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CIRCUITS THROUGH SPECIFIED EDGES

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We prove a theorem implying the conjecture of Woodall [14] that, given any k independent edges in a $(k+1)$-connected graph, there is a circuit containing all of them. This implies the truth of a conjecture of Berge [1, p. 214] and provides strong evidence to a conjecture of Lovász [8].

1. Introduction

A well-known result of Dirac [5] states that, given any k vertices in a k-connected graph, there is a circuit containing all of them. Bondy and Lovász [4] proved that the set of circuits through k specified vertices in a $(k+1)$-connected graph generates the cycle space of the graph and deduced that any $(k+1)$-connected non-bipartite graph contains an odd circuit through any k specified vertices as conjectured by Toft [12].

If L is a set of k independent edges in a k-connected graph G, k odd, such that $G-L$ is disconnected, then clearly G has no circuit containing all edges of L. Lovász [8] and, independently, Woodall [14] conjectured that, if k is even or $G-L$ is connected, then G has a circuit containing all edges of L. Woodall [14] also stated the weaker conjecture that any k independent edges in a $(k+1)$-connected graph are contained in a circuit of the graph and pointed out that this would imply the truth of a conjecture of Berge [1, p. 214]. As an important step towards a proof of his conjecture, Woodall [14] proved that, if L is a set of k edges in a $(k+1)$-connected graph G and $G-\{a, b\}$ has a circuit containing all edges of $L\setminus\{(a, b)\}$, where $(a, b)\in L$, then G has a circuit containing all edges of L, and he deduced immediately that, given any set L of k independent edges in a $(2k-2)$-connected graph G, $k\geq 2$, there is a circuit containing all of them. Thomassen [11] proved that the same conclusion holds under the weaker condition that G is $[\frac{3}{2}k-\frac{1}{2}]$-connected and the referee has informed us that Peter L. Erdös and Ervin Györi have shown that it is even sufficient to assume that the connectivity of G is at least $\frac{8}{7}(k+1)$.

The purpose of this paper is to derive the same conclusion under the weaker assumption that any two vertices which are incident with L are connected by $k+1$ internally disjoint paths. This proves the above conjecture of Woodall also for infinite graphs. The proof involves a refinement of Woodall's Hopping Lemma, which was introduced in [13] and applied in [7, 13, 14].

2. Terminology

The terminology is the same as in [11] except that we denote two edges with no common end as *independent*. We shall consider *mixed graphs*, i.e. graphs such that some edges are directed. We regard a *path* in a graph or mixed graph to be oriented, i.e., we distinguish between the path $P: x_1x_2\cdots x_{m-1}x_m$ and its *reverse path* $x_mx_{m-1}\cdots x_2x_1$. If any directed edge which have an end on P is included in P, we say that P is *admissible*. If P is admissible and, in addition, all directed edges of P are of the form (x_i, x_{i+1}) (resp. (x_{i+1}, x_i)), we say that P is a *forward* (resp. *backward*) path.

3. Circuits through specified edges

The following lemma plays a crucial role in the proof.

Lemma 1. *Let m and r be integers, $m \geq 1$, $r \geq 0$. Let G be a graph and L a set of at most r independent edges of G. If x and y are two vertices of G connected by $m+r$ internally disjoint paths and G' is the mixed graph obtained from G by deleting all edges of L incident with x or y and directing all other edges of L, then G' has m internally disjoint forward paths from x to y.*

Proof (by induction on r). Let $P_1, P_2, \ldots, P_{m+r}$ be internally disjoint paths from x to y in G. If some edge e of L has an end in common with some P_i, $1 \leq i \leq m+r$, and with no P_j, $j \neq i$, then we delete e and all intermediate vertices of P_i and use the induction hypothesis. So we can assume that each edge of L joins distinct paths P_i and P_j. We form a new graph H whose edges are L and whose vertices are obtained by identifying the intermediate vertices of each P_i into a vertex. If some component H' of H has a circuit, we delete from G those edges of L and the intermediate vertices of those paths P_i which correspond to H', and the result follows by induction. So we can assume that H is a forest, in particular, there is an edge e of L having an end in a P_i such that P_i is incident with no other edge of L. We now delete e and all interior vertices of P_i and obtain, by the induction hypothesis, a collection of m internally disjoint forward paths from x to y. If one of these, say Q, contains the other end of e, we replace an appropriate segment of Q by e and a segment of P_i and the result follows.

Theorem 1. *If L is a set of k independent edges in a graph G such that any two vertices incident with L are connected by $k+1$ internally disjoint paths, then G has a circuit containing all edges of L.*

Proof. If $e=(x,y)$ is an edge of L, then by Lemma 1, $G-e$ has a forward path from x to y with respect to any orientation of the edges of $L\setminus\{e\}$. So G contains a circuit C such that for each edge e of L, either e is contained in C or no end of e is on C. Put $L'=C\cap L$ and $L''=L\setminus L'$ and let $m=|L'|$ and $r=|L''|$. We can assume $r>0$. We assign an orientation to each edge of L'' and we let (b,a) (oriented in that direction) be one of the edges of L''. Let Z be the set of vertices such that $G-(V(C)\cup\{b\})$ has a forward path from a to z and $G-(V(C)\cup\{a\})$ has a backward path from b to z. We assume that C is chosen such that m is maximum and, subject to that condition, $|Z|$ is minimum. If $X\subseteq V(C)$, we consider all maximal segments of $C-L'$ connecting two vertices of X. Following [14], the union of the vertex sets of these segments is denoted $\mathrm{Cl}(X)$, the endvertices of the segments constitute $\mathrm{Fr}(X)$ and finally $\mathrm{Int}(X)=\mathrm{Cl}(X)\setminus\mathrm{Fr}(X)$. We define the sequence $A_{-1}\subseteq A_0\subseteq A_1\subseteq\cdots$ of subsets of $V(C)$ as follows: $A_{-1}=\emptyset$ and A_0 is the set of vertices z of C such that $G-b$ has a forward path from a to z having only z in common with C. For each $p\geq 1$, A_p is the union of A_{p-1} and the set of vertices z such that G contains a forward path P from $\mathrm{Int}(A_{p-1})$ to z having only its ends in common with C. (Note that if P contains a or b it contains (b,a) and then z is even in A_0.) The sequence $\emptyset=B_{-1}\subseteq B_0\subseteq B_1\subseteq\cdots$ is defined analogously except that we consider backward paths instead of forward paths and B_0 is the set of vertices of C which can be reached from b in $G-a$ by a backward path. Extending Woodall's proof [14] we consider the following statement:

$X(p,q)$: There exists a path $R_{p,q}$ in $G-\{a,b\}$ starting at a_p in A_p and terminating at b_q in B_q such that conditions (C_1)–(C_4) below are satisfied.

(C_1) $R_{p,q}$ contains all edges of L' and all vertices of $\mathrm{Int}(A_{p-1})\cup\mathrm{Int}(B_{q-1})$.

(C_2) If Q is a segment of $R_{p,q}$ from u to v say, having precisely u and v in common with C, then either Q is a forward path or a backward path or both (meaning that it contains no end of an edge of L''). If Q contains edges of L'' and is a forward path, then $u\notin B_q$ and $v\notin A_p$; if Q contains edges of L'' and is a backward path, then $u\notin A_p$ and $v\notin B_q$. Finally, if Q contains no edge of L'', then one of u and v is outside A_p and the other is outside B_q.

(C_3) If $y\in\mathrm{Int}(X)\cap R_{p,q}$, where $X=A_{p'}$, $p'\leq p-1$, or $X=B_{q'}$, $q'\leq q-1$, and T denotes the segment of $C-L'$ which starts and terminates at $\mathrm{Fr}(X)$ and contains y, then $R_{p,q}$ contains T (or the reverse of T).

(C_4) No vertex of $V(C)\setminus V(R_{p,q})$ can be reached by a forward path from a in $G-(V(R_{p,q})\cup\{b\})$ or a backward path from b in $G-(V(R_{p,q})\cup\{a\})$.

We first prove that $X(p,q)$ holds for some p and q. For suppose this is not the case. Then we put $A=\bigcup_{i=0}^{\infty}A_i$ and $B=\bigcup_{i=0}^{\infty}B_i$ and we conclude that none of the m paths of $C-L'$ intersects both A and B unless it contains precisely one vertex from $A \cup B$. Assume w.l.o.g. that $|\mathrm{Fr}(A)| \leq |\mathrm{Fr}(B)|$. Then $|\mathrm{Fr}(A)| \leq m$ and C contains a vertex z which is incident with L and not in $\mathrm{Cl}(A)$. Now every forward path in $G-b$ from a to z intersects $\mathrm{Fr}(A)$. On the other hand $G-b$ has, by Lemma 1 and the assumption of Theorem 1, a set of $m+1$ internally disjoint forward paths from a to z. This contradiction proves that $X(p, q)$ holds for some p and q.

We choose p and q such that $X(p, q)$ holds and such that $p+q$ is minimum under this restriction. Assume w.l.o.g. that $p \geq q \geq 0$. We shall prove that $p=q=0$. For suppose $p>0$. Let $R_{p,q}$ and a_p and b_q be as in the statement of $X(p, q)$. By the minimality of $p+q$, $a_p \in A_p \setminus A_{p-1}$ and $b_q \in B_q \setminus B_{q-1}$. Now G contains a forward path S from a vertex y_{p-1} in $\mathrm{Int}(A_{p-1})$ to a_p having only its ends in common with C. We claim that S has only its ends in common with $R_{p,q}$. For otherwise, S would intersect one of the segments Q of $R_{p,q}$ satisfying (C_2). We now go along S from y_{p-1} towards a_p and we stop at the first vertex in such a segment Q. We then go along Q towards an end d_p, say, of Q in C, and by (C_2), we can do it in such a way that the resulting path from y_{p-1} to d_p is a forward path and such that d_p is not in A_p. But since y_{p-1} is in $\mathrm{Int}(A_{p-1})$ we conclude that d_p is, in fact, in A_p.

This contradiction proves that S has only its ends in common with $R_{p,q}$. Let U denote the segment of A_{p-1} contained in $C-L'$ and containing y_{p-1}. Then U or its reverse segment is a segment of $R_{p,q}$ and, since $X(p-1, q)$ does not hold, U does not intersect B_q. Let U' denote the segment of U which forms the intersection of U with the segment of $R_{p,q}$ from a_p to y_{p-1} Let p' be the smallest integer such that $U'-y_{p-1}$ intersects $A_{p'}$ and let $a_{p'}$ be the vertex such that no intermediate vertex on the segment of U' from $a_{p'}$ to y_{p-1} is contained in $A_{p'}$. We now let $R_{p',q}$ denote the path obtained by forming the union of the reverse path of $R_{p,q}$ from $a_{p'}$ to a_p, the reverse path of S, and the segment of $R_{p,q}$ from y_{p-1} to b_q. It is now easy to see that $R_{p',q}$ satisfies $X(p', q)$. This contradiction shows that assertion $X(0, 0)$ holds.

Consider a path $R_{0,0}$ from a_0 in A_0 to b_0 in B_0 such that $X(0, 0)$ holds. Let T_a (resp. T_b) be a forward (resp. backward) path from a (resp. b) to a_0 (resp. b_0) in $G-b$ (resp. $G-a$) having only a_0 (resp. b_0) in common with C. Since $R_{0,0}$ satisfies condition (C_2), T_a (resp. T_b) has only a_0 (resp. b_0) in common with $R_{0,0}$. If T_a and T_b are disjoint, we get a circuit C' containing all those edges of L that have an end on C' and containing $L' \cup \{(b, a)\}$, a contradiction to the maximality of m. So assume $T_a \cap T_b \neq \emptyset$. We now walk along the reverse path of T_b from b_0 and we stop when we meet the first vertex z on T_a and we then follow T_a from z to a_0. In this way we extend $R_{0,0}$ to a circuit C'' containing L' and no end of an edge L'' (by the maximality of m). We now consider C'' instead of C. Since $R_{0,0}$ satisfies condition (C_4), the set of vertices that can be reached from a by a forward path in $G-(V(C'') \cup \{b\})$ and by a backward path from b in $G-(V(C'') \cup \{a\}) \cup$ is a subset of $Z \setminus \{z\}$. But this contradicts the minimality of $|Z|$ and the proof is complete.

As a corollary of the proof of Theorem 1 we get the following extension of Woodall's result [14].

Corollary 1. *Let L be a set of k independent edges of a graph G and suppose C is an L-admissible cycle of G, i.e. for each edge e of L, C contains e if C contains an end of e. If* $C \cap L \neq \emptyset$ *and* $(a, b) \in L \setminus E(C)$ *such that a* (resp. *b*) *is connected to each vertex incident with* $L \cap E(C)$ *by* $k+1$ *internally disjoint paths, then G has an admissible circuit containing* $(L \cap E(C)) \cup \{(a, b)\}$.

In the proof of Theorem 1 it is assumed that G is finite. However, any infinite graph satisfying the assumption of Theorem 1 contains a finite subgraph with the same property so Theorem 1 extends to infinite graphs.

4. A research problem

G being a graph, $\alpha(G)$ denotes the maximum number of independent vertices of G. The afore-mentioned conjecture of Berge [1, p. 214] can be formulated as follows:

Theorem 2. *If G is* $(\alpha(G)+k)$*-connected, then any system of disjoint paths in G of total length at most k can be extended into a Hamiltonian circuit.*

As pointed out by Woodall [14] it is easy to reduce Theorem 2 to the following statement: If G is a $(\alpha(G)+k)$-connected graph, then any set of k independent edges of G is contained in a circuit. This statement is clearly a consequence of Theorem 1. We offer a stronger conjecture:

Conjecture 1. If G is an $\alpha(G)$-connected graph and L is a set of independent edges of G such that $G-L$ is connected, than G has a circuit containing all edges of L.

If true, Conjecture 1 combined with a result of Bondy [3] would imply the following recent result of K. Berman (private communication):

Theorem 3. *If G is a graph with n vertices such that the degree sum of any two non-adjacent vertices is at least* $n+1$, *then any set L of independent edges is contained in a circuit of G.*

This result was conjectured by Häggkvist [6] who verified it in the case where L is a 1-factor. The case where L has only one edge was treated by Ore [10].

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