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Advances in Mathematics 243 (2013) 232–261

**ADVANCES IN
Mathematics**www.elsevier.com/locate/aim

Orlov spectra as a filtered cohomology theory[☆]

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Received 7 November 2012; accepted 3 April 2013

Available online 23 May 2013

Communicated by Tony Pantev

Abstract

This paper presents a new approach to the dimension theory of triangulated categories by considering invariants that arise in the pretriangulated setting.

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Keywords: Differential graded categories; Homological dimension; Triangulated categories; *A*-infinity modules

1. Introduction

In [17], Rouquier gave several results on the dimension theory of triangulated categories. Following this paper, Orlov computed the dimension of the derived category of coherent sheaves on an arbitrary smooth curve and found it to equal one in [16]. Orlov then advanced a more general perspective on dimension theory by defining the spectrum of a triangulated category, now called the Orlov spectrum, which includes the generation times of all strong generators. The relevance of strong generators in triangulated categories and their connection to algebraic geometry was thoroughly established in the seminal paper [3] by Bondal and Van den Bergh. As the Orlov spectrum compares the generation times amongst all strong generators, it serves as a more nuanced invariant than dimension.

In the important recent work [1] of Ballard, Favero and Katzarkov, gaps in the Orlov spectrum were found to depend on the existence of algebraic cycles. To further this line of reasoning,

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they stated the following conjectures which link large gaps in the Orlov spectrum to birational invariants.

Conjecture 1.1. *Let X be a smooth algebraic variety. If $\langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ is a semi-orthogonal decomposition of \mathcal{T} , then the length of any gap in $D^b(X)$ is at most the maximal Rouquier dimension amongst the \mathcal{A}_i .*

Conjecture 1.2. *Let X be a smooth algebraic variety. If \mathcal{A} is an admissible subcategory of $D^b(X)$, then the length of any gap of \mathcal{A} is at most the maximal length of any gap of $D^b(X)$. Conversely, if \mathcal{A} has a gap of length at least s , then so does $D^b(X)$.*

These have many important corollaries connecting birational geometry to triangulated categories and their Orlov spectrum. We recall again from [1] two such results.

Corollary 1.3. *Suppose Conjectures 1.1 and 1.2 hold. Let X and Y be birational smooth proper varieties of dimension n . The category, $D^b(X)$, has a gap of length n or $n - 1$ if and only if $D^b(Y)$ has a gap of the same length i.e. the gaps of length greater than $n - 2$ are a birational invariant.*

Corollary 1.4. *Suppose Conjectures 1.1 and 1.2 hold. If X is a rational variety of dimension n , then any gap in $D^b(X)$ has length at most $n - 2$.*

Establishing a procedure for computing the Orlov spectrum of $D^b(X)$ would also allow us to pursue, for example, the following.

Conjecture 1.5. *Let X be a generic smooth four dimensional cubic. Then the gap of the spectra of the derived category of this cubic is equal to two.*

From the considerations above, this conjecture implies that generic smooth four dimensional cubic is not rational, a standing question in algebraic geometry.

While the triangulated setting serves as an accessible model for homological invariants, it is generally accepted that triangulated categories are inadequate for giving a natural characterization of homotopy theory for derived categories. Instead of working in this setting, it is advisable to lift to a pretriangulated category, or $(\infty, 1)$ -category framework, where several constructions are more natural [15,7]. In this paper, we study the Orlov spectra of triangulated categories by lifting to pretriangulated DG or A_∞ -categories.

When the category \mathcal{T} is strongly generated by a compact object G , we upgrade several classical results in dimension theory of abelian categories to the pretriangulated setting and find that the natural filtration given by the bar construction plays a determining role in the calculus of dimension. Indeed, if G is such a generator, using a result of Lefèvre-Hasegawa, we can regard \mathcal{T} as the homotopy category of perfect modules over an A_∞ algebra $A_G = \text{Hom}^*(G, G)$. In addition to being a DG category, the category of perfect A_∞ modules over A_G is enhanced over filtered chain complexes, where the filtration is obtained through the bar construction. This filtration descends to the triangulated level. The first main result, [Theorem 3.12](#), in this paper is that the generation time of a strong generator G equals the maximal length of this filtration.

Theorem 1.6. *The generation time of $G \in \mathcal{T}$ equals the supremum over all $M, N \in A_G\text{-mod}_\infty$ of the lengths of $\text{Hom}_{A_G\text{-mod}_\infty}(M, N)$ with respect to the filtration induced by the bar construction.*

As a result, we develop a filtered cohomology theory which yields the generation times that occur in Orlov spectra. The lengths referred to in this theorem are those of the filtrations induced

on the cohomology of the complexes, or the Ext groups, by the pretriangulated filtrations. In practice, it is possible to compute these lengths by calculating their spectral sequences which will converge under very mild assumptions.

Another filtration that occurs naturally from the bar construction is on the tensor product. This filtration is especially useful as one may define change of base as a tensor product with an appropriate bimodule. After establishing basic adjunction results in the next section, we generalize the classical change of base formula for dimension to the A_∞ algebra setting in [Theorem 3.20](#). A new multiplicative constant appears in this version which is related to the speed at which a spectral sequence associated to the tensor product filtration converges.

Theorem 1.7. *Let P be a (B, A) -bimodule and M a left A -module. Suppose the spectral sequence of $P \overset{\infty}{\otimes}_A M$ degenerates at the $(s + 1)$ -st page. If the convolution functor $P \overset{\infty}{\otimes}_-$ is faithful, then*

$$\text{lvl}_A(M) \leq \text{lvl}_A(P) + s \cdot \text{lvl}_B(P \overset{\infty}{\otimes}_A M).$$

Here $\text{lvl}_A(M)$ plays the role of homological, or projective, dimension of a module M . If the algebra A is formal, the constant s is 1 and we see the classical formula. If higher products are relevant, one must modify the classical inequality.

2. A_∞ constructions

This section will review many definitions and constructions related to A_∞ algebras and modules. The aim of our treatment is to approach this subject with a special emphasis on the filtrations arising from the bar constructions. These filtrations are the main technical structure we use in the dimension theory for pretriangulated categories.

After reviewing some standard definitions, we will give the definitions of filtered tensor product, filtered internal Hom and duals in the category of A_∞ -bimodules. The mantra that all constructions in the A_∞ setting are derived constructions will be continually reinforced. Moreover, the above functors will land in the category of lattice filtered A_∞ -modules, which preserves the relevant data for a study of dimension. The $\otimes - \mathcal{H}om$ adjunction, usually written in either the abelian or derived setting, will be formed as an adjunction between filtered DG functors. The categorical formulation of this statement is that the category Alg_∞ is a biclosed bicategory enriched over filtered cochain complexes. We will utilize this to update classical results on the relationship between flat and projective dimensions for perfect modules.

2.1. Fundamental notions

We take a moment to lay out some basic notation and fix our sign conventions. All algebras and vector spaces will be over a fixed field \mathbb{K} and categories will be \mathbb{K} -linear categories. Let gr be the category of graded vector spaces over \mathbb{K} and finite sums of homogeneous maps. We take Ch to be the category of cochain complexes of vector spaces over \mathbb{K} and finite sums of homogeneous maps. We will identify Hom_{Ch} with the internal Hom whose differential of

$$f \in \text{Hom}_{Ch}^k((C, d_C), (C', d_{C'}))$$

is the usual one, namely,

$$df := f \circ d_C - (-1)^k d_{C'} \circ f.$$

Finally, we take K to be the category of chain complexes and cochain maps. In other words, maps which are cocycles relative to d in Ch . For most of the paper, we will assume our chain

complexes are \mathbb{Z} -graded, but there will be examples of the $(\mathbb{Z}/2\mathbb{Z})$ -graded case. This should cause no difficulty as the proofs will be independent of this choice.

We view Ch as a closed category with respect to the tensor product along with the Koszul sign rule $\gamma_{V,W} : V \otimes W \rightarrow W \otimes V$ given by:

$$\gamma_{V,W}(v \otimes w) = (-1)^{|v||w|} w \otimes v. \tag{2.1}$$

We will need to implement this sign convention when discussing tensor products of maps as well. For this we follow the usual convention. Namely, given homogeneous maps $f \in \text{Hom}_{gr}^*(V_1, V_2)$ $g \in \text{Hom}_{gr}^*(W_1, W_2)$ then we define $f \otimes g \in \text{Hom}_{gr}(V_1 \otimes W_1, V_2 \otimes W_2)$ via $(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w)$.

By a differential graded, or DG, category \mathcal{D} we mean a category enriched in Ch . We let

$$h_- : \mathcal{D} \rightarrow Ch^{\mathcal{D}^{op}}$$

be the Yoneda functor given by $h_E(E') = \text{Hom}_{\mathcal{D}}(E', E)$.

In categories gr , Ch and K , we have the shift functor s which sends V^* to V^{*+1} . On morphisms we have $s(f) = (-1)^{|f|} f$. There is also a (degree 1) natural transformation $\sigma : I \rightarrow s$ defined as $\sigma(v) = (-1)^{|v|} v$. One can utilize σ to translate the signs occurring in various bar constructions given in this text and those in the ordinary desuspended case. In particular, given a map $f : V^{\otimes n} \rightarrow W^{\otimes m}$ in Ch we define $s_{\otimes}(f) : (sV)^{\otimes n} \rightarrow (sW)^{\otimes m}$ to be $\sigma^{\otimes m} \circ f \circ (\sigma^{-1})^{\otimes n}$. We will often use this notation to write the equations defining various structures without mentioning the elements of our algebras or modules. A nice account of the various choices and techniques used in sign conventions can be found in [6].

Filtrations will occur throughout this paper and our initial approach will be rather general. We partially order \mathbb{Z}^k for any $k \in \mathbb{N}$ with the product order. A lattice filtered complex will consist of the data $\mathbf{V} = (V, \{V_{\alpha}\}_{\alpha \in \mathbb{Z}^k})$ for some $k \in \mathbb{N}$, where V is an object in Ch and $\{V_{\alpha}\}_{\alpha \in \mathbb{Z}^k}$ is a collection of subcomplexes partially ordered by inclusion. If $k = 1$, we simply call V filtered. Given two lattice filtered complexes $\mathbf{V} = (V, \{V_{\alpha}\}_{\alpha \in \mathbb{Z}^k})$ and $\mathbf{W} = (W, \{W_{\beta}\}_{\beta \in \mathbb{Z}^l})$, we define the lattice filtered tensor product and internal hom as follows.

$$\mathbf{V} \otimes \mathbf{W} = (V \otimes W, \{V_{\alpha} \otimes W_{\beta}\}_{(\alpha, \beta) \in \mathbb{Z}^{k+l}})$$

and

$$\mathcal{H}om(\mathbf{V}, \mathbf{W}) = (\mathcal{H}om(V, W), \{\mathcal{H}om_{-\alpha, \beta}(V, W)\}_{(\alpha, \beta) \in \mathbb{Z}^{k+l}})$$

where $\mathcal{H}om_{-\alpha, \beta}(V, W) = \{\phi : V \rightarrow W \mid \phi(V_{\alpha}) \subseteq W_{\beta}\}$. The category of lattice filtered complexes and filtered complexes will be denoted Ch^{lf} and Ch^f respectively. We note that the above constructions make Ch^{lf} a closed symmetric monoidal category.

Given a DG category \mathcal{D} , we define the category \mathcal{D}^{lf} to have objects consisting of the data $\mathbf{E} = (E, \{E_{\alpha}\}_{\alpha \in \mathbb{Z}^k})$ where $(h_E(E'), \{(h_{E_{\alpha}}(E'))\}_{\alpha \in \mathbb{Z}^k}) \in Ch^{lf}$ for every object $E' \in \mathcal{D}$. The cochain complex of morphisms between \mathbf{D} and \mathbf{E} is simply $\text{Hom}_{\mathcal{D}}(D, E)$. Restricting to the case of $k = 1$ yields the definition of \mathcal{D}^f .

The total filtration functor $\text{Tot} : Ch^{lf} \rightarrow Ch^f$ is defined as

$$\text{Tot}(V, \{V_{\alpha}\}_{\alpha \in \mathbb{Z}^k}) = (V, \{\cup_{|\alpha|=n} V_{\alpha}\}_{n \in \mathbb{Z}})$$

for $k \neq 0$ where $|\alpha| = a_1 + \dots + a_k$ for $\alpha = (a_1, \dots, a_k)$. One needs to deal with $k = 0$ a bit differently and define $\text{Tot}(V, \{V_0\}) = (V, \{V'_n\})$ with $V'_n = 0$ for $n < 0$ and $V'_n = V_0$ otherwise.

Now suppose $\mathbf{V} \in Ch^f$ is a filtered complex. Letting Z_n be the subspace of cocycles in V_n , we have that the cohomology

$$H^*(\mathbf{V}) = \left(H^*(V), \left\{ H^*(V)_n = \frac{Z_n}{\text{im}(d) \cap Z_n} \right\}_{n \in \mathbb{Z}} \right)$$

is then a filtered object in gr . We define the upper and lower length of the filtration as follows. If $\cup_n H^*(\mathbf{V}) \neq H^*(V)$ we take $\ell_+(\mathbf{V}) = \infty$ and if $\cap_n H^*(\mathbf{V}) \neq 0$ then $\ell_-(\mathbf{V}) = -\infty$. Otherwise, we define these lengths as

$$\ell_+(\mathbf{V}) = \inf\{n : H^*(V)_n = H^*(V)\} \quad \ell_-(\mathbf{V}) = \inf\{n : H^*(V)_n \neq 0\}. \tag{2.2}$$

By the length $\ell(\mathbf{V})$ of \mathbf{V} we will mean the maximum of $|\ell_+(\mathbf{V})|$ and $|\ell_-(\mathbf{V})|$. We extend these definitions to $\mathbf{V} \in Ch^{lf}$ by taking length of $\text{Tot}(\mathbf{V})$.

Given a DG category \mathcal{D} and an object $\mathbf{E} \in \mathcal{D}^{lf}$, we define the lengths of \mathbf{E} as

$$\begin{aligned} \ell_+(\mathbf{E}) &= \sup\{\ell_+(h_{\mathbf{E}}(E')) : E' \in \mathcal{D}\}, \\ \ell_-(\mathbf{E}) &= \inf\{\ell_-(h_{\mathbf{E}}(E')) : E' \in \mathcal{D}\}, \\ \ell(\mathbf{E}) &= \sup\{\ell(h_{\mathbf{E}}(E')) : E' \in \mathcal{D}\}. \end{aligned}$$

Given two DG categories $\mathcal{D}, \tilde{\mathcal{D}}$, a DG functor $F : \mathcal{D} \rightarrow \tilde{\mathcal{D}}^f$ and $E \in \mathcal{D}$, we take $\ell_{\pm}^F(E) = \ell_{\pm}(F(E))$ and $\ell^F = \sup\{\ell^F(E) : E \in \mathcal{D}\}$. One can consider ℓ^F as a generalization of the cohomological dimension of a functor between abelian categories. Note that in the DG category Ch the two notions of length are equal. In other words, the definition given by Eqs. (2.2) yield the same quantities as the definition above using the Yoneda embedding h_{\cdot} .

A motivating example for the above definitions is the case where \mathcal{D} and $\tilde{\mathcal{D}}$ are categories of bounded below cochain complexes of injective objects in abelian categories D and \tilde{D} . Note that these categories admit embeddings into their filtered versions by sending any complex E^* to $(E^*, \{\tau_n(E^*)\}_{n \in \mathbb{Z}})$ where $\tau_n(E^*)^k = E^k$ for $k \leq n$ and zero otherwise. Assuming D and \tilde{D} have enough injectives, any functor $F : D \rightarrow \tilde{D}$ has the (pre)derived DG functor $RF : \mathcal{D} \rightarrow \tilde{\mathcal{D}}$ and after composition with the embedding above one has a DG functor $\mathcal{F} : \mathcal{D} \rightarrow \tilde{\mathcal{D}}^f$. It is then plain to see that $\ell^{\mathcal{F}}$ equals the cohomological dimension of F .

2.2. A_{∞} -algebras

One of the fundamental structures in our study is an A_{∞} -algebra.

Definition 2.1. A non-unital A_{∞} -algebra A is an object $A \in Ch$ and a collection of degree 1 maps $\mu_A^n : (sA)^{\otimes n} \rightarrow sA$ for $n > 0$ satisfying the relation

$$\sum_{k=0}^n \left[\sum_{r=0}^{n-k} \mu_A^{n-k+1} \circ (\mathbf{1}^{\otimes r} \otimes \mu_A^k \otimes \mathbf{1}^{\otimes(n-r-k)}) \right] = 0$$

for every n .

We note that it is common to see the definition utilizing the desuspended maps $s_{\otimes}^{-1}(\mu_A^n)$ which involves more intricate signs.

In this paper we will assume that our A_∞ -algebras come equipped with a strict unit. We recall that this means there exists a unit map $u : \mathbb{K} \rightarrow A[1]$ where

$$\mu_A^2(u \otimes \mathbf{1}) = \mathbf{1} = -\mu_A^2(\mathbf{1} \otimes u) \tag{2.3}$$

$$\mu^n(\mathbf{1}^{\otimes r} \otimes u \otimes \mathbf{1}^{\otimes(n-r-1)}) = 0 \quad \text{for } n \neq 2. \tag{2.4}$$

We will normally write e_A for $u(1)$ (or e if the algebra is implicit).

If A is an A_∞ -algebra, we take A^{op} to be the algebra with structure maps $\mu_{A^{op}}^k = \mu_A^k \circ \sigma_k$ where $\sigma_k : (sA)^{\otimes k} \rightarrow (sA)^{\otimes k}$ reverses the ordering of the factors via the symmetric monoidal transformation γ in Eq. (2.1).

It is immediate that the cohomology $H^*(A)$ defined with respect to μ_A^1 is a graded \mathbb{K} -algebra with multiplication induced by μ_A^2 . However, the higher products determine more structure than the cohomology algebra can express on its own. In order to see this we need to be able to compare two different algebras. A homomorphism of A_∞ -algebras is defined as follows.

Definition 2.2. If (A, μ_A^*) and (B, μ_B^*) are A_∞ -algebras then a collection of graded maps $\phi^n : (sA)^{\otimes n} \rightarrow sB$ for $n \geq 1$ is an A_∞ -map if

$$\sum_{k=1}^n \left[\sum_{r=1}^{n-k} \phi^{n-k+1} \circ (\mathbf{1}^{\otimes r} \otimes \mu_A^k \otimes \mathbf{1}^{\otimes(n-r-k)}) \right] \\ = \sum_{j=1}^n \left[\sum_{i_1+\dots+i_j=n} \mu_B^j \circ (\phi^{\otimes i_1} \otimes \dots \otimes \phi^{\otimes i_j}) \right].$$

A strictly unital homomorphism is also required to preserve the unit as well as satisfying the identities

$$\phi^{r+s+1} \circ (\mathbf{1}^{\otimes r} \otimes u \otimes \mathbf{1}^{\otimes s}) = 0$$

for all $r + s > 0$. The category of unital and non-unital A_∞ -algebras will be denoted Alg_∞ and Alg_∞^{nu} respectively.

When all maps $\phi^k = 0$ except ϕ^1 , we call $\{\phi^k\}$ strict. If there is an A_∞ -map $\epsilon_A : A \rightarrow \mathbb{K}$ we will call A augmented. Any augmented, strictly unital A_∞ algebra is required to satisfy the equation $\epsilon_A u = 1_{\mathbb{K}}$.

It is important to observe that $[\phi^1]$ induces an algebra homomorphism $H^*(A)$ to $H^*(B)$ so that cohomology is a functor from A_∞ -algebras to ordinary algebras. When the induced map $[\phi^1]$ is an isomorphism, we call ϕ^* a quasi-isomorphism. The following proposition can be found in any of the basic references given above.

Proposition 2.3. *Given a quasi-isomorphism $\phi^* : A \rightarrow B$ there exists a quasi-isomorphism $\psi^* : B \rightarrow A$ for which $[\phi^1]$ and $[\psi^1]$ are inverse.*

Some of the A_∞ -algebras discussed in this paper satisfy additional conditions.

Definition 2.4. (i) An A_∞ -algebra is formal if it is quasi-isomorphic to its cohomology algebra.
 (ii) An A_∞ -algebra is compact if its cohomology algebra is finite dimensional.

While it is rarely the case that an A_∞ -algebra is formal, there is an A_∞ -structure on its cohomology, called the minimal model, which yields a quasi-isomorphic A_∞ -algebra. It is a

well known fact that, for (A, μ_A^*) , this is a uniquely defined A_∞ -structure $(H^*(A), \tilde{\mu}_A^*)$ with $\tilde{\mu}_A^1 = 0$ (here $\tilde{\mu}_A^2 = [\mu_A^2]$ and the higher $\tilde{\mu}_A^*$ are determined by a tree level expansion formula). Let us state this as a proposition.

Proposition 2.5. *For any A_∞ -algebra (A, μ_A^*) there is an A_∞ -algebra $(H^*(A), \tilde{\mu}_A^*)$, uniquely defined up to A_∞ -isomorphism, and a quasi-isomorphism $\phi_A : A \rightarrow H^*(A)$. We call $(H^*(A), \tilde{\mu}_A^*)$ a minimal model of (A, μ_A^*) .*

It will be important to have at our disposal another equivalent definition, the algebra bar construction, for which we closely follow [13,10]. First, given $V \in gr$ we denote the tensor algebra and coalgebra by $T^a V$ and $T^c V$ respectively. As graded vector spaces, both are equal to

$$TV = \bigoplus_{n=0}^{\infty} V^{\otimes n}.$$

For space considerations, we will use bar notation and write $[v_1 | \cdots | v_n]$, or simply \mathbf{v} , for $v_1 \otimes \cdots \otimes v_n$ for an arbitrary element of TV . These spaces are bigraded, with one grading denoting the length of a tensor product, and the other denoting the total degree. Our notation conventions for these gradings will be

$$(TV)^{r,s} = \{ [v_1 | \cdots | v_r] : \sum |v_i| = s \}.$$

In many situations, we will be interested only in the length grading, in which case we use the notation

$$(TV)_n = \bigoplus_{k=0}^n (TV)^{k,\bullet} \quad (TV)_{>n} = \bigoplus_{k>n} (TV)^{k,\bullet}.$$

The algebra map for $T^a V$ is the usual product and the coalgebra map $\Delta : T^c V \rightarrow T^c V \otimes_{\mathbb{K}} T^c V$ is defined as

$$\Delta[v_1 | \cdots | v_n] = \sum_{i=0}^n [v_1 | \cdots | v_i] \otimes [v_{i+1} | \cdots | v_n]$$

where the empty bracket $[\]$ denotes the identity in \mathbb{K} .

The tensor coalgebra naturally lives in the category of coaugmented, counital, dg coalgebras Cog' . The objects in this category consist of data (C, d, η, ϵ) where C is a coalgebra, d is a degree 1, square zero, coalgebra derivation, $\eta : C \rightarrow \mathbb{K}$ and $\epsilon : \mathbb{K} \rightarrow C$ are the counit and coaugmentation satisfying $\eta\epsilon = 1_{\mathbb{K}}$. However, this category is too large for our purposes and we instead consider a subcategory Cog consisting of cocomplete objects. To define these objects, take $\pi : C \rightarrow \bar{C} = C/\mathbb{K}$ to be the cokernel of ϵ . Consider the kernel $C_n = \ker(\tilde{\Delta}^n)$ where

$$\tilde{\Delta}^n : C \xrightarrow{\Delta^n} C^{\otimes n} \xrightarrow{\pi^{\otimes n}} \bar{C}^{\otimes n}.$$

Elements of C_n are called n -primitive and C_1 is referred to as the coaugmentation ideal. They form an increasing sequence

$$C_0 \subset C_1 \subset \cdots$$

called the canonical filtration.

This defines a natural inclusion $Cog' \rightarrow (Cog')^f$ and we say that the augmented coalgebra C is cocomplete if $C = \lim C_n$. One easily observes that the tensor coalgebra is an object of Cog as

$$(T^c V)_n = (TV)^{n, \bullet}.$$

Moreover, the tensor coalgebra $T^c V$ is cofree in the category Cog (i.e. the tensor coalgebra functor is right adjoint to the forgetful functor).

Now we recall the (coaugmented) bar functor

$$\mathcal{B} : Alg_{\infty}^{nu} \rightarrow Cog$$

which takes any non-unital A_{∞} -algebra (A, μ_A) to $\mathcal{B}A = (T^c(sA), b_A, \eta_{\mathcal{B}A}, \epsilon_{\mathcal{B}A})$. The definitions of the counit and coaugmentation are clear. We define $b_A : T^c(sA) \rightarrow T^c(sA)$ via its restriction to $(sA)^{\otimes n}$ as

$$b_A|_{(sA)^{\otimes n}} = \sum_{k=1}^n \left(\sum_{r=0}^{n-k} \mathbf{1}^{\otimes r} \otimes \mu_A^k \otimes \mathbf{1}^{\otimes (n-r-k)} \right).$$

There are several variants of this construction, most importantly the ordinary bar construction $\overline{\mathcal{B}}A = (T(sA)_{>0}, b_A)$ which takes values in cocomplete coalgebras. We fix notation for the inclusion to be

$$\iota_A : \overline{\mathcal{B}}A \hookrightarrow \mathcal{B}A. \tag{2.5}$$

The differential is simply the restriction of the one defined in the coaugmented case. It is helpful to understand $\overline{\mathcal{B}}A$ when A is an ordinary algebra A . In this case, we see that $\overline{\mathcal{B}}A$ is just the augmented bar resolution for A (and hence, acyclic).

The bar construction of A inherits the increasing filtration

$$\mathcal{B}_n A := \bigoplus_{i \leq n} (\mathcal{B}A)^{i, \bullet} = (\mathcal{B}A)_n.$$

We refer to this, and the module variants to come, as the length filtration.

Remark 2.6. We note here that one advantage of the bar construction is the ease at which one can discuss structures that are more difficult to define in the category Alg_{∞} . One example of this is the tensor product of two A_{∞} -algebras A, A' which has more than one fairly intricate definition. In Cog we define the tensor product of $\mathcal{B}A \otimes \mathcal{B}A'$ in the usual way. We then say that $B \in Alg_{\infty}$ is quasi-isomorphic to the tensor product if $B = A \otimes A'$ and $\mathcal{B}B$ is quasi-isomorphic to $\mathcal{B}A \otimes \mathcal{B}A'$ in Cog^f . See [14] for an article comparing various constructions of a natural quasi-isomorphism.

2.3. A_{∞} polymodules

We start this section with a general definition of a module over several A_{∞} -algebras which we call a polymodule. It is both useful and correct to think of a polymodule as a bimodule with respect to the tensor product of several algebras or, even more simply, as a module over the tensor product of algebras and their opposites. This is analogous to defining an (R, S) bimodule as opposed to an $R \otimes S^{op}$ module. We take this approach at the outset to avoid some of the cumbersome notation and uniqueness issues surrounding the tensor product of multiple A_{∞} -algebras. This is accomplished utilizing the bar construction and working in the category of comodules where many structures are more accessible. The definitions and results in this section are adaptations of those for modules and bimodules which can be found in [13]. We add the

caveat that Lefèvre-Hasegawa uses the different term *polymodules* to define what we would call a module.

For this section, we fix A_∞ -algebras A_1, \dots, A_r and B_1, \dots, B_s and write $(\mathfrak{a}, \mathfrak{b})$ for the data $(A_1, \dots, A_r | B_1, \dots, B_s)$. Let P be a graded vector space and write

$$\mathbf{B}^{(\mathfrak{a}, \mathfrak{b})} P = \mathcal{B}A_1 \otimes \dots \otimes \mathcal{B}A_r \otimes P \otimes \mathcal{B}B_1 \otimes \dots \otimes \mathcal{B}B_s \tag{2.6}$$

for the bar construction of P .

When $(\mathfrak{a}, \mathfrak{b})$ is fixed or understood from the context, we simply write $\mathbf{B}P$. We make a note that $\mathbf{B}P$ is naturally an object of Ch^{lf} where the lattice is \mathbb{Z}^{r+s} and the filtration is induced by the length filtrations on the bar constructions. Given any $\gamma \in \mathbb{Z}^{r+s}$, we denote the γ filtered piece of $\mathbf{B}P$ by $\mathbf{B}_\gamma P$. Observe also that $\mathbf{B}P$ is a cofree left comodule over the coalgebras $\mathcal{B}A_i$ and a cofree right comodule over coalgebras $\mathcal{B}B_i$ where $\Delta_{i,P} : \mathbf{B}P \rightarrow \mathcal{B}A_i \otimes \mathbf{B}P$ and $\Delta_{P,j} : \mathbf{B}P \rightarrow \mathbf{B}P \otimes \mathcal{B}B_j$ are the comodule maps. These are defined by repeatedly applying γ from Eq. (2.1) to permute the left factor of $\mathcal{B}A_i$ and right factor of $\mathcal{B}B_j$ to the left and right respectively, after having applied their comultiplications. We take,

$$\Delta_P = \Delta_{P,s} \circ \dots \circ \Delta_{P,1} \circ \Delta_{r,P} \circ \dots \circ \Delta_{1,P}$$

as the polymodule comultiplication from $\mathbf{B}P$ to $\mathcal{B}A_1 \otimes \dots \otimes \mathcal{B}A_r \otimes \mathbf{B}P \otimes \mathcal{B}B_1 \otimes \dots \otimes \mathcal{B}B_s$. The differentials on each coalgebra tensor to define the differential $d'_P : \mathbf{B}P \rightarrow \mathbf{B}P$.

Definition 2.7. A non-unital $(\mathfrak{a}, \mathfrak{b}) = (A_1, \dots, A_r | B_1, \dots, B_s)$ polymodule (P, μ_P) is a graded vector space P along with a degree 1 map

$$\mu_P : \mathbf{B}^{(\mathfrak{a}, \mathfrak{b})} P \rightarrow P \tag{2.7}$$

satisfying the equation

$$\mu_P \circ [(\mathbf{1}_{\mathcal{B}A_1} \otimes \dots \otimes \mathbf{1}_{\mathcal{B}A_r} \otimes \mu_P \otimes \mathbf{1}_{\mathcal{B}B_1} \otimes \dots \otimes \mathbf{1}_{\mathcal{B}B_s}) \circ \Delta + d'_P] = 0. \tag{2.8}$$

We call the P a polymodule if μ_P satisfies the unital conditions for every i and j ,

$$\begin{aligned} \mu_P \circ (\epsilon_{\mathcal{B}A_1} \otimes \dots \otimes u_{\mathcal{B}A_i} \otimes \dots \otimes \epsilon_{\mathcal{B}A_r} \otimes \mathbf{1}_P \otimes \epsilon_{\mathcal{B}B_1} \otimes \dots \otimes \epsilon_{\mathcal{B}B_r}) &= \mathbf{1}_P, \\ \mu_P \circ (\epsilon_{\mathcal{B}A_1} \otimes \dots \otimes \epsilon_{\mathcal{B}A_r} \otimes \mathbf{1}_P \otimes \epsilon_{\mathcal{B}B_1} \otimes \dots \otimes u_{\mathcal{B}B_j} \otimes \dots \otimes \epsilon_{\mathcal{B}B_r}) &= \mathbf{1}_P, \\ \mu_P \circ (\iota_{A_1} \otimes \dots \otimes u_{\mathcal{B}A_i} \otimes \dots \otimes \iota_{A_r} \otimes \mathbf{1}_P \otimes \iota_{B_1} \otimes \dots \otimes \iota_{B_r}) &= 0, \\ \mu_P \circ (\iota_{A_1} \otimes \dots \otimes \iota_{A_r} \otimes \mathbf{1}_P \otimes \iota_{B_1} \otimes \dots \otimes u_{\mathcal{B}B_j} \otimes \dots \otimes \iota_{B_r}) &= 0, \end{aligned}$$

where ϵ is the coaugmentation and ι the inclusion from Eq. (2.5).

A bimodule is a polymodule for which $r = 1 = s$. A left module is a bimodule for which $B_1 = \mathbb{K}$ and similarly, a right module is a bimodule for which $A_1 = \mathbb{K}$. A non-unital morphism from the polymodule P to P' is defined as any gr map $\phi : \mathbf{B}P \rightarrow P'$. The collection of these maps forms a complex $\text{Hom}_{(\mathfrak{a}, \mathfrak{b})\text{-Mod}_\infty}^{nu}(P, P')$ with differential defined as

$$\begin{aligned} d\phi &= \phi \circ (\mathbf{1}_{\mathcal{B}A_1} \otimes \dots \otimes \mathbf{1}_{\mathcal{B}A_r} \otimes \mu_P \otimes \mathbf{1}_{\mathcal{B}B_1} \otimes \dots \otimes \mathbf{1}_{\mathcal{B}B_s}) \circ \Delta_P \\ &\quad + \phi \circ d'_P - (-1)^{|\phi|} \mu_{P'} \circ (\mathbf{1}_{\mathcal{B}A_1} \otimes \dots \otimes \mathbf{1}_{\mathcal{B}A_r} \otimes \phi \otimes \mathbf{1}_{\mathcal{B}B_1} \otimes \dots \otimes \mathbf{1}_{\mathcal{B}B_s}) \circ \Delta_P. \end{aligned} \tag{2.9}$$

A morphism $\phi \in \text{Hom}_{(\mathfrak{a}, \mathfrak{b})\text{-Mod}_\infty}(P, P')$ is a non-unital morphism satisfying the unital conditions

$$\phi \circ (\mathbf{1}_{\mathcal{B}A_1} \otimes \dots \otimes u_{\mathcal{B}A_i} \otimes \dots \otimes \mathbf{1}_{\mathcal{B}A_r} \otimes \mathbf{1}_P \otimes \mathbf{1}_{\mathcal{B}B_1} \otimes \dots \otimes \mathbf{1}_{\mathcal{B}B_r}) = 0,$$

$$\phi \circ (\mathbf{1}_{\mathcal{B}_{A_1}} \otimes \cdots \otimes \mathbf{1}_{\mathcal{B}_{A_r}} \otimes \mathbf{1}_P \otimes \mathbf{1}_{\mathcal{B}_{B_1}} \otimes \cdots \otimes u_{\mathcal{B}_{B_j}} \otimes \cdots \otimes \mathbf{1}_{\mathcal{B}_{B_r}}) = 0.$$

The fact that $\text{Hom}_{(\mathfrak{a}, \mathfrak{b})\text{-Mod}_\infty}(P, P')$ is indeed a subcomplex follows from the unital condition on algebras and modules. A morphism $\phi \in \text{Hom}_{(\mathfrak{a}, \mathfrak{b})\text{-Mod}_\infty}(P, P')$ is called strict if $\phi|_{\mathbf{B}_{>0}P} = 0$. A homomorphism is defined to be a cocycle in this complex. A homomorphism ϕ for which $\phi|_{\mathbf{B}_0P} : P \rightarrow P'$ is a quasi-isomorphism will be called a quasi-isomorphism. Given $\psi \in \text{Hom}_{(\mathfrak{a}, \mathfrak{b})\text{-Mod}_\infty}(P', P'')$ we define composition as

$$\psi\phi = \psi \circ (\mathbf{1}_{\mathcal{B}_{A_1}} \otimes \cdots \otimes \mathbf{1}_{\mathcal{B}_{A_r}} \otimes \phi \otimes \mathbf{1}_{\mathcal{B}_{B_1}} \otimes \cdots \otimes \mathbf{1}_{\mathcal{B}_{B_s}}) \circ \Delta_P.$$

It is a straightforward, albeit tedious check to see that these definitions make $(\mathfrak{a}, \mathfrak{b})$ polymodules into a DG category which we label $(\mathfrak{a}, \mathfrak{b})\text{-Mod}_\infty$, or just Mod_∞ . We write $H^0(\text{Mod}_\infty)$ ($H^*(\text{Mod}_\infty)$) for the zeroth (graded) cohomology category. The next proposition follows immediately from Remark 2.6 and the naturality of γ . A rigorous proof is omitted but can be assembled from results in [13].

Proposition 2.8. *The category of filtered $(\mathfrak{a}, \mathfrak{b}) = (A_1, \dots, A_r | B_1, \dots, B_s)$ polymodules is quasi-equivalent to the category of filtered left $A_1 \otimes \cdots \otimes A_r \otimes B_1^{op} \otimes \cdots \otimes B_s^{op}$ -modules.*

From this, or from a direct argument, one obtains the following corollary which will be applied often implicitly.

Corollary 2.9. *The category of filtered $(A_1, \dots, A_r | B_1, \dots, B_s)$ polymodules is naturally equivalent to the category of filtered $(A_1, \dots, A_r, B_1^{op}, \dots, B_s^{op} | \mathbb{K})$ polymodules.*

Following [4], we observe that Mod_∞ is a pretriangulated category with sums and shifts defined in the obvious way and the natural cone construction $\text{cone}(\phi)$ given in the usual way. Namely, $\text{cone}(\phi)$ is the graded vector space $P \oplus sP'$ and its structure morphism is

$$\mu_{\text{cone}(\phi)} = \begin{bmatrix} \mu_P & \sigma \circ \phi \\ 0 & \mu_{sP'} \end{bmatrix}.$$

Given $(\mathfrak{a}, \mathfrak{b})$ we let

$$\mathbf{U}_{(\mathfrak{a}, \mathfrak{b})} = A_1 \otimes \cdots \otimes A_r \otimes B_1 \otimes \cdots \otimes B_s$$

be the trivial polymodule whose structure map is induced by suspension, γ and the algebra structure maps. A free polymodule is defined as a direct sum of copies of $\mathbf{U}_{(\mathfrak{a}, \mathfrak{b})}$ and a projective polymodule as a direct summand of a free polymodule. A projective polymodule will be called finitely generated if it is a submodule of a finite sum of copies of $\mathbf{U}_{(\mathfrak{a}, \mathfrak{b})}$.

Definition 2.10. The subcategory of perfect $(\mathfrak{a}, \mathfrak{b})$ polymodules is the category mod_∞ of all polymodules quasi-isomorphic to a module built by finitely many cones of finitely generated projective polymodules.

The concept of a polymodule is derived from the more natural notion of a differential comodule over several coalgebras in *Cog*. From this point of view, we have taken a backwards approach by defining the polymodule first, as the structure maps and definitions of morphisms are more transparent in the comodule setting. Nevertheless, we continue along our path full circle towards a realization of this structure as the bar construction of a polymodule.

Given an $(\mathfrak{a}, \mathfrak{b}) = (A_1, \dots, A_r | B_1, \dots, B_s)$ polymodule (P, μ_P) , we take the free comodule $\mathbf{B}^{(\mathfrak{a}, \mathfrak{b})}P$ as its bar construction (note that this is not free as a DG comodule). We define its

differential b_P as

$$b_P = (\mathbf{1}_{\mathcal{B}A_1} \otimes \cdots \otimes \mathbf{1}_{\mathcal{B}A_r} \otimes \mu_P \otimes \mathbf{1}_{\mathcal{B}B_1} \otimes \cdots \otimes \mathbf{1}_{\mathcal{B}B_s}) \circ \Delta_P + d'_P.$$

Then it follows from the defining equation (2.8) that $(\mathbf{B}^{(a,b)}P, b_P)$ is a left and right differential comodule over the coalgebras $\mathcal{B}A_i$ and $\mathcal{B}B_j$ respectively. We denote the DG category of such DG comodules with comodule morphisms as $(a, b)\text{-}cmo d_\infty$ or simply $cmo d_\infty$.

Given a morphism $\phi \in \text{Hom}_{(a,b)\text{-}Mod_\infty}(P, P')$ we take $b_\phi : \mathbf{B}P \rightarrow \mathbf{B}P'$ to be the map $b_\phi = (\mathbf{1}_{\mathcal{B}A_1} \otimes \cdots \otimes \mathbf{1}_{\mathcal{B}A_r} \otimes \phi_P \otimes \mathbf{1}_{\mathcal{B}B_1} \otimes \cdots \otimes \mathbf{1}_{\mathcal{B}B_s}) \circ \Delta_P$. It then becomes an exercise that the bar construction gives a full and faithful functor from Mod_∞ to $cmo d_\infty$ whose essential image consists of free comodules.

For our purposes, this is not enough as we wish to keep track of the length filtration throughout. The category $(a, b)\text{-}cmo d_\infty$ has a natural embedding into $((a, b)\text{-}cmo d_\infty)^{lf}$ given by the primitive filtration. More concretely, given $(\mathbf{i}, \mathbf{j}) = (i_1, \dots, i_r, j_1, \dots, j_s) \in \mathbb{Z}^{r+s}$ we define

$$\mathbf{B}_{(\mathbf{i}, \mathbf{j})}^{(a,b)}P = \mathcal{B}_{i_1}A_1 \otimes \cdots \otimes \mathcal{B}_{i_r}A_r \otimes P \otimes \mathcal{B}_{j_1}B_1 \otimes \cdots \otimes \mathcal{B}_{j_s}B_s.$$

This induces an embedding

$$\mathbf{B} : (a, b)\text{-}Mod_\infty \rightarrow ((a, b)\text{-}cmo d_\infty)^{lf}. \tag{2.10}$$

The induced length filtration on polymodule morphisms is then given by

$$\mathcal{F}^{(\mathbf{i}, \mathbf{j})}\text{Hom}_{Mod_\infty}(P, P') = \{\phi : \phi|_{\mathbf{B}_{(\mathbf{i}, \mathbf{j})}P} = 0\}. \tag{2.11}$$

An advantage of the bar construction is the ease at which one sees the following proposition.

Proposition 2.11. *The category $(a, b)\text{-}Mod_\infty$ is enriched over \mathbb{Z}^{r+s} -lattice filtered complexes.*

In other words, morphism composition respects the total filtration on the tensor product. As stated above, this follows immediately from the definition of comodule morphism in the category $cmo d_\infty$.

Remark 2.12. One should make certain not to confuse this enrichment with the notion that the objects of $(a, b)\text{-}Mod_\infty$ are lattice filtered, as this only occurs if we resolve the polymodules.

2.4. Filtered constructions

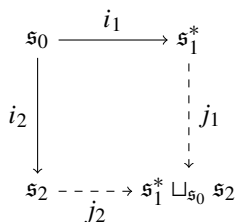
In this section we define tensor products and inner homs of polymodules. To do this effectively, it is helpful to have a picture in mind as well as the appropriate notation associated to this picture. We will say $\mathfrak{s} = (S^+, S^-, \kappa)$ is a labelled set if S^+ and S^- are finite sets and κ is a function from $S^+ \sqcup S^-$ to the objects of Alg_∞ . We will write $A \in \mathfrak{s}$ (or $A \in \mathfrak{s}^\pm$) if there is $s \in S^+ \sqcup S^-$ (or $s \in S^\pm$) such that $\kappa(s) = A$. Given a labelled set $\mathfrak{s} = (S^+, S^-, \kappa)$, we write \mathfrak{s}^* for the labelled set (S^-, S^+, κ) . We take \mathcal{L} to be the category of labelled sets with morphisms that are injective maps respecting the labelling. Note that \mathcal{L} is closed under finite direct limits.

Given a labelled set $\mathfrak{s} = (\{t_1^+, \dots, t_r^+\}, \{t_1^-, \dots, t_s^-\}, \kappa)$ we take $\mathfrak{s}\text{-}Mod_\infty$ to denote the category of $(\kappa(t_1^+), \dots, \kappa(t_r^+)|\kappa(t_1^-), \dots, \kappa(t_s^-))$ polymodules. We abbreviate the differential coalgebra

$$\mathcal{B}[\kappa(t_1^+)] \otimes \cdots \otimes \mathcal{B}[\kappa(t_r^+)] \otimes \mathcal{B}[\kappa(t_1^-)] \otimes \cdots \otimes \mathcal{B}[\kappa(t_s^-)]$$

by $\mathcal{B}A_\mathfrak{s}$. Any morphism $i : \mathfrak{s}_1 \rightarrow \mathfrak{s}_2$ induces a forgetful functor $i^* : \mathfrak{s}_2\text{-}Mod_\infty \rightarrow \mathfrak{s}_1\text{-}Mod_\infty$.

By gluing data $t = (s_0, s_1, s_2, i_1, i_2, j_1, j_2)$, we mean a pushout diagram as below in \mathcal{L}



and we abbreviate $s_1 \#_{s_0} s_2$ for the labelled set $[s_1 - i_1(s_0^*)] \sqcup [s_2 - i_2(s_0)]$.

Given gluing data $t = (s_0, s_1, s_2, i_1, i_2, j_1, j_2)$, we define the tensor product as a functor

$$\overset{\infty}{\otimes}_{s_0-} : s_1\text{-Mod}_{\infty} \times s_2\text{-Mod}_{\infty} \rightarrow (s_1 \#_{s_0} s_2\text{-Mod}_{\infty})^{lf}. \tag{2.12}$$

As usual, this product is given by first passing through the bar construction, applying the cotensor product and then recognizing the result as the bar construction of a polymodule. The details of this are now given.

Let P_1, P_2 be s_1, s_2 polymodules respectively. Then we let

$$P_1 \overset{\infty}{\otimes}_{s_0} P_2 = P_1 \otimes \mathcal{B}A_{s_0} \otimes P_2.$$

To simplify the definition of the structure map, we write $\Delta_1 = \Delta_{i_1^*(P_1)}$ and $\Delta_2 = \Delta_{i_2^*(P_2)}$ as partial comultiplications. These are the comultiplications obtained when considering $\mathbf{B}P_1$ and $\mathbf{B}P_2$ as comodules over $\mathcal{B}A_{s_0}$. Then we see that there is an isomorphism of graded vector spaces:

$$\alpha : \mathbf{B}^{s_1 \#_{s_0} s_2} (P_1 \overset{\infty}{\otimes}_{s_0} P_2) \rightarrow \mathbf{B}^{s_1} P_1 \square_{\mathcal{B}A_{s_0}} \mathbf{B}^{s_2} P_2$$

where $\square_{\mathcal{B}A_{s_0}}$ is the cotensor product (see, e.g. [9]). Recall that this is the kernel of

$$\Delta_1 \otimes 1 - 1 \otimes \Delta_2 : \mathbf{B}^{s_1} P_1 \otimes \mathbf{B}^{s_2} P_2 \rightarrow \mathbf{B}^{s_1} P_1 \otimes \mathcal{B}A_{s_0} \otimes \mathbf{B}^{s_2} P_2.$$

Restricting α to $P_1 \overset{\infty}{\otimes}_{s_0} P_2$, it is defined as $\alpha(p_1 \otimes a \otimes p_2) = p_1 \otimes \Delta_{\mathcal{B}A_{s_0}}(a) \otimes p_2$ where, as always, we implicitly use the symmetric monoidal map γ . It is extended to the bar construction by tensoring with the remaining coalgebras. Utilizing α , one pulls back the differential from the cotensor product to obtain a differential d on $\mathbf{B}^{s_1 \#_{s_0} s_2} (P_1 \overset{\infty}{\otimes}_{s_0} P_2)$. As this differential is a square zero comodule coderivation, it is induced by its composition with the projection

$$\pi : \mathbf{B}^{s_1 \#_{s_0} s_2} (P_1 \overset{\infty}{\otimes}_{s_0} P_2) \rightarrow P_1 \overset{\infty}{\otimes}_{s_0} P_2$$

and one obtains the A_{∞} -module map $\mu_{P_1 \overset{\infty}{\otimes}_{s_0} P_2} = \pi \circ d$.

Given morphisms $\phi_i : P_i \rightarrow P'_i$ in s_i , we have that the cotensor product of the bar constructions

$$b_{\phi_1} \square_{\mathcal{B}A_{s_0}} b_{\phi_2} : \mathbf{B}^{s_1 \#_{s_0} s_2} (P_1 \overset{\infty}{\otimes}_{s_0} P_2) \rightarrow \mathbf{B}^{s_1 \#_{s_0} s_2} (P'_1 \overset{\infty}{\otimes}_{s_0} P'_2)$$

yields a natural map $\phi_1 \overset{\infty}{\otimes}_{s_0} \phi_2$ in $s_1 \#_{s_0} s_2\text{-Mod}_{\infty}$. When considering $P_1 \overset{\infty}{\otimes}_{s_0-}$ as a functor, we take ϕ_2 to $1_{P_1 \overset{\infty}{\otimes}_{s_0} \phi_2}$. Note that it follows from the definitions above and that of the cone that $P_1 \overset{\infty}{\otimes}_{s_0-}$ is an exact functor.

Since the coalgebra $\mathcal{B}A_{\mathfrak{s}_0}$ is $\mathbb{Z}^{|\mathfrak{s}_0|}$ -filtered by the primitives of $\mathcal{B}A$ for $A \in \mathfrak{s}_0$, we have that $P_1 \overset{\infty}{\otimes}_{\mathfrak{s}_0} P_2$ is lattice filtered by $\mathbb{Z}^{|\mathfrak{s}_0|}$. We will preserve this filtration in the definition and write

$$P_1 \otimes_{\mathfrak{s}_0}^{[\gamma]} P_2 = P_1 \otimes \left(\otimes_{A \in \mathfrak{s}_0^+} \mathcal{B}_{k_i} A^{op} \right) \otimes \left(\otimes_{B \in \mathfrak{s}_0^-} \mathcal{B}_{l_j} B \right) \otimes P_2$$

where $\gamma = (k_1, \dots, k_a, l_1, \dots, l_b) \in \mathbb{Z}^{|\mathfrak{s}_0|}$. Thus, we have obtained the DG functor (2.12).

It will be useful to have notation for filtered quotients in this setting. For this, we write

$$P_1 \odot_{\mathfrak{s}_0}^{\gamma} P_2 := \frac{P_1 \overset{\infty}{\otimes}_{\mathfrak{s}_0} P_2}{P_1 \otimes_{\mathfrak{s}_0}^{[\gamma]} P_2}. \tag{2.13}$$

As expected, the tensor product of a given polymodule with the diagonal polymodule yields a quasi-equivalent polymodule. However, the filtration is added structure which will be exploited later in the paper. For now, we simply define the natural quasi-equivalence and its inverse. Fix a labelled set $\mathfrak{s} = (S^+, S^-, \kappa)$, let $2\mathfrak{s} = \mathfrak{s}^* \sqcup \mathfrak{s}$ and $\mathfrak{t} = (\mathfrak{s}, 2\mathfrak{s}, \mathfrak{s}, i_1, i_2, j_1, j_2)$ the natural gluing data. We take $\mathbf{D}_{\mathfrak{s}}$ to be the diagonal $2\mathfrak{s}$ polymodule

$$\otimes_{t \in S^+ \cup S^-} \kappa(t).$$

The structure maps for $\mathbf{D}_{\mathfrak{s}}$ are simply the tensor products of the A_{∞} algebra maps composed with the shift for the various labelling algebras. Then we define the natural equivalences

$$\xi_P : \mathbf{D}_{\mathfrak{s}} \overset{\infty}{\otimes}_{\mathfrak{s}} P \rightarrow P \quad \epsilon_P : P \rightarrow \mathbf{D}_{\mathfrak{s}} \overset{\infty}{\otimes}_{\mathfrak{s}} P. \tag{2.14}$$

Here ξ_P is defined as the map induced by tensor multiplication m , the shift σ and the polymodule multiplication map μ_P ,

$$\xi_P = \mu_P \circ (m \otimes 1_P) \circ (1_{\mathcal{B}A_{\mathfrak{s}}} \otimes \sigma \otimes 1_{\mathcal{B}A_{\mathfrak{s}}} \otimes 1_P).$$

Letting $\mu_{\mathcal{B}A_{\mathfrak{s}}} : \mathbb{K} \rightarrow \mathcal{B}A_{\mathfrak{s}}$ send 1 to $e_{\mathfrak{s}} = \otimes_{t \in S^+ \cup S^-} e_{\mathcal{B}[\kappa(\mathfrak{s})]}$, we take

$$\epsilon_P = \sigma_{\otimes}^{-1}(e_{\mathfrak{s}}) \otimes 1_{\mathcal{B}A_{\mathfrak{s}}} \otimes 1_P.$$

Using the unital conditions, it is easy to verify that ξ_P and ϵ_P are quasi-inverse maps. As a consequence, we obtain the following basic lemma which is instructive as to the bar construction of a module.

Lemma 2.13. *Suppose A is an A_{∞} algebra and denote A regarded as a right module over itself as A^r . Let P be a left A module, then the vector space $A^r \overset{\infty}{\otimes} P$ is naturally quasi-isomorphic to $H^*(P)$.*

Proof. Let $\mathfrak{s} = (S^+, S^-, \kappa)$ be the labelled set with $S^+ = \emptyset$, $S^- = \{t\}$ and $\kappa(t) = A$. Then the lemma follows from the basic observation that, by definition,

$$A^r \overset{\infty}{\otimes} P = \mathbf{D}_{2\mathfrak{s}} \overset{\infty}{\otimes}_{\mathfrak{s}} P$$

as a complex. Using the natural equivalence in (2.14), latter is quasi-isomorphic to P which has minimal model $H^*(P)$. \square

Combining this lemma with earlier remarks, we obtain the following important fact.

Proposition 2.14. *Let $\mathfrak{t} = (\mathfrak{s}_0, \mathfrak{s}_1, \mathfrak{s}_2, i_1, i_2, j_1, j_2)$ be gluing data such that the algebras labelled by \mathfrak{s}_0 are compact. If P_i are perfect \mathfrak{s}_i polymodules then $P_1 \overset{\infty}{\otimes}_{\mathfrak{s}_0} P_2$ is a perfect $\mathfrak{s}_1 \overset{\#}{\mathfrak{s}}_{\mathfrak{s}_0} \mathfrak{s}_2$ polymodule.*

Proof. The previous lemma implies that there is a quasi-isomorphism

$$\phi : \mathbf{U}_{\mathfrak{s}_1} \overset{\infty}{\otimes}_{\mathfrak{s}_0} \mathbf{U}_{\mathfrak{s}_2} \xrightarrow{q.i.} \mathbf{U}_{\mathfrak{s}_1 \overset{\#}{\mathfrak{s}}_{\mathfrak{s}_0} \mathfrak{s}_2} \otimes \left(\otimes_{A \in \mathfrak{s}_0} H^*(A) \right).$$

By the compactness assumption, this implies that tensor products of finitely generated projective polymodules are finitely generated projective polymodules. Together with the definition of perfect modules and the fact that tensor product $\overset{\infty}{\otimes}_{\mathfrak{s}_0}$ is exact, we have the result. \square

To define the internal Hom, we again follow the approach for the tensor product and pass to coalgebras and comodules. There is an additional notion needed here from classical homotopy theory, that of a twisting cochain which we recall here. If C is a DG coalgebra and A a DG algebra, a map $\rho : C \rightarrow A$ is called a twisting cochain if $\partial\rho + \rho \cdot \rho = 0$ where $\partial\rho = d_A\rho - (-1)^{|\rho|}\rho d_C$ and $\rho \cdot \rho := m \circ \rho \otimes \rho \circ \Delta_C$ where m is multiplication in A .

One of the central features of twisted cochains is that they allow one to define twisted tensor products [5,13]. We take a moment to recall this construction for the case of a left module.

Definition 2.15. Given a dg coalgebra C , a dg algebra A , a dg C bicomodule M , a left dg A module N and a twisting cochain $\rho : C \rightarrow A$, the twisted tensor product $M \otimes_{\rho} N$ (or $N \otimes_{\rho} M$) is defined as the ordinary tensor product of vector spaces with chain map $d_M \otimes 1_N + 1_M \otimes d_N + \rho \cap -$ where

$$\rho \cap - = (1_M \otimes m_N) \circ (1_M \otimes \rho \otimes 1_N) \circ (\Delta_M \otimes 1_N).$$

The result is a left (or right) C comodule.

The case of right module and bimodule is analogous.

Now, let $\mathfrak{s} = \mathfrak{s}' \sqcup \mathfrak{s}''$ in \mathcal{L} and $i : \mathfrak{s}' \rightarrow \mathfrak{s}, j : \mathfrak{s}'' \rightarrow \mathfrak{s}$ the inclusion maps. Given a \mathfrak{s} polymodule P , we define a map

$$\rho_j : \mathcal{B}A_{\mathfrak{s}'} \rightarrow \text{Hom}_{\mathfrak{s}''\text{-Mod}_{\infty}}(j^*(P), j^*(P))$$

as $[\rho_j(\mathbf{c})](\mathbf{a} \otimes p \otimes \mathbf{b}) = \mu_P(\mathbf{c} \otimes \mathbf{a} \otimes p \otimes \mathbf{b})$ where $\mathbf{a} \otimes p \otimes \mathbf{b} \in \mathbf{B}^{\mathfrak{s}''} P$. It follows from Eqs. (2.8) and (2.9) that ρ_j is a twisting cochain from the DG coalgebra $\mathcal{B}A_{\mathfrak{s}'}$ to the DG algebra $\text{Hom}_{\mathfrak{s}''\text{-Mod}_{\infty}}(j^*(P), j^*(P))$.

Suppose $\mathfrak{t} = (\mathfrak{s}_0, \mathfrak{s}_1, \mathfrak{s}_2, i_1, i_2, j_1, j_2)$ is gluing data and P_1, P_2 are $\mathfrak{s}_1^*, \mathfrak{s}_2$ polymodules respectively. Then, as a graded vector space, we define $\mathcal{H}om_{\mathfrak{s}_0}(P_1, P_2)$ as $\text{Hom}_{\mathfrak{s}_0\text{-Mod}_{\infty}}(i_1^*(P_1), i_2^*(P_2))$. The structure map

$$\mu_{\mathcal{H}om_{\mathfrak{s}_0}(P_1, P_2)} : \mathbf{B}^{\mathfrak{s}_1 \overset{\#}{\mathfrak{s}}_{\mathfrak{s}_0} \mathfrak{s}_2} \mathcal{H}om_{\mathfrak{s}_0}(P_1, P_2) \rightarrow \mathcal{H}om_{\mathfrak{s}_0}(P_1, P_2)$$

is set to equal the differential on the twisted tensor product composed with the projection $\pi : \mathbf{B}^{\mathfrak{s}_1 \overset{\#}{\mathfrak{s}}_{\mathfrak{s}_0} \mathfrak{s}_2} \mathcal{H}om_{\mathfrak{s}_0}(P_1, P_2) \rightarrow \mathcal{H}om_{\mathfrak{s}_0}(P_1, P_2)$, where the former is induced by the isomorphism

$$\mathbf{B}^{\mathfrak{s}_1 \overset{\#}{\mathfrak{s}}_{\mathfrak{s}_0} \mathfrak{s}_2} \mathcal{H}om_{\mathfrak{s}_0}(P_1, P_2) = \mathcal{B}A_{\mathfrak{s}_2 - i_2(\mathfrak{s}_0)} \otimes_{\rho_{i_2}} \mathcal{H}om_{\mathfrak{s}_0}(P_1, P_2) \otimes_{\rho_{i_1}} \mathcal{B}A_{\mathfrak{s}_1 - i_1(\mathfrak{s}_0)}.$$

Again we keep track of the lattice filtration so that $\mathcal{H}om_{\mathfrak{s}_0}(P_1, P_2)$ is a $\mathbb{Z}^{|\mathfrak{s}_0|}$ filtered polymodule.

As in the case of the tensor product, for any \mathfrak{s} polymodule P , the diagonal polymodule $\mathbf{D}_{\mathfrak{s}}$ plays the role of a unit for $\mathcal{H}om_t(\mathbf{D}_{\mathfrak{s}}, P)$. Again we define the natural transformations

$$\chi_P : \mathcal{H}om_{\mathfrak{s}}(\mathbf{D}_{\mathfrak{s}}, P) \rightarrow P \quad \nu_P : P \rightarrow \mathcal{H}om_{\mathfrak{s}}(\mathbf{D}_{\mathfrak{s}}, P).$$

Where $\chi_P(\mathbf{a} \otimes \phi \otimes \mathbf{b}) = (-1)^{|\phi||\mathbf{a}|} \phi(\mathbf{a} \otimes u_{\mathcal{B}A_{\mathfrak{s}}(1)} \otimes \mathbf{b})$ and ν_P is the strict map sending p to the morphism ϕ_p defined as $\phi_p(\mathbf{a} \otimes \mathbf{q} \otimes \mathbf{b}) = \mu_P(\mathbf{a} \otimes \mathbf{q}^+ \otimes p \otimes \mathbf{q}^- \otimes \mathbf{b})$ where \mathbf{q}^{\pm} is the tensor factor of q in $\mathbf{D}_{\mathfrak{s}^{\pm}}$.

2.5. Filtered adjunction

In this section we observe the classic adjunction between tensor product and internal Hom for polymodules. This leads to elementary, but powerful, observations on dual A_{∞} -modules. We will be concerned with preserving the lattice filtrations naturally throughout.

To state the theorem, we need to specify the gluing data between three categories of polymodules. Assume \mathfrak{s}_i are labelled sets for $i = 1, 2, 3$. We say that the data $\tau = (t_{12}, t_{23}, t_{31})$ form a gluing cycle if t_{ij} are the gluing data

$$\begin{aligned} t_{12} &= (\mathfrak{s}_{12}, \mathfrak{s}_1, \mathfrak{s}_2, i_{12}, i'_{12}, j_{12}, j'_{12}), \\ t_{23} &= (\mathfrak{s}_{23}, \mathfrak{s}_2^*, \mathfrak{s}_3, i_{23}, i'_{23}, j_{23}, j'_{23}), \\ t_{31} &= (\mathfrak{s}_{31}, \mathfrak{s}_3, \mathfrak{s}_1^*, i_{31}, i'_{31}, j_{31}, j'_{31}), \end{aligned}$$

and $\text{im}(i_{kl})$ is disjoint from $\text{im}(i'_{mk})$. A gluing cycle can be represented graphically as a directed graph with three vertices. Vertices v_1, v_2 have incoming and outgoing edges \mathfrak{s}_i^{\mp} and v_3 has incoming and outgoing edges \mathfrak{s}_3^{\pm} . Those edges that connect vertices v_i and v_j form the labelled set \mathfrak{s}_{ij} . This is depicted in Fig. 1.

We take \mathfrak{s}_{τ} to be the labelled set $(\mathfrak{s}_1 \#_{\mathfrak{s}_{12}} \mathfrak{s}_2)^* \#_{\mathfrak{s}_{23} \sqcup \mathfrak{s}_{31}} \mathfrak{s}_3$, i.e. \mathfrak{s}_{τ} consists of the half edges in Fig. 1. With this notation, we can prove the following classic adjunction:

Theorem 2.16. *Given a gluing cycle τ and polymodules $P_i \in \mathfrak{s}_i\text{-Mod}_{\infty}$, there is a natural isomorphism Φ in $(\mathfrak{s}_{\tau}\text{-Mod}_{\infty})^{lf}$,*

$$\Phi : \mathcal{H}om_{\mathfrak{s}_{31} \sqcup \mathfrak{s}_{23}}(P_1 \overset{\infty}{\otimes}_{\mathfrak{s}_{12}} P_2, P_3) \rightarrow \mathcal{H}om_{\mathfrak{s}_{31} \sqcup \mathfrak{s}_{12}}(P_1, \mathcal{H}om_{\mathfrak{s}_{23}}(P_2, P_3)). \tag{2.15}$$

Proof. This is simply an exercise in the definitions of the last section and the observation that gr^{lf} is a closed category. Recall that closed means that there is an internal Hom and tensor with the usual adjunction. Letting $\star = \mathcal{H}om_{\mathfrak{s}_{31} \sqcup \mathfrak{s}_{23}}(P_1 \overset{\infty}{\otimes}_{\mathfrak{s}_{12}} P_2, P_3)$ we have the following natural isomorphisms of $\mathbb{Z}^{|\mathfrak{s}_{31}|+|\mathfrak{s}_{12}|+|\mathfrak{s}_{23}|}$ filtered graded vector spaces

$$\begin{aligned} \star &= \text{Hom}_{(\mathfrak{s}_{31} \sqcup \mathfrak{s}_{23})\text{-Mod}_{\infty}}(P_1 \overset{\infty}{\otimes}_{\mathfrak{s}_{12}} P_2, P_3) \\ &= \text{Hom}_{gr}(\mathcal{B}A_{\mathfrak{s}_{31}} \otimes P_1 \otimes \mathcal{B}A_{\mathfrak{s}_{12}} \otimes P_2 \otimes \mathcal{B}A_{\mathfrak{s}_{23}}, P_3) \\ &\simeq \text{Hom}_{gr}(\mathcal{B}A_{\mathfrak{s}_{31}} \otimes P_1 \otimes \mathcal{B}A_{\mathfrak{s}_{12}}, \text{Hom}_{gr}(P_2 \otimes \mathcal{B}A_{\mathfrak{s}_{23}}, P_3)), \\ &= \mathcal{H}om_{\mathfrak{s}_{31} \sqcup \mathfrak{s}_{12}}(P_1, \mathcal{H}om_{\mathfrak{s}_{23}}(P_2, P_3)). \end{aligned}$$

The first equality follows from the definition A_{∞} -morphism, the second from the closedness of gr^{lf} and the third from the definitions of internal Hom and A_{∞} -morphism. To complete the proof, one must show that the isomorphisms above respect the differentials, which follows immediately from the definitions. \square

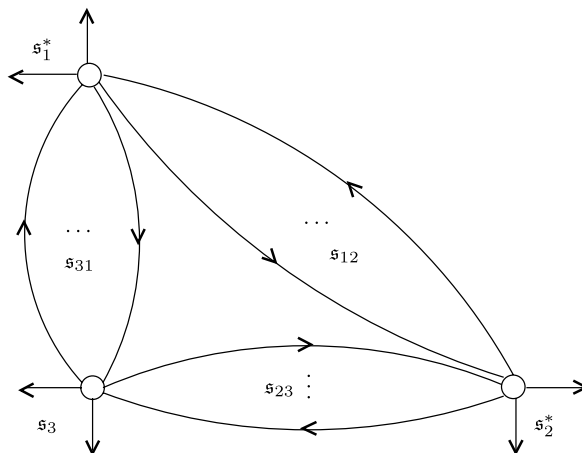


Fig. 1. The gluing cycle τ .

The same proof gives a natural equivalence

$$\Phi^j : \text{Hom}_{\mathfrak{s}_{31} \sqcup \mathfrak{s}_{23}}(P_1 \overset{\infty}{\otimes}_{\mathfrak{s}_{12}} P_2, P_3) \rightarrow \text{Hom}_{\mathfrak{s}_{32} \sqcup \mathfrak{s}_{23}}(P_2, \text{Hom}_{\mathfrak{s}_{13}}(P_1, P_3)),$$

making the bicategory of A_∞ -algebras and bimodules into a biclosed bicategory (see [12]). We apply this theorem to a simple gluing cycle to obtain the following corollary.

Corollary 2.17. *Suppose $A \in \text{Alg}_\infty$ and $P \in A\text{-Mod}_\infty$. Then:*

$$\ell^{-\infty}_{\otimes_A P} \leq \ell^{\text{Hom}_A(P, \cdot)}. \tag{2.16}$$

Proof. Here we take $\mathfrak{s}_1 = \mathfrak{s}_2^*$ to be the labelled set $S^- = \{A\}$ and $S^+ = \emptyset$ while \mathfrak{s}_3 is just the empty labelled set. We take $P_1 = Q$ to be any A module and $P_2 = P, P_3 = \mathbb{K}$. Then the filtered adjunction (2.15) reads

$$\text{Hom}_{\mathbb{K}}(Q \overset{\infty}{\otimes}_A P, \mathbb{K}) \simeq \text{Hom}_A(P, \text{Hom}_{\mathbb{K}}(Q, \mathbb{K})).$$

By universal coefficients, the left hand side has length equal to $\ell(Q \overset{\infty}{\otimes}_A P)$. The right hand side has length $\ell(\text{Hom}_A(P, \text{Hom}_{\mathbb{K}}(Q, \mathbb{K})))$. As the Q is arbitrary, we have then that the supremum $\ell^{-\infty}_{\otimes_A P}$ is less than or equal to the supremum $\ell^{\text{Hom}_A(P, \cdot)}$ verifying the claim. \square

For formal algebras concentrated in degree zero, the above corollary is the elementary fact that flat dimension is less than or equal to projective dimension. We note that for arbitrary (formal and non-formal) algebras A , it is not the case that all left modules are quasi-isomorphic to $\text{Hom}_{\mathbb{K}}(Q, \mathbb{K})$ for some Q , so just as in the formal setting, this inequality can be strict. We will observe conditions for which this inequality is an equality below.

The dual P^\vee of an \mathfrak{s} polymodule P is the \mathfrak{s}^* polymodule $\text{Hom}_{\mathbb{K}}(P, \mathbb{K})$. We start with an elementary lemma for perfect polymodules over compact algebras.

Proposition 2.18. *Suppose \mathfrak{s} labels compact algebras. Then*

$${}^\vee : \mathfrak{s}\text{-mod}_\infty \rightarrow \mathfrak{s}^*\text{-mod}_\infty$$

is an equivalence of categories and there is a natural isomorphism $\Theta : I \rightarrow (I^\vee)^\vee$.

Proof. We prove this for the case of \mathfrak{s} labelling a single compact algebra A as the general case is the same. Every perfect A module P has a finite dimensional minimal model P_{\min} defined uniquely up to isomorphism. Thus there is the usual graded vector space natural isomorphism $\Theta_{gr} : P_{\min} \rightarrow (P_{\min}^\vee)^\vee$ defined in the usual way $[\Theta_{gr}(p)](l) = (-1)^{|l||p|}l(p)$. It is immediate from the definition of internal hom that $\Theta := \Theta_{gr}$ is indeed a strict A -module homomorphism. \square

To generalize this proposition, we fix gluing data \mathfrak{t} between \mathfrak{s}_1^* and \mathfrak{s}_2 . The following proposition, which was observed early in homological algebra, is stated below in terms of polymodules.

Proposition 2.19. *Suppose P_i is a \mathfrak{s}_i polymodule and P_2 is a perfect \mathfrak{s}_2 polymodule. If the algebras labelled by $\mathfrak{s}_2 - i_2(\mathfrak{s}_0)$ are compact, then there is a natural filtered quasi-equivalence*

$$P_1^\vee \overset{\infty}{\otimes}_{\mathfrak{s}_0} P_2 \simeq \mathcal{H}om_{\mathfrak{s}_0}(P_2, P_1)^\vee.$$

Proof. First we define a morphism $\Psi : P_1^\vee \overset{\infty}{\otimes}_{\mathfrak{s}_0} P_2 \rightarrow \mathcal{H}om_{\mathfrak{s}_0}(P_2, P_1)^\vee$ of filtered $\mathfrak{s}_1^* \#_{\mathfrak{s}_0} \mathfrak{s}_2$ polymodules by

$$\begin{aligned} & [\Psi(\mathbf{a} \otimes \phi \otimes \mathbf{b} \otimes p \otimes \mathbf{c})](\psi) \\ &= (-1)^{|\psi|(|p|+|\mathbf{b}|+|\mathbf{c}|)+|\phi||\mathbf{a}|} \phi(\mu_{\mathcal{H}om_{\mathfrak{s}_0}(P_2, P_1)}(\mathbf{a} \otimes \psi \otimes \mathbf{c})(\mathbf{b} \otimes p)). \end{aligned}$$

It is plain to see that Ψ preserves the lattice filtrations and that Ψ is a natural transformation.

Now we check to see that Ψ is a quasi-isomorphism for $P_2 = \mathbf{U}_{\mathfrak{s}_2}$. Write \mathfrak{s}'_2 for $\mathfrak{s}_2 - i_2(\mathfrak{s}_0)$ and note that $\mathbf{U}_{\mathfrak{s}_2} = \mathbf{U}_{\mathfrak{s}_0} \otimes \mathbf{U}_{\mathfrak{s}'_2}$. By choosing minimal models for the algebras labelled by \mathfrak{s}'_2 we may assume $\mathbf{U}_{\mathfrak{s}'_2}$ is a finite dimensional vector space over \mathbb{K} . This gives

$$P_1^\vee \overset{\infty}{\otimes}_{\mathfrak{s}_0} \mathbf{U}_{\mathfrak{s}_2} = (P_1^\vee \overset{\infty}{\otimes}_{\mathfrak{s}_0} \mathbf{U}_{\mathfrak{s}_0}) \otimes \mathbf{U}_{\mathfrak{s}'_2}.$$

While on the other side we obtain

$$\begin{aligned} \mathcal{H}om_{\mathfrak{s}_0}(\mathbf{U}_{\mathfrak{s}_2}, P_1)^\vee &= \mathcal{H}om_{\mathfrak{s}_0}(\mathbf{U}_{\mathfrak{s}_0}, P_1)^\vee \otimes (\mathbf{U}_{\mathfrak{s}'_2}^\vee)^\vee, \\ &= \mathcal{H}om_{\mathfrak{s}_0}(\mathbf{U}_{\mathfrak{s}_0}, P_1)^\vee \otimes \mathbf{U}_{\mathfrak{s}'_2}, \end{aligned}$$

where the last equality follows from the compactness assumption. It is easy to see that Ψ factors through this tensor decomposition of $\mathbf{U}_{\mathfrak{s}_2}$, so we may cancel the $\mathbf{U}_{\mathfrak{s}'_2}$ factor and show the equivalence on the $\mathbf{U}_{\mathfrak{s}_0}$ factor.

For this, observe that the tensor product and internal Hom with $P_2 = \mathbf{U}_{\mathfrak{s}_0}$ yields the same complex as $P_2 = \mathbf{D}_{\mathfrak{s}_0}$ we restrict ξ to obtain the quasi-commutative diagram in Fig. 2.

By exactness of $P_1^\vee \overset{\infty}{\otimes}_{\mathfrak{s}_0}$ and $\mathcal{H}om_{\mathfrak{s}_0}(-, P_1)^\vee$ and naturality of Ψ , we have that Ψ induces a quasi-isomorphism on perfect \mathfrak{s}_2 polymodules. As was observed above, Ψ respects filtrations which yields the claim. \square

As a corollary, we have the following important fact

Corollary 2.20. *Suppose $A \in \text{Alg}_\infty$ and $P \in A\text{-mod}_\infty$. Then*

$$\ell^{\text{Hom}(P, \cdot)} = \ell^{-\overset{\infty}{\otimes}} P.$$

This equality motivates the following definition.

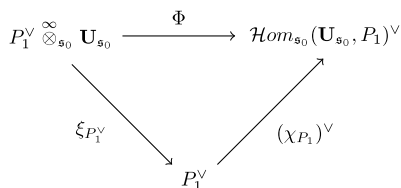


Fig. 2. Quasi-commutative diagram.

Definition 2.21. For $P \in A\text{-mod}_\infty$ we define the length of P to be

$$\ell(P) := \ell^{\text{Hom}(P, -)} = \ell^{\otimes P},$$

and define the global length of $A\text{-mod}_\infty$ to be the supremum

$$\ell(A\text{-mod}_\infty) := \sup\{\ell(P) : P \in A\text{-mod}_\infty\}.$$

3. Dimensions of A_∞ categories

In this section we lift many of the definitions and theorems of the dimension theory for triangulated categories to the pretriangulated setting. After recalling some definitions and results on triangulated and pretriangulated categories from [1,13,16,17], we prove our first main theorem that equates filtered length of internal homs with the generation time of a given object. We follow this with a proof of the base change formula for A_∞ -algebras.

3.1. Generators in triangulated categories

We take a moment to recall some definitions and notation from [17]. Given a triangulated category \mathcal{C} and a subcategory \mathcal{I} , we define $\langle \mathcal{I} \rangle$ to be the smallest full subcategory of \mathcal{C} closed under direct summands, finite direct sums and shifts. Given two subcategories $\mathcal{I}_1, \mathcal{I}_2 \subset \mathcal{T}$, we define $\mathcal{I}_1 * \mathcal{I}_2$ to be the category of objects N such that there exists a distinguished triangle

$$M_1 \rightarrow N \rightarrow M_2 \rightarrow$$

in \mathcal{T} with $M_1 \in \mathcal{I}_1$ and $M_2 \in \mathcal{I}_2$. We take $\mathcal{I}_1 \diamond \mathcal{I}_2 := \langle \mathcal{I}_1 * \mathcal{I}_2 \rangle$. It follows from the octahedral axiom that \diamond is an associative operation, so the category $\mathcal{I}^{\diamond d}$ is well defined. With this notation in hand, the following definitions can be stated.

Definition 3.1. Let $\mathcal{T} \subseteq \mathcal{C}$.

- (i) \mathcal{I} generates \mathcal{T} if given $N \in \mathcal{T}$ with $\text{Hom}_{\mathcal{T}}(M[i], N) = 0$ for all $M \in \mathcal{I}$ and all $i \in \mathbb{Z}$, then $N = 0$.
- (ii) \mathcal{I} is a d -step generator of \mathcal{T} if $\mathcal{T} = \mathcal{I}^{\diamond d}$.
- (iii) \mathcal{T} is finitely generated if there exists $G \in \mathcal{T}$ which generates \mathcal{T} . In this case we call G a generator for \mathcal{T} .
- (iv) \mathcal{T} is strongly finitely generated if there exists $M \in \mathcal{T}$ which is a d -step generator.

We utilize the above definitions to define level and dimension as follows.

Definition 3.2. If G generates \mathcal{T} and $M \in \mathcal{T}$ we say the level of M with respect to G is

$$\text{lvl}_G(M) = \min \left\{ d : M \in \langle G^{\diamond(d-1)} \rangle \right\}$$

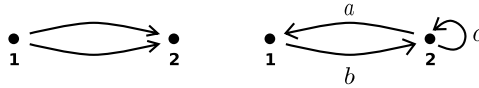


Fig. 3. Quivers of generators for $\mathcal{D}^b(\mathbb{P}^1)$.

and the generation time of G is

$$t(G) = \min \left\{ d : \mathcal{T} = \left\langle G^{\otimes(d-1)} \right\rangle \right\} = \max \{ \text{lvl}_G(M) : M \in \mathcal{T} \}.$$

The dimension of a category \mathcal{T} with generators is defined to be the smallest generation time. The Orlov spectrum of \mathcal{T} is the set of all generation times.

The central theme of this paper is to enhance the above definitions into the language of DG and A_∞ -categories. Thus we will assume our category \mathcal{T} is always a subcategory of the homotopy category $H^0(\mathcal{A})$ for some pretriangulated A_∞ -category \mathcal{A} . If \mathcal{T} is an algebraic triangulated category, this is implied by a theorem of Lefèvre-Hasegawa which we cite below. Recall that a triangulated category is algebraic if it is the stable category of a k -linear Frobenius category (see [11]). First we fix notation and, in a triangulated category \mathcal{T} , write $\text{Hom}_{\mathcal{T}}^*(M, N)$ for the algebra $\bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\mathcal{T}}(M, N[n])$.

Theorem 3.3 (7.6.0.4, [13]). *If \mathcal{T} is an algebraic triangulated category which is strongly generated by an object G , then there is an A_∞ structure on $A_G := \text{Hom}_{\mathcal{T}}^*(G, G)$ such that the Yoneda functor evaluated at G from \mathcal{T} to $H^0(A_G\text{-mod}_\infty)$ is a triangulated equivalence.*

We say that a pretriangulated A_∞ -subcategory \mathcal{B} (strongly) generates if $H^0(\mathcal{B})$ does in $H^0(\mathcal{A})$. We also use the same language and notation as above for level, generation time and dimension.

Before proceeding with this discussion, we take a moment to illustrate this theorem with some examples.

Example 3.4. For \mathbb{P}^n , Beilinson showed [2] that $\langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n) \rangle$ forms a full exceptional collection for $\mathcal{D}^b(\mathbb{P}^n)$. Taking $G = \bigoplus_{i=0}^n \mathcal{O}(i)$ then gives a generator. From grading considerations, the endomorphism algebra A_G has no higher products so $\mathcal{D}^b(\mathbb{P}^n) \simeq H^0(A_G\text{-mod}_\infty)$. In the case of $n = 1$, A_G is the path algebra of the Kronecker quiver illustrated in Fig. 3.

Exceptional collections in the dimension theory of triangulated categories were studied in [1]. In general, one can mutate an exceptional collection to obtain a new exceptional collection and thereby a new generator. Below we examine a generator which does not arise from such mutations and in fact is not defined as the direct sum of objects in an exceptional collection.

Example 3.5. Let $n = 1$ in the previous example and let $G' = \mathcal{O} \oplus \mathcal{O}_p$ where $p \in \mathbb{P}^1$ is any point. The algebra $A_{G'}$ is the quiver algebra with relations given in the middle of Fig. 3 where $\text{deg}(a) = 1 = \text{deg}(c)$ and $\text{deg}(b) = 0$ and $ba = c$. Here, the grading does not preclude the existence of higher products, but it is not hard to exhibit a quasi-isomorphism from this algebra to the DG algebra endomorphism algebra of the mutated objects in $A_G\text{-mod}_\infty$. Again we obtain the isomorphism $\mathcal{D}^b(\mathbb{P}^1) \simeq H^0(A_{G'}\text{-mod}_\infty)$ from coherent sheaves to graded modules over the graded algebra $A_{G'}$.

Example 3.6. Another studied example is the category of matrix factorizations for the function $f_n : \mathbb{C} \rightarrow \mathbb{C}$ via $f_n(z) = z^n$, or equivalently the derived category of singularities $D_{sg}^b(f_n^{-1}(0))$.

It was observed in [1] that every non-zero object of $MF(\mathbb{C}[[z]], f_n)$ is a strong generator and that the generator

$$\left[\mathbb{C} \begin{array}{c} \xrightarrow{z^{n-1}} \\ \xleftarrow{z} \end{array} \mathbb{C} \right] \in MF(\mathbb{C}[[z]], f_n)$$

which is equivalent to $\mathcal{O}_0 \in D_{sg}^b(f_n^{-1}(0))$ had maximal generation time. Also, in [8], the computation of a minimal model for A_G as a $\mathbb{Z}/2\mathbb{Z}$ graded A_∞ -algebra was performed and found to equal $A_G = k[\theta]/(\theta^2)$ where $\deg(\theta) = 1$ and all higher products vanish except $\mu^n(\theta, \theta, \dots, \theta) = 1$. Again, we have $D_{sg}^b(f_n^{-1}(0)) \simeq H^0(A_G\text{-mod}_\infty)$.

Now, starting with an A_∞ pretriangulated category \mathcal{A} , let $G \in \mathcal{A}$ be a generator and $A_G = \text{Hom}^*(G, G)$ its A_∞ endomorphism algebra. We define the A_∞ -functor $\text{ev}_G : \mathcal{A} \rightarrow A_G\text{-Mod}_\infty$ via

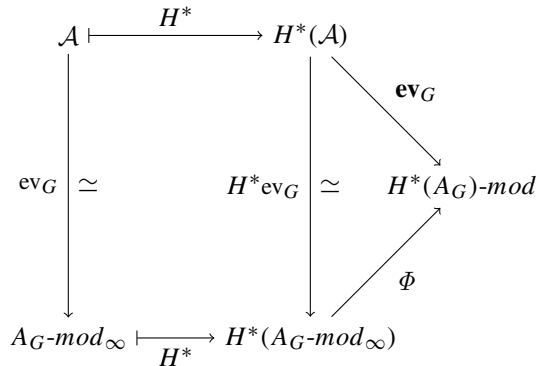
$$\text{ev}_G(B) = \text{Hom}^*(G, B).$$

The map T^1 on morphisms is composition and T^k is defined using higher multiplication. Clearly, ev_G factors through the Yoneda embedding and can be thought of as evaluation of Yoneda at the point G . From Theorem 3.3, if G is a strong generator, we have that ev_G is an equivalence with $A_G\text{-mod}_\infty$. In particular, given any two objects, $M, N \in \mathcal{A}$, the associated map

$$H^*\text{ev}_G : H^*(\text{Hom}^*(M, N)) \rightarrow H^*(\text{Hom}^*(\text{ev}_G M, \text{ev}_G N)) \tag{3.1}$$

is an isomorphism. In a moment, we will examine the right hand side of Eq. (3.1).

Were we to have started out in the triangulated setting, we could have defined the functor $\text{ev}_G : H^*\mathcal{A} \rightarrow H^*(A_G)\text{-mod}$. It is well known that the natural functor $\Phi : H^*(A_G\text{-mod}_\infty) \rightarrow H^*(A_G)\text{-mod}$ is not an equivalence of categories. However, all of these categories and functors fit into the diagram of categories below.



The kernel (i.e. all morphisms sent to zero) of ev_G is defined to be the G -ghost ideal. We write this ideal as \mathcal{G}_G and its n -th power to be \mathcal{G}_G^n . The following lemma will be fundamental for what follows and can be found in [1].

Lemma 3.7 (The Ghost Lemma). *If \mathcal{T} is an algebraic triangulated category with strong generator G such that A_G is compact. Then $M \in \langle G^{\odot(d)} \rangle$ and $M \notin \langle G^{\odot(d-1)} \rangle$ if and only if there exists an $N \in \mathcal{T}$ such that $\mathcal{G}_G^d \text{Hom}^*(M, N) \neq 0$.*

An important conceptual point about this perspective is that, by choosing a generating object G , the homotopy category of \mathcal{A} has been enhanced to a filtered category. This is not an invariant of the A_∞ -category \mathcal{A} , nor is it an invariant of the triangulated category $H^*\mathcal{A}$. It is an additional structure introduced by the choice of generator which provides homological information relative to G .

3.2. Ghosts and length

We will now establish the link between generation time and filtration length. The following lemma is straightforward, but we supply a proof to establish some notation.

Lemma 3.8. *In $A\text{-Mod}_\infty$ we have $\ell(\mathbf{U}_A) = 0$.*

Before we begin the proof, we set up the more general comodule notation and define a weaker class of maps $\text{Map}_C^k(P, P')$ between two DG comodules of a coalgebra C . First, given a DG coalgebra $C \in \text{Cog}$ and DG comodule P , we have a canonical filtration on P given by

$$P_{[k]} := \ker(\overline{\Delta}_P^k).$$

The analogue of the length filtration in (2.11) for DG comodules is then

$$\mathcal{F}^i \text{Hom}(P, P') = \{\phi : \phi(P_{[k]}) \subseteq P_{[k-i]}\}.$$

This induces a filtration F^\bullet on the cohomology of $\text{Hom}(P, P')$. We will often abuse notation and write $\phi \in F^k$ to indicate that the cohomology class $[\phi] \in F^k$. In the context of the DG category of A -modules Mod_∞ for some A_∞ algebra A , it follows from the definition that the length filtration in (2.11) equals that of the \mathcal{F}^\bullet filtration on the bar constructions

$$\mathcal{F}^\bullet \text{Hom}_{\text{Mod}_\infty}(M, N) = \mathcal{F}^\bullet \text{Hom}(\mathbf{B}M, \mathbf{B}N).$$

This filtration is on morphisms in the pretriangulated category Mod_∞ , while the filtration F^\bullet is on morphisms $H^*(\text{Hom}_{\text{Mod}_\infty}(M, N))$ or $H^0(\text{Hom}_{\text{Mod}_\infty}(M, N))$ in the derived category $H^*(\text{Mod}_\infty)$ or $H^0(\text{Mod}_\infty)$ respectively.

For any map of comodules $f : P \rightarrow P'$, we take

$$[\Delta, f] := \Delta_{P'} f - (1_C \otimes f) \Delta_P$$

and note that

$$[\Delta, fg] = (1_{C'} \otimes f \otimes 1_C)[\Delta, g] + [\Delta, f]g.$$

For $k \geq 0$, define

$$\text{Map}_C^k(P, P') = \{f : \text{image}([\Delta, f]) \subset C \otimes P'_{[k]}\}.$$

It is immediate that the vector spaces $\text{Map}_C^\bullet(P, P')$ form an increasing filtration. These classes of maps will be useful when defining homotopies. Indeed, they naturally appear in the cobar complex of morphisms from the cobar of P to the cobar of P' satisfying filtration properties on their differential in that complex. A straightforward generalization $\text{Map}_{C_0, C_1}^{k,l}(P, P')$ of the above definition to (C_0, C_1) bicomodules P and P' will also be used. We leave the elementary proof of the following properties to the reader.

- Lemma 3.9.** (i) If $f \in \text{Map}_C^k(P, P')$, $g \in \mathcal{F}^i \text{Hom}(P', P'')$ with $k \leq i$, then $gf \in \mathcal{F}^{i-k} \text{Hom}(P, P'')$.
 (ii) If $f \in \text{Map}_C^k(P, P')$ then $f(P_{[n]}) \subseteq P'_{[n+k]}$.
 (iii) If $f \in \text{Map}_C^k(P, P')$, $g \in \text{Map}_C^i(P', P'')$ then $gf \in \text{Map}_C^{k+i}(P, P'')$.
 (iv) If $f \in \text{Map}_C^k(P, P')$ then $\partial f \in \text{Map}_C^k(P, P')$.

Utilizing these properties, we proceed with the proof of Lemma 3.8.

Proof (Proof of Lemma 3.8). To prove the lemma, we define a homotopy contraction

$$h_A : \overline{\mathbf{BU}}_A \rightarrow \overline{\mathbf{BU}}_A$$

as $\sum_{m=1}^\infty 1^{\otimes m} \otimes \eta$ where η is the insertion of the identity. More concretely,

$$h_A([a_1 | \dots | a_m]) = (-1)^{|a_1| + \dots + |a_m|} [a_1 | \dots | a_m | e]$$

where $|a_i|$ is the degree of a_i in $A[1]$. A quick computation shows that indeed

$$h_A b_A + b_A h_A = 1$$

so that h_A is a vector space contracting homotopy of \mathbf{BU}_A .

Note that h_A is not a \mathcal{BA} -comodule morphism of \mathbf{BU}_A (otherwise, the entire category $A\text{-Mod}_\infty$ would be zero). Indeed, we have, for any $a \in \mathbf{BU}_A$,

$$(\Delta h_A - (1 \otimes h_A) \Delta)(a) = (-1)^{|a|} a \otimes [e]$$

This implies that $h_A \in \text{Map}_{\mathcal{BA}}^1(\mathbf{BU}_A, \mathbf{BU}_A)$. By Lemma 3.9, we have that if $\phi \in \mathcal{F}^1 \text{Hom}_{\text{Mod}_\infty}(U_A, M)$ then $b_\phi \circ h_A \in \text{Map}_{\mathcal{BA}}^0(\mathbf{BU}_A, \mathbf{BU}_A)$ is a comodule morphism. Thus, if $\phi \in \mathcal{F}^1 \text{Hom}_{\text{Mod}_\infty}(A, M)$ is a homomorphism, then $\partial(b_\phi h_A) = b_\phi \partial h_A = b_\phi$ implying that it is a boundary and therefore $F^1 \text{Hom}(U_A, M) = 0$. \square

Applying this lemma yields the following corollary.

Corollary 3.10. For any A_∞ -algebra A , $\mathcal{G}_A = F^1$.

Proof. Clearly, if $\phi : M \rightarrow N$ is in \mathcal{F}^1 , then $\phi_* : \text{Hom}_{\text{Mod}_\infty}(A, M) \rightarrow \mathcal{F}^1 \text{Hom}(A, N)$ so $[\phi]_* = 0$. Conversely, using the homotopy retract above, one sees that there exists a map

$$\text{Hom}_{\text{Mod}_\infty}(A, K) \rightarrow \text{Hom}_{\text{Mod}_\infty}(A, K) / \mathcal{F}^1 \text{Hom}_{\text{Mod}_\infty}(A, K)$$

which is natural with respect to $K \in \text{Mod}_\infty$. This induces a natural inclusion

$$\begin{aligned} \text{Hom}(A, K) &\hookrightarrow \text{Hom}_{\text{mod}}(H(A), H(K)), \\ &\simeq H(K). \end{aligned}$$

Thus if $[\phi]_* = 0$ then $[\phi^0] = 0$ implying $[\phi] \in F^1 \text{Hom}(M, N)$. \square

Indebted to the compatibility of the length filtration with composition, we also easily obtain.

Corollary 3.11. For all r we have $\mathcal{G}_A^r \subseteq F^r$.

The following theorem asserts that this inclusion is an equality.

Theorem 3.12. For any A_∞ -algebra A , $\mathcal{G}_A^r = F^r$.

Proof. We start this proof by writing down two homotopies of the diagonal (A, A) -bimodule

$$h_{diag}^{\pm} : \mathbf{B}A \rightarrow \mathbf{B}A$$

where

$$h_{diag}^+([\mathbf{a}|a|\mathbf{a}']) = (-1)^{|\mathbf{a}|+|a|}[[\mathbf{a}|a]|e|\mathbf{a}']$$

$$h_{diag}^-([\mathbf{a}|a|\mathbf{a}']) = (-1)^{|\mathbf{a}|}[\mathbf{a}|e|[a|\mathbf{a}']].$$

While these maps fail to be bicomodule morphisms, it is the case that $h_{diag}^+ \in \text{Map}_{\mathbf{B}A, \mathbf{B}A}^{1,0}(\mathbf{D}_A, \mathbf{D}_A)$ and $h_{diag}^- \in \text{Map}_{\mathbf{B}A, \mathbf{B}A}^{0,1}(\mathbf{D}_A, \mathbf{D}_A)$. Indeed, we have

$$[\Delta, h_{diag}^+](\mathbf{a}|a|\mathbf{a}') = [\mathbf{a}|a] \otimes [e] \otimes [\mathbf{a}']$$

and

$$[\Delta, h_{diag}^-](\mathbf{a}|a|\mathbf{a}') = [\mathbf{a}] \otimes [e] \otimes [a|\mathbf{a}'].$$

Furthermore, letting τ^{\pm} be the translation maps

$$\tau_+(\mathbf{a}|a|[a'_1] \cdots |a'_m]) = (-1)^{1+|\mathbf{a}|+|a|}[[\mathbf{a}|a]|a'_1|[a'_2] \cdots |a'_m]],$$

$$\tau_-([a_1] \cdots |a_n]|a|\mathbf{a}') = (-1)^{1+|a_1|+\cdots+|a_{n-1}|}[[a_1] \cdots |a_{n-1}||a_n|[a|\mathbf{a}']],$$

our homotopies bound to

$$\partial h_{diag}^{\pm} = 1 - \tau_{\pm}.$$

Thus $\tau_- \in \text{Map}_{\mathbf{B}A, \mathbf{B}A}^{0,1}(\mathbf{D}_A, \mathbf{D}_A)$ by part (iv) of Lemma 3.9. More generally, we have

$$\partial \left[h_{diag}^{\pm} (1 + \tau_{\pm} + \tau_{\pm}^2 + \cdots + \tau_{\pm}^{k-1}) \right] = 1 - \tau_{\pm}^k.$$

We observe that, from the fact that $\text{Map}^{0,\bullet}(\mathbf{D}_A, \mathbf{D}_A)$ is an increasing filtration, and by part (iii) of Lemma 3.9,

$$\sigma_k^- := h_{diag}^{\pm} (1 + \tau_{\pm} + \tau_{\pm}^2 + \cdots + \tau_{\pm}^{k-1}) \in \text{Map}_{(\mathbf{B}A, \mathbf{B}A)}^{0,k}(\mathbf{B}D_A, \mathbf{B}D_A). \tag{3.2}$$

Finally, we note that for any l , as a map in Ch the translation map satisfies

$$\tau_-^k(\mathbf{B}_{(k,l)}\mathbf{D}_A) = 0. \tag{3.3}$$

We now use induction to prove our theorem. It suffices to show that if $\phi \in \mathcal{F}^r \text{Hom}_{\text{Mod}_{\infty}}(M, N)$, then there exists a module K and homomorphisms $\pi : M \rightarrow K, \psi : K \rightarrow N$ such that $\pi \in F^1, \psi \in F^{r-1}$ and $\phi = \psi \circ \pi$. Recall from Eq. (2.13) for an $(A|A)$ bimodule P_1 and a left A module P_2 , the module $P_1 \circlearrowleft_k P_2$ is defined as the quotient

$$\frac{P_1 \otimes^{\infty} P_2}{P_1 \otimes^{[k]} P_2}.$$

We consider the diagram in Fig. 4 which is commutative up to homotopy. The map $\epsilon_{l,M}$ was defined in Eq. (2.14) and is a quasi-isomorphism. In particular, a simple examination of the map

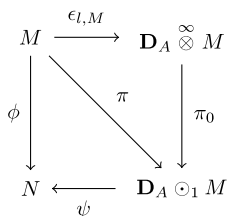


Fig. 4. Diagram factoring ϕ .

shows that

$$\left(\mathbf{D}_A \overset{\infty}{\otimes} M, \left\{ \mathbf{D}_A \otimes^{[n-1]} M \right\}_{n \in \mathbb{Z}} \right)$$

has length 0 as a filtered module. We note that the proof of Lemma 3.19 is independent of the results of this section and apply it here. Taking $p = 0$, $N = \mathbf{D}_A \overset{\infty}{\otimes} M$, $N_t = \mathbf{D}_A \otimes^{[t]} M$ and $n = 0$, Lemma 3.19 implies that $\pi \in F^1$.

On the other hand, as ψ is the restriction of $\xi_{l,M} \circ (1 \otimes \phi)$, we can write it out concretely. It is a strict map whose restriction to $A \otimes A[1]^{\otimes n} \otimes M$ is

$$\psi_n^0([a|a_1| \cdots |a_n|m]) = \sum_{i=0}^n (-1)^{|a_1| + \cdots + |a_i|} \mu_N^{i+1}([a|a_1| \cdots |a_i| \phi^{n-i}([a_i| \cdots |a_n|m])])$$

for $n > 1$. As $\phi \in F^r$, we see in particular that $\psi_n^0 = 0$ for $n \leq r$. Thus ψ factors as a composition

$$A \circlearrowleft M \xrightarrow{\pi} A \circlearrowleft_r M \xrightarrow{\tilde{\psi}} N$$

where $\tilde{\psi}$ is a strict homomorphism. Utilizing equation (3.2), a direct calculation shows that $\sigma_{r-1} \otimes 1_M : \mathbf{B}\mathbf{D}_A \otimes_{\mathbb{K}} \mathbf{B}M \rightarrow \mathbf{B}\mathbf{D}_A \otimes_{\mathbb{K}} \mathbf{B}M$ restricts to a well defined A_{∞} -module morphism

$$\sigma_{r-1}^- \circlearrowleft 1_M : A \circlearrowleft M \rightarrow A \circlearrowleft_r M.$$

Composing with $\tilde{\psi}$ and applying the differential gives

$$\begin{aligned}
 \partial[(-1)^{|\psi|} \tilde{\psi} \circ (\sigma_{r-1}^- \circlearrowleft 1_M)] &= \tilde{\psi} \circ ((\partial \sigma_{r-1}^-) \circlearrowleft 1_M) \\
 &= \tilde{\psi} \circ ((1_A - \tau_{r-1}^{r-1}) \circlearrowleft 1_M) \\
 &= \tilde{\psi} \circ (1_A \circlearrowleft 1_M) - \tilde{\psi} \circ (\tau_{r-1}^{r-1} \circlearrowleft 1_M) \\
 &= \tilde{\psi} \circ \pi - \tilde{\psi} \circ (\tau_{r-1}^{r-1} \circlearrowleft 1_M) \\
 &= \psi - \tilde{\psi} \circ (\tau_{r-1}^{r-1} \circlearrowleft 1_M).
 \end{aligned}$$

Thus ψ is cohomologous to $\tilde{\psi} \circ (\tau_{r-1}^{r-1} \circlearrowleft 1_M)$. Yet, by Eq. (3.3) we have that

$$(\tau_{r-1}^{r-1} \circlearrowleft 1_M) (\mathbf{B}_{(r-1,0)} \mathbf{D}_A \otimes_{\mathbb{K}} \mathbf{B}M) = 0$$

and since $\tilde{\psi}$ is strict, this implies that

$$\tilde{\psi} \circ (\tau_{r-1}^{r-1} \circlearrowleft 1_M) (\mathbf{B}_{(r-1,0)} \mathbf{D}_A \otimes_{\mathbb{K}} \mathbf{B}M) = 0.$$

Thus, $\psi \simeq \tilde{\psi} \circ (\tau_{r-1}^{r-1} \circlearrowleft 1_M) \in F^{r-1}$. \square

Combining this theorem with the Ghost Lemma of the previous section, we have the following homological criteria for generation time.

Corollary 3.13. *Given an A_∞ -algebra A , the generation time of an A_∞ -module U_A in $H^0(A\text{-Mod}_\infty)$ is the global length $\ell(A\text{-mod}_\infty)$.*

Coupling this to the theory of enhanced triangulated categories, we also obtain the corollary below.

Corollary 3.14. *If \mathcal{A} is a pretriangulated A_∞ -category and $G \in \mathcal{A}$ is a generator, then $t(G) = \ell(A_G\text{-Mod}_\infty)$.*

More refined statements on the level $\text{lvl}_G(M)$ of an object with respect to a given generator G are also of use. We write the result in the A_∞ -module category as opposed to concentrating on the A_G -module case.

Corollary 3.15. *If M is an A -module then $\text{lvl}_A(M) = \ell(M)$.*

Example 3.16. As was mentioned at the end of Section 2.1, when A_G is an ordinary algebra, the global length of $\ell(A_G\text{-Mod}_\infty)$ is precisely its homological dimension. For the cases of the Beilinson exceptional collection $\langle \mathcal{O}, \dots, \mathcal{O}(n) \rangle$, one may use Beilinson’s resolution of the diagonal to see that this dimension is n .

Example 3.17. For the generator $\mathcal{O} \oplus \mathcal{O}_p$ of \mathbb{P}^1 , we again have formality, but $A_{G'}$ is now a graded algebra. Viewing G' as a quiver with relations whose vertices correspond to \mathcal{O} and \mathcal{O}_p , one observes that the graded simple modules S_1 and S_2 arise from considering the idempotents at the vertices while the graded projective modules P_1, P_2 from considering all arrows mapping out of each vertex. The projective resolutions below for the simple objects give the homological dimension of $A_{G'}$ as 2.

$$\begin{aligned} \dots 0 \rightarrow P_1 \rightarrow P_2 \rightarrow P_1 \rightarrow S_1 \rightarrow 0 \\ \dots 0 \rightarrow P_1 \rightarrow P_2 \rightarrow S_2 \rightarrow 0. \end{aligned}$$

The final example explores a case where higher products have a significant effect on generation time.

Example 3.18. From Example 3.6, we recalled that $MF(\mathbb{C}[[z]], z^n)$ had a generator G with $A_G = k[\theta]/(\theta^2)$ with a single higher product $\mu^n(\theta, \dots, \theta) = 1$. To describe $H^0(A_G\text{-mod}_\infty)$, we examine the A_∞ -relation for the products of a minimal A_G -module M . First, we recall that M is $\mathbb{Z}/2\mathbb{Z}$ graded and the usual A_∞ -module map $\mu_M^r : A_G^r \otimes M \rightarrow M$ is degree $r + 1$ (due to the desuspension of A_G). Since we assume M is unital, μ_M^r is completely determined by $\mu_M^r([\theta] \cdots [\theta|m])$. Writing $L_r = \mu_M^r([\theta] \cdots [\theta|-]) \in \text{Hom}_{gr}^1(M, M)$, we may condense μ_M into a power series $\mathbf{L} = \sum_{r=1}^\infty L_r u^r \in \text{Hom}_{gr}^1(M, M) \otimes \mathbb{C}[[u]]$. It is not hard to see that the A_∞ -relation on μ_M^r translates into the equality

$$\mathbf{L} \cdot \mathbf{L} = 1_M \cdot u^n.$$

Decomposing M into its graded summands $M = M_0 \oplus M_1$, we may split $L_r = L_r^0 \oplus L_r^1$ where $L_r^0 : M_0 \rightarrow M_1$ and $L_r^1 : M_1 \rightarrow M_0$. Summing, we write $\mathbf{L}^i = \sum_{r=1}^\infty L_r^i u^r$ and after tensoring

M with $\mathbb{C}[[u]]$ we then have

$$M_0 \otimes \mathbb{C}[[u]] \begin{array}{c} \xrightarrow{\mathbf{L}^0} \\ \xleftarrow{\mathbf{L}^1} \end{array} M_1 \otimes \mathbb{C}[[u]]$$

with $\mathbf{L}^0 \mathbf{L}^1 = u^n = \mathbf{L}^1 \mathbf{L}^0$. This returns us full circle to the setting of matrix factorizations, but with the added presence of the length filtration. Indeed, as above, given another A_G module $(N, \tilde{\mathbf{L}})$ we may write any morphism $\phi : M \rightarrow N$ as a power series $\mathbf{T} = \sum_{r=0}^{\infty} T_r u^r \in \text{Hom}_{gr}^*(M, N) \otimes \mathbb{C}[[u]]$ where $T_r(m) = \phi(|\theta| \cdots |\theta|m)$. The differential on $\text{Hom}_{mod\infty}^*(M, N)$ is the usual matrix factorization differential $d\mathbf{T} = \tilde{\mathbf{L}}\mathbf{T} - (-1)^{|\mathbf{T}|}\mathbf{T}\mathbf{L}$. It is obvious from this representation that $\phi \in \mathcal{F}^k \text{Hom}_{mod\infty}(M, N)$ if and only if $\text{deg}(\mathbf{T}) \geq k$.

For $1 \leq m \leq \lfloor \frac{n}{2} \rfloor$, and define M_m to be the module corresponding to

$$\mathbb{C}[[u]] \begin{array}{c} \xrightarrow{u^m} \\ \xleftarrow{u^{n-m}} \end{array} \mathbb{C}[[u]]$$

These make up the irreducible modules. One can show that the maximal filtered homomorphism between any two such modules is attained by $\phi : M_m \rightarrow M_m$, for $m = \lfloor \frac{n}{2} \rfloor$, and ϕ corresponding to $\mathbf{T} = u^{m-1}$. This implies the generation time of G is $\text{deg}(\mathbf{T}) = \lfloor \frac{n}{2} \rfloor - 1$ in agreement with results in [1].

The last example raises interesting questions on which filtrations arise as length filtrations on the category of matrix factorizations. In the above example, we obtained the \mathfrak{m} -adic filtration on matrices by considering the generator R/\mathfrak{m} where $R = \mathbb{C}[[u]]$ and $\mathfrak{m} = (u)$. It is natural then to hope that a similar phenomena occurs in general. In particular, take $W \in \mathbb{C}[[x_1, \dots, x_n]] = R$ to have an isolated singularity at 0, $I \triangleleft R$ an ideal and $G \in MF(R, W)$ is the generator corresponding to $R/I \in D_{sg}^b(W^{-1}(0))$. Given $M, N \in MF(R, W)$, we conjecture that the length filtration on $\text{Hom}(M, N)$ equals the I -adic filtration. One obstruction to proving this as we did for the case of z^n is that the application of the homological perturbation lemma to the Yoneda algebra of R/I involves very complicated computations. However, it is not hard to observe that the I -adic filtration refines the ghost filtration.

3.3. Change of base formula

In this subsection we generalize the classical change of base formula for dimension to the case of dimensions of A_∞ -algebras. We see that a new multiplicative factor appears in this formula that measures the formality of the algebras involved.

We start by obtaining a general lemma on filtered A_∞ -modules. To simplify the exposition and some proofs, we will work with modules as opposed to polymodules. Suppose M is an A -module and $(N, \mathcal{G}^*) \in (A\text{-mod}_\infty)^f$ is a filtered A -module of finite filtration length and $\phi \in \text{Hom}_{A\text{-Mod}_\infty}(M, N)$ is any map. We wish to obtain a finite approximation of ϕ relative to both the internal filtration on N and the filtration on $\text{Hom}_{A\text{-Mod}_\infty}(M, N)$. A surprising parameter that emerges in this pursuit is the degeneration time of the spectral sequence associated to (N, \mathcal{G}^*) . For the following lemma, assume $\mathcal{G}^{-1}N = 0 \neq \mathcal{G}^0N$, let $N_t = N/\mathcal{G}^t N$ and $\pi_t : N \rightarrow N_t$ be the projection.

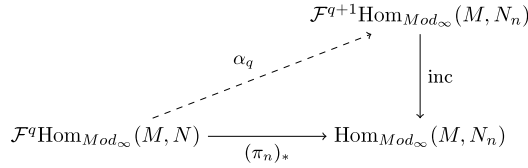
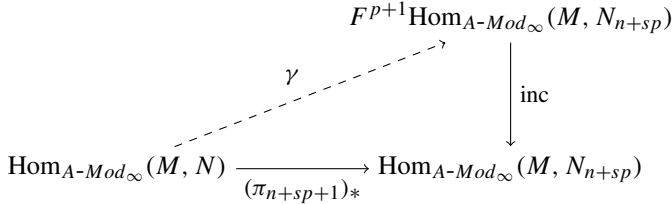


Fig. 5. Diagram factoring $(\pi_n)_*$.

Lemma 3.19. *Suppose the spectral sequence associated to (N, \mathcal{G}^*) degenerates on the $(s + 1)$ -page and $\ell(N) = n$. Then for every p there exists a lift γ such that the following diagram commutes up to homotopy*



Before proving this lemma, let us set up some basic notation. Take

$$(\mathbf{B}M)^q := A[1]^{\otimes q} \otimes M \subset \mathbf{B}M$$

and define

$$\rho_q : \mathcal{F}^q\mathrm{Hom}_{\mathrm{Mod}_\infty}(M, N) \rightarrow \mathrm{Hom}_{\mathrm{Ch}}((\mathbf{B}M)^q, N) = \mathrm{Hom}_{\mathrm{Ch}}(A[1]^{\otimes q} \otimes M, N)$$

to be the restriction map. Here, the right hand side is the complex of morphisms from $((\mathbf{B}M)^q, b_M|_{(\mathbf{B}M)^q})$ to (N, μ_N^1) . It is worthwhile to note that ρ_q is a map of cochain complexes (i.e. $d\rho_q = 0$ in Ch).

Let us also introduce the general “strictification” map

$$\sigma_q : \mathrm{Hom}_{\mathrm{Ch}}(A[1]^{\otimes q} \otimes M, N) \rightarrow \mathrm{Hom}_{\mathrm{Mod}_\infty}(M, N).$$

This is the map $\sigma_q(\phi) = \{\phi^k\}$ where

$$\phi^k = \begin{cases} \phi & \text{if } k = q \\ 0 & \text{otherwise.} \end{cases}$$

We note that this is not in general a cochain complex map. Nevertheless, it is clear that, for every q ,

$$\rho_q \circ \sigma_q = 1. \tag{3.4}$$

Proof. We start by proving the following claim.

Claim. *With the assumptions of the lemma, for every q , the map $(\pi_n)_*$ has a lift α_q which commutes up to homotopy in Fig. 5.*

We first observe that since $\mathcal{G}^{-1}N = 0 \neq \mathcal{G}^0N$ and $\ell(N) = n$ the map $\pi_n : N \rightarrow N_n$ is contractible. Thus, restricting to $(\mathbf{B}M)^q$, the induced map on chain complexes

$$(\tilde{\pi}_n)_* : \mathrm{Hom}_{\mathrm{Ch}}^*(A[1]^{\otimes q} \otimes M, N) \rightarrow \mathrm{Hom}_{\mathrm{Ch}}^*(A[1]^{\otimes q} \otimes M, N_n)$$

is also contractible. Here the differential associated to $A[1]^{\otimes q} \otimes M$ is the restriction of b_M . We use the notation of $\tilde{\pi}_n$ above in order to distinguish it from the map in the claim, but both are obtained through composition and the equation

$$\rho_q \circ (\pi_n)_* = (\tilde{\pi}_n)_* \circ \rho_q \tag{3.5}$$

holds. Let

$$\tau : \text{Hom}_{Ch}^*(A[1]^{\otimes q} \otimes M, N) \rightarrow \text{Hom}_{Ch}^{*-1}(A[1]^{\otimes q} \otimes M, N_n)$$

be a cochain bounding $(\tilde{\pi}_n)_*$ (i.e. $(\tilde{\pi}_n)_* = d\tau$ in Ch) and take

$$\alpha_q(\phi) = [(\pi_n)_* - d(\sigma_q \circ \tau \circ \rho_q)](\phi).$$

Observe that, for every ϕ , this is a cocycle by virtue of $(\pi_n)_*$ being a cochain map and the fact that df is a cochain map for any f in Ch . It is equally obvious that the diagram in Fig. 5 then commutes up to homotopy. So the only point left to prove for the claim is that any module homomorphism $\phi \in \text{Hom}_{Mod\infty}(M, N)$ has image in $\mathcal{F}^{q+1}\text{Hom}_{Mod\infty}(M, N_n)$. This is true iff $\rho_q(\alpha_q(\phi)) = 0$. Since ρ_q is a chain map, we have $\rho_q(dg) = d(\rho_q(g))$, and by Eqs. (3.4), (3.5)

$$\begin{aligned} \rho_q(\alpha_q(\phi)) &= \rho_q([(\pi_n)_* - d(\sigma_q \circ \tau \circ \rho_q)](\phi)) \\ &= \rho_q \circ (\pi_n)_*(\phi) - \rho_q[d(\sigma_q \circ \tau \circ \rho_q)(\phi)] \\ &= (\tilde{\pi}_n)_* \circ \rho_q(\phi) - d[(\rho_q \circ \sigma_q \circ \tau \circ \rho_q)(\phi)] \\ &= (\tilde{\pi}_n)_* \circ \rho_q(\phi) - d[(\tau \circ \rho_q)(\phi)] \\ &= (\tilde{\pi}_n)_* \circ \rho_q(\phi) - (d\tau) \circ \rho_q(\phi) \\ &= (\tilde{\pi}_n)_* \circ \rho_q(\phi) - (\tilde{\pi}_n)_* \circ \rho_q(\phi) \\ &= 0. \end{aligned}$$

One now uses the claim to prove the lemma by observing that if (C^*, \mathcal{G}) is any filtered chain complex whose length is r and whose spectral sequence converges at the $(p + 1)$ -th page, then $\ell(C/\mathcal{G}^r C, \mathcal{G}) \leq p$. This argument relies on simply unravelling the definition of the spectral sequence associated to a filtration. We recall that the page $E_k^q = Z_k^q/B_k^q$ is the subquotient of $\mathcal{G}^k C/\mathcal{G}^{k-1} C$ where

$$Z_k^q = \{[c] : c \in \mathcal{G}^k C, dc \in \mathcal{G}^{k-q} C\}$$

and

$$B_k^q = \{[dc] : c \in \mathcal{G}^{k+q-1} C, dc \in \mathcal{G}^k C\}.$$

Note then that E_k^{r+q} is the same as \tilde{E}_k^q for $q > p$ where the later is the spectral sequence for $(C/\mathcal{G}^r C, \mathcal{G}^{*-r})$. In particular, $\tilde{E}_k^q = 0$ for all $q > p$ implying the length $\ell(C/\mathcal{G}^r C, \mathcal{G}) \leq p$. To finish the proof, just inductively apply the claim above and this observation with (N, \mathcal{G}^*) . \square

The following theorem is a result of 3.19.

Theorem 3.20. *Let P be a (B, A) -bimodule and M a left A -module. Suppose the spectral sequence of $P \overset{\infty}{\otimes}_A M$ degenerates at the $(s + 1)$ -st page. If the convolution functor $P \overset{\infty}{\otimes}_-$ is faithful, then*

$$\text{lvl}_A(M) \leq \text{lvl}_A(P) + s \cdot \text{lvl}_B(P \overset{\infty}{\otimes}_A M)$$

Proof. Assume that this is not the case. Then there exists a nonzero morphism $f \in F^r \text{Hom}^*(M, N)$ with $r > \text{lvl}_A P^\vee + s \cdot \text{lvl}_B(P \overset{\infty}{\otimes}_A M)$. Then by definition, $1_P \overset{\infty}{\otimes} f$ is zero on $P \otimes^{[r-1]} M$ implying $1_P \overset{\infty}{\otimes} f = \psi \circ \pi_r = \pi_r^*(\psi)$ where

$$\pi_r : P \overset{\infty}{\otimes} M \rightarrow P \odot_r M.$$

Now, by assumption, the spectral sequence associated to $P \overset{\infty}{\otimes}_A M$ degenerates at $(s + 1)$ and by 2.20,

$$\ell(P \overset{\infty}{\otimes}_A M) \leq \ell(P^\vee) = \text{lvl}_A(P).$$

Letting $n = \text{lvl}_A P$, the following lifting problem is solvable for all p by Lemma 3.19.

$$\begin{array}{ccc} & & F^{p+1} \text{Hom}_{\text{Mod}_\infty}(P \overset{\infty}{\otimes} M, P \odot_{n+sp+1} M) \\ & \nearrow \gamma & \downarrow \text{inc} \\ \text{Hom}_{\text{Mod}_\infty}(P \overset{\infty}{\otimes} M, P \overset{\infty}{\otimes} M) & \xrightarrow{(\pi_{n+sp+1})^*} & \text{Hom}_{\text{Mod}_\infty}(P \overset{\infty}{\otimes} M, P \odot_{n+sp+1} M) \end{array}$$

In particular, if $p = \text{lvl}_B(P \overset{\infty}{\otimes}_A M)$ we have that $\pi_{n+sp+1} \simeq 0$. This implies that for all

$$t \geq n + sp + 1 = \text{lvl}_A P^\vee + s \cdot \text{lvl}_B(P \overset{\infty}{\otimes}_A M)$$

we must have $\pi_t \simeq 0$ so that $\pi_r \simeq 0$ and therefore $1_P \overset{\infty}{\otimes} f \simeq 0$. This contradicts the assumption that $P \overset{\infty}{\otimes} _$ is faithful. \square

We observe that in the case of formal algebras with formal modules, this theorem reproduces the classical change of base theorem in the dimension theory of rings. In more generality, it is possible to relate the constant s with matric Massey products of the algebra A and module M . This quantifies a lack of formality and ties it directly to the Orlov spectrum of a category.

Acknowledgments

Support was provided by NSF Grant DMS0600800, NSF FRG Grant DMS-0652633, FWF Grant P20778, and an ERC Grant — GEMIS. Both authors would like to thank Matt Ballard, Colin Diemer, David Favero, Maxim Kontsevich, Dima Orlov, Pranav Pandit, Tony Pantev and Paul Seidel for helpful comments and conversation during the preparation of this work. The second author would like to thank David Favero in particular for his patient explanations of results on the dimension theory of triangulated categories.

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