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Communication

Positivity of second order linear recurrent sequences

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Abstract

We give a decision method for the Positivity Problem for second order recurrent sequences: it is decidable whether or not a recurrent sequence defined by $u_n = au_{n-1} + bu_{n-2}$ has only nonnegative terms.

Keywords: Positivity Problem; Recurrent sequence; Integer matrix; Decidability

1. Introduction

In general, a (homogeneous) *linear recurrence equation* over integers has the form

$$u_n = a_1 u_{n-1} + a_2 u_{n-2} + \cdots + a_k u_{n-k}, \tag{1}$$

for $n \geqslant k$, where $a_1, a_2, \ldots, a_k \in \mathbb{Z}$ are integer constants. The linear recurrence equation (1) defines a unique integer sequence $(u_n)_{n=0}^{\infty}$ after the first k initial terms $u_0, u_1, \ldots, u_{k-1}$ are given. A sequence $(u_n)_{n=0}^{\infty}$ is said to be *recurrent* if it is defined by a linear recurrence equation. The integer k in (1) is called the *order* of the recurrence and also of the defined recurrent sequence.

In the following we are mainly interested in second order recurrent sequences, i.e., those defined by a linear recurrence equation (1) in the form

$$u_n = au_{n-1} + bu_{n-2} \tag{2}$$

for constants $a, b \in \mathbb{Z}$. The Fibonacci numbers form probably the most well known recurrent sequence of second order. The linear recurrence equation defining these numbers is $u_n = u_{n-1} + u_{n-2}$ and the initial terms are $u_0 = u_1 = 1$.

We shall consider the following problem of recurrent sequences.

Positivity Problem. Let a linear recurrence equation (1) be given together with the initial terms u_i for $i = 0, 1, \ldots, k-1$. Is the recurrent sequence $(u_n)_{n=0}^{\infty}$ *nonnegative*, i.e., does it hold that $u_n \geqslant 0$ for all n?

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If in the above $k=2$, then the problem is referred to as the *second order Positivity Problem*. We shall show that this problem has an algorithmic solution, that is, *the second order Positivity Problem is decidable.*

The proof uses only elementary techniques of the complex numbers.

As an example, consider the recurrent sequence $(u_n)_{n=0}^{\infty}$, where

$$u_n = 9u_{n-1} - 21u_{n-2}$$

with $u_0 = 1$ and $u_1 = 14$. The first negative member of this sequence is $u_{17} = -344\,532\,183\,345$ while the maximum value before u_{17} is as high as $u_{15} = 17\,954\,992\,251$.

The Positivity Problem is also well known for matrices, and it is through matrix theory that it has many connections to other fields of mathematics; see, for instance, [7,8]. Indeed, let $M \in \mathbb{Z}^{k\times k}$ be a $k \times k$ integer matrix and $\mathbf{v} = (1, 0, \ldots, 0)$ and $\mathbf{u} = (0, \ldots, 0, 1)$ two vectors in $\mathbb{Z}^k$. It is straightforward to show that the sequence $(u_n)_{n=0}^{\infty}$ defined by

$$u_n = \mathbf{v}M^n\mathbf{u}^{\mathrm{T}}$$

is a recurrent sequence of order k. Therefore it follows from our main theorem that it is also decidable whether or not the right upper-corner entries $(M^n)_{1,2}$ are nonnegative for all powers of a given 2×2 integer matrix M. This connection between matrices and recurrent sequences works also in the other direction, but then the dimension of the matrix may grow. Indeed, for any recurrent sequence $(u_n)_{n=0}^{\infty}$ of order k, there exists a matrix $M \in \mathbb{Z}^{(k+1)\times(k+1)}$ such that $u_n = (M^n)_{1,k+1}$ for all $n \geqslant 1$.

Before going into more details of the proof, we mention shortly some other related questions on recurrent sequences. In *Skolem's Problem* it is required to determine whether or not a given recurrent sequence $(u_n)_{n=0}^{\infty}$ has a zero: $u_n = 0$ for some $n \geqslant 0$. It is known, but highly nontrivial to prove, that Skolem's Problem is decidable for recurrent sequences of order 5 or less; see [4]. It is also known that if the Positivity Problem is decidable for all orders $k \geqslant 1$, then also Skolem's Problem is decidable in the general case. Indeed, this follows from the fact that if $(u_n)_{n=0}^{\infty}$ is recurrent, then so is $(u_n^2 - 1)_{n=0}^{\infty}$; for the details of the construction of $(u_n^2 - 1)_{n=0}^{\infty}$ we refer to [2]. On the other hand, it is decidable whether or not a linear recurrence equation (of any order) produces infinitely many zeros; see [1].

Finally, we also mention an old open problem for 2×2 integer matrices. In the *Mortality Problem* a finite set $M_1, M_2, \ldots, M_k$ of 2×2 integer matrices is given, and the task is to determine whether or not $M_{i_1}M_{i_2}, \ldots, M_{i_n} = 0$ for some product of these matrices. It is not known whether this problem is decidable; see [9]. However, if we increase the size of the matrices to 3, then the problem becomes undecidable. This result is due to Paterson [6]; see also [3].

We end this section with the well known result giving the solutions of a linear recurrence equation of second order; for the general case, see cf. [5, Theorem 3.1.1]. We denote as usual by $\overline{z}$ the complex conjugate of complex number z.

Lemma 1. *Let*

$$u_n = au_{n-1} + bu_{n-2} \tag{3}$$

be a linear recurrence equation with $a, b \neq 0$, and let $p(x) = x^2 - ax - b$ be its characteristic polynomial with discriminant $D = a^2 + 4b$.

(1) *If $D > 0$, then $u_n = A\lambda_1^n + B\lambda_2^n$, where $\lambda_1 \neq \lambda_2$ are the real roots of $p(x)$, $A = \frac{u_1 - u_0\lambda_2}{\sqrt{D}}$ and $B = \frac{u_0\lambda_1 - u_1}{\sqrt{D}}$.*
(2) *If $D = 0$, then $u_n = (A + Bn)\lambda^n$, where $\lambda = a/2$ is the double root of $p(x)$, $A = u_0$, and $B = \frac{2u_1 - u_0 a}{a}$.*
(3) *If $D < 0$, then $u_n = A\lambda^n + \overline{A}\overline{\lambda}^n$, where λ and $\overline{\lambda}$ are the complex roots of $p(x)$, and A is as in Case* 1.

Proof. To determine A and B in Case 1, the system of equations

$$\begin{cases} u_0 = A + B \\ u_1 = A\lambda_1 + B\lambda_2 \end{cases}$$

can be solved by Cramer's rule, since its determinant $-\sqrt{D}$ is nonzero. The solutions are as in the statement. It is then a simple task to show by induction on n that the claimed solution indeed satisfies the recurrence. The same conclusion holds for Case 3, where, in addition, $B = \overline{A}$, since in this case $\sqrt{D}$ is a pure imaginary number.

In Case 2 we have $b = -a^2/4$ and the system of equations becomes

$$\begin{cases} u_0 = A \\ u_1 = A\lambda + B\lambda \end{cases}$$

Since $a \neq 0$, the determinant (which is now equal to λ) is again nonzero and Cramer's rule applies. The solutions of the system are as stated, and it is again easy to verify that the recurrence is satisfied by these solutions. □

The expression of the terms u_n in the three cases of Lemma 1 is usually called the *solution* of the recurrent sequence $(u_n)_{n=0}^{\infty}$. Note that the roots are nonzero, since $b \neq 0$.

2. Solving the Positivity Problem

We begin with the easiest cases not covered by Lemma 1, and which, actually, corresponds to the Positivity Problem of the first order recurrence. Indeed, the Positivity Problem can be solved for first order recurrent sequences in a trivial way, since then $u_n = a^n u_0$ for any $n \geqslant 0$, and $(u_n)_{n=0}^{\infty}$ is nonnegative if and only if $a \geqslant 0$ and $u_0 \geqslant 0$.

Lemma 2. *If $a = 0$ or $b = 0$ in* (3), *then the Positivity Problem can be effectively solved.*

Proof. If $b = 0$, then $u_n = a^{n-1}u_1$ for all $n \geqslant 1$, and thus the sequence is nonnegative if and only if $u_0, u_1 \geqslant 0$ and $a \geqslant 0$.

If $a = 0$, i.e., $u_n = bu_{n-2}$, then we divide $(u_n)_{n=0}^{\infty}$ into two sequences, namely

$$u_n = \begin{cases} b^m u_0 & \text{if } n = 2m, \\ b^m u_1 & \text{if } n = 2m + 1. \end{cases}$$

Hence the sequence is nonnegative if and only if $u_0, u_1 \geqslant 0$ and $b \geqslant 0$. □

We now turn to the cases of Lemma 1: we assume that $a \neq 0$ and $b \neq 0$ in the following. In Case 1 the asymptotic behaviour of the sequence $(u_n)_{n=0}^{\infty}$ is essentially determined by the characteristic root with the maximum absolute value. Notice first that, in the terms of Lemma 1, $p(x) = x^2 + (\lambda_1 + \lambda_2)x + \lambda_1\lambda_2$, i.e.,

$$a = \lambda_1 + \lambda_2 \quad \text{and} \quad b = \lambda_1\lambda_2. \tag{4}$$

Lemma 3. *Assume that, for all $n \geqslant 0$, $u_n = A\lambda_1^n + B\lambda_2^n$, where $\lambda_1 \neq \lambda_2$ are real and nonzero, and $A, B \in \mathbb{R}$. Then the Positivity Problem of $(u_n)_{n=0}^{\infty}$ can be effectively solved.*

Proof. By (4), $|\lambda_1| \neq |\lambda_2|$, since the roots are supposed to be real, and $a \neq 0$. Assume that $|\lambda_1| > |\lambda_2|$. Now $u_n = \lambda_1^n \alpha_n$, where

$$\alpha_n = A + B\left(\frac{\lambda_2}{\lambda_1}\right)^n \tag{5}$$

with $|\frac{\lambda_2}{\lambda_1}| < 1$. If $A = 0 = B$, then the problem is trivial. Therefore, assume that $AB \neq 0$. We have then two cases according to the sign of λ_1.

If $\lambda_1 < 0$, then $u_n \geqslant 0$ if and only if, for each n, $\alpha_n = 0$ or its sign equals the sign of λ_1^n. Since $\lim_{n\to\infty}(\frac{\lambda_2}{\lambda_1})^n = 0$, there can be only one index n with $\alpha_n = 0$, and the sequence (5) alternates in sign only if $A = 0$, $\lambda_2 > 0$ and $B \neq 0$. Since $\frac{\lambda_2}{\lambda_1} < 0$, necessarily $B > 0$. Therefore, in this case, the sequence $(u_n)_{n=0}^{\infty}$ is nonnegative if and only if $\lambda_2 > 0$, $A = 0$ and $B > 0$.

If $\lambda_1 > 0$, then the sequence $(u_n)_{n=0}^{\infty}$ is nonnegative if and only if $\alpha_n \geqslant 0$ for all n. If $B = 0$, then $(u_n)_{n=0}^{\infty}$ is nonnegative when $A \geqslant 0$; otherwise $\alpha_n \geqslant 0$ is equivalent to

$$\left(\frac{\lambda_2}{\lambda_1}\right)^n \geqslant \frac{-A}{B}. \tag{6}$$

We have two cases according to whether $\lambda_2 > 0$ or not. First, if $\lambda_2 > 0$, then (6) is satisfied if and only if $\frac{-A}{B} \leqslant 0$, since $(\frac{\lambda_2}{\lambda_1})^n$ tends to zero. In the second case, where $\lambda_2 < 0$, again $(\frac{\lambda_2}{\lambda_1})^n$ tends to zero, but with alternating signs. Now it is easy to see that $(u_n)_{n=0}^{\infty}$ is nonnegative, i.e., (6) holds for all $n \geqslant 0$, if and only if (6) holds for $n = 1$. Indeed, in this case $(\frac{\lambda_2}{\lambda_1})^n \geqslant (\frac{\lambda_2}{\lambda_1})$ for all $n \geqslant 0$.

Clearly, each of the above cases provides a method for deciding the nonnegativity of the sequence $(u_n)_{n=0}^{\infty}$. □

Next we consider the Positivity Problem for Case 2 of Lemma 1.

Lemma 4. *Assume that $u_n = (A + Bn)\lambda^n$ for all $n \geqslant 0$. Then the Positivity Problem of the sequence $(u_n)_{n=0}^{\infty}$ can be effectively solved.*

Proof. If $B = 0$, then $(u_n)_{n=0}^{\infty}$ is nonnegative if and only if $A \geqslant 0$ and $\lambda \geqslant 0$. We assume thus that $B \neq 0$. Note that the sign of $A + Bn$ can change only once when n grows. Therefore, if $\lambda < 0$, then $(u_n)_{n=0}^{\infty}$ cannot be nonnegative. On the other hand, if $\lambda > 0$, then $(u_n)_{n=0}^{\infty}$ is nonnegative if and only if $A + Bn \geqslant 0$ for all $n \geqslant 0$, which is equivalent to the condition $n \geqslant \frac{-A}{B}$, and thus to the condition $\frac{-A}{B} \leqslant 0$. □

There remains the case of two complex roots, i.e., Case 3. For a complex number α, we use the representation $\alpha = |\alpha| e^{i\theta}$, where $\theta \in [-\pi, \pi)$ is the *phase* of α.

Assume that

$$u_n = A\lambda^n + \overline{A}\overline{\lambda}^n,$$

for all $n \geqslant 0$, and let $\lambda = |\lambda| e^{i\theta}$ and $A = |A| e^{i\phi}$ for some $\theta, \phi \in [-\pi, \pi)$. Then

$$\begin{aligned} u_n &= |A| e^{i\phi}(|\lambda| e^{i\theta})^n + |A| e^{-i\phi}(|\lambda| e^{-i\theta})^n \\ &= |A||\lambda|^n (e^{i(\phi+n\theta)} + e^{-i(\phi+n\theta)}) \\ &= 2|A||\lambda|^n \cos(\phi + n\theta). \end{aligned} \tag{7}$$

We need only to study the sign of $\cos(\phi + n\theta)$ in order to solve the Positivity Problem. Note that in Case 3 the root λ in not real and hence $\theta \notin \{-\pi, 0\}$.

Lemma 5. *If $u_n = A\lambda^n + \overline{A}\overline{\lambda}^n$ for $n \geqslant 0$ and $\lambda \notin \mathbb{R}$, then $(u_n)_{n=0}^{\infty}$ has negative elements.*

Proof. By the above considerations, we need only to consider the signs of $\cos(\phi + n\theta)$ in (7). Now $\phi, \theta \in [-\pi, \pi)$ with $\theta \notin \{-\pi, 0\}$. Assume contrary to the claim that $u_n \geqslant 0$ for all n, i.e., $\cos(\phi + n\theta) \geqslant 0$ for all $n \geqslant 0$, which implies that for each $n \geqslant 0$,

$$\phi + n\theta \in [-\pi/2 + k2\pi, \pi/2 + k2\pi],$$

for some $k \in \mathbb{Z}$. However, we must have $\phi \in [-\pi/2, \pi/2]$, and then $|\theta| < \pi$ implies that also $\phi + \theta \in [-\pi/2, \pi/2]$. Therefore inductively we obtain that $\phi + n\theta \in [-\pi/2, \pi/2]$ for all $n \geqslant 0$. But $\theta \neq 0$ implies $\lim_{n\to\infty} |\phi + n\theta| = \infty$; a contradiction. Therefore, we must have $u_n < 0$ for some n. □

For our algorithm, Lemma 5 yields that for Case 3 the answer for Positivity Problem is always negative.

From the previous lemmas, our main theorem follows:

Theorem 1. *The Positivity Problem is decidable for second order recurrent sequences.*

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