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Lineability of the set of bounded linear non-absolutely summing operators

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ABSTRACT

In this note we solve, except for extremely pathological cases, a question posed by Puglisi and Seoane-Sepúlveda on the lineability of the set of bounded linear non-absolutely summing operators. We also show how the idea of the proof can be adapted to several related situations.

1. Introduction and notation

Henceforth E, F and G will stand for infinite-dimensional (real or complex) Banach spaces. The topological dual of F is represented by F^*.

According to [2,7,10] and others, a subset A of an infinite-dimensional vector space X is said to be *lineable* if $A \cup \{0\}$ contains an infinite-dimensional subspace of X.

The space of absolutely (r,s)-summing linear operators from E to F will be denoted by $\Pi_{r,s}(E;F)$ ($\Pi_r(E;F)$ if $r=s$) and the space of bounded linear operators from E to F will be represented by $\mathcal{L}(E;F)$. For details on the theory of absolutely summing operators we refer to [6].

Recently, D. Puglisi and J. Seoane-Sepúlveda [15] proved, among other interesting results, that if E has the two series property and $G = F^*$ for some F, then the set

$$\mathcal{L}(E;G) \setminus \Pi_1(E;G)$$

is lineable. In the same paper the authors pose the following question:

Problem 1.1. If E is superreflexive and $p \geqslant 1$, is it true that

$$\mathcal{L}(E;F) \setminus \Pi_p(E;F)$$

is lineable for every Banach space F?

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M.A. Sofi, in a private communication to the authors, kindly pointed out that the following situation should be settled first: given operator ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ and Banach spaces E and F, is it always true that $\mathcal{I}_1(E;F) \setminus \mathcal{I}_2(E;F)$ is either empty of lineable? Quite surprisingly, we have:

Example 1.2. Let SS denote the ideal of strictly singular linear operators and E be a hereditarily indecomposable complex Banach space. Let us show that the set $\mathcal{L}(E;E) \setminus SS(E;E)$, which is not empty because of the identity operator, does not contain a two-dimensional subspace. Let u_1, u_2 be arbitrary linearly independent operators in $\mathcal{L}(E;E) \setminus SS(E;E)$. By [13, Theorem 6] there are scalars λ_1, λ_2 and strictly singular operators $v_1, v_2 \in SS(E;E)$ such that $u_1 = \lambda_1 \, id_E + v_1$ and $u_2 = \lambda_2 \, id_E + v_2$. It is clear that $\lambda_1 \neq 0 \neq \lambda_2$ because u_1 and u_2 are not strictly singular. Letting $u = \lambda_2 u_1 - \lambda_1 u_2$ we have that $u \neq 0$ because u_1 and u_2 are linearly independent; from $u = \lambda_2 v_1 - \lambda_1 v_2$ we conclude that u is strictly singular. Hence u belongs to the subspace generated by u_1 and u_2 but $u \notin (\mathcal{L}(E;E) \setminus SS(E;E)) \cup \{0\}$, proving that $(\mathcal{L}(E;E) \setminus SS(E;E)) \cup \{0\}$ does not contain a two-dimensional subspace.

In the absence of a general result, particular situations must be investigated by *ad hoc* arguments. The aim of this short note is to answer Problem 1.1 in the positive, except for very particular quite pathological cases, and to extend the idea of the proof to related situations.

2. Superreflexive spaces

By $\mathcal{K}$ we denote the ideal of compact operators.

Theorem 2.1. *Let $p \geqslant 1$ and E be superreflexive. If either E contains a complemented infinite-dimensional subspace with unconditional basis or F contains an infinite unconditional basic sequence, then $\mathcal{K}(E;F) \setminus \Pi_p(E;F)$ (hence $\mathcal{L}(E;F) \setminus \Pi_p(E;F)$) is lineable.*

Proof. Assume that E contains a complemented infinite-dimensional subspace E_0 with unconditional basis $(e_n)_{n=1}^{\infty}$. First consider

$$\mathbb{N} = A_1 \cup A_2 \cup \cdots \tag{2.1}$$

a decomposition of $\mathbb{N}$ into infinitely many infinite pairwise disjoint subsets $(A_j)_{j=1}^{\infty}$. Since $\{e_n; n \in \mathbb{N}\}$ is an unconditional basis, it is well known (e.g., combine [12, Proposition 1.c.6] and [1, Proposition 1.1.9]) that $\{e_n; n \in A_j\}$ is an unconditional basic sequence for every $j \in \mathbb{N}$. Let us denote by E_j the closed span of $\{e_n; n \in A_j\}$. As a subspace of a superreflexive space, E_j is superreflexive as well, so from [5, Theorem] it follows that for each j there is an operator

$$u_j : E_j \longrightarrow F$$

belonging to $\mathcal{K}(E_j;F) \setminus \Pi_p(E_j;F)$.

Denoting by ϱ the unconditional basis constant of $(e_n)_{n=1}^{\infty}$ we know that

$$\left\| \sum_{j=1}^{\infty} \varepsilon_j a_j e_j \right\| \leqslant \varrho \left\| \sum_{j=1}^{\infty} a_j e_j \right\|$$

for every $\varepsilon_j = \pm 1$ and scalars a_j. For each i we denote by $P_i : E_0 \longrightarrow E_i$ the canonical projection onto E_i. For

$$y = \sum_{j=1}^{\infty} a_j e_j \in E_0 \quad \text{and} \quad x = P_i(y) \in E_i$$

we have

$$2x = \sum_{j \in A_i} 2a_j e_j = \sum_{j=1}^{\infty} \varepsilon_j a_j e_j + \sum_{j=1}^{\infty} \varepsilon'_j a_j e_j$$

for a convenient choice of signs ε_j and ε'_j. Thus

$$2\|P_i(y)\| = \|2x\| \leqslant \left\| \sum_{j=1}^{\infty} \varepsilon_j a_j e_j \right\| + \left\| \sum_{j=1}^{\infty} \varepsilon'_j a_j e_j \right\| \leqslant 2\varrho \|y\|.$$

So each projection $P_i : E_0 \longrightarrow E_i$ is continuous and has norm $\leqslant \varrho$. This also implies that each E_i is a complemented subspace of E_0.

If $\pi_0 : E \longrightarrow E_0$ denotes the projection onto E_0, for each $j \in \mathbb{N}$ we can define the operator

$$\widetilde{u_j} : E \longrightarrow F, \qquad \widetilde{u_j} := u_j \circ P_j \circ \pi_0.$$

Since $(P_j \circ \pi_0)(x) = x$ for every $x \in E_j$, it is plain that $\widetilde{u_j}$ belongs to $\mathcal{K}(E;F) \setminus \Pi_p(E;F)$. Given $n \in \mathbb{N}$ and scalars $a_1, \ldots, a_n$, with at least one $a_k \neq 0$, $1 \leqslant k \leqslant n$, since $\widetilde{u_k}$ fails to be absolutely p-summing, there is a weakly p-summable sequence (x_j) in E_k such that $\sum_j \|u_k(x_j)\|^p = +\infty$. It is clear that (x_j) is weakly p-summable in E and $\widetilde{u_k}(x_j) = u_k(x_j)$ for every j. But $A_k \cap A_i = \emptyset$ for $i = 1, \ldots, n$, $i \neq k$, so it follows that $\widetilde{u_i}(x_j) = 0$ for every $i = 1, \ldots, n$, $i \neq k$ and $j \in \mathbb{N}$. So,

$$\sum_j \bigl\| a_1 \widetilde{u_1}(x_j) + \cdots + a_n \widetilde{u_n}(x_j) \bigr\|^p = \sum_j \bigl\| a_k u_k(x_j) \bigr\|^p = +\infty,$$

proving that $a_1 \widetilde{u_1} + \cdots + a_n \widetilde{u_n}$ is not absolutely p-summing. This proves that the span of $\{\widetilde{u_j};\, j \in \mathbb{N}\}$ is contained in $\mathcal{K}(E;F) \setminus \Pi_p(E;F)$.

Let us see now that the set $\{\widetilde{u_j};\, j \in \mathbb{N}\}$ is linearly independent. Let $n \in \mathbb{N}$ and $a_1, \ldots, a_n$ be scalars such that

$$a_1 \widetilde{u_1} + \cdots + a_n \widetilde{u_n} = 0.$$

For every $k \in \{1, \ldots, n\}$ we can choose $x_k \in E_k$ such that $\widetilde{u_k}(x_k) \neq 0$ because $\widetilde{u_k} \neq 0$. But $(P_j \circ \pi_0)(x_k) = P_j(x_k) = 0$ for every $j = 1, \ldots, n$, $j \neq k$. So,

$$a_k \widetilde{u_k}(x_k) = 0 + \cdots + 0 + a_k \widetilde{u_k}(x_k) + 0 + \cdots + 0 = a_1 \widetilde{u_1}(x_k) + \cdots + a_n \widetilde{u_n}(x_k) = 0.$$

It follows that $a_k = 0$. Hence the span of $\{\widetilde{u_j};\, j \in \mathbb{N}\}$ is an infinite-dimensional subspace contained in $\mathcal{K}(E;F) \setminus \Pi_p(E;F)$.

Now, suppose that F contains a subspace G with unconditional basis $\{e_n; n \in \mathbb{N}\}$ with unconditional basis constant ϱ. Still considering the subsets (A_n) of $\mathbb{N}$ as above, define F_j as the closed span of $\{e_n; n \in A_j\}$ and let $P_j : G \longrightarrow F_j$ be the corresponding projections. Proceeding as above we conclude that $\|P_j\| \leqslant \varrho$. From [5, Theorem] we know that for each j there is an operator

$$u_j : E \longrightarrow F_j$$

belonging to $\mathcal{K}(E;F_j) \setminus \Pi_p(E;F_j)$.

Recall that $F_i \cap F_j = \{0\}$ if $i \neq j$. So, if $y_i \in F_i$ and $y_j \in F_j$ (with $i \neq j$), we have

$$\|y_i\| = \bigl\| P_i(y_i + y_j) \bigr\| \leqslant \varrho \|y_i + y_j\|. \tag{2.2}$$

Now by $\widetilde{u_j}$ we mean the composition of u_j with the inclusion from F_j to F. It is clear that $\widetilde{u_j}$ is compact and fails to be absolutely p-summing. From (2.2) it follows that

$$\bigl\| \widetilde{u_i}(x) + \widetilde{u_j}(x) \bigr\| \geqslant \varrho^{-1} \bigl\| \widetilde{u_i}(x) \bigr\|$$

for every $x \in E$. Hence

$$\widetilde{u_i} + \widetilde{u_j} \in \mathcal{K}(E;F) \setminus \Pi_p(E;F) \quad \text{for all } i, j,$$

and so we can easily deduce that the span of $\{\widetilde{u_j};\, j \in \mathbb{N}\}$ is contained in $(\mathcal{K}(E;F) \setminus \Pi_p(E;F)) \cup \{0\}$. A reasoning similar to the first case shows that the vectors $\widetilde{u_j}$, $j \in \mathbb{N}$, are linearly independent, therefore $\mathcal{K}(E;F) \setminus \Pi_p(E;F)$ is lineable. $\square$

Remark 2.2. Note that Theorem 2.1 solves the problem posed by Puglisi and Seoane-Sepúlveda except when E is a superreflexive Banach space not containing an infinite-dimensional complemented subspace with unconditional basis (such a space was constructed by V. Ferenczi [8,9]) *and* F does not contain an infinite-dimensional subspace with unconditional basis (for example, hereditarily indecomposable spaces). It is in this sense we claim that Theorem 2.1 solves the problem modulo extremely pathological cases.

We proved that $\mathcal{L}(E;F) \setminus \Pi_p(E;F)$ is $\aleph_0$-lineable under the assumptions of Theorem 2.1, where $\aleph_0$ is the cardinality of $\mathbb{N}$. The anonymous referee kindly pointed out the following interesting question:

Problem 2.3. Under what circumstances is $\mathcal{L}(E;F) \setminus \Pi_p(E;F)$ μ-lineable for $\mu > \aleph_0$?

3. Non-necessarily superreflexive spaces

Examining the proof of Theorem 2.1 it becomes clear that the result holds if: (i) E contains a sequence $(E_n)_{n=1}^{\infty}$ of complemented infinite-dimensional subspaces such that $E_n \cap E_m = \{0\}$ if $m \neq n$; (ii) $\mathcal{L}(E_n;F) \setminus \Pi_p(E_n;F) \neq \emptyset$ for every $n \in \mathbb{N}$. Having this in mind, the argument of the proof can be adapted to many other circumstances, even for spaces of operators on non-superreflexive spaces.

We start by adapting the proof of Theorem 2.1 to spaces of operators on spaces containing complemented copies of ℓ_1 or c_0 (observe that in these cases the domain spaces are not even reflexive):

Proposition 3.1.

(a) *If E contains a complemented copy of ℓ_1 and F is not isomorphic to a Hilbert space, then $\mathcal{L}(E;F)\setminus\Pi_1(E;F)$ is lineable.*
(b) *If E contains a complemented copy of c_0 and $1 \leqslant p < 2$, then $\mathcal{L}(E;F)\setminus\Pi_p(E;F)$ is lineable for every Banach space F.*

Proof. Up to the composition with the corresponding projections, it suffices to work with $E = \ell_1$ in (a) and $E = c_0$ in (b).

(a) Decomposing $\mathbb{N}$ as in (2.1) we have that the closed span of each $\{e_n; n \in A_j\}$, denoted by E_j, is a complemented copy of ℓ_1 which is isometrically isomorphic to ℓ_1. From [11] we know that $\mathcal{L}(\ell_1;F)\setminus\Pi_1(\ell_1;F) \neq \emptyset$, so $\mathcal{L}(E_n;F)\setminus\Pi_1(E_n;F) \neq \emptyset$ for every n. Now proceed as in the proof of Theorem 2.1 to complete the proof.

(b) Using that c_0 enjoys the same property of ℓ_1 we used above and that $\mathcal{L}(c_0;F)\setminus\Pi_p(c_0;F)$ is nonvoid for every F (see [3,14]), the proof of (a) can be repeated line by line. $\square$

An adaptation of the proof of Theorem 2.1 combined with [4, Corollary 2.2] yields:

Proposition 3.2. *If $p \geqslant 1$, then $\mathcal{L}(E;F)\setminus\Pi_p(E;F)$ is lineable for every Banach space E and every Banach space F containing a copy of c_0.*

4. Non-absolutely $(q,1)$-summing linear operators

In this section we turn our attention to the lineability of the set of non-absolutely $(q,1)$-summing operators, which is, *a priori*, a more delicate matter. Absolutely $(q,1)$-summing operators are closely connected to the cotypes of the underlying spaces; for this reason, given a Banach space F, we define $\cot F = \inf\{q \geqslant 2\colon\ F \text{ has cotype } q\}$.

If E has unconditional basis $(x_n)_{n=1}^{\infty}$, define

$$\mu_{E,(x_n)} = \inf\left\{t\colon\ (a_j)_{j=1}^{\infty} \in \ell_t \text{ whenever } x = \sum_{j=1}^{\infty} a_j x_j \in E\right\}.$$

By adapting the arguments we used so far with [3, Corollary 2.1] as starting point, it is not difficult to prove that:

Proposition 4.1. *If $1 \leqslant q < \cot F$ and $p > q$, then $\mathcal{L}(\ell_p;F)\setminus\Pi_{q,1}(\ell_p;F)$ is lineable.*

We shall improve substantially both Proposition 4.1 (in the sense that ℓ_p can be replaced by spaces E with unconditional basis (x_n) such that $\mu_{E,(x_n)} > q$) and [3, Corollary 2.1] (in the sense that $\mathcal{L}(E;F)\setminus\Pi_{q,1}(E;F)$ is actually lineable). We will need the following result:

Lemma 4.2. *(See [15, Lemma 1.1].) Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive real numbers. If $\sum_{j=1}^{\infty} a_n = \infty$, then there is a sequence of sets of positive integers $(A_j)_{j=1}^{\infty}$ so that:*

(i) $\mathbb{N} = A_1 \cup A_2 \cup \cdots$.
(ii) *Each A_j has the same cardinality of $\mathbb{N}$.*
(iii) *The sets A_j are pairwise disjoint.*
(iv) $\sum_{j \in A_k} a_j = \infty$ *for each k.*

Theorem 4.3. *If $1 \leqslant q < \cot F$, E has an unconditional normalized basis $(x_n)_{n=1}^{\infty}$ and $\mu_{E,(x_n)} > q$, then $\mathcal{L}(E;F)\setminus\Pi_{q,1}(E;F)$ is lineable.*

Proof. Since $\mu_{E,(x_n)} > q$, we can find $(a_j)_{j=1}^{\infty}$ and $\varepsilon > 0$ so that

$$x = \sum_{j=1}^{\infty} a_j x_j \in E \quad \text{and} \quad \sum_{j=1}^{\infty} |a_j|^{q+\varepsilon} = \infty. \tag{4.1}$$

Let $(A_j)_{j=1}^{\infty}$ be the sets of Lemma 4.2 associated to the divergent series $\sum_{j=1}^{\infty} |a_j|^{q+\varepsilon}$. For each positive integer k, define $E_k = \overline{\operatorname{span}\{x_j;\ j \in A_k\}}$. From the proof of Theorem 2.1 we know that each $\{x_n; n \in A_k\}$ is an unconditional basic sequence and E_k is a complemented subspace of E. From the choice of A_k we have that $\mu_{E_k,(x_n)} > q$, so [3, Corollary 2.1] gives that $\mathcal{L}(E_k;F)\setminus\Pi_{q,1}(E_k;F) \neq \emptyset$ for every k. The result follows by repeating once more the procedure of the proof of Theorem 2.1. $\square$

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