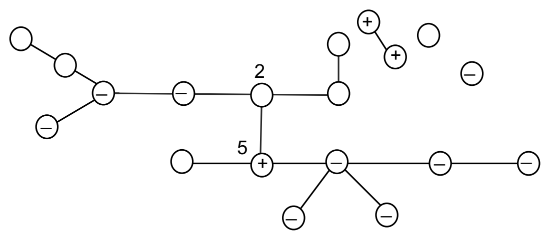

Fig. 2.2. The signed forest $\widehat{\mathscr{G}}(Z) - Q_1$ resulting from the first iteration of Algorithm 2.4.

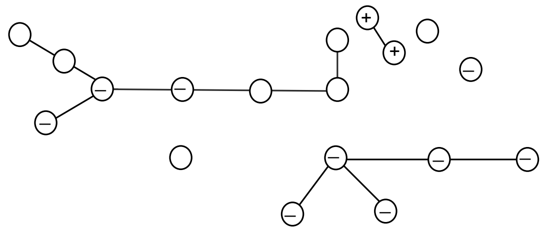

Fig. 2.3. The signed forest $\widehat{\mathscr{G}}(Z) - Q$.

The set $W = \bigcup W_i$ produced by Algorithm 2.4, realizes $\Delta(\widehat{T}) = \widehat{M}(\widehat{T})$ where $\widehat{T} = \widehat{\mathscr{G}}(S)$. As a comparison, note that for the simple tree $\widehat{T}$ associated with Fig. 2.1, $W = \{1, 2, 4, 6\}$ and $\widehat{T} - Q$ consists of 11 paths, so $\widehat{M}(\widehat{T}) = 11 - 4 = 7$ and $\widehat{\text{mr}}(\widehat{T}) = 20 - 7 = 13$ (note only 2 of the 11 paths allow singularity when the diagonal entries are restricted as shown in Fig. 2.1). In this example, W is the same as the set of deleted vertices produced by the Johnson–Saiago Algorithm [13], although the two algorithms (Johnson–Saiago and using Algorithm 2.4 to generate the set W) do not always produce the same set of deleted vertices for simple trees (allowing the diagonal to be unrestricted).

Example 2.6. Let Z be the symmetric tree sign pattern shown in Fig. 2.1. We compute $M_{-1}(Z)$. Initially, $Q = \emptyset$, $i = 1$ and $H = \{1, 2, 3, 4, 5, 6\}$ is the set of high degree vertices.

For the first iteration of Algorithm 2.4, $\widehat{T_1} = \widehat{\mathscr{G}}(Z)$, and $W_1 = \{1, 4, 6\}$.

Deletion of vertex 1 leaves two H-free components of $\mathscr{S}(Z)$ that allow a negative eigenvalue, since z_{89} is nonzero and $z_{77} = -$. Thus $1 \in Q_1$.

Deletion of vertex 4 leaves three H-free components, two of which allow a negative eigenvalue. Thus $4 \in Q_1$.

Deletion of vertex 6 leaves three H-free components that allow a negative eigenvalue, so $6 \in Q_1$.

Vertices 3 and 5 are no longer high degree, and so are removed from H also.

Now $Q = Q_1 = \{1, 4, 6\}$, $H = \{2\}$, and the signed forest $\widehat{\mathscr{G}}(Z) - Q_1$ is shown in Fig. 2.4 (the only labels now shown are for vertices currently in H).

For the second iteration of Algorithm 2.4, $\widehat{T_2}$ is the component that contains 2, and $W_2 = \{2\}$. $\widehat{T_2} - 2$ has three H-free components. The components $Z[\{10\}]$, $Z[\{3, 11\}]$, and $Z[\{5, 16\}]$ each allow a negative eigenvalue, so $2 \in Q_2$.

Thus $Q = \{1, 2, 4, 6\}$ and the forest $\widehat{\mathscr{G}}(Z) - Q$ (with signs of diagonal entries) is shown in Fig. 2.5. It is clear from this figure and Lemma 1.8 that $Z(Q)$ has ten components that allow a negative eigenvalue. Since $|Q| = 4$, by Theorems 2.1 and 2.8, $M_{-1}(Z) = \mathscr{C}_{-1}(Z) = 10 - 4 = 6$.

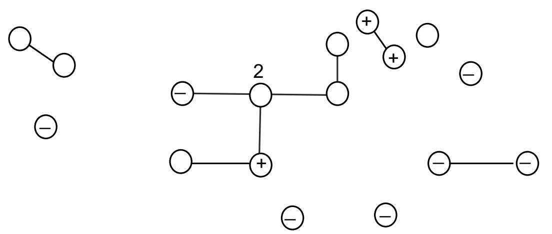

Fig. 2.4. The signed forest $\widehat{\mathscr{G}}(Z) - Q_1$.

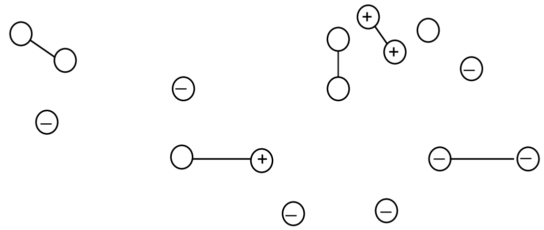

Fig. 2.5. The signed forest $\widehat{\mathscr{G}}(Z) - Q$.

Construction of a specific symmetric integer matrix $A \in \mathscr{S}(Z)$ with $m_A(-1) = 6$ is discussed in Example 3.10.

Example 2.7. We apply Algorithm 2.4 to compute the minimum rank of the tree T shown in Fig. 2.6 by computing $M_0(T)$. Here $S = \mathscr{S}(T)$ and the simple tree in Algorithm 2.4 is actually $\widehat{T}$, but the components generated by the algorithm must be examined in T itself, so we refer to the components of T rather than the components of $\mathscr{S}(T)$. Initially, $Q = \emptyset$, $i = 1$ and $H = \{1, 2, 3, 4, 5, 6, 7, 8\}$ is the set of high degree vertices.

For the first iteration of Algorithm 2.4, $T_1 = T$, and $W_1 = \{1, 3, 6, 7\}$.

Deletion of vertex 1 leaves two H-free components both of which require nonsingularity, so $1 \notin Q_1$.

Deletion of vertex 3 leaves four H-free components, three of which, $\langle 13, 14, 15\rangle$, $\langle 16\rangle$, and $\langle 17, 18, 19, 20\rangle$ allow singularity, as can be verified by Theorem 1.13. Thus $3 \in Q_1$.

Deletion of vertex 6 leaves two H-free components, both of which allow singularity, so $6 \in Q_1$.

Deletion of vertex 7 leaves two H-free components, both of which require nonsingularity, so $7 \notin Q_1$.

Now $Q = Q_1 = \{3, 6\}$, $H = \{2, 4, 5, 8\}$ and the forest $T - Q_1$ is shown in Fig. 2.7 (the only labels shown are for vertices currently in H).

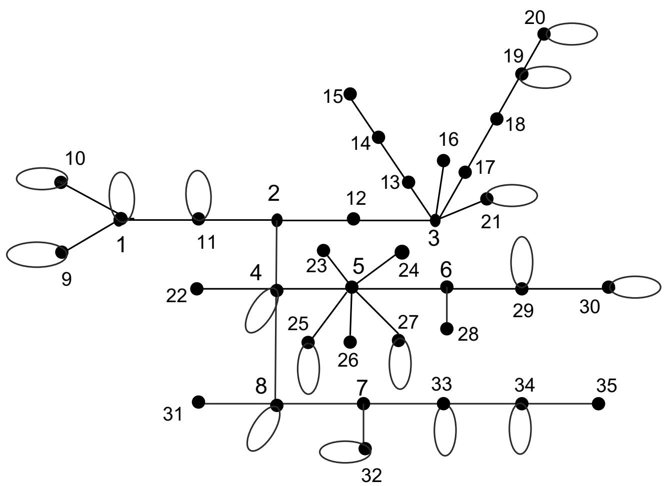

Fig. 2.6. The tree $T = T_1$.

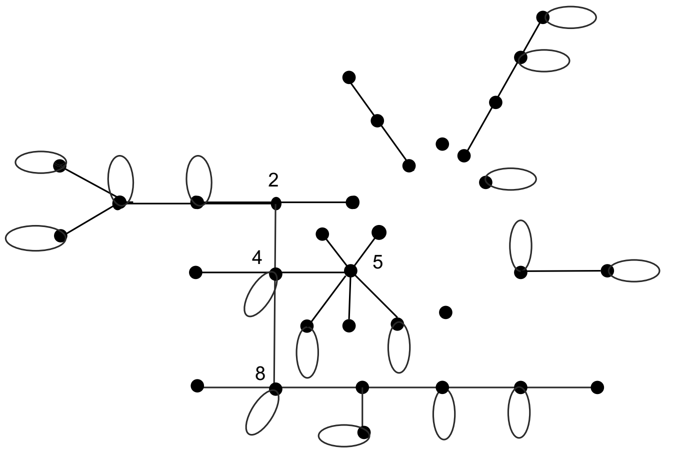

Fig. 2.7. The forest $T - Q_1$ resulting from the first iteration of Algorithm 2.4.

For the second iteration of Algorithm 2.4, T_2 is the component that contains 2, 4, 5, 8, and $W_2 = \{2, 5, 8\}$:

$T_2 - 2$ has two H-free components, both of which allow singularity. The fact that the component that contains vertex 1 (look at Fig. 2.6 in order to see that label) allows singularity follows from Theorem 1.13. Thus $2 \in Q_2$.

$T_2 - 5$ has five H-free components, three of which allow singularity, so $5 \in Q_2$.

$T_2 - 8$ has two H-free components, both of which allow singularity, so $8 \in Q_2$.

Thus $Q = \{2, 3, 5, 6, 8\}$ and $T - Q$ is shown in Fig. 2.8. There is no third iteration since the only vertex remaining in H after the removal of W_2, i.e., 4, no longer has high degree, and so is removed from H also.

Since $T - Q$ has twelve components which allow singularity, by Theorems 2.1 and 2.8, $M_0(T) = \mathscr{C}_0(T) = 12 - 5 = 7$. Thus $\operatorname{mr}(T) = 35 - 7 = 28$. Construction of a specific symmetric integer matrix $A \in \mathscr{S}(T)$ of rank 28 is discussed in Example 3.6.

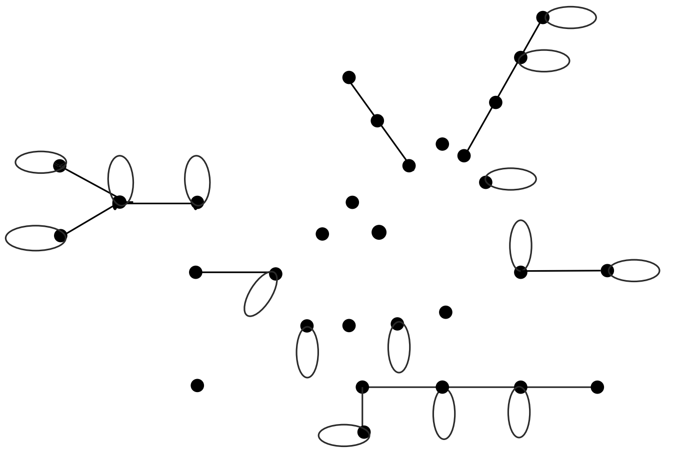

Fig. 2.8. The forest $T - Q$.

We now prove that the set Q produced by Algorithm 2.4 realizes $\mathscr{C}_\lambda(S)$.

Theorem 2.8. *Let Z be a symmetric tree sign pattern and let T be a tree. For $S \in \{\mathscr{S}(Z), \mathscr{S}(T)\}$, $\mathscr{C}_\lambda(S) = c_\lambda(Q) - |Q|$ for Q the set of vertices determined by Algorithm 2.4.*

Proof. Let $S \in \{\mathscr{S}(Z), \mathscr{S}(T)\}$. Perform Algorithm 2.4, recording the number r of iterations performed and the sets Q_i and W_i produced in iteration i. Let $W = \bigcup_{j=1}^r W_j$ and let $W_0 = U_0 = Q_0 = \emptyset$. For $i = 1, \ldots, r$, $\widehat{T_i}$ is the tree used in the ith iteration of the algorithm, and we let $\widehat{T}_{r+1} = \emptyset$.

Now we partition the set $U = \iota(S) - W$ into subsets U_i. Note first that $\widehat{T} - W$ is a disjoint union of paths, because if a vertex v has high degree in $\widehat{T} - W$, then the algorithm would not have terminated after r steps. Since $\widehat{T}$ is connected, each path P of $\widehat{T} - W$ has one or more vertices having neighbor(s) in W. Define $\omega(P)$ to be the maximum of the indices i such that a vertex of P has a neighbor in W_i. Then define U_i to be the set of all vertices in all paths P such that $\omega(P) = i$. Note $U = \bigcup_{j=1}^r U_j$ and $\widehat{T} = \langle W \cup U \rangle$ (the graph induced by $W \cup U$).

Let X be a set of vertices of $\widehat{T}$. We say

- X has property α at level i if $\left(\bigcup_{j=1}^i U_j\right) \cap X = \emptyset$,
- X has property β at level i if $\left(\bigcup_{j=1}^i (W_j - Q_j)\right) \cap X = \emptyset$,
- X has property γ at level i if $\bigcup_{j=1}^i Q_j \subseteq X$.

If X has property φ at level i, then X has property φ at level j for $j < i$ ($\varphi \in \{\alpha, \beta, \gamma\}$). For $v \in X$, define $X(v)$ to be the set obtained from X by removing v from X. If X has property φ at level i and $v \notin Q$, then clearly $X(v)$ also has property φ at level i.

Let $v \in W_{i+1} \cup U_{i+1}$. By construction, $v \in \widehat{T}_{i+1}$. If X has property γ at level i, the component C of $\widehat{T} - X(v)$ (or the component C of $\widehat{T} - X$ if $v \notin X$) that contains v is contained in $\widehat{T}_{i+1}$, because $\widehat{T}_{i+1}$ is a connected component of $\widehat{T} - \bigcup_{j=1}^i Q_j$ and $\bigcup_{j=1}^i Q_j \subseteq X$.

Note that any set X has properties α, β and γ at level 0, because $U_0 = W_0 = Q_0 = \emptyset$. Assume that X has properties α, β and γ at level $i < r$. We show that we can find a set X_γ of vertices of $\widehat{T}$ such that X_γ has properties α, β and γ at level $i + 1$ and $c_X - |X| \leqslant c_{X_\gamma} - |X_\gamma|$. Note that if Y has properties α, β and γ at level r, then $Y = Q$, so repeated application of this step shows $c_X - |X| \leqslant c_Q - |Q|$, i.e., $\mathscr{C}_\lambda(S) = c_Q - |Q|$.

Suppose that X has properties α, β, γ at level i, but does not have property α at level $i + 1$. Then there is a vertex u in U_{i+1} that is in X. By the algorithm, u has degree 2 or less in $\widehat{T}_{i+1}$. Since the component C of $\widehat{T} - X(u)$ that contains u is contained in $\widehat{T}_{i+1}$, $\deg_C u \leqslant 2$, so removing u from C creates at most one additional component. Thus $c_X - |X| \leqslant c_{X(u)} + 1 - (|X(u)| + 1) = c_{X(u)} - |X(u)|$. So if X_α is obtained from X by removing every vertex of U_{i+1} that is in X, then X_α has property α at level $i + 1$ and properties β and γ at level i, and $c_X - |X| \leqslant c_{X_\alpha} - |X_\alpha|$.

Suppose that X_α does not have property β at level $i + 1$. Then there is a vertex $w \in W_{i+1} - Q_{i+1}$ that is in X_α. Let C be the component of $\widehat{T} - X_\alpha(w)$ that contains w. Since X_α has properties β and γ at level i and property α at level $i + 1$, any component of $C - w$ that is not in $\widehat{T}_{i+2}$ is a component of $\widehat{T}_{i+1} - w$. Since $w \notin Q_{i+1}$, at most one such component of S allows eigenvalue λ, i.e., $S[V(C - w)]$ has at most one component not in $\widehat{T}_{i+2}$ that allows eigenvalue λ, and so at most two components that allow eigenvalue λ. Then $c_{X_\alpha} - |X_\alpha| \leqslant c_{X_\alpha(w)} + 1 - (|X_\alpha(w)| + 1) = c_{X_\alpha(w)} - |X_\alpha(w)|$. So if X_β is obtained from X_α by removing every vertex of $W_{i+1} - Q_{i+1}$

that is in X_α, then X_β has properties α and β at level $i+1$ and property γ at level i, and $c_{X_\alpha} - |X_\alpha| \leqslant c_{X_\beta} - |X_\beta|$.

Suppose that X_β does not have property γ at level $i+1$. Then there is a vertex $q \in Q_{i+1}$ that is not in X_β. Let C be the component of $\widehat{T} - X_\beta$ that contains q. Since X_β has properties α and β at level $i+1$ and γ at level i, any component of $C - q$ that is not in $\widehat{T}_{i+2}$ is a component of $\widehat{T}_{i+1} - q$. So $S[V(C-q)]$ has at least two components that allow eigenvalue λ. Then, $c_{X_\beta \cup \{q\}} - |X_\beta \cup \{q\}| \geqslant c_{X_\beta} + 1 - (|X_\beta| + 1) = c_{X_\beta} - |X_\beta|$. So if X_γ is obtained from X_β by adding every vertex of Q_{i+1} that is not in X_β, then X_γ satisfies properties α, β and γ at level $i+1$, and $c_{X_\beta} - |X_\beta| \leqslant c_{X_\gamma} - |X_\gamma|$. $\square$

3. Finding a symmetric integer matrix realizing minimum rank for trees and tree sign patterns

In this section, we show how to use Algorithm 2.4 to obtain an integer matrix realizing the minimum rank of a tree sign pattern or a tree that allows loops. This algorithm can be applied to a forest or forest sign pattern by executing it on each component separately.

For a diagonal sign pattern D, let $D^{(1)}$ denote the real diagonal matrix obtained from D by replacing $+$ by 1 and $-$ by -1. Before performing Algorithm 3.1, a tree sign pattern Z should be preprocessed by applying Lemma 1.4 to determine a nonsingular diagonal sign pattern D_1 and a symmetric tree sign pattern Z_1 such that $Z = Z_1 D_1$. When an integer matrix $A_1 \in \mathscr{S}(Z_1)$ with rank $A_1 = \text{mr}(Z_1)$ is obtained, then $A = A_1 D_1^{(1)}$ is a matrix having the desired properties.

Algorithm 3.1. Let $S \in \{\mathscr{S}(Z), \mathscr{S}(T)\}$, where T is a tree and Z is a symmetric tree sign pattern. To construct an integer matrix $A \in S$ having rank $A = \text{mr}(S)$:

1. Apply Algorithm 2.4 to S to find the subset Q of indices to be deleted. Let the indices of the components of $S(Q)$ be denoted by R_i, $i = 1, \ldots, h$.
2. For each i, construct a rational symmetric singular matrix $A_i \in S[R_i]$.
3. Construct a matrix A such that $A[R_i] = A_i$ and $A \in S$, using 0, 1, or -1 for any as yet unspecified entry.
4. If necessary, multiply by a positive scalar to obtain an integer matrix.

It is clear how to perform each of the steps in Algorithm 3.1 except step 2. Method 3.2 (respectively, Algorithm 3.4) gives a procedure for finding a rational singular matrix in $\mathscr{S}(Z)$ (respectively, $\mathscr{S}(T)$) that is usually simple to use in practice. We prove that Algorithm 3.4 (for trees) always produces a rational singular matrix having the given tree as its graph. We prove (in Lemma 3.7) that it is always theoretically possible to find a rational singular matrix having a given symmetric tree sign pattern that allows singularity; we do not prove Method 3.2 will always produce such a rational singular matrix, cf. Example 3.8.

Method 3.2. Let Z be a symmetric tree sign pattern that allows but does not require singularity. To construct a rational singular matrix A having $\mathscr{Z}(A) = Z$:

1. Apply the method given in the proof of Lemma 1.2 to compute a nonsingular diagonal sign pattern D such that $Z_1 = DZD$ has all nonzero off-diagonal entries equal to $+$.

2. Construct rational $A_1 \in \mathscr{S}(Z_1)$ as follows:
   (a) Set all nonzero off-diagonal entries of A_1 equal to 1.
   (b) For $j = 1, \ldots, r$, where $r = o(Z_1)$, set the jth diagonal entry to the jth diagonal entry of Z_1 times x_j, where the x_j are independent indeterminates.
   (c) Compute $\det A_1 = p(x_1, \ldots, x_r)$.
   (d) Select a variable x_s that appears in one of a pair of terms of opposite sign and not in the other.
   (e) Express p as

$$p(x_1, \ldots, x_r) = \pm\big(x_s q_1(x_1, \ldots, x_{s-1}, x_{s+1}, \ldots, x_r) - q_2(x_1, \ldots, x_{s-1}, x_{s+1}, \ldots, x_r)\big),$$

   where q_1 and q_2 each contain at least one positive term.
   (f) If possible, choose positive rational values of $x_1, \ldots, x_{s-1}, x_{s+1}, \ldots, x_r$ to make both q_1 and q_2 positive; otherwise the method does not produce the desired matrix.
   (g) With the chosen values of the x_j, set

$$x_s = \frac{q_2(x_1, \ldots, x_{s-1}, x_{s+1}, \ldots, x_r)}{q_1(x_1, \ldots, x_{s-1}, x_{s+1}, \ldots, x_r)}.$$

3. $A = \big(D^{(1)} A_1 D^{(1)}\big)$.

We illustrate Algorithm 3.1 and Method 3.2 in the next example. Method 3.2 calls for setting all nonzero off-diagonal elements to one. The *adjacency matrix* $\mathscr{A}(\widehat{T})$ of a simple graph $\widehat{T}$ is a 0, 1-matrix that has 1's in exactly the off-diagonal entries corresponding to the edges of the graph. Thus it is convenient to describe each matrix constructed by giving only its diagonal, since the matrix $A[R]$ is the sum of the adjacency matrix for $\widehat{\mathscr{G}}(Z[R])$ and a diagonal matrix.

Example 3.3. Let Z be the symmetric tree sign pattern shown in Fig. 2.1 (assuming the nonzero off-diagonal entries of Z are already +). Algorithm 2.4 has been applied to this sign pattern in Example 2.5. For each of the components $Z[\{1, 2, 3, 7, 8, 9, 10, 11\}]$, $Z[\{6, 17, 18, 19, 20\}]$, and $Z[\{12, 13\}]$, we will produce a rational singular matrix $A \in \mathscr{S}(Z)$ that is the sum of the $\mathscr{A}(\widehat{\mathscr{G}}(Z[R]))$ and a rational diagonal matrix. Let d_i denote the ith diagonal element of the matrix A. Note that choices are involved and many other matrices could be obtained from the algorithm.

We illustrate steps 2(a) to 2(g) of Algorithm 3.2 on $Z[\{6, 17, 18, 19, 20\}]$, shown in Fig. 1.1. The matrix produced by steps 2(a) and 2(b) is

$$Z_x = \begin{bmatrix} -x_6 & 1 & 1 & 1 & 0 \\ 1 & -x_{17} & 0 & 0 & 0 \\ 1 & 0 & -x_{18} & 0 & 0 \\ 1 & 0 & 0 & -x_{19} & 1 \\ 0 & 0 & 0 & 1 & -x_{20} \end{bmatrix}.$$

Step 2(c) yields

$$\det Z_x = -x_{17} - x_{18} + x_{17}x_{18}x_{20} + x_{17}x_{19}x_{20} + x_{18}x_{19}x_{20} + x_{17}x_{18}x_6 - x_{17}x_{18}x_{19}x_{20}x_6.$$

We select x_{17} as our chosen variable in step 2(d), and step 2(e) yields

$$\det Z_x = -\big(x_{17}(1 - x_{18}x_{20} - x_{19}x_{20} - x_{18}x_6 + x_{18}x_{19}x_{20}x_6) - (-x_{18} + x_{18}x_{19}x_{20})\big),$$
$$q_1(x_6, x_{18}, x_{19}, x_{20}) = 1 - x_{18}x_{20} - x_{19}x_{20} - x_{18}x_6 + x_{18}x_{19}x_{20}x_6,$$
$$q_2(x_6, x_{18}, x_{19}, x_{20}) = -x_{18} + x_{18}x_{19}x_{20}.$$

In step 2(f), we choose $x_6 = 2$, $x_{18} = 1$, $x_{19} = 2$, $x_{20} = 2$, so $\det Z_x = 3 - x_{17}$. In step 2(g), $x_{17} = 3$, and thus $d_6 = -2$, $d_{17} = -3$, $d_{18} = -1$, $d_{19} = -2$, $d_{20} = -2$.

For $Z[\{12, 13\}]$, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is singular, so let $d_{12} = d_{13} = 1$.

For $Z[\{1, 2, 3, 7, 8, 9, 10, 11\}]$, $\det Z_x = 1 - x_1x_7$, so we choose $x_1 = 1, x_7 = 1, x_{10} = 1$. The resulting diagonal entries are $d_1 = -1$, $d_7 = -1$, $d_{10} = -1$ (the latter value is irrelevant to the determinant, but must have the correct sign).

The only remaining undetermined diagonal entries are d_5 and d_{15}. Step 3 of Algorithm 3.1 sets $d_5 = 1, d_{15} = -1$. Then the matrix we have constructed is

$$A = \mathscr{A}(\widehat{\mathscr{G}}(Z)) + \operatorname{diag}(-1, 0, 0, 0, 1, -2, -1, 0, 0, -1, 0, 1, 1, 0, -1, 0, -3, -1, -2, -2),$$

and rank $A = 17$.

The algorithm for trees is simpler.

Algorithm 3.4. Let T be a tree that allows but does not require singularity. To construct a rational singular matrix A having $\mathscr{G}(A) = T$:

(a) Set all nonzero off-diagonal entries of A equal to 1.
(b) For $j = 1, \ldots, r$, where $r = |V(T)|$, if T has a loop at vertex j, set the jth diagonal entry to x_j; otherwise set the jth diagonal entry to zero.
(c) Compute $\det A = p(x_1, \ldots, x_r)$. Since T allows but does not require singularity, there are at least two nonzero terms.
(d) Select a variable x_s that appears in one of the nonzero terms and not in another.
(e) Express p as

$$p(x_1, \ldots, x_r) = x_s q_1(x_1, \ldots, x_{s-1}, x_{s+1}, \ldots, x_r) - q_2(x_1, \ldots, x_{s-1}, x_{s+1}, \ldots, x_r),$$

where both q_1 and q_2 contain at least one nonzero term.
(f) Choose rational values of $x_1, \ldots, x_{s-1}, x_{s+1}, \ldots, x_r$ to make both q_1 and q_2 nonzero.
(g) With the chosen values of the x_j, set

$$x_s = \frac{q_2(x_1, \ldots, x_{s-1}, x_{s+1}, \ldots, x_r)}{q_1(x_1, \ldots, x_{s-1}, x_{s+1}, \ldots, x_r)}.$$

Lemma 3.5. *Let T be a tree that allows singularity. If T allows but does not require singularity, then Algorithm 3.4 will produce a singular symmetric rational matrix $A \in \mathscr{S}(T)$. If T requires singularity, then any symmetric rational matrix with graph T is a singular matrix.*

Proof. Let T_x be the symmetric matrix such that all nonzero off-diagonal entries are equal to one, and having x_i as its ith diagonal entry if T has a loop at vertex i, where the x_i are independent indeterminates. Since T allows but does not require singularity, $\det T_x$ has at least two nonzero terms. Then by Lemma 1.12, there is a loop ss that is in one nonzero term that is not in another nonzero term. So we can write $\det T_x = x_s q_1(x_i) - q_2(x_i)$, where both q_1 and q_2 are nonzero polynomials in the variables x_i, $i \neq s$. We can choose rational values a_{ii} for the variables x_i, $i \neq s$ that make $q_1(a_{ii}) \neq 0$ and $q_2(a_{ii}) \neq 0$. Let $a_{ss} = \frac{q_2(a_{ii})}{q_1(a_{ii})}$. Then the matrix A having nonzero diagonal entries a_{ii} is a rational symmetric singular matrix with $\mathscr{G}(A) = T$. $\square$

Example 3.6. In Example 2.7, Algorithm 2.4 was applied to the tree in Fig. 2.6. The components are shown in Fig. 2.8. It is not difficult to apply Algorithm 3.4 to each component to choose integer values for the diagonal that when added to the adjacency matrix produce a singular matrix. One particular set of choices to produce such singular matrices yields $A = \mathscr{A}(\widehat{\mathscr{G}}(Z)) + \operatorname{diag}(3, 0, 0, 1, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 0, 1, 0, 1, 1, 0, 1, -1, 1, 0)$, and rank $A = 28$. There are many other possible choices that achieve this rank.

We now prove it is always theoretically possible to find a singular symmetric rational matrix having a given tree sign pattern that allows singularity.

Lemma 3.7. *If Z is a symmetric tree sign pattern that allows singularity and has all nonzero off-diagonal entries equal to $+$, then there exists a singular symmetric rational matrix $A \in \mathscr{S}(Z)$.*

Proof. If Z requires singularity then any symmetric rational matrix with sign pattern Z may be chosen, so assume Z does not require singularity. Note $o(Z) \geqslant 2$.

We say a tree sign pattern Z is *minimally singular* if for every index $s \in \iota(Z)$ such that $z_{ss} \neq 0$, $Z(s)$ is nonsingular. Any nondiagonal sign pattern of size two that allows singularity is minimally singular. We first show that it is possible to find the desired singular rational matrix if Z is minimally singular.

Let Z_x be a matrix having all nonzero off-diagonal entries equal to one and having $z_{ii}x_i$ as the ith diagonal entry, where the x_i are independent indeterminates. Since Z allows but does not require singularity, as in the proof of Lemma 3.5 there is a variable x_s that appears in one term and does not appear in another term. Then $\det Z_x = x_s q_1(x_i) - q_2(x_i)$, where both $q_1(x_i)$ and $q_2(x_i)$ are nonzero polynomials in the variables x_i, $i \neq s$. By Lemma 1.3, there is a singular matrix $\widetilde{A} = [\tilde{a}_{ij}]$ in $\mathscr{S}(Z)$ all of whose nonzero off-diagonal entries are one, so there are values $\tilde{a}_i = |\tilde{a}_{ii}|$ that make $\tilde{a}_s q_1(\tilde{a}_i) - q_2(\tilde{a}_i) = \det \widetilde{A} = 0$. Note that $\det \widetilde{A}(s) = \pm q_1(\tilde{a}_i)$ and $\widetilde{A}(s) \in \mathscr{S}(Z(s))$, so by the hypothesis that Z is minimally singular, $q_1(\tilde{a}_i) \neq 0$. Since $\tilde{a}_s > 0$, $\operatorname{sgn}(q_2(\tilde{a}_i)) = \operatorname{sgn}(q_1(\tilde{a}_i))$. Thus we can perturb the $\tilde{a}_i$, $i \neq s$, slightly to rational values a_i so that $\operatorname{sgn}(q_j(a_i)) = \operatorname{sgn}(q_j(\tilde{a}_i))$, $j = 1, 2$. Let $a_s = \frac{q_2(a_i)}{q_1(a_i)}$. Then the matrix A with diagonal defined by $a_{ii} = z_{ii}a_i$ and having all nonzero off-diagonal entries equal to one is the desired singular rational matrix.

Now we consider the case where Z is not assumed minimally singular. Let X_Z be the matrix of independent indeterminates defined in Section 1.5. Let Z' be the symmetric forest sign pattern obtained from Z by changing to zero the vw and wv entries whenever $v < w$ and x_{vw} is not in any nonzero term of $\det X_Z$. Every nonzero off-diagonal entry of $X_{Z'}$ appears in at least one nonzero term of $\det X_{Z'}$. An edge of $G' = \mathscr{G}(Z')$ is called *isolated* if the component of G' that contains the edge has only two vertices. If $v < w$ and x_{vw}^2 appears in every nonzero term of $\det X_{Z'}$, then vw is isolated.

Choose a minimally singular principal subpattern $Z'[R]$ of Z'. Carry out the procedure described above to find index s, polynomials q_j, $j = 1, 2$, and a symmetric singular rational matrix $A[R] \in Z'[R]$ such that if $a_i = |a_{ii}|$, then $\operatorname{sgn}(q_1(a_i)) = \operatorname{sgn}(q_2(a_i)) \neq 0$ and $a_s = \frac{q_2(a_i)}{q_1(a_i)}$. Even if $\mathscr{G}(Z'[R])$ is not a component of $\mathscr{G}(Z')$, there must exist a nonzero term in $\det X_{Z'[\overline{R}]}$, since for any edge vw (with $v < w$) that is not isolated, x_{vw}^2 is not required to appear in every nonzero term of $\det X_{Z'}$. So $Z'[\overline{R}]$ does not require singularity, and we can choose a matrix $A[\overline{R}] \in Z'[\overline{R}]$ such that $0 \neq \det A[\overline{R}] = f(a_j)$, $j \in \overline{R}$, $a_j = |a_{jj}|$. Now all diagonal elements of A have been determined. Set to one any off-diagonal entry of A that has not yet been assigned a value that corresponds to an edge of $\mathscr{G}(Z)$ between two vertices in R or to an edge between two vertices in

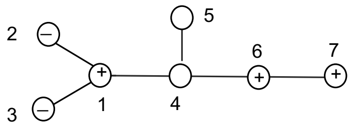

Fig. 3.1. The tree $\widehat{\mathscr{G}}(Z)$ in Example 3.8.

$\overline{R}$; the values of these entries are irrelevant in computing the determinant of A by the construction of Z'. Assign all remaining nonzero off-diagonal entries to be ϵ. Then there exists a polynomial $g(x_i, x_j)$ with $i \in R$, $j \in \overline{R}$, such that $\det A = f(a_j)(a_s q_1(a_i) - q_2(a_i)) + \epsilon^2 g(a_i, a_j) = a_s f(a_j) q_1(a_i) - (f(a_j) q_2(a_i) - \epsilon^2 g(a_i, a_j))$. Choose ϵ rational and sufficiently small so that $\operatorname{sgn}(f(a_j) q_2(a_i) - \epsilon^2 g(a_i, a_j)) = \operatorname{sgn}(f(a_j) q_2(a_i))$. □

Although it works well in practice, we have not proved that step 2(f) of Algorithm 3.2 will always produce values for $x_1, \ldots, x_{s-1}, x_{s+1}, \ldots, x_r$ that make q_1, q_2 both positive; in fact, for some choice of x_s that may be impossible, as is demonstrated in the next example.

Example 3.8. Let Z be the tree sign pattern shown in Fig. 3.1, with all nonzero off-diagonal positions being +. Then $\det Z_x = (1 - x_6 x_7)(x_1 x_2 x_3 + x_2 + x_3)$, so it is not possible to use any of x_1, x_2, x_3 as x_s in Algorithm 3.2, even though each of these variables appears in both a positive and a negative term. In this example, if either x_6 or x_7 is chosen as x_s, the algorithm will produce the desired matrix.

We now turn our attention to constructing a rational matrix having maximum multiplicity for a nonzero rational eigenvalue.

Algorithm 3.9. Let $S \in \{\mathscr{S}(Z), \mathscr{S}(T)\}$, where T is a tree and Z is a symmetric tree sign pattern. Given a rational number λ, to construct a symmetric rational matrix $A \in S$ having $m_A(\lambda) = M(S)$:

1. Apply Algorithm 2.4 to S to find the subset Q of indices to be deleted. Let the indices of the components of $S(Q)$ be denoted by R_i, $i = 1, \ldots, h$.
2. For each i, construct a rational symmetric matrix $A_i \in S[R_i]$ having eigenvalue λ.
3. Construct a matrix A such that $A[R_i] = A_i$ and $A \in S$, using 0, 1, or -1 for any as yet unspecified entry.

Again, it is clear how to perform each of the steps in Algorithm 3.9 except step 2. Although we do not present formal algorithms for step 2 for the nonzero case, it is usually not hard to construct a rational matrix having the desired rational eigenvalue, as illustrated in the next example.

Example 3.10. Let Z be the symmetric tree sign pattern shown in Fig. 2.1 (assuming the nonzero off-diagonal entries of Z are already +). Algorithm 2.4 has been applied to this sign pattern for eigenvalue -1 in Example 2.6 (see Fig. 2.4). Table 3.1 lists matrices having eigenvalue -1 and components for which they should be used to assemble a matrix $A \in \mathscr{S}(Z)$ having $m_A(-1) = 6$. For nonzero eigenvalues, it is not always possible to have all the nonzero off-diagonal entries be one, so we are no longer using the sum of the adjacency matrix and a diagonal matrix. Instead, one embeds the matrices shown in Table 3.1 in the appropriate places.

Table 3.1
The matrices $A[R]$ used to construct A realizing $M_{-1}(Z)$ for the sign pattern Z in Fig. 2.1

Matrix	R
$[-1]$	{7}, {10}, {15}, {17}, {18}
$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$	{12, 13}
$\begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$	{5, 16}
$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	{3, 11}, {8, 9}
$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$	{19, 20}

4. Conclusions

In this section, we summarize our main results.

Theorem 4.1. *For any symmetric tree sign pattern Z, $\mathscr{C}_\lambda(Z) = M_\lambda(Z)$. The following parameters can be computed by using Algorithm 2.4 to compute $\mathscr{C}_\lambda(Z)$:*

- *The maximum multiplicity of any positive eigenvalue, which is equal to $M_1(Z)$.*
- *The maximum multiplicity of any negative eigenvalue, which is equal to $M_{-1}(Z)$.*
- *The maximum multiplicity of eigenvalue zero, $M_0(Z)$.*
- *The minimum rank* $\operatorname{mr}(Z) = o(Z) - M_0(Z)$.

There is an integer matrix in $\mathscr{S}(Z)$ realizing the minimum rank, and a rational matrix in $\mathscr{S}(Z)$ realizing $M_0(Z)$.

Furthermore, the minimum rank of any tree sign pattern (not necessarily symmetric) is equal to the minimum rank of the symmetric tree sign pattern obtained by replacing each off-diagonal $-$ by $+$.

Theorem 4.2. *For any tree T, $\mathscr{C}_\lambda(T) = M_\lambda(T)$. The following parameters can be computed by using Algorithm 2.4 to compute $\mathscr{C}_\lambda(T)$:*

- *The maximum multiplicity of any nonzero eigenvalue, which is equal to $M_1(T)$.*
- *The maximum multiplicity of eigenvalue zero, $M_0(Z)$.*
- *The minimum rank* $\operatorname{mr}(Z) = o(Z) - M_0(Z)$.

There exists a matrix $A \in \mathscr{S}(T)$ such that every off-diagonal element of A is 0 *or* 1, *the diagonal of A is rational*, rank $A = \operatorname{mr}(T)$ *and $m_A(0) = M_0(T)$.*

References

[1] F. Barioli, S. Fallat, L. Hogben, Computation of minimal rank and path cover number for graphs, Linear Algebra Appl. 392 (2004) 289–303.

[2] F. Barioli, S. Fallat, L. Hogben, On the difference between the maximum multiplicity and path cover number for tree-like graphs, Linear Algebra Appl. 13 (2005) 387–404.

[3] W.W. Barrett, H. van der Holst, R. Loewy, Graphs whose minimal rank is two, Electron. J. Linear Algebra 11 (2004) 258–280.

[4] R. Brualdi, B. Shader, Matrices of Sign-Solvable Linear Systems, Cambridge University Press, Cambridge, 1995.

[5] G. Chen, F.J. Hall, Z. Li, B. Wei, On ranks of matrices associated with trees, Graphs Combin. 19 (3) (2003) 323–334.

[6] C.A. Eschenbach, C.R. Johnson, Sign patterns that require repeated eigenvalues, Linear Algebra Appl. 190 (1993) 169–179.

[7] F.J. Hall, Z. Li, Sign Pattern Matrices, Handbook of Linear Algebra, CRC Press, Boca Raton, 2006, in press.

[8] F.J. Hall, Z. Li, D. Wang, Symmetric sign pattern matrices that require unique inertia, Linear Algebra Appl. 338 (2001) 153–169.

[9] D. Hershkowitz, H. Schneider, Ranks of zero patterns and sign patterns, Linear and Multilinear Algebra 34 (1993) 3–19.

[10] R. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1985.

[11] C.R. Johnson, A. Leal Duarte, The maximum multiplicity of an eigenvalue in a matrix whose graph is a tree, Linear and Multilinear Algebra 46 (1999) 139–144.

[12] C.R. Johnson, A. Leal Duarte, C.M. Saiago, The Parter–Wiener theorem: refinement and generalization, SIAM J. Matrix Anal. Appl. 25 (2003) 311–330.

[13] C.R. Johnson, C.M Saiago, Estimation of the maximum multiplicity of an eigenvalue in terms of the vertex degrees of the graph of the matrix, Electron. J. Linear Algebra 9 (2002) 27–31.

[14] Z. Li, L. Harris, Sign patterns that require all distinct eigenvalues, JP J. Algebra Number Theory Appl. 2 (2002) 161–179.

[15] S. Parter, On the eigenvalues and eigenvectors of a class of matrices, J. Soc. Indust. Appl. Math. 8 (1960) 376–388.

[16] G. Wiener, Spectral multiplicity and splitting results for a class of qualitative matrices, Linear Algebra Appl. 61 (1984) 15–29.