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Quarter-regular biembeddings of Latin squares

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ABSTRACT

We apply a recursive construction for biembeddings of Latin squares to produce a new infinite family of biembeddings of cyclic Latin squares of even side having a high degree of symmetry. Reapplication of the construction yields two further classes of biembeddings.

1. Introduction

In this paper we apply a recursive construction for biembeddings of Latin squares to produce a new infinite family of biembeddings of cyclic Latin squares of even side. The only previously known infinite family of biembeddings of Latin squares representing Cayley tables of groups was the class of regular biembeddings. A regular biembedding of a pair of Latin squares of side n has the greatest possible symmetry, with full automorphism group of order $12n^2$, the maximum possible value. This class of biembeddings was presented in [5] using a voltage graph based on a dipole with n parallel edges embedded in a sphere, and later in [3] directly from the cyclic Latin squares. For $n \geq 6$, the new biembeddings in this paper have full automorphism group of order $3n^2$ and so we call them *quarter-regular*. By reapplying the construction, further biembeddings are also obtained. Finally, a generalization of the recursive construction is presented which yields a class of biembeddings of cyclic Latin squares with non-group based squares. We start by giving some basic definitions.

A *Latin square* L of side n is an $n \times n$ array in which each element of $N = \{0, 1, \ldots, n-1\}$ occurs once in each row and once in each column of L. If the rows and columns of L are indexed by N, an equivalent representation of L as a subset of $N \times N \times N$ is obtained by listing the n^2 (row, column, entry) triples. In this representation, the defining property is as follows:

if $(u_r, u_c, u_e), (v_r, v_c, v_e) \in L$, where $u_a = v_a$, $u_b = v_b$, and $\{a, b\} \subset \{r, c, e\}$, then $(u_r, u_c, u_e) = (v_r, v_c, v_e)$.

Thus each (row, column) pair appears in precisely one triple, as does each (row, entry) pair and each (column, entry) pair.

A *transversal design* of order n and block size 3, TD(3, n), is a triple $(V, \mathcal{G}, \mathcal{B})$, where V is a $3n$-element set (the points), $\mathcal{G}$ is a partition of V into three parts (the groups) each of cardinality n, and $\mathcal{B}$ is a collection of 3-element subsets (the blocks) of V such that each 2-element subset of V is either contained in exactly one block of $\mathcal{B}$ or in exactly one group of $\mathcal{G}$, but not

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both. Alternatively, a TD(3, n) may be regarded as a decomposition of the complete tripartite graph $K_{n,n,n}$ into triangles with the tripartition defining the groups of the design. Two TD(3, n)s $(V, \{G_1, G_2, G_3\}, \mathcal{B})$ and $(V', \{G'_1, G'_2, G'_3\}, \mathcal{B}')$ are said to be *isomorphic* if, for some permutation π of $\{1, 2, 3\}$, there exist bijections $\alpha_i : G_i \mapsto G'_{\pi(i)}$, $i = 1, 2, 3$ that map blocks of $\mathcal{B}$ to blocks of $\mathcal{B}'$. Automorphisms of a TD(3, n) are defined similarly.

Given a Latin square of side n, a TD(3, n) may be obtained by taking the three groups as the sets of row numbers, column numbers and entries, and the blocks as the n^2 triples of the Latin square. Conversely, given a TD(3, n) a Latin square of side n may be formed by taking the three groups as copies of N and assigning them, in any one of six possible orders, to be the row numbers, column numbers and entries of the square. Thinking of Latin squares as transversal designs, two Latin squares P and Q, of side n, are said to belong to the same *main class* if the corresponding transversal designs are isomorphic.

Starting with a face 2-colourable triangular embedding M of the complete tripartite graph $K_{n,n,n}$, the faces in each colour class (black and white) give a decomposition of $K_{n,n,n}$ into triangles and hence a TD(3, n) or, equivalently, a Latin square of side n. Such an embedding must necessarily be in an orientable surface [3]. Thus a face 2-colourable triangular embedding of $K_{n,n,n}$ may be regarded as a *biembedding* of two Latin squares P and Q of side n. The triangles of the embedding will correspond to (row, column, entry) triples of the Latin squares and may be denoted by ordered triples (u_r, u_c, u_e). Note from now on we drop the subscript notation and assume the triples of the Latin square refer to (row, column, entry). An *automorphism* of M is a bijection ϕ on the vertices of M that preserves the edges and faces of M, and the *automorphism group* of M is the set of all automorphisms which map M to M.

Of interest throughout this paper will be the main classes of Latin squares corresponding to the Cayley tables of cyclic groups and in particular the biembedding given by the two Latin squares

$$C_n = \{(i, j, i + j \pmod n) \mid 0 \le i, j \le n - 1\}, \quad \text{and}$$
$$(C + 1)_n = \{(i, j, i + j + 1 \pmod n) \mid 0 \le i, j \le n - 1\}.$$

In future we will use the notation $i \oplus_m j$ to denote the integer $k \equiv i + j \pmod m$, $0 \le k \le m - 1$. Similarly, $i \ominus_m j$ denotes the integer $k \equiv i - j \pmod m$, $0 \le k \le m - 1$.

The supporting surface is formed by taking these triples as triangles and sewing them together along their common edges. This biembedding is a so-called *regular embedding*. By saying that an orientable embedding M of a graph G is regular we mean that for every two flags, that is ordered triples of vertices, edges and faces, say (v_1, e_1, f_1) and (v_2, e_2, f_2), where e_i is an edge incident with vertex v_i and face f_i, $1 \le i \le 2$, there exists an automorphism of M which maps v_1 to v_2, e_1 to e_2, and f_1 to f_2. Note that this definition includes automorphisms which reverse the global orientation of the surface and, in the present situation of face 2-colourability, interchange the colour classes. A regular embedding has the greatest possible number of automorphisms because the image of any one flag under an automorphism is sufficient to determine the automorphism completely. Thus the total number of automorphisms in a regular orientable triangular embedding of $K_{n,n,n}$ is just the number of flags, which is easily seen to be $12n^2$. Conversely, an orientable triangular embedding M of $K_{n,n,n}$ having $12n^2$ automorphisms must be regular. As an aside, we remark that some authors use the term "regular" with a slightly different meaning; for a discussion of the terminology the reader is referred to [1].

The biembedding of the two Latin squares C_n and $(C+1)_n$ mentioned above is, to within isomorphism, the unique regular triangular embedding of the complete tripartite graph $K_{n,n,n}$ in an orientable surface [4]. Until the results of the present paper, these embeddings were the only known infinite class of biembeddings of Latin squares formed as the Cayley tables of cyclic groups; indeed as the Cayley tables of any groups.

In the next section we first recall Construction 10.3 of [2], and then apply it to the regular biembedding of C_n and $(C+1)_n$ to obtain a further infinite class of biembeddings of Latin squares obtained from the Cayley tables of cyclic groups of even order. Then in Section 3 we determine the full automorphism groups of these biembeddings. In Section 4 we show that reapplication of the construction yields two further classes of biembeddings.

2. Construction

Starting with a biembedding of two Latin squares of side n we may construct a biembedding of two Latin squares of side $2n$ as follows.

Construction [2]: Take any biembedding of two Latin squares of side n in a surface S. Next take two copies of the given biembedding in disjoint surfaces S^0 and S^1 with the colour classes and orientation in S^1 reversed so that a white triangle (u^0, v^0, w^0) in S^0 corresponds to a black triangle (u^1, w^1, v^1) in S^1. Octahedral bridges joining these two surfaces are formed from copies of a face 2-colourable triangular embedding M of $K_{2,2,2}$ in a sphere having vertex parts $\{a^0, a^1\}$, $\{b^0, b^1\}$, $\{c^0, c^1\}$, a black face (a^0, c^0, b^0) and a white face (a^1, b^1, c^1). For each white triangular face (u, v, w) in S we bridge S^0 and S^1 using a copy of M, obtained by renaming a^i, b^i and c^i as u^i, v^i and w^i respectively. The black face (u^0, w^0, v^0) from the copy of M is glued to the white face (u^0, v^0, w^0) in S^0 and the white face (u^1, v^1, w^1) from the copy of M is glued to the black face (u^1, w^1, v^1) in S^1. It is now routine to check that the resulting embedding represents a biembedding of two Latin squares of side $2n$. Indeed, the bridging operation supplies all adjacencies of the forms u^0v^1, u^0w^1, v^0w^1, v^0u^1, w^0u^1, w^0v^1, so that the embedded graph is $K_{2n,2n,2n}$, while all faces are triangular and face 2-colourability is preserved.

In order to determine the structure of the resulting Latin squares of side $2n$, label the Latin squares corresponding to the biembedding in S as P and Q. Then the Latin squares of side $2n$ obtained from the above construction are as follows, with

$\mathcal{P}$

P^1	P^0
P^0	Q^1

$\mathcal{Q}$

Q^0	P^1
P^1	P^0

Fig. 1. The construction.

the triples recorded in row, column, entry order.

$$\mathcal{P} = \{(u^0, v^0, p^1), (u^0, v^1, p^0), (u^1, v^0, p^0), (u^1, v^1, q^1) \mid (u, v, p) \in P \text{ and } (u, v, q) \in Q\},$$
$$\mathcal{Q} = \{(u^0, v^0, q^0), (u^0, v^1, p^1), (u^1, v^0, p^1), (u^1, v^1, p^0) \mid (u, v, p) \in P \text{ and } (u, v, q) \in Q\}.$$

Diagrammatically this can be represented as in Fig. 1 where the rows and columns are ordered naturally by superscript 0 followed by superscript 1, and the entry P^i or Q^i denotes that entries are taken respectively from P or Q with superscript i.

Theorem 2.1. *The Latin squares $\mathcal{P}$ and $\mathcal{Q}$ belong to the same main class.*

Proof. Define π to be a mapping such that

$$\pi(x^i) = x^j \quad \text{where } x \in \{u, v, p, q\},\ i, j \in \{0, 1\},\ i \neq j.$$

Then π is an involution which maps the triples of $\mathcal{P}$ to the triples of $\mathcal{Q}$, and vice versa. Thus π determines a colour reversing automorphism of the biembedding. $\square$

Applying the construction to the regular biembedding where $P = C_n$ and $Q = (C + 1)_n$ and writing u^0 as i, u^1 as $i + n$, v^0 as j and v^1 as $j + n$ we obtain the Latin squares

$$W = \{(i, j, (i \oplus_n j) + n), (i, j + n, i \oplus_n j), (i + n, j, i \oplus_n j), (i + n, j + n, (i \oplus_n j \oplus_n 1) + n) \mid 0 \le i, j \le n - 1\},$$
$$B = \{(i, j, i \oplus_n j \oplus_n 1), (i, j + n, (i \oplus_n j) + n), (i + n, j, (i \oplus_n j) + n), (i + n, j + n, i \oplus_n j) \mid 0 \le i, j \le n - 1\}.$$

For the remainder of this paper W and B will refer to the Latin squares described above. The symmetry of the construction is emphasized in the diagram given in Fig. 2. Here C_n and $(C + 1)_n$ are the Latin squares described above, while C'_n and $(C + 1)'_n$ represent isomorphic Latin squares where the set of entries is $\{n, \dots, 2n - 1\}$.

W

C'_n	C_n
C_n	$(C+1)'_n$

B

$(C+1)_n$	C'_n
C'_n	C_n

Fig. 2. The Latin squares W and B.

We show that W and B are in the same main class as C_{2n}, the Cayley table of the cyclic group. In order to do this first define bijections f and g on $2N = \{0, 1, \dots, 2n - 1\}$ given by:

$$f(x) = \begin{cases} 2x, & \text{if } 0 \le x \le n - 1 \\ 2(x - n) + 1, & \text{otherwise.} \end{cases}$$

$$g(x) = \begin{cases} 2x + 1, & \text{if } 0 \le x \le n - 1 \\ 2(x - n), & \text{otherwise.} \end{cases}$$

Further let

$$\mathcal{F}(W) = \{(f(i), f(j), g(k)) \mid (i, j, k) \in W\},$$
$$\mathcal{G}(B) = \{(g(i), g(j), f(k)) \mid (i, j, k) \in B\}.$$

Clearly $\mathcal{F}(W)$ and $\mathcal{G}(B)$ are Latin squares in the same main class as W and B, respectively. We then have the following theorems, but first it will be useful to note that for $0 \le x, y \le n - 1$, if $x + y = z + kn$ then $2x + 2y = 2z + 2kn$ and so

$$z = x \oplus_n y \Rightarrow 2z = 2x \oplus_{2n} 2y. \tag{1}$$

Theorem 2.2. *For all* $n \geq 2$, $\mathcal{F}(W) = C_{2n}$.

Proof. We need to show that for all $(x, y, z) \in W$, $(f(x), f(y), g(z)) \in C_{2n}$. Let $0 \leq i, j \leq n-1$ and consider possible combinations of row and column numbers.

For $(i, j, k) \in W$, $(i \oplus_n j) + n = k$ and so $n \leq k \leq 2n-1$. Hence $g(k) = 2((i \oplus_n j) + n - n) = 2(i \oplus_n j)$ or equivalently by (1) we have $g(k) = 2i \oplus_{2n} 2j$. Since $f(i) = 2i$ and $f(j) = 2j$, it follows that $(f(i), f(j), g(k)) \in C_{2n}$.

For $(i, j+n, k) \in W$ then $(i+n, j, k) \in W$, and vice versa, and each implies that $k = i \oplus_n j$ and so $0 \leq k \leq n-1$. It follows that $g(k) = 2(i \oplus_n j) + 1$ or equivalently by (1) we have both $g(k) = (2i+1) \oplus_{2n} 2j$ and $g(k) = 2i \oplus_{2n} (2j+1)$. Since $f(i) = 2i, f(i+n) = 2i+1, f(j) = 2j$ and $f(j+n) = 2j+1$, in both cases we have $(f(i+n), f(j), g(k)) \in C_{2n}$ and $(f(i), f(j+n), g(k)) \in C_{2n}$.

For $(i+n, j+n, k) \in W$, $k = (i \oplus_n j \oplus_n 1) + n$ and so $n \leq k \leq 2n-1$. It follows that $g(k) = 2((i \oplus_n j \oplus_n 1) + n - n)$ which by (1) implies that $g(k) = (2i \oplus_{2n} 2j \oplus_{2n} 2)$. Since $f(i+n) = 2i+1$ and $f(j+n) = 2j+1$, we have $(f(i+n), f(j+n), g(k)) \in C_{2n}$. □

Theorem 2.3. *For all* $n \geq 2$, $\mathcal{G}(B) = C_{2n}$.

Proof. The proof is analogous to the proof of the previous theorem with the roles of the bijections f and g reversed. □

In the next section we will study the automorphism group of the biembedding.

3. Automorphism group

The full automorphism group of the regular biembedding of Latin squares of side $2n$ has order $12(2n)^2$. After applying the above construction to the regular biembedding of the cyclic group of order n, we obtain a biembedding of Latin squares of side $2n$. We now determine the full automorphism group of the resulting biembedding and show that for $n \geq 3$ it has order $3(2n)^2$. To do this, rather than working directly with W and B, we work with the biembedding of $\mathcal{W} = \mathcal{F}(W)$ with $\mathcal{B} = \mathcal{F}(B)$, where

$$\mathcal{F}(B) = \{(f(i), f(j), g(k)) \mid (i, j, k) \in B\}.$$

We have already shown, in Theorem 2.2, that for all $n \geq 2$, $\mathcal{W} = C_{2n}$; that is,

$$\mathcal{W} = \{(i, j, i \oplus_{2n} j) \mid 0 \leq i, j \leq 2n-1\}.$$

The structure of $\mathcal{B}$ is given in the next theorem.

Theorem 3.1. *For all* $n \geq 2$, $\mathcal{B} = \{(i, j, k) \mid 0 \leq i, j \leq 2n-1\}$, *where*

$$k = i * j = \begin{cases} i \oplus_{2n} j \oplus_{2n} 3 & \text{if } i \text{ and } j \text{ are both even} \\ i \oplus_{2n} j \ominus_{2n} 1 & \text{otherwise.} \end{cases}$$

Proof. Take i, j satisfying $0 \leq i, j \leq n-1$.

For $(i, j, k) \in B$, $k = i \oplus_n j \oplus_n 1$ and so $0 \leq k \leq n-1$. Hence $g(k) = 2k+1 = 2(i \oplus_n j \oplus_n 1) + 1$, which by (1) implies that $g(k) = 2i \oplus_{2n} 2j \oplus_{2n} 3$. Since $f(i) = 2i$ and $f(j) = 2j$, it follows that $(2i, 2j, 2i \oplus_{2n} 2j \oplus_{2n} 3) \in \mathcal{B}$.

For $(i, j+n, k) \in B$, $k = (i \oplus_n j) + n$ and so $n \leq k \leq 2n-1$. Hence $g(k) = 2(k-n) = 2((i \oplus_n j) + n - n)$, which by (1) implies $g(k) = 2i \oplus_{2n} 2j$. Since $f(i) = 2i$ and $f(j+n) = 2j+1$, it follows that $(2i, 2j+1, 2i \oplus_{2n} 2j) \in \mathcal{B}$. Similarly, $(2i+1, 2j, 2i \oplus_{2n} 2j) \in \mathcal{B}$.

For $(i+n, j+n, k) \in B$, $k = i \oplus_n j$ and so $0 \leq k \leq n-1$. Hence $g(k) = 2(i \oplus_n j) + 1$, and by (1) $g(k) = 2i \oplus_{2n} 2j \oplus_{2n} 1$. Since $f(i+n) = 2i+1$ and $f(j+n) = 2j+1$, $(2i+1, 2j+1, 2i \oplus_{2n} 2j \oplus_{2n} 1) \in \mathcal{B}$. □

In order to identify all bijections θ which form an automorphism of the biembedding of $\mathcal{W}$ with $\mathcal{B}$, we divide the analysis into four cases. Case C1 corresponds to automorphisms that preserve the colour classes, the tripartition and the orientation. Case C2 corresponds to automorphisms that preserve the colour classes and the orientation but cyclically permute the tripartition. Case C3 corresponds to automorphisms that preserve the colour classes but reverse the orientation by non-cyclically permuting the tripartition. Finally case C4 corresponds to automorphisms that exchange the colour classes. It is easy to see that any automorphism of the biembedding must be a composition of automorphisms of these types.

Since we are working with Latin squares of side $2n$, in this section all arithmetic is computed modulo $2n$, and for brevity we will omit the terms (mod $2n$). In each case we take $0 \leq i, j \leq 2n-1$ and $n \geq 2$, so that $(i, j, i+j) \in \mathcal{W}$ and $(i, j, i*j) \in \mathcal{B}$, where $i * j = i + j + 3$ if i and j are both even, and $i * j = i + j - 1$ otherwise. We will think of θ as consisting of three bijections α, β and γ each acting on $2N = \{0, 1, 2, \ldots, 2n-1\}$. The action of $\theta = (\alpha, \beta, \gamma)$ in each of the four cases is as follows.

C1: for $(i, j, i+j) \in \mathcal{W}$ and $(i, j, i*j) \in \mathcal{B}$, $(\alpha(i), \beta(j), \gamma(i+j)) \in \mathcal{W}$ and $(\alpha(i), \beta(j), \gamma(i*j)) \in \mathcal{B}$;
C2: for $(i, j, i+j) \in \mathcal{W}$ and $(i, j, i*j) \in \mathcal{B}$, $(\gamma(i+j), \alpha(i), \beta(j)) \in \mathcal{W}$ and $(\gamma(i*j), \alpha(i), \beta(j)) \in \mathcal{B}$;
C3: for $(i, j, i+j) \in \mathcal{W}$ and $(i, j, i*j) \in \mathcal{B}$, $(\beta(j), \alpha(i), \gamma(i+j)) \in \mathcal{W}$ and $(\beta(j), \alpha(i), \gamma(i*j)) \in \mathcal{B}$;

C4: for $(i, j, i+j) \in \mathcal{W}$ and $(i, j, i*j) \in \mathcal{B}$, $(\beta(j), \alpha(i), \gamma(i*j)) \in \mathcal{W}$ and $(\beta(j), \alpha(i), \gamma(i+j)) \in \mathcal{B}$.

We analyze each case in turn.

Case C1: Assume that there exists bijections $\alpha,\ \beta,\ \gamma$ of $2N$ such that, $(\alpha(i), \beta(j), \gamma(i+j)) \in \mathcal{W}$ and $(\alpha(i), \beta(j), \gamma(i*j)) \in \mathcal{B}$, and so

$$\alpha(i) + \beta(j) = \gamma(i+j), \tag{2}$$

$$\alpha(i) * \beta(j) = \gamma(i*j). \tag{3}$$

For $i = 0$, (2) implies that for all $j \in 2N$, $\beta(j) = \gamma(j) - \alpha(0)$. Similarly for $j = 0$, and for all $i \in 2N$, $\alpha(i) = \gamma(i) - \beta(0)$. Thus there exist constants b and c such that for all $i \in 2N$,

$$\beta(i) = \alpha(i) + b, \tag{4}$$

$$\gamma(i) = \alpha(i) + c. \tag{5}$$

Consequently (2) can be rewritten as

$$\alpha(i+j) = \alpha(i) + \alpha(j) + a, \tag{6}$$

for some constant a. Taking $j = 1$ in (6) gives $\alpha(i+1) = \alpha(i) + \alpha(1) + a$, or more generally there exists a constant $d = \alpha(1) + a$, such that

$$\alpha(1) = \alpha(0) + d$$

$$\alpha(2) = \alpha(1) + d = \alpha(0) + 2d$$

$$\vdots$$

$$\alpha(2n-1) = \alpha(0) + (2n-1)d.$$

Hence for all $i \in 2N$, $\alpha(i) = \alpha(0) + id$. This implies that d is coprime with $2n$, since otherwise α is not a bijection, and hence d must be odd. Substituting in (4) gives $\beta(i) = \alpha(i) + b = \alpha(0) + id + b = \beta(0) + id$ and, similarly, (5) implies that $\gamma(i) = \gamma(0) + id$.

$\alpha(0)$	$\beta(0)$	i	j	$\alpha(0)+id$	$\beta(0)+jd$	r	s	$r = sd$
E	O	E	E	E	O	-1	3	$-1 = 3d$
E	O	O	O	O	E	-1	-1	$-1 = -d$
E	O	E	O	E	E	3	-1	$3 = -d$
E	O	O	E	O	O	-1	-1	$-1 = -d$
O	O	E	E	O	O	-1	3	$-1 = 3d$
O	O	O	O	E	E	3	-1	$3 = -d$
O	O	E	O	O	E	-1	-1	$-1 = -d$
O	O	O	E	E	O	-1	-1	$-1 = -d$
E	E	E	E	E	E	3	3	$3 = 3d$
E	E	O	O	O	O	-1	-1	$-1 = -d$
E	E	E	O	E	O	-1	-1	$-1 = -d$
E	E	O	E	O	E	-1	-1	$-1 = -d$

Fig. 3. Summary of possible parities for terms given in (8).

Applying these results under the assumption that $(\alpha(i), \beta(j), \gamma(i*j)) \in \mathcal{B}$ gives $(\alpha(0)+id, \beta(0)+jd, \gamma(0)+(i*j)d) \in \mathcal{B}$, with the additional information that for all $0 \le i, j \le 2n-1$,

$$(\alpha(0) + id) * (\beta(0) + jd) = \gamma(0) + (i*j)d. \tag{7}$$

Hence there exists constants $r, s \in \{-1, 3\}$ such that

$$(\alpha(0) + id) + (\beta(0) + jd) + r = \gamma(0) + (i + j + s)d. \tag{8}$$

However, since $\alpha(0) + \beta(0) = \gamma(0)$ it follows that $r = sd$. The table given in Fig. 3 gives the possible parities of the terms in (8) and calculates the corresponding values for r and s, (E represents even parity and O represents odd parity). There are four cases depending on the parities of $\alpha(0)$ and $\beta(0)$ but, by symmetry, the case where $\alpha(0)$ is odd and $\beta(0)$ is even is identical to that where $\alpha(0)$ is even and $\beta(0)$ is odd. Thus the table presents just three cases.

In the first two of the three cases, a solution exists if and only if $2n = 4$ and $d = 1$. In the last of the three cases there is no restriction on n, however $d = 1$. Thus for all $n \ge 3$, if $\alpha(0)$ and $\beta(0)$ are both even we obtain an automorphism of the biembedding of $\mathcal{W}$ with $\mathcal{B}$. Hence for $n \ge 3$ there are precisely $(2n)^2/4 = n^2$ automorphisms of the form

$$\phi_{a,b} : (i, j, k) \longrightarrow (i + 2a, j + 2b, k + (2a + 2b))$$

where $a, b \in \{0, 1, \ldots, n-1\}$. These automorphisms preserve the colour classes, the tripartition and the orientation.

It is easy to check that if $n = 2$, then there are no restrictions on $\alpha(0)$ and $\beta(0)$ and we obtain 16 such automorphisms. There is just one biembedding of Latin squares of side 4, the regular one [3], so no new biembedding can be obtained.

In the remaining cases C2-4 below, we make the assumption that $n \geq 3$.

Case C2: We require bijections α, β, γ of $2N$ such that $(\gamma(i+j), \alpha(i), \beta(j)) \in \mathcal{W}$ and $(\gamma(i*j), \alpha(i), \beta(j)) \in \mathcal{B}$, where $\gamma(i+j) + \alpha(i) = \beta(j)$, and $\gamma(i*j) * \alpha(i) = \beta(j)$. One such set of bijections is given by

$$\alpha(i) = i, \qquad \beta(j) = 2n - j - 1, \qquad \gamma(k) = 2n - k - 1.$$

This is easily checked: $(\gamma(i+j), \alpha(i), \beta(j)) = (2n-i-j-1, i, 2n-j-1)$ and $(2n-i-j-1)+i = 2n-j-1$, while $(\gamma(i*j), \alpha(i), \beta(j)) = (2n-(i*j)-1, i, 2n-j-1)$ and $(2n-(i*j)-1)*i = 2n-j-1$ whatever the parities of i and j. So now define χ_0 by

$$\chi_0 : (i, j, k) \longrightarrow (2n-k-1, i, 2n-j-1).$$

Then $\chi_0^2(i, j, k) = (j, 2n-k-1, 2n-i-1)$ and $\chi_0^3(i, j, k) = (i, j, k)$, so that χ_0 has order 3. By composing each $\phi_{a,b}$ from case C1 with χ_0 and χ_0^2, we obtain a further $2n^2$ automorphisms of the biembedding, giving a running total of $3n^2$ automorphisms that preserve the colour classes and orientation. Now assume that π is any such automorphism. Then one of π, $\pi\chi_0$, $\pi\chi_0^2$ must be one of the n^2 mappings $\phi_{a,b}$. But then π is one of $\phi_{a,b}$, $\phi_{a,b}\chi_0$, $\phi_{a,b}\chi_0^2$. Hence there are precisely $3n^2$ automorphisms that preserve the colour classes and orientation.

Case C3: We require bijections α, β and γ on $2N$ such that $(\beta(j), \alpha(i), \gamma(i+j)) \in \mathcal{W}$ and $(\beta(j), \alpha(i), \gamma(i*j)) \in \mathcal{B}$, where $\beta(j) + \alpha(i) = \gamma(i+j)$ and $\beta(j) * \alpha(i) = \gamma(i*j)$. One such set of bijections is given by

$$\alpha(i) = i, \qquad \beta(j) = j, \qquad \gamma(k) = k.$$

This is trivially checked, so now define μ_0 by

$$\mu_0 : (i, j, k) \longrightarrow (j, i, k).$$

Since $\mathcal{W}$ and $\mathcal{B}$ are both symmetric it follows that μ_0 is an automorphism of order 2, which preserves colour classes but reverses the orientation. The automorphism μ_0 can be composed with the earlier permutations giving $3n^2$ automorphisms of this type. As in Case C2 it is easy to show that there are no further automorphisms of this type. Altogether there are $6n^2$ automorphisms that preserve the colour classes.

Case C4: We require bijections α, β and γ on $2N$ such that $(\beta(j), \alpha(i), \gamma(i*j)) \in \mathcal{W}$ and $(\beta(j), \alpha(i), \gamma(i+j)) \in \mathcal{B}$, where $\beta(j) + \alpha(i) = \gamma(i*j)$ and $\beta(j) * \alpha(i) = \gamma(i+j)$. One such set of bijections is given by

$$\alpha(i) = \beta(i) = \gamma(i) = \begin{cases} i+1 & \text{if } i \text{ is even,} \\ i-1 & \text{if } i \text{ is odd.} \end{cases}$$

To check this, note that

$$\begin{aligned}(\beta(j), \alpha(i), \gamma(i*j)) &= \begin{cases} (j+1, i+1, i+j+3-1) & \text{if } i, j \text{ both even,} \\ (j-1, i+1, i+j-1+1) & \text{if } i \text{ even, } j \text{ odd,} \\ (j+1, i-1, i+j-1+1) & \text{if } i \text{ odd, } j \text{ even,} \\ (j-1, i-1, i+j-1-1) & \text{if } i, j \text{ both odd,} \end{cases} \\ &= (\beta(j), \alpha(i), \beta(j) + \alpha(i)),\end{aligned}$$

and

$$\begin{aligned}(\beta(j), \alpha(i), \gamma(i+j)) &= \begin{cases} (j+1, i+1, i+j+1) & \text{if } i, j \text{ both even,} \\ (j-1, i+1, i+j-1) & \text{if } i \text{ even, } j \text{ odd,} \\ (j+1, i-1, i+j-1) & \text{if } i \text{ odd, } j \text{ even,} \\ (j-1, i-1, i+j+1) & \text{if } i, j \text{ both odd,} \end{cases} \\ &= (\beta(j), \alpha(i), \beta(j) * \alpha(i)).\end{aligned}$$

So define ν_0 by

$$\nu_0 : (i, j, k) \longrightarrow (\beta(j), \alpha(i), \gamma(k))$$

and note that $\nu_0^2(i, j, k) = (i, j, k)$, so that ν_0 has order 2. Taking compositions of ν_0 with the $6n^2$ automorphisms that preserve the colour classes and arguing as before, we obtain precisely $6n^2$ automorphisms that reverse the colour classes.

We now state our conclusions in the following theorem which asserts the existence of quarter-regular biembeddings of Latin squares of even side.

Theorem 3.2. *The full automorphism group of the biembedding of the cyclic Latin squares W and B, each of side $2n \geq 6$, has order $12n^2$. Furthermore, there are $3n^2$ automorphisms that preserve the squares and the orientation, $3n^2$ that preserve the squares but reverse the orientation, $3n^2$ that exchange the squares but preserve the orientation, and $3n^2$ that exchange the squares and reverse the orientation.*

$\overline{W}$

C'_{2n}	C_{2n}
C_{2n}	$\mathcal{B}'$

$\bar{B}$

$\mathcal{B}$	C'_{2n}
C'_{2n}	C_{2n}

Fig. 4. The Latin squares $\overline{W}$ and $\bar{B}$.

$\mathcal{S}$

P^1	P^0
P^0	R^1

$\mathcal{T}$

Q^0	P^1
P^1	P^0

Fig. 5. The generalized construction.

4. Reapplication of the construction

Having obtained a new infinite family of biembeddings of cyclic Latin squares of even side, we can apply the construction to two Latin squares of side $2n$ in this family to obtain further biembeddings. These will be Latin squares of side $4n$. Details are given below. Further, we prove that the biembedded Latin squares so obtained, are not in the same main class as the Cayley table of the cyclic group; indeed of any group. Hence we have a further infinite family of biembeddings of Latin squares of side $4n$ for $n \geq 3$.

Referring to Fig. 1, let $P = \mathcal{W} = C_{2n}$ and $Q = \mathcal{B}$, and denote the resulting Latin squares by $\overline{W}$ and $\bar{B}$, where

$$\overline{W} = \{(i, j, (i \oplus_{2n} j) + 2n), (i, j + 2n, i \oplus_{2n} j), (i + 2n, j, i \oplus_{2n} j), (i + 2n, j + 2n, (i * j) + 2n) \mid 0 \leq i, j \leq 2n - 1\},$$
$$\bar{B} = \{(i, j, i * j), (i, j + 2n, (i \oplus_{2n} j) + 2n), (i + 2n, j, (i \oplus_{2n} j) + 2n), (i + 2n, j + 2n, i \oplus_{2n} j) \mid 0 \leq i, j \leq 2n - 1\}.$$

Recall that in this context $i * j = i \oplus_{2n} j \oplus_{2n} 3$, if i and j are both even, and $i * j = i \oplus_{2n} j \ominus_{2n} 1$, otherwise.

Diagrammatically, this is shown in Fig. 4; here C'_{2n} and $\mathcal{B}'$ represent Latin squares isomorphic to C_{2n} and $\mathcal{B}$ respectively where the set of entries is $\{2n, \ldots, 4n - 1\}$.

In the next theorem we prove that the Latin squares $\overline{W}$ and $\bar{B}$ do not belong to the same main class as the Cayley table of a group. To prove this result we recall the statement of the quadrangle criterion and use the fact that if a Latin square or any of its conjugates corresponds to the Cayley table of a group, then it must satisfy the quadrangle criterion (a consequence of associativity).

A Latin square L of order n satisfies the *quadrangle criterion* if for $e, f, g, h, i, j, k, l, w, x, y, z \in N$, whenever $(e, f, w), (g, h, w), (i, f, x), (j, h, x), (i, k, y), (j, l, y), (e, k, z) \in L$ then $(g, l, z) \in L$.

Theorem 4.1. *The Latin squares $\overline{W}$ and $\bar{B}$ do not belong to the same main class as the Cayley table of a group.*

Proof. First recall from Theorem 2.1 that $\overline{W}$ and $\bar{B}$ belong to the same main class. So it is sufficient to prove the theorem for $\overline{W}$.

It can be easily seen that $(0, 0, 2n), (0, 2n + 2, 2), (2n, 0, 0), (1, 2n - 1, 2n), (1, 2n + 1, 2), (2n + 1, 2n - 1, 0) \in \overline{W}$; however $(2n, 2n + 2, 2n + 5) \in \overline{W}$, while $(2n + 1, 2n + 1, 2n + 5) \notin \overline{W}$ since $(2n + 1, 2n + 1, 2n + 1) \in \overline{W}$. Hence $\overline{W}$ does not satisfy the quadrangle criterion. $\square$

Finally, we note that the construction described in Section 2 may be generalized. Suppose that P, Q and R are Latin squares of side n and that the surface S^0 supports a biembedding M^0 of P (white triangles) with Q (black triangles), while the surface S^1 supports a biembedding M^1 of R (white triangles) with P (black triangles). Then the white triangles of M^0 may be bridged to the black triangles of M^1 exactly as before to form a biembedding of two Latin squares $\mathcal{S}$ and $\mathcal{T}$ of side $2n$. Diagrammatically this generalized construction is presented in Fig. 5.

The Latin squares of side $2n$ so obtained are

$$\mathcal{S} = \{(u^0, v^0, p^1), (u^0, v^1, p^0), (u^1, v^0, p^0), (u^1, v^1, r^1) \mid (u, v, p) \in P \text{ and } (u, v, r) \in R\},$$
$$\mathcal{T} = \{(u^0, v^0, q^0), (u^0, v^1, p^1), (u^1, v^0, p^1), (u^1, v^1, p^0) \mid (u, v, p) \in P \text{ and } (u, v, q) \in Q\}.$$

Applying the generalized construction to the quarter-regular biembedding M^0 and the regular biembedding M^1, by taking $P = C_{2n}$, $Q = \mathcal{B}$ and $R = (C + 1)_{2n}$, we obtain the biembedding represented in Fig. 6. It was shown in Theorem 2.2 that

$\mathcal{S}$

C'_{2n}	C_{2n}
C_{2n}	$(C+1)'_{2n}$

$\mathcal{T}$

$\mathcal{B}$	C'_{2n}
C'_{2n}	C_{2n}

Fig. 6. A biembedding supporting Latin squares in different main classes.

the Latin square $\mathcal{S}$ belongs to the same main class as the Cayley table of the cyclic group, while Theorem 4.1 shows that $\mathcal{T}$ does not belong to the same main class as the Cayley table of a group. Hence in this new construction we see that the Latin squares so obtained are in different main classes.

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