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Characterizing and recognizing weak visibility polygons

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Abstract

A polygon is said to be a weak visibility polygon if every point of the polygon is visible from some point of an internal segment. In this paper we derive properties of shortest paths in weak visibility polygons and present a characterization of weak visibility polygons in terms of shortest paths between vertices. These properties lead to the following efficient algorithms: (i) an $O(E)$ time algorithm for determining whether a simple polygon P is a weak visibility polygon and for computing a visibility chord if it exists, where E is the size of the visibility graph of P and (ii) an $O(n^2)$ time algorithm for computing the maximum hidden vertex set in an n-sided polygon weakly visible from a convex edge.

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1. Introduction

Visibility in geometric objects is a central issue in many computing applications including robotics [14, 26], motion planning [15, 17], vision [23, 27], graphics [4, 16], CAD/CAM [3, 6], computer-aided architecture [5, 20] and pattern recognition [1, 24]. The notion of visibility of a polygon from an internal segment arose when Avis and Toussaint [2] considered variations of the following art gallery problem: to place minimum number of stationary guards in an art gallery so that, together they can see every point in the interior of the gallery. In formal setting, the art gallery can be viewed as a simple polygon and guards as some points in the polygon. Avis and Toussaint [2] considered the case when number of guards is restricted to one but the guard is allowed to move along an edge of the polygon. Formally, this corresponds to finding an edge of the polygon such that every point in the polygon is visible from some point on the edge. Avis and Toussaint referred to visibility of a polygon from an edge as weak visibility of the polygon. A more general notion of (weak) visibility is one which allows visibility of the polygon from an internal segment, not necessary an edge. We refer to polygons, which have such an internal segment as weak visibility polygons. It is not hard to see that all polygons are not weak visibility polygons. As we see in this paper and other related work [9, 19], a weak visibility polygon has several interesting geometric properties, which allow simple and efficient algorithms for the class of weak visibility polygons.

In the recent past, the notion of shortest paths has been used to compute the region of a polygon P, weakly visible from a given segment [11, 25] or a set [8] inside P. If a polygon is weakly visible from an edge $v_k v_{k+1}$, then it is known that for any vertex v_i, the shortest path from v_k to v_i, the shortest path from v_{k+1} to v_i and the segment $v_k v_{k+1}$ form a funnel, where $v_k v_{k+1}$ is the base of the funnel and v_i is the apex of the funnel (Fig. 1). In this paper we derive properties of shortest paths in weak visibility polygons and present a characterization of weak visibility polygons in terms of shortest paths between vertices. We develop two efficient algorithms based on these properties.

In [2] Avis and Toussaint posed the following problem. Given an n-sided polygon P, determine in $o(n^2)$ time whether P is weakly visible from an edge of P. Sack and Suri [18] and Shin and Woo [22] proposed $O(n)$ time algorithms for this problem. Here we consider the following general problem. Given an n-sided polygon P, find an internal segment (if it exists) such that P is weakly visible from the segment. The best known algorithm for this problem runs in $O(n \log n)$ time and is given by Ke [13]. Here we propose an $O(E)$ time algorithm for this problem where E is the size of the visibility graph of P. The algorithm is based on the properties of shortest paths in weak visibility polygons and also uses properties for the pair of edges containing the endpoints of a visibility chord. Based completely on polygonal geometry and simple data structures, our

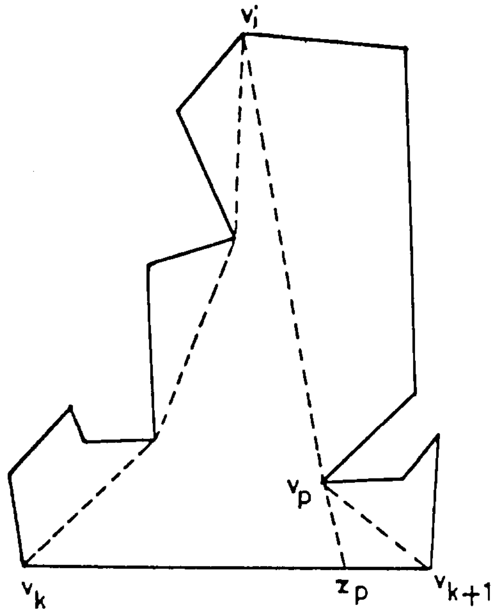

Fig. 1. $SP(v_k, v_i)$, $SP(v_{k+1}, v_k)$ and the convex edge v_kv_{k+1} form a funnel.

algorithm is conceptually simpler than the algorithm of Ke [13]. For the class of polygons where $E < o(n \log n)$, it is asymptotically better.

We also propose an $O(n^2)$ time algorithm for computing the maximum hidden vertex set in an n-sided polygon P weakly visible from a convex edge. The maximum hidden vertex set of a polygon is the maximum cardinality set of vertices such that no two vertices of the set are mutually visible. The problem of finding the maximum hidden set in an arbitrary polygon is known to be NP-hard [21]. Everett [7] has proposed linear time algorithms for spiral polygons and convex polygons with one hole.

We assume that the simple polygon P is given as a counterclockwise sequence of *vertices* $v_1, v_2, \ldots, v_n$ with their respective x and y coordinates and no three vertices of P are collinear. The symbol P is used to denote the region of the plane enclosed by P and bd(P) denotes the boundary of P. If p and q are two points on bd(P) then the counterclockwise bd(P) from p to q is denoted as bd(p, q). An edge v_iv_{i+1} of P is called a *convex* edge if both v_i and v_{i+1} are convex vertices. Two points are said to be *visible* if the segment joining them lies inside P. If the segment joining two points touches bd(P), they are still considered to be visible. The *visibility graph* of P is the graph defined with the set of vertices of P as the vertex set and the set of visible pairs of vertices of P as the edge set. A point p is said to be *weakly visible* from an edge or an internal segment st, if there is a point z in the *interior* of st such that p and z are visible. If every point in P is weakly visible from st then P is said to be *weakly visible from st*. P is called a *weak* *visibility polygon* if it is weakly visible from some internal segment. If a polygon P is weakly visible from a convex edge $v_k v_{k+1}$, we call the edge $v_k v_{k+1}$ a *convex visibility edge.* Let $\mathrm{VP}(P, v_i v_{i+1})$ denote the region of P weakly visible from $v_i v_{i+1}$ and $\mathrm{BVP}(P, v_i v_{i+1})$ denote the boundary of $\mathrm{VP}(P, v_i v_{i+1})$. Let $\mathrm{SP}(u, v)$ denote the Euclidean shortest path inside P from a point u to another point v. For any vertex u of P the *shortest path tree of P rooted at u*, denoted as $\mathrm{SPT}(u)$, is the union of the shortest paths from u to all vertices of P. Let st be an internal segment of P such that s and t belong to $\mathrm{bd}(P)$. If st cannot be extended either from s or from t without intersecting the exterior of P, we call it a *chord* of P. If a polygon P is weakly visible from a chord st, st is called a *visibility chord* of P. Given any three points $p_i = (x_i, y_i)$, $p_j = (x_j, y_j)$, and $p_k = (x_k, y_k)$, let $S = x_k(y_i - y_j) + y_k(x_j - x_i) + y_j x_i - y_i x_j$. If $S < 0$ then $p_i p_j p_k$ is a right turn. If $S > 0$ then $p_i p_j p_k$ is a left turn. If S is zero then the three points are collinear. We sometimes denote $p_i p_j p_k$ by p_i, p_j, p_k for clarity.

The paper is organized as follows. In Section 2 we derive properties of shortest paths between vertices of weak visibility polygons and present the characterization result (Theorem 2). In Section 3 we propose an $O(E)$ time algorithm for determining whether a polygon is a weak visibility polygon and for computing a visibility chord. In Section 4 we propose an $O(n^2)$ time algorithm for computing the maximum hidden vertex set in an n-sided polygon weakly visible from a convex edge. In Section 5 we conclude the paper with a few remarks.

2. Properties of shortest paths in weak visibility polygons

We begin by stating the known properties of shortest paths in weak visibility polygons [11, 25]. If a vertex v_i of P is weakly visible from a convex edge $v_k v_{k+1}$ of P, then the following properties hold (Fig. 1).

(1) $\mathrm{SP}(v_{k+1}, v_i)$ makes a right turn at every vertex in the path.

(2) $\mathrm{SP}(v_k, v_i)$ makes a left turn at every vertex in the path.

(3) $\mathrm{SP}(v_k, v_i)$ and $\mathrm{SP}(v_{k+1}, v_i)$ are two disjoint paths and they meet only at v_i.

(4) The region enclosed by $\mathrm{SP}(v_k, v_i)$, $\mathrm{SP}(v_{k+1}, v_i)$ and $v_k v_{k+1}$ is totally contained inside P.

Note that for the above properties to hold, the convexity of $v_k v_{k+1}$ is essential. There are polygons which are weakly visible from a nonconvex edge, for which the above properties do not hold (Fig. 2).

It can be seen that the above properties are of shortest paths between the vertices v_k and v_{k+1}, and any other vertex v_i of P. They do not suggest any property of the shortest path between two arbitrary vertices v_i and v_j of P. However, the shortest path between two vertices of a weak visibility polygon satisfies several nice properties as shown in the following lemmas.

Before we state the lemmas, we need some notations. For any two vertices v_i and v_j, $\mathrm{chain}(v_i, v_j)$ denotes either $\mathrm{bd}(v_i, v_j)$ or $\mathrm{bd}(v_j, v_i)$. If $\mathrm{chain}(v_i, v_j)$

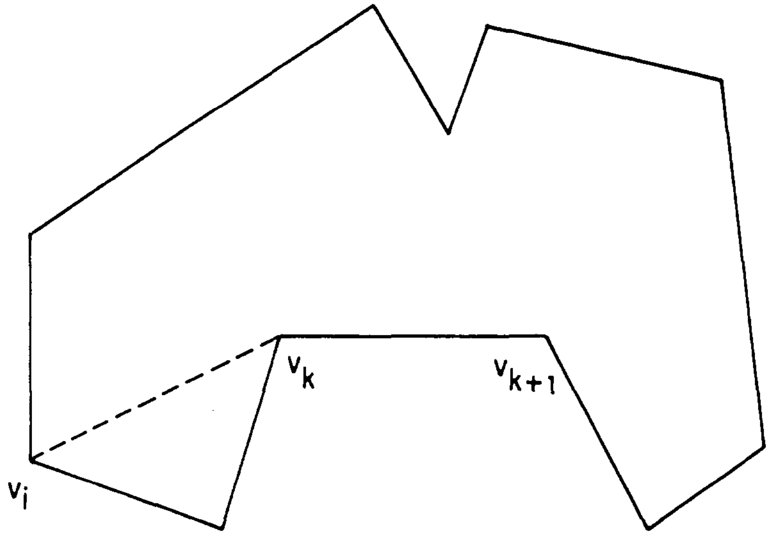

Fig. 2. P is weakly visible from a nonconvex edge $v_k v_{k+1}$.

contains an edge $v_k v_{k+1}$, we call it same chain$(v_i, v_j, v_k v_{k+1})$. If chain(v_i, v_j) does not contain $v_k v_{k+1}$, we call it opposite chain$(v_i, v_j, v_k v_{k+1})$. In Fig. 3 the vertices of same chain$(v_2, v_6, v_3 v_4)$ are v_2, v_3, v_4, v_5, v_6 and the vertices of opposite chain$(v_2, v_6, v_3 v_4)$ are v_6, v_7, v_1, v_2. We now state our lemmas.

Lemma 1. *If P is weakly visible from a convex edge $v_k v_{k+1}$, then for any two vertices v_i and v_j, all vertices of* SP(v_i, v_j) *belong to* opposite chain$(v_i, v_j, v_k v_{k+1})$.

Proof. We assume without loss of generality that v_i belongs to bd(v_{k+1}, v_j). We know that v_i and v_j are weakly visible from $v_k v_{k+1}$. Therefore v_i (respectively, v_j) is visible from some point z_i (respectively, z_j) of $v_k v_{k+1}$ (Fig. 4). Consider SP(v_i, v_j). Since v_i is visible from z_i, SP(v_i, v_j) cannot intersect $v_i z_i$. Therefore SP(v_i, v_j) cannot pass through any vertex of bd(v_{k+1}, v_i). Analogously, SP(v_i, v_j) cannot intersect $v_j z_j$ and therefore, it cannot pass through any vertex of bd(v_j, v_k). Hence SP(v_i, v_j) passes only through vertices of bd(v_i, v_j), which is opposite chain $(v_i, v_j, v_k v_{k+1})$. □

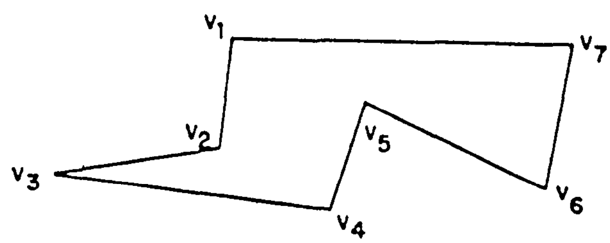

Fig. 3. Same chain$(v_2, v_6, v_3 v_4)$ = bd(v_2, v_6) and opposite chain$(v_2, v_6, v_3 v_4)$ = bd(v_6, v_2).

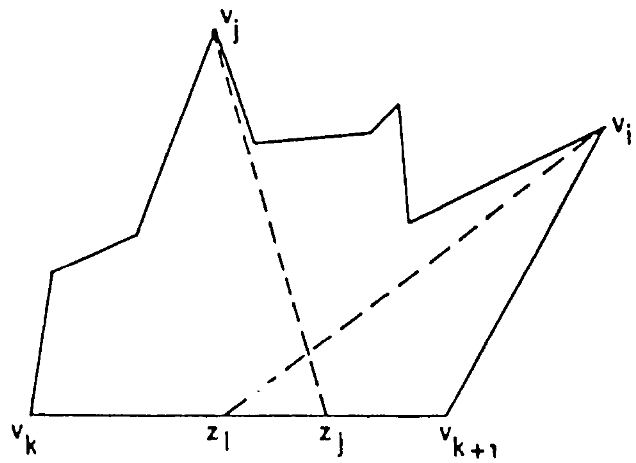

Fig. 4. Shortest path vertices lie on opposite chain.

Lemma 2. *Let $v_k v_{k+1}$ be a convex edge of a polygon P. For any two vertices v_i and v_j of P where v_i belongs to* $\mathrm{bd}(v_{k+1}, v_j)$, *if all vertices of* $\mathrm{SP}(v_i, v_j)$ *belong to opposite chain $(v_i, v_j, v_k v_{k+1})$, then* $\mathrm{SP}(v_i, v_j)$ *makes a right turn at every vertex in the path.*

Proof. Let v_i and v_j be any two vertices of P where $v_i \in \mathrm{bd}(v_{k+1}, v_j)$. We first show that $\mathrm{SP}(v_i, v_j)$ makes a right turn at every vertex in the path. We know that $\mathrm{SP}(v_i, v_j)$ passes only through vertices of opposite chain$(v_i, v_j, v_k v_{k+1})$. Assume that $\mathrm{SP}(v_i, v_j)$ makes a left turn at v_m. Consider the convex angle at v_m formed by

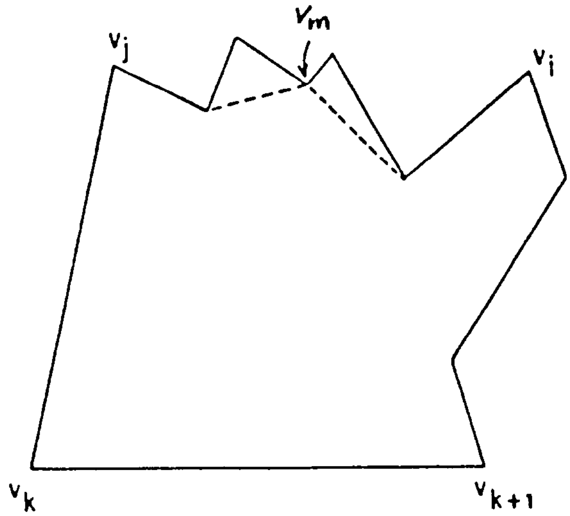

Fig. 5. The shortest path is convex.

$\mathrm{SP}(v_i, v_j)$. If the angle is *facing* toward the interior of P then by triangle inequality $\mathrm{SP}(v_i, v_j)$ does not pass through v_m, a contradiction (Fig. 5). If the angle if *facing* toward the exterior of P then v_m belongs to the same chain$(v_i, v_j, v_k v_{k+1})$, a contradiction. Therefore $\mathrm{SP}(v_i, v_j)$ makes a right turn at v_m. So $\mathrm{SP}(v_i, v_j)$ makes a right turn at every vertex in the path. □

Lemma 3. *Let $v_k v_{k+1}$ be a convex edge of a polygon P. If for every vertex v_i of P, $\mathrm{SP}(v_{k+1}, v_i)$ makes a right turn at every vertex in the path and $\mathrm{SP}(v_k, v_i)$ makes a left turn at every vertex in the path, then P is weakly visible from $v_k v_{k+1}$.*

Proof. To show that P is weakly visible from $v_k v_{k+1}$ it is enough to show that each vertex of P is weakly visible from $v_k v_{k+1}$ [2]. Consider any vertex v_i of P. Let v_p be the vertex preceding v_i on $\mathrm{SP}(v_{k+1}, v_i)$. Extend $v_i v_p$ from v_p till it meets a point z_p on $\mathrm{bd}(P)$ (Fig. 1). Since $\mathrm{SP}(v_{k+1}, v_i)$ makes a right turn at every vertex in the path, $z_p \in \mathrm{bd}(v_i, v_{k+1})$. If z_p lies on $v_k v_{k+1}$, then v_i is weakly visible from $v_k v_{k+1}$. Consider the other case when z_p does not lie on $v_k v_{k+1}$. We show that this case cannot arise. If z_p does not lie on $v_k v_{k+1}$, then $v_i z_p$ partitions P into subpolygons such that v_i and $v_k v_{k+1}$ are in different subpolygons. Since z_p, v_p and v_i are collinear, $\mathrm{SP}(v_k, v_i)$ cannot cross $v_p z_p$ and hence the last turn in $\mathrm{SP}(v_k, v_i)$ is a right turn. This contradicts the hypothesis that $\mathrm{SP}(v_k, v_i)$ makes a left turn at every vertex in the path. □

Theorem 1. *Let $v_k v_{k+1}$ be a convex edge of a polygon P. The following statements are equivalent.*

(i) *P is weakly visible from $v_k v_{k+1}$.*

(ii) *For any two vertices v_i and v_j of P where v_i belongs to $\mathrm{bd}(v_{k+1}, v_j)$, $\mathrm{SP}(v_i, v_j)$ passes only through vertices of opposite chain$(v_i, v_j, v_k v_{k+1})$.*

(iii) *For any two vertices v_i and v_j of P, where v_i belongs to $\mathrm{bd}(v_{k+1}, v_j)$, $\mathrm{SP}(v_i, v_j)$ makes a right turn at every vertex in the path.*

(iv) *For any vertex v_i of P, $\mathrm{SP}(v_{k+1}, v_i)$ makes a right turn at every vertex in the path and $\mathrm{SP}(v_k, v_i)$ makes a left turn at every vertex in the path.*

Proof. (i) implies (ii) by Lemma 1, (ii) implies (iii) by Lemma 2, (iii) implies (iv) as a special case and (iv) implies (i) by Lemma 3. □

Using Theorem 1, weak visibility polygons can be characterized in terms of shortest paths as follows.

Theorem 2. *A polygon P is a weak visibility polygon if and only if there is a chord st inside P dividing P into subpolygons P' and P'' where s precedes (respectively, succeeds) t in P' (respectively, P'') in counterclockwise order. Further, the following equivalent conditions hold for P' and analogously for P''.*

(1) *For any two vertices v_i and v_j of P' where v_i belongs to $\mathrm{bd}(t, v_j)$, $\mathrm{SP}(v_i, v_j)$ passes only through vertices of opposite chain(v_i, v_j, st).*

(2) *For any two vertices v_i and v_j of P', where v_i belongs to* $\mathrm{bd}(t, v_j)$, $\mathrm{SP}(v_i, v_j)$ *makes a right turn at every vertex in the path.*

(3) *For any vertex v_i of P',* $\mathrm{SP}(t, v_i)$ *makes a right turn at every vertex in the path and* $\mathrm{SP}(s, v_i)$ *makes a left turn at every vertex in the path.*

Proof. If P is a weak visibility polygon from a chord st, then the chord st divides P into subpolygons P' and P'' such that st is a convex edge of P' and P''. Using Theorem 1, it follows that the three equivalent conditions hold for P' and P''.

Conversely, if there is a chord st in P such that the three equivalent conditions hold for subpolygons P' and P'', then using Theorem 1, it follows that both P' and P'' are weakly visible from st. Therefore, P is weakly visible from st. □

From Theorem 1, we know that the shortest path between any two vertices in a polygon weakly visible from a convex edge has only left turns or only right turns. However, if a polygon is weakly visible from a chord, shortest paths between its vertices may contain *eaves*. An edge v_iv_j of $\mathrm{SP}(v_k, v_m)$ is an eave if $\mathrm{SP}(v_k, v_m)$ makes a right (or left) turn at v_i and makes a left (respectively, right) turn at v_j where v_i, v_j, v_k and v_m are distinct vertices.

Lemma 4. *If st is a visibility chord of a polygon, then the shortest path between two vertices in the same subpolygon of has no eaves.*

Proof. Follows from Theorem 1. □

Lemma 5. *If the shortest path between two arbitrary vertices in a weak visibility polygon has an eave, then every visibility chord of the polygon intersects the eave.*

Proof. Let uu' be an eave of $\mathrm{SP}(v_i, v_j)$. If a visibility chord st does not intersect uu' then both u and u' belong to the same subpolygon P_1 of st. If v_i and v_j belong to P_1 then $\mathrm{SP}(v_i, v_j)$ containing uu' lies inside P_1, contradicting Lemma 4. If v_i and v_j belong to different subpolygons then either $\mathrm{SP}(s, v_i)$ or $\mathrm{SP}(s, v_j)$ or $\mathrm{SP}(t, v_i)$ or $\mathrm{SP}(t, v_j)$ must contain the eave uu', contradicting Lemma 4. □

Lemma 6. *In a weak visibility polygon, the shortest path between two arbitrary vertices has at most one eave.*

Proof. Since any chord in P can intersect only one eave of $\mathrm{SP}(v_i, v_j)$, by Lemma 5, $\mathrm{SP}(v_i, v_j)$ can have at most one eave. □

It can be seen that even if the shortest path between any two vertices of a polygon has no eave, the polygon need not be a weak visibility polygon (for example, a spiral polygon).

3. Recognizing weak visibility polygons

In this section we propose an O(E) time algorithm to determine whether an n-sided polygon P is a weak visibility polygon. The algorithm computes a visibility chord in P by searching for the endpoints of the visibility chord on the polygonal boundary. We show that the pair of edges containing the endpoints of a visibility chord satisfies three properties, which follow from the properties of shortest paths mentioned in Section 2. The algorithm finds such a pair of edges satisfying three properties by computing the weak visibility polygon of P from every edge.

A visibility chord is either an edge of P or a chord inside P. Now we state the procedure for determining whether P is weakly visible from an edge of P. If P is weakly visible from an edge v_iv_{i+1}, then $\text{bd}(P) = \text{BVP}(P, v_iv_{i+1})$. The procedure computes $\text{BVP}(P, v_iv_{i+1})$ for all i and determines whether an edge of P is a visibility chord. The procedure computes the visibility polygon from each edge as follows. It first computes $\text{SPT}(v_1)$ [9, 11]. Then it uses the algorithm of Hershberger [12] which successively computes $\text{BVP}(P, v_1v_2)$, $\text{SPT}(v_2)$, $\text{BVP}(P, v_2v_3)$, $\text{SPT}(v_3), \ldots, \text{SPT}(v_n)$, $\text{BVP}(P, v_nv_1)$. If $\text{BVP}(P, v_iv_{i+1} = \text{bd}(P)$ for some i, a visibility edge has been found. In particular, if P is a convex polygon, the procedure stops after computing $\text{BVP}(P, v_1v_2)$ and takes O(n) time even though $E = \text{O}(n^2)$.

We now consider the case when P is not weakly visible from an edge. Before we state the procedure for computing a visibility chord, we introduce three properties that are used in the procedure. If there is a visibility chord st in P where $s \in v_iv_{i+1}$ and $t \in v_jv_{j+1}$, then the following properties hold for v_iv_{i+1} and v_jv_{j+1}.

(1) The edges v_iv_{i+1} and v_jv_{j+1} are *eligible* edges. An edge v_kv_{k+1} is called an *eligible* edge if there exists a point p on v_kv_{k+1} such that the shortest path from p to any vertex of P is convex (i.e. makes only left turns or only right turns).

(2) The edges v_iv_{i+1} and v_jv_{j+1} form a *potential pair.* The edges v_iv_{i+1} and v_jv_{j+1} are said to form a *potential pair* if the shortest path between any two vertices of $\text{bd}(v_{i+1}, v_j)$ as well as the shortest path between any two vertices of $\text{bd}(v_{j+1}, v_i)$ are convex.

(3) The point s belongs to both *left* and *right intervals of* v_iv_{i+1} *for* v_jv_{j+1} and t belongs to both *left* and *right intervals of* v_jv_{j+1} *for* v_iv_{i+1}. We define left and right intervals later in this section.

Before we establish the properties, we need some definitions. An edge of $\text{BVP}(P, v_iv_{i+1})$ is called a *constructed edge* if only its endpoints are on $\text{bd}(P)$. Note that one of two endpoints of any constructed edge is a vertex of P. For a constructed edge v_kq, if v_k precedes q in clockwise order on $\text{BVP}(P, v_iv_{i+1})$, then we say v_kq is a left constructed edge (Fig. 6(i)) and a *right* constructed edge (Fig. 6(ii)), otherwise.

We now establish the above properties and wherever a property is established,

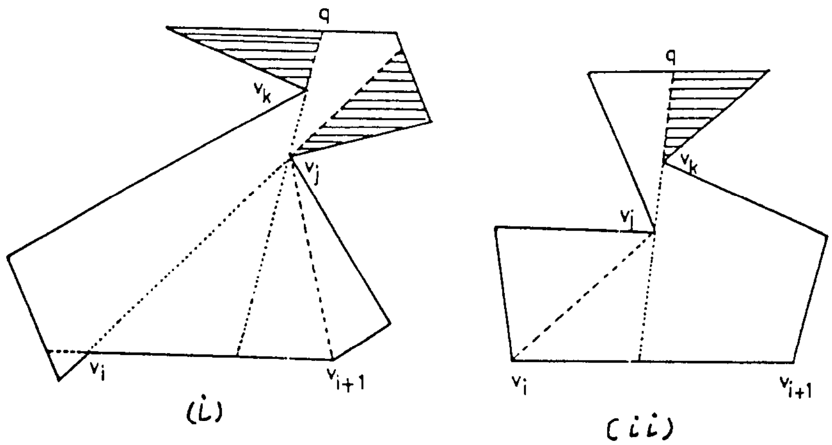

Fig. 6. Finding eligible edges.

we also give the outline of the procedure for finding those edges of P satisfying the property. We have the following two lemmas for the first property.

Lemma 7. *Let v_kq be any left (or right) constructed edge on* BVP(P, v_iv_{i+1}) *and v_j be the parent of v_k in* SPT(v_{i+1}) *(respectively,* SPT(v_i)). *If v_j is not v_{i+1} (respectively, v_i) then there is no visibility chord ending at any edge of* bd(v_{i+i}, v_j) *(respectively,* bd(v_j, v_i)) *(Fig.* 6).

Proof. It follows from Lemma 5 that any visibility chord must intersect v_kv_j because v_kv_j is an eave. So, one endpoint of any visibility chord must belong to bd(v_k, v_{i+1}) and the other endpoint must belong to bd(v_j, q). Therefore there is no visibility chord ending at any edge of bd(v_{i+1}, v_j). □

Initially we call all edges of P *eligible* edges. Using Lemma 7 for each constructed edge on BVP(P, v_iv_{i+1}) for all i, we remove every edge v_iv_{i+1} from the list of eligible edges if there is no visibility chord ending at v_jv_{j+1}. It follows from Lemma 7 that if there is a visibility chord between two edges, then both edges are eligible edges. If there is no eligible edge then P does not have a visibility chord. In the following lemma we establish the property of an eligible edge in terms of shortest paths.

Lemma 8. *Let v_iv_{i+1} be an eligible edge and v_kq be a constructed edge of* BVP(P, v_iv_{i+1}). *If v_kq is a left constructed edge then for every vertex $v_m \in$* bd(q, v_k), SP(v_i, v_m) *makes only left turns. If v_kq is a right constructed edge then for every vertex $v_m \in$* bd(v_k, q), SP(v_{i+1}, v_m) *makes only right turns.*

Proof. Let v_kq be a left constructed edge of BVP(P, v_iv_{i+1}). We show that for any vertex $v_m \in$ bd(q, v_k) (Fig. 7), SP(v_i, v_m) makes only left turns. Since v_k is

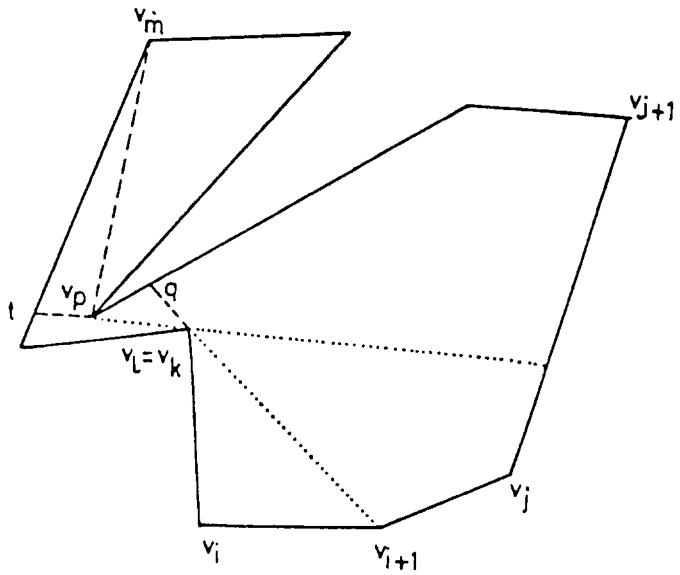

Fig. 7. Pockets have convex shortest paths.

weakly visible from v_iv_{i+1}, SP(v_i, v_k) makes only left turns. Assume that SP(v_i, v_m) makes the first right turn at v_p. Let v_l be the previous vertex of v_p in SP(v_i, v_m). We extend v_pv_l from v_p to bd(P) till it meets a point t on bd(P). We also extend v_pv_l from v_l to bd(P) till it meets an edge v_jv_{j+1} where $v_jv_{j+1} \in$ bd(v_{i+1}, v_p). It can be seen that v_pt is a right constructed edge of BVP(P, v_jv_{j+1}) and v_l is the parent of v_p in SPT(v_j). By Lemma 7, no edge of bd(v_l, v_j) is an eligible edge, a contradiction. Since $v_iv_{i+1} \in$ bd(v_l, v_j), v_iv_{i+1} cannot be an eligible edge. Therefore SP(v_i, v_m) makes only left turns. Analogous arguments show that if v_kq is a right constructed edge then for every vertex $v_m \in$ bd(v_k, q), SP(v_{i+1}, v_m) makes only right turns. □

We establish the second property as follows. An edge v_jv_{j+1} is said to be a *potential edge* of v_iv_{i+1} if for each vertex v_k of bd(v_{j+1}, v_i), SP(v_i, v_k) makes only left turns, and for each vertex v_k of bd(v_{i+1}, v_j), SP(v_{i+1}, v_k) makes only right turns (Fig. 8). If an edge v_jv_{j+1} is a potential edge of v_iv_{i+1} and vice versa, we say that v_iv_{i+1} and v_jv_{j+1} form a *potential pair* and we denote it as potential pair(v_iv_{i+1}, v_jv_{j+1}). If a chord has one endpoint on an edge v_iv_{i+1} and the other endpoint on an edge v_jv_{j+1}, we say that the chord is *between* v_iv_{i+1} and v_jv_{j+1}. In the following lemma we show the relationship between a visibility chord and a potential pair.

Lemma 9. *If a visibility chord is between* v_iv_{i+1} *and* v_jv_{j+1} *then* (v_iv_{i+1}, v_jv_{j+1}) *is a potential pair.*

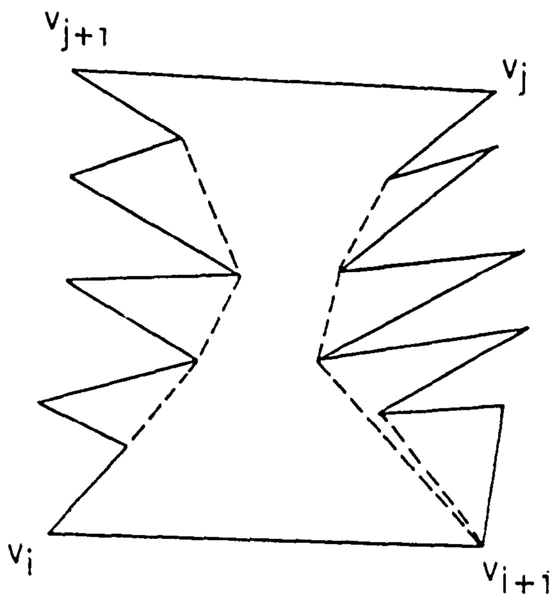

Fig. 8. Potential pair of edges.

Proof. Assume that st is a visibility chord where $s \in v_iv_{i+1}$ and $t \in v_jv_{j+1}$. Since st is a visibility chord, it follows from Theorem 2 that for any vertex $v_k \in \mathrm{bd}(v_{j+1}, v_i)$, $\mathrm{SP}(v_i, v_k)$ makes only left turns, and for any vertex $v_m \in \mathrm{bd}(v_{i+1}, v_j)$, $\mathrm{SP}(v_{i+1}, v_m)$ makes only right turns. Therefore v_jv_{j+1} is a potential edge of v_iv_{i+1}. Analogous arguments show that v_iv_{i+1} is a potential edge of v_jv_{j+1}. Hence, (v_iv_{i+1}, v_jv_{j+1}) is a potential pair. □

In order to find a potential pair, find potential edges for each eligible edge of P as follows. From the definition of a potential edge and Lemma 8, an edge v_jv_{j+1} is a potential edge of v_iv_{i+1} if all left constructed edges of $\mathrm{BVP}(P, v_iv_{i+1})$ are on $\mathrm{bd}(v_{j+1}, v_i)$ and all right constructed edges of $\mathrm{BVP}(P, v_iv_{i+1})$ are on $\mathrm{Bd}(v_{i+1}, v_j)$.

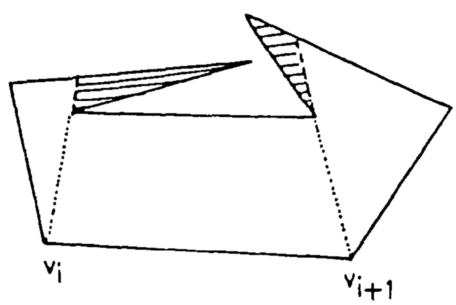

Fig. 9. v_iv_{i+1} has no potential edges.

We scan BVP(P, v_iv_{i+1}) in clockwise order starting at v_i and the following three cases arise.

Case 1: If there is no constructed edge then BVP(P, v_iv_{i+1}) = bd(P) i.e. the edge v_iv_{i+1} is a visibility chord of P.

Case 2: If a right constructed edge is scanned before a left constructed edge then there is no potential edge of v_iv_{i+1}. (Fig. 9).

Case 3: If all the left constructed edges are scanned before all the right constructed edges then all edges of P between the last left constructed edge and the first right constructed edge are potential edges of v_iv_{i+1} (Fig. 10). Note that the potential edges of v_iv_{i+1} are consecutive edges on bd(P).

Once potential edges of v_iv_{i+1} are found, check whether v_iv_{i+1} is already found to be a potential edge of v_jv_{j+1}. If so, a potential pair(v_iv_{i+1}, v_jv_{j+1}) is found. Note that if (v_iv_{i+1}, v_jv_{j+1}) and (v_iv_{i+1}, v_kv_{k+1}) are potential pairs, where $j > k$, then for any edge $v_sv_{s+1} \in$ bd(v_{k+1}, v_j), (v_iv_{i+1}, v_sv_{s+1}) is a potential pair.

We establish the third property as follows. Let v_jv_{j+1} be a potential edge of v_iv_{i+1}. Let S be the set of all points z on v_iv_{i+1} such that for any vertex v_k of bd(v_{j+1}, v_i), SP(z, v_k) makes only left turns (Fig. 11). Observe that S is an interval in v_iv_{i+1} ending at v_i. We call it the *left interval of* v_iv_{i+1} *for* v_jv_{j+1} and denote it by the segment $v_ir(i, j)$, where $r(i, j)$ is a point on v_iv_{i+1}. Similarly we define *right interval of* v_iv_{i+1} *for* v_jv_{j+1} and denote it by $l(i, j)v_{i+1}$ where $l(i, j)$ is a point on v_iv_{i+1}. Now we have the following lemma.

Lemma 10. *For any chord st where* $s \in v_iv_{i+1}$, $t \in v_jv_{j+1}$ *and* (v_iv_{i+1}, v_jv_{j+1}) *is a potential pair, st is a visibility chord if and only if s belongs to both left and right intervals of* v_iv_{i+1} *and t belongs to both left and right intervals of* v_jv_{j+1}.

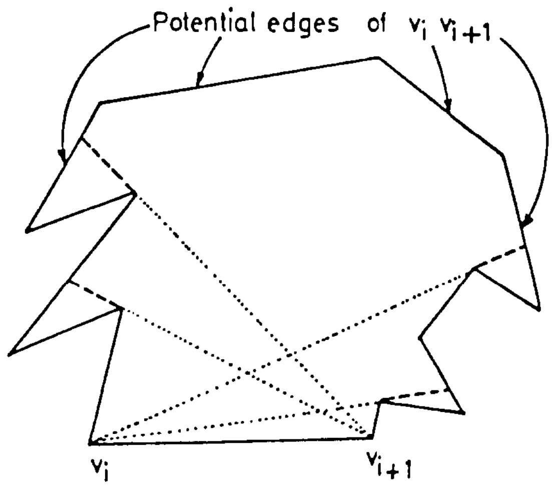

Fig. 10. Potential edges exist.

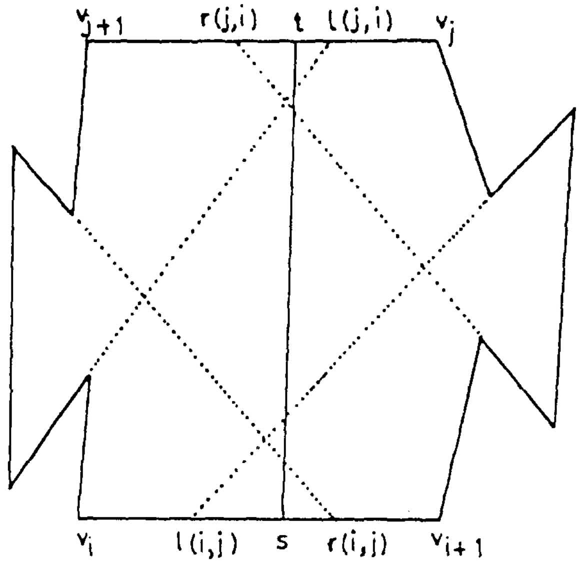

Fig. 11. The third property.

Proof. Assume that st is a visibility chord between v_iv_{i+1} and v_jv_{j+1}. It follows from Theorem 2 that for any vertex $v_k \in \mathrm{bd}(v_{j+1}, v_i)$, $\mathrm{SP}(s, v_k)$ makes only left turns. So s belongs to the left interval of v_iv_{i+1}. Similarly, for any vertex $v_m \in \mathrm{bd}(v_{i+1}, v_j)$, $\mathrm{SP}(s, v_m)$ makes only right turns. So s belongs to the right interval of v_iv_{i+1}. Hence s belongs to both intervals of v_iv_{i+1}. Analogous arguments show that t belongs to both intervals of v_jv_{j+1}.

To show the converse, assume that s belongs to left and right intervals of v_iv_{i+1} and t belongs to left and right intervals of v_jv_{j+1}. Therefore, for any vertex $v_k \in \mathrm{bd}(v_{j+1}, v_i)$, $\mathrm{SP}(s, v_k)$ (respectively, $\mathrm{SP}(t, v_k)$) makes only left (respectively, right) turns, and for any vertex $v_m \in \mathrm{bd}(v_{i+1}, v_j)$, $\mathrm{SP}(s, v_m)$ (respectively, $\mathrm{SP}(t, v_m)$) makes only right (respectively, left) turns. Hence, by Theorem 2 st is a visibility chord of P. □

We state our procedure for computing left and right intervals of v_iv_{i+1} as follows. Let v_jv_{j+1} be a potential edge of v_iv_{i+1}. For each vertex $v_k \in \mathrm{bd}(v_{j+1}, v_i)$ and $\mathrm{BVP}(P, v_iv_{i+1})$, find the intersection of v_iv_{i+1} and the ray drawn from v_k through the parent of v_k in $\mathrm{SPT}(v_{i+1})$. Let $r(i, j)$ be the intersection point closest to v_i among all the intersection points on v_iv_{i+1} (Fig. 12). So $v_ir(i, j)$ is the left interval of v_iv_{i+1} for v_jv_{j+1}. Analogously, the right interval of v_iv_{i+1} for v_jv_{j+1} can be computed. If v_iv_{i+1} has two potential edges then the left and right intervals of v_iv_{i+1} for each potential edge can be computed as follows. Let v_jv_{j+1} and v_kv_{k+1} be potential edges of v_iv_{i+1} where v_kv_{k+1} belongs to $\mathrm{bd}(v_i, v_j)$. By scanning $\mathrm{BVP}(P, v_iv_{i+1})$ in clockwise order from v_i to v_{j+1}, compute the left interval of v_iv_{i+1} for v_jv_{j+1}. Starting with this left interval of v_iv_{i+1}, compute the

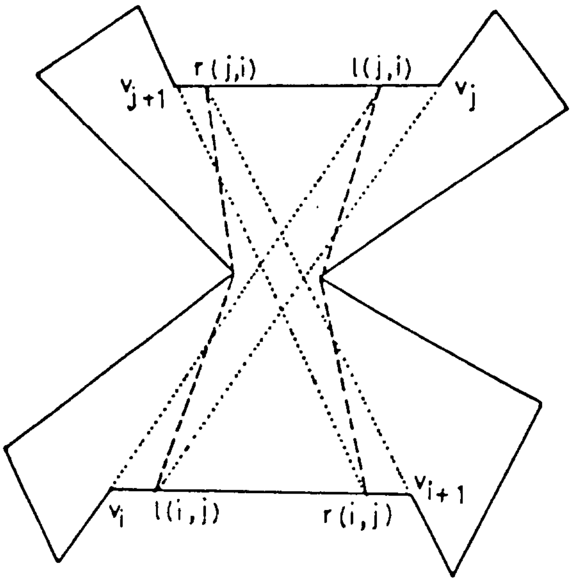

Fig. 12. Construction of a visibility chord.

left interval of v_iv_{i+1} for v_kv_{k+1} by scanning BVP(P, v_iv_{i+1}) in clockwise order from v_j to v_{k+1}. So the left interval of v_iv_{i+1} for each potential edge of v_iv_{i+1} can be computed by scanning BVP(P, v_iv_{i+1}) once in clockwise order from v_i to v_{i+1}. Similarly, the right interval of v_iv_{i+1} for each potential edge of v_iv_{i+1} can be computed by scanning BVP(P, v_iv_{i+1}) once in counterclockwise order from v_{i+1} to v_i.

Now we state our procedure for constructing a visibility chord between the edges v_iv_{i+1} and v_jv_{j+1} satisfying the three properties. It follows from Lemma 10 that any chord from a point $s \in l(i, j)r(i, j)$ to a point $t \in l(j, i)r(j, i)$ is a visibility chord. Now we locate points s and t. Consider SP($r(i, j)$, $l(j, i)$) and SP($l(i, j)$, $r(j, i)$). Assume that they do not share a vertex (Fig. 12). Let v_k and v_m be consecutive vertices on SP($l(i, j)$, $l(i, j)$) such that $v_k \in$ SP($l(i, j)$, $r(j, i)$) and $v_m \in$ SP($r(i, j)$, $l(j, i)$). Note that v_kv_m is a tangent between the convex paths SP($l(i, j)$, $r(j, i)$) and SP($r(i, j)$, $l(j, i)$). We extend v_kv_m from both ends to v_iv_{i+1} and v_jv_{j+1} to obtain s and t respectively. Thus we construct a visibility chord in P.

Consider the case where SP($r(i, j)$, $l(j, i)$) and SP($l(i, j)$, $r(j, i)$) share a vertex (Fig. 13). Since there is no chord joining $l(i, j)r(i, j)$ and $l(j, i)r(j, i)$, by Lemma 10 there is no visibility chord between v_iv_{i+1} and v_kv_{k+1}. Moreover, there is no visibility chord in P as shown in the following lemma.

Lemma 11. *Let* (v_iv_{i+1}, v_jv_{j+1}) *be a potential pair such that the left interval* $v_ir(i, j)$ *and the right interval* $l(i, j)v_{i+1}$ *of* v_iv_{i+1} *overlap and the left interval* $v_jr(j, i)$ *and the right interval* $l(j, i)v_{j+1}$ *of* v_jv_{j+1} *overlap. If* SP($l(i, j)$, $r(j, i)$) *and* SP($r(i, j)$, $l(j, i)$) *share a vertex then there is no visibility chord in* P.

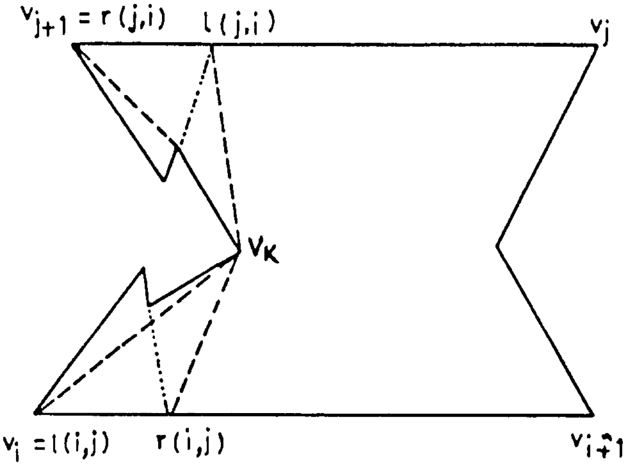

Fig. 13. P does not have a visibility chord.

Proof. Let v_k be a vertex common to $\mathrm{SP}(l(i, j))$ and $\mathrm{SP}(r(i, j), l(j, i))$. We assume without loss of generality that $v_k \in \mathrm{bd}(v_{j+1}, v_i)$ (Fig. 13). Since $\mathrm{SP}(r(i, j), l(j, i))$ and $\mathrm{SP}(l(i, j), r(j, i))$ meet at v_k, there is no chord between $\mathrm{bd}(l(j, i), v_k)$ and $\mathrm{bd}(v_k, r(i, j))$. Observe that any visibility chord must have one endpoint on $\mathrm{bd}(v_k, r(i, j))$ and the other on $\mathrm{bd}(l(j, i), v_k)$. Therefore, there is no visibility chord in P. □

In the following we state the major steps for computing a visibility chord ending at any eligible edge v_iv_{i+1}.

Step 1: Find the potential edges of v_iv_{i+1} by scanning $\mathrm{BVP}(P, v_iv_{i+1})$ in clockwise order from v_i to v_{i+1}.

Step 2: Scan $\mathrm{BVP}(P, v_iv_{i+1})$ once in clockwise order and compute the left interval of v_iv_{i+1} for each potential edge of v_iv_{i+1}.

Step 3: Scan $\mathrm{BVP}(P, v_iv_{i+1})$ once in counterclockwise order and compute the right interval of v_iv_{i+1} for each potential edge of v_iv_{i+1}.

Step 4: If the left and right intervals of v_iv_{i+1} do not overlap for a potential edge v_jv_{j+1}, then remove v_jv_{j+1} from the list of potential edges of v_iv_{i+1}.

Step 5: If v_jv_{j+1} is a potential edge of v_iv_{i+1} and v_iv_{i+1} is already found to be a potential edge of v_jv_{j+1}, then compute a visibility chord between v_iv_{i+1} and v_jv_{j+1}. If there is no visibility chord between v_iv_{j+1} and v_jv_{j+1} then report 'there is no visibility chord in P'.

Now we analyze the time complexity of the algorithm. The visibility polygon from each edge of P can be computed by computing $\mathrm{SPT}(v_1)$, $\mathrm{SPT}(v_2), \ldots,$ $\mathrm{SPT}(v_n)$. $\mathrm{SPT}(v_1)$ can be computed in $O(n)$ time [9, 11]. We wish to note that the algorithm of Ghosh et al. [9], which computes the shortest path tree for a class of polygons, suffices for our purpose; moreover, it uses simple data structures and does not require triangulation as a preprocessing step. The remaining shortest path trees can be computed in $O(E)$ time as shown in [12]. All eligible edges can also be found in $O(E)$ time. To find a potential pair, the algorithm scans the boundary of each visibility polygon three times. Therefore, time required to find a potential pair is $O(E)$. Once a potential pair is found, constructing a visibility chord between the edges of the potential pair requires $O(n)$ time. Therefore, the overall time complexity of the algorithm is $O(E)$. Now we summarize our result in the following theorem.

Theorem 3. *A visibility chord in a polygon P can be constructed in $O(E)$ time, where E is the size of the visibility graph of P.*

4. Computing the maximum hidden set

In this section we propose an $O(n^2)$ time algorithm for computing the maximum hidden vertex set in an n-sided polygon weakly visible from a convex edge. A maximum *hidden vertex set* is the maximum cardinality set of vertices in which no two vertices are mutually visible. The problem of finding the maximum hidden vertex set in an arbitrary polygon is known to be NP-hard [21].

Without loss of generality, we assume that P is weakly visible from the convex edge $v_n v_1$. For any two vertices v_i and v_j where $1 \leqslant j \leqslant i \leqslant n$, the maximum hidden set of opposite chain$(v_i, v_j, v_n v_1)$ is denoted as $\mathrm{mhs}(i, j)$. The cardinality of $\mathrm{mhs}(i, j)$ is denoted as $\mathrm{smhs}(i, j)$. In the following lemma, we present the main idea used in the algorithm.

Lemma 12. *Assume that P is weakly visible from the convex edge $v_n v_1$. For any two vertices v_i and v_j where $1 \leqslant j \leqslant i \leqslant n$,*

$$\mathrm{smhs}(i, j) = \max(\mathrm{smhs}(i, k+1) + \mathrm{smhs}(k-1, j), \mathrm{smhs}(i-1, j)),$$

where v_k is the vertex following v_i in $\mathrm{SP}(v_i, v_j)$.

Proof. If $\mathrm{mhs}(i, j)$ does not contain v_i, then

$$\mathrm{mhs}(i, j) = \mathrm{mhs}(i-1, j) \text{ and } \mathrm{smhs}(i, j) = \mathrm{smhs}(i-1, j).$$

If $\mathrm{mhs}(i, j)$ contains v_i then

$$\mathrm{mhs}(i, j) = \mathrm{mhs}(i, k+1) \cup \mathrm{mhs}(k-1, j)$$

under the assumption that no vertex of $\mathrm{mhs}(i, k+1)$ is visible from any vertex of $\mathrm{mhs}(k-1, j)$. Now we show that no vertex of $\mathrm{mhs}(i, k+1)$ is visible from any vertex of $\mathrm{mhs}(k-1, j)$ (Fig. 14).

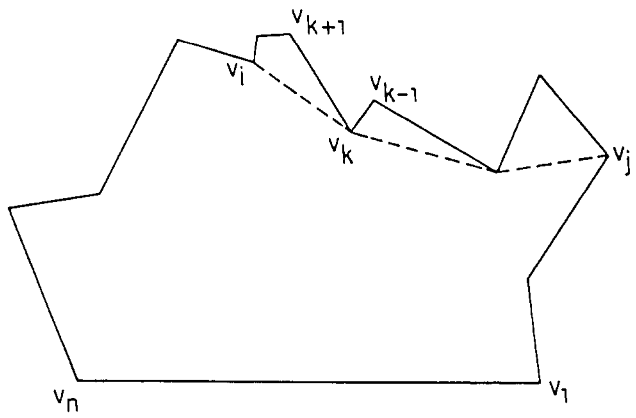

Fig. 14. Illustrating the lemma.

Let u be a vertex of mhs$(i, k+1)$ and w be a vertex of mhs$(k-1, j)$. Since SP(v_i, v_j) passes through v_k and by Theorem 1 it makes a left turn at every vertex in the path, SP(u, w) passes through v_k and makes a left turn at v_k. Therefore u and w are not visible. □

Corollary 1. *If* v_j *is the vertex following* v_i *on* SP(v_i, v_j) *then* smhs$(i, j) =$ max(smhs$(i, j+1)$, smhs$(i-1, j)$).

Lemma 12 suggests a simple procedure for computing the maximum hidden vertex set of P using dynamic programming. It can be seen that mhs$(n, 1)$ is the maximum hidden vertex set of P. In the algorithm, for all values of i and j where $1 \leq j \leq i \leq n$, smhs(i, j) is computed using Lemma 12 and is stored in the location $A(i, j)$ of the array A. We trace the path of computation backward from the location $A(n, 1)$ and reach a set of locations of the array. Each such location corresponds to a vertex of P and these vertices together form the maximum hidden vertex set of P. Now we formally state our algorithm *hidden-set* for computing the maximum hidden vertex set of P.

Algorithm *hidden-set*
begin
  for $i := 1$ **to** n **do** $A[i, i] := 1$
  for $i := 2$ **to** n **do**
  for $j := i-1$ **downto** 1 **do**
  begin let v_m be the previous vertex of v_{j+1} on SP(v_i, v_{j+1})
    if $v_m v_{j+1} v_j$ is a left turn **then**
    SP$(v_i, v_j) :=$ SP$(v_i, v_{j+1}) \cup v_j$
    else SP$(v_i, v_j) :=$ SP$(v_i, v_s) \cup v_j$
    where $v_j v_s$ is the tangent from v_j to SP(v_i, v_{j+1});

if v_j is the vertex following v_i on SP(v_i, v_j) **then** **v_i is visible from v_j**
$A[i, j] := \max(A[i, j+1], A[i-1, j])$
else let v_k be the vertex following v_i on SP(v_i, v_j)
$A[i, j] := \max(A[i, k+1] + A[k-1, j], A[i-1, j])$;
end
end

Theorem 4. *Given a polygon P of n vertices weakly visible from a convex edge, the maximum hidden vertex set of P can be computed in* $O(n^2)$ *time.*

Proof. The correctness of the algorithm follows from Lemma 12. We now analyze the time complexity of the algorithm. The value of i in the algorithm ranges from 2 to n. The value of j ranges from 1 to $i-1$ for each value of i. For each i, computing SP(v_i, v_j) for all $j < i$ requires $O(i)$ time. Thus the algorithm runs in $O(n^2)$ time. □

5. Concluding remarks

We have shown that properties of shortest paths between vertices in weak visibility polygons lead to two efficient algorithms and give characterization of weak visibility polygons. The properties also help in solving other problems on visibility and shortest path problems (see [9, 10, 19]).

The algorithm in Section 3 runs in $O(E)$ times and the algorithm of Ke [13] for the same problem runs in $O(n \log n)$ time. So it remains open whether a weak visibility polygon can be recognized in $O(n)$ time. The technique in our recognition algorithm of scanning only the visibility polygon from each edge has also been used in [10] to recognize a palm polygon. It will be interesting if the same technique can be used to give algorithms for related problems such as partitioning a polygon into the minimum number of weak visibility polygons.

We have proposed an algorithm for computing the maximum hidden vertex set in a polygon weakly visible from a convex edge. Everett [7] has proposed linear time algorithms for spiral polygons and convex polygons with one hole. If a polygon is weakly visible from an internal segment, the problem is open. Even for a star-shaped polygon it is not known whether the maximum hidden vertex set can be computed in polynomial time.

Acknowledgments

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