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A CONVEX CHARACTERIZATION OF THE GRAPHS OF THE DODECAHEDRON AND ICOSAHEDRON

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Let Γ be a 3-polytopal graph such that every face of Γ is convex. We prove that if the set of proper convex subgraphs of Γ is equal to the set of proper convex subgraphs of the dodecahedron (resp. icosahedron), then Γ is isomorphic to the dodecahedron (resp. icosahedron).

1. Introduction

In this paper, we always consider finite undirected graphs Γ which we assume to be connected. For the usual definitions and notations, we follow the terminology of [2]. Let $V(\Gamma)$ denote the set of vertices of Γ. If $x \in V(\Gamma)$, we denote by $d(x)$ the degree of x. If x is adjacent to y, we write $x \sim y$.

A path joining two vertices x and y of Γ is called *geodesic* if it is a path of minimum length joining x to y. An induced subgraph S of Γ is called *convex* if for any two vertices x and y of S, every geodesic path joining x to y is contained in S. A convex subgraph S is *proper* if it is not the empty graph, a vertex, an edge or the whole graph. If X is any subset of $V(\Gamma)$, the *convex closure* $\langle X\rangle$ is the smallest convex subgraph of Γ containing X (that is the intersection of all convex subgraphs of Γ containing X). If $x_1, x_2, \ldots, x_n$ are the vertices of X, we write $\langle X\rangle = \langle x_1, x_2, \ldots, x_n\rangle$. The set of all pairwise non-isomorphic proper convex subgraphs of Γ will be denoted by $C(\Gamma)$.

A finite graph Γ is 3-*polytopal* if Γ is isomorphic to the graph of vertices and edges of some convex 3-polytope in 3-dimensional Euclidean space. Moreover any cycle of Γ corresponding to a face of the 3-polytope is called a *face* of Γ. By a well-known theorem of Steinitz [1], a graph is 3-polytopal if and only if it is finite, planar and 3-connected.

Note that two convex 3-polytopes corresponding to the same 3-polytopal graph have necessarily the same combinatorial type [1]. Let P be a convex 3-polytope; in the following, for the convenience of the reader, we shall also denote by P the 3-polytopal graph corresponding to P.

We have proved in [3] that if Γ is a 3-polytopal graph, the set of proper convex subgraphs of Γ coincides with the set of faces of Γ if and only if Γ is the

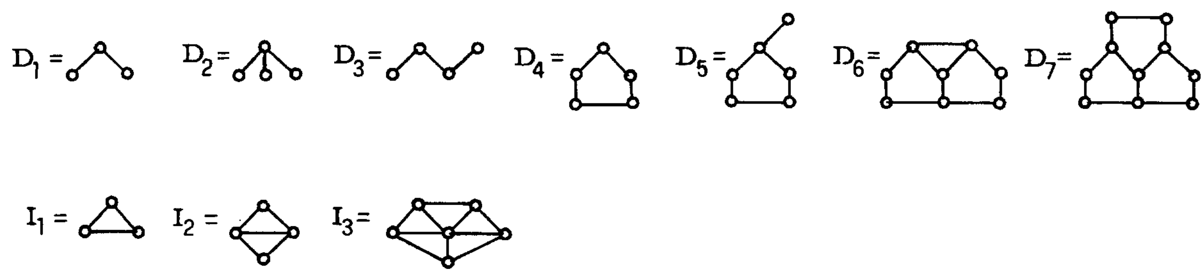

Fig. 1.

tetrahedron, the triangular prism, the cube or the k-gonal bipyramid ($k \geq 4$). Two of the 5 regular polyhedra are excluded by this property. Indeed, the dodecahedron and icosahedron contain some proper convex subgraphs which are not faces. If $\mathscr{D} = C$ (Dodecahedron) and $\mathscr{I} = C$ (Icosahedron), we have $\mathscr{D} = \{D_1, D_2, D_3, D_4, D_5, D_6, D_7\}$ and $\mathscr{I} = \{I_1, I_2, I_3\}$. See Fig. 1.

The purpose of this paper is to prove the following theorems:

Theorem 1. *If Γ is a 3-polytopal graph such that every face of Γ is convex and $C(\Gamma) = \mathscr{D}$, then Γ is isomorphic to the dodecahedron.*

Theorem 2. *If Γ is a 3-polytopal graph such that every face of Γ is convex and $C(\Gamma) = \mathscr{I}$, then Γ is isomorphic to the icosahedron.*

We don't know any example of a 3-polytopal graph Γ with $C(\Gamma) = \mathscr{D}$ or $C(\Gamma) = \mathscr{I}$ and such that at least one face of Γ is a non convex subgraph. We have the feeling that, when $C(\Gamma) = \mathscr{D}$, the convexity condition on the faces could be removed.

Conjecture. If Γ is a 3-polytopal graph such that $C(\Gamma) = \mathscr{D}$, then Γ is isomorphic to the dodecahedron.

2. Proof of Theorem 1

Lemma 1.1. *Let Γ be a 3-polytopal graph. If Γ is regular of degree 3 and if every face of Γ is a 5-cycle, then Γ is isomorphic to the dodecahedron.*

Proof. Let $F_1 = x_1x_2x_3x_4x_5x_1$ be a face of Γ. (See Fig. 2.) The edge $\{x_1, x_2\}$, which is already in the face F_1, must be in another face, say F_2. Since x_1, x_2 are the only two vertices of Γ incident with both F_1 and F_2, there exist three new vertices, x_6, x_7 and x_8, such that $F_2 = x_1x_2x_6x_7x_8x_1$. The edge $\{x_2, x_3\}$ belongs to F_1 and to another face F_3. Since $d(x_2) = 3$ and since x_2, x_3 are the only two vertices incident

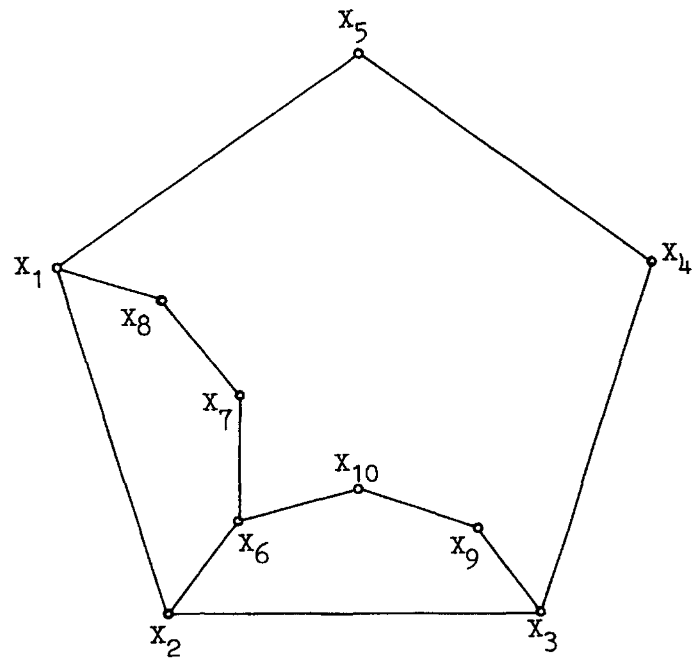

Fig. 2.

with F_1 and F_3, the faces F_2 and F_3 have exactly x_2 and x_6 in common. This implies that there are two new vertices x_9 and x_{10}, such that $F_3 = x_2x_3x_9x_{10}x_6x_2$. The edge $\{x_3, x_4\}$ belongs to F_1 and to another face F_4. Note that $x_4 \not\sim x_7$, $x_4 \not\sim x_8$, $x_9 \not\sim x_7$ and $x_9 \not\sim x_8$. Indeed, if $x_4 \sim x_7$ (resp. $x_4 \sim x_8, x_9 \sim x_7$), the cycle $x_3x_4x_7x_6x_{10}x_9x_3$ (resp. $x_4x_8x_1x_5x_4$, $x_7x_6x_{10}x_9x_7$) would be a face of Γ, contradicting the fact that every face is a 5-cycle; if $x_9 \sim x_8$, since x_1, x_3, x_4 are incident with exactly one face (namely F_1), the path $x_4x_3x_9x_8x_7$ would be incident with a face of Γ, which is impossible since every face is a 5-cycle and $x_4 \not\sim x_7$. Since $d(x_3) = 3$ and since x_3, x_4 are the only two vertices incident with F_1 and F_4, x_9 is necessarily incident with F_4. we conclude that there are two new vertices x_{11} and x_{12} such that $F_4 = x_3x_4x_{11}x_{12}x_9x_3$. Similar arguments show that there exist three new vertices x_{13}, x_{14} and x_{15}, such that the edge $\{x_4, x_5\}$ (resp. $\{x_5, x_1\}$) belongs to the faces F_1 and $F_5 = x_4x_5x_{13}x_{14}x_{11}x_4$ (resp. $F_6 = x_5x_1x_8x_{15}x_{13}x_5$). The vertices x_7, x_{10}, x_{12}, x_{14} and x_{15} are obviously pairwise non adjacent (otherwise it would force Γ to contain a triangular face). It follows easily that there are five vertices, x_{16}, x_{17}, x_{18}, x_{19}, x_{20} such that the 5-cycles $F_7 = x_6x_7x_{16}x_{17}x_{10}x_6$, $F_8 = x_{10}x_9x_{12}x_{18}x_{17}x_{10}$, $F_9 = x_{12}x_{11}x_{14}x_{19}x_{18}x_{12}$, $F_{10} = x_{14}x_{13}x_5x_{20}x_{19}x_{14}$, $F_{11} = x_5x_8x_7x_{16}x_{20}x_5$ and $F_{12} = x_{16}x_{17}x_{18}x_{19}x_{20}$ are faces of Γ. We conclude that Γ contains a subgraph D isomorphic to the dodecahedron. Since D and Γ are regular of degree 3 and since Γ is connected, Γ must be isomorphic to D. $\square$

Lemma 1.2. *Let Γ be a 3-polytopal graph such that every face of Γ is convex and $C(\Gamma) = \mathfrak{D}$. If for any two vertices x, y at distance 2 in Γ, the convex closure $\langle x, y\rangle$ is proper (Condition (*)), then Γ is isomorphic to the dodecahedron.*

Proof. Since every face of Γ is convex and since the only cycle in the set $\mathscr{D}$ is a 5-cycle, every face of Γ is necessarily a 5-cycle. By Steinitz's theorem, $d(x) \geqslant 3$ for every vertex x of Γ. If there is a vertex x with $d(x) \geqslant 4$, then Γ contains a 4-claw. This 4-claw is necessarily a proper convex subgraph (otherwise, by Condition (*), Γ would contain a 3-cycle, which would be a proper convex subgraph not in $\mathscr{D}$). This is a contradiction, because no element of $\mathscr{D}$ is a 4-claw. Therefore Γ is a regular graph of degree 3. Since Γ satisfies the hypotheses of Lemma 1.1, Γ is isomorphic to the dodecahedron. □

Proof of Theorem 1. Let Γ be a graph satisfying the hypotheses of Theorem 1. In view of Lemma 1.2, it suffices to prove that Γ satisfies Condition (*). Suppose that there are two vertices x_1, x_3 at distance 2 in Γ such that $\langle x_1, x_3\rangle = \Gamma$. Let x_2 be a vertex adjacent to both x_1 and x_3, and let $F = x_2x_3x_4x_5x_6x_2$ be a face of Γ containing the edge $\{x_2, x_3\}$ (by the same argument as in the proof of Lemma 1.2, every face of Γ is a 5-cycle). Finally, let Γ' be a planar representation of Γ in which F is the infinite face.

Since $\langle x_1, x_3\rangle = \Gamma$ is not reduced to the set $\{x_1, x_2, x_3\}$, there is at least a 4-cycle containing the path $x_1x_2x_3$, and so in particular there is at least a 4-cycle passing through the edge $\{x_2, x_3\}$. Let $C = x_2x_3x_7x_8x_2$ be the unique 4-cycle of Γ' such that any other 4-cycle passing through the edge $\{x_2, x_3\}$ is inside C. Let S be the subgraph of Γ' induced on the set of vertices lying on C or inside C. We claim that S is a proper convex subgraph of Γ'. S is obviously proper because x_5 is not in S. If S is not convex, there are two vertices a, b of S and a geodesic path P joining a and b, such that P is not contained in S. Since P is geodesic and since Γ' has no 3-cycle, there is a vertex x outside C, such that P contains the path x_3xx_8 or the path x_2xx_7. The existence of one of the 4-cycles $x_2x_3xx_8x_2$ or $x_2x_3x_7xx_2$ contradicting the choice of C, we conclude that S is convex.

We have seen before that there is a 4-cycle containing the path $x_1x_2x_3$. It follows that x_1 is a vertex of S. Since S is convex, the convex closure $\langle x_1, x_3\rangle = \Gamma'$ is contained in S, contradicting the fact that S is a proper subgraph of Γ'. □

3. Proof of Theorem 2

Lemma 2. *Let Γ be a 3-polytopal graph such that every face of Γ is convex and $C(\Gamma) = \mathscr{I}$. If for any two vertices x, y at distance 2 in Γ, the convex closure $\langle x, y\rangle$ is proper (Condition (*)), then Γ is the icosahedron.*

Proof. Since every face of Γ is convex and since the only cycle in the set $\mathscr{I}$ is a 3-cycle, every face of Γ is a 3-cycle. Moreover Condition (*) implies that for any two vertices x, y at distance 2 in Γ, $\langle x, y\rangle$ is necessarily isomorphic to the subgraph I_2 (this follows from the fact that the convex closure of two vertices at distance 2

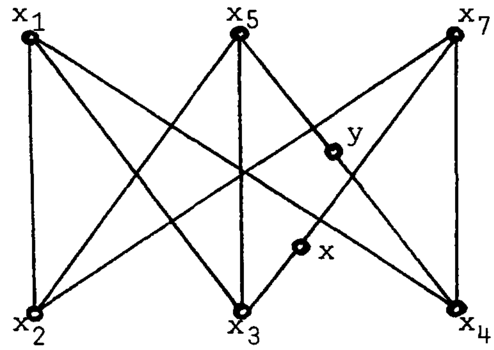

Fig. 3.

in I_3 is isomorphic to I_2). Let $T_1 = x_1x_2x_3x_1$ be a face of Γ and let Γ' be a planar representation of Γ in which T_1 is the infinite face. The edge $\{x_1, x_2\}$, which is in T_1, must be in another face $T_2 = x_1x_2x_4x_1$. Since the complete graph K_4 is not an element of $\mathcal{I}$, we have $x_3 \not\sim x_4$ and so there is necessarily a new vertex x_5 such that the edge $\{x_2, x_3\}$ is in the faces T_1 and $T_3 = x_2x_3x_5x_2$, with $x_5 \not\sim x_1$. For similar reasons, there is a new vertex x_6 such that $\{x_1, x_3\}$ is in the faces T_1 and $T_4 = x_1x_3x_6x_1$, with $x_6 \not\sim x_2$.

Since $\langle x_3, x_4\rangle$ and $\langle x_2, x_6\rangle$ are isomorphic to I_2, we know that $x_4 \not\sim x_5$, $x_4 \not\sim x_6$ and $x_5 \not\sim x_6$. Therefore there is a new vertex x_7 such that the edge $\{x_2, x_4\}$ is in the faces T_2 and $T_5 = x_2x_4x_7x_2$. Note that $x_7 \not\sim x_1$ (otherwise there would be a subgraph isomorphic to K_4), $x_7 \not\sim x_3$ (otherwise $\langle x_1, x_5\rangle$ would not be isomorphic to I_2) and $x_7 \not\sim x_6$ (otherwise $\langle x_4, x_6\rangle$ would not be isomorphic to I_2). On the other hand, $x_7 \sim x_5$. Indeed, if $x_5 \not\sim x_7$, since $\langle x_7, x_3\rangle$ (resp. $\langle x_4, x_5\rangle$) is not isomorphic to I_2, there is a new vertex x (resp. y) adjacent to x_2, x_3, x_7 (resp. x_2, x_4, x_5) ($x \neq y$, otherwise $x \sim x_4$ and $\langle x_3, x_4\rangle$ would not be isomorphic to I_2) and Γ' contains the subgraph K shown in Fig. 3. But this contradicts the planarity of Γ' by Kuratowski's theorem. Therefore $x_5 \sim x_7$.

We prove in the same way the existence of a new vertex x_8 (resp. x_9) such that the edge $\{x_1, x_4\}$ (resp. $\{x_3, x_6\}$) is in the faces T_2 (resp. T_4) and $T_6 = x_1x_4x_8x_1$ (resp. $T_7 = x_3x_6x_9x_3$) with $x_8 \sim x_6$ (resp. $x_9 \sim x_5$) and $x_7 \not\sim x_8$, $x_7 \not\sim x_9$, $x_8 \not\sim x_9$. Using the convex closures $\langle x_7, x_8\rangle$, $\langle x_7, x_9\rangle$, $\langle x_8, x_9\rangle$ and similar arguments, it is now easy to prove the existence of three new vertices x_{10}, x_{11}, x_{12} such that the subgraph S of Γ' induced on the set $\{x_1, x_2, \ldots, x_{12}\}$ is isomorphic to the icosahedron.

Suppose that Γ' is not isomorphic to S. Then, since Γ' is connected there is a vertex x not in S such that x is adjacent to some vertex x_i of S and is inside a 3-cycle $x_ix_jx_kx_i$ of S. Since S is isomorphic to the icosahedron, there is a vertex x_t of S with $x_t \sim x_i$, $x_t \not\sim x_j$ and $x_t \not\sim x_k$. Because of the planarity of Γ', $\langle x, x_t\rangle$ cannot be isomorphic to I_2, a contradiction. Therefore Γ' must be isomorphic to S. □

Proof of Theorem 2. In view of Lemma 2, it suffices to prove that any graph satisfying the hypotheses of Theorem 2 satisfies necessarily Condition (*). Let us

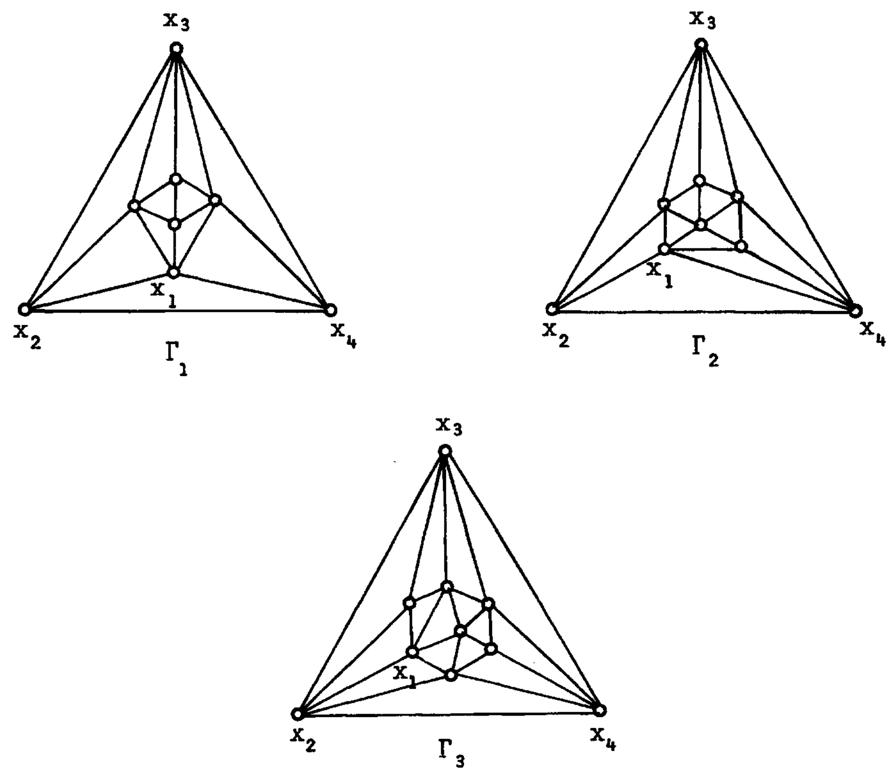

Fig. 4.

suppose that there exists a graph Γ which satisfies the hypotheses of Theorem 2 but not Condition (*). It is easy to see that Γ contains necessarily two vertices x_1, x_3 at distance 2 such that there are at least three paths of length 2 joining x_1 to x_3 (in particular $\langle x_1, x_3\rangle = \Gamma$). Let x_2 be a vertex adjacent to both x_1 and x_3, and let $F = x_2x_3x_4x_2$ be a face of Γ containing the edge $\{x_2, x_3\}$. Finally, let Γ' be a planar representation of Γ in which F is the infinite face.

Using the following properties of Γ': each face of Γ' is a 3-cycle, each edge of Γ' is in exactly two faces, Γ' has no subgraph isomorphic to K_4, the convex closure of two vertices at distance 2 in Γ' is never reduced to a unique path of length 2 and $\langle x_1, x_3\rangle = \Gamma'$, it is easy but tedious to construct Γ' edge by edge. We shall not give the details here. It turns out that Γ' is isomorphic either to a k-gonal bipyramid, with $k \geqslant 4$, or to one of the three graphs Γ_1, Γ_2, Γ_3 represented in Fig. 4, where, for convenience of the reader, we have marked the vertices x_1, x_2, x_3, x_4.

A k-gonal bipyramid ($k \geqslant 4$) does not contain I_2 as proper convex subgraph, the graphs Γ_1 and Γ_2 do not contain I_3 as proper convex subgraph, and the graph Γ_3 has a proper convex subgraph which is not an element of $\mathcal{I}$ (namely the graph of a 4-gonal bipyramid). We conclude that Γ does not satisfy the hypotheses of Theorem 2, a contradiction. □

Acknowledgement

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References

[1] B. Grünbaum, Convex Polytopes (Wiley, New York, 1967).
[2] F. Harary, Graph Theory (Addison-Wesley, MA, 1969).
[3] P. Vanden Cruyce, A convexity problem in 3-polytopal graphs, submitted to Archiv der Mathematik.