On a sharp bound for the regularity index of any set of fat points

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Abstract

We propose an upper bound for the regularity index of fat points of $\mathbb{P}^n$ with no geometric conditions on the points. Whenever the conjecture is true, the bound is sharp. It is, in fact, reached when there are points with high multiplicities either on a line or on some rational curve. Besides giving an easy proof of the conjecture in $\mathbb{P}^2$, we prove it in $\mathbb{P}^3$, by using some preliminary results which hold, more generally, in $\mathbb{P}^n$. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

It is a fairly difficult problem to determine the Hilbert function of fat points of $\mathbb{P}^n$ so one tries, at least, to determine their regularity index or, even less, an upper bound for it.

An obvious upper bound comes from collinear points, but, since it characterizes collinear points, it is not a good bound for a more general set of points.

A sharp bound was given by Segre in 1961 for points of $\mathbb{P}^2$ in general position [7] and was extended to points of $\mathbb{P}^n$ in general position by Catalisano et al. [2] and to any set of fat points of $\mathbb{P}^2$ by Fatabbi [4] and recently by Thien [9], using different methods.

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We conjecture a sharp bound for any set of fat points in \( \mathbb{P}^n \) (which extends all the previous bounds) in terms of a ‘stratification’ of the one given by Fatabbi, in the sense that we look at certain integers derived from the multiplicities of those points which lie on lines, planes, etc.: the regularity index is then conjectured to be bounded by the maximum of all these integers.

We give some preliminary results which hold in \( \mathbb{P}^n \) and use them to prove the conjecture in \( \mathbb{P}^3 \).

The same conjecture was also given independently by Trung and proved by Thien in the case \( n = 3 \), using different methods (see [8]).

2. About the conjecture

Let \( P_1, \ldots, P_s \) be a set of distinct points of \( \mathbb{P}^n \) and let \( p_1, \ldots, p_s \) be the associated homogeneous prime ideals in \( R = k[X_0, \ldots, X_n] \). Given a set of non-negative integers \( m_1, \ldots, m_s \), we consider the set of all hypersurfaces of \( \mathbb{P}^n \) passing through each \( P_i \) with multiplicity at least \( m_i \) or, in algebraic terms, the (saturated) ideal \( p_1^{m_1} \cap \cdots \cap p_s^{m_s} \). We denote by \( X = \{(P_1, m_1), \ldots, (P_s, m_s)\} \) the (zero-dimensional) subscheme of \( \mathbb{P}^n \) defined by \( p_1^{m_1} \cap \cdots \cap p_s^{m_s} \), and call it a set of fat points. We also denote \( I_X = p_1^{m_1} \cap \cdots \cap p_s^{m_s} \).

We recall that the Hilbert function of \( X \), \( H(X, t) = \dim_k(R/I_X) \), strictly increases until it reaches the degree of \( X \), \( \delta(X) = \sum \dim_k(R_m) \), and it keeps constant thereafter.

The least integer \( t \) such that the Hilbert function of \( X \) reaches \( \delta(X) \), or, more geometrically, the least integer \( t \) such that the points (with their multiplicities) impose independent conditions to the hypersurfaces of degree \( t \), is called the regularity index of \( X \) and is denoted by \( \tau(X) \), or also by \( \tau(I_X) \).

It was proved in [3, Corollary 2.3] that \( \tau(X) = (\sum_{i=1}^{s} m_i) - 1 \) when (and only when) the points are all on a line.

If \( \text{Supp}(X) \) is in general position and the multiplicities are ordered non-increasingly (after relabelling the points, if necessary), then [2] proves

\[
\tau(X) \leq \max \left\{ m_1 + m_2 - 1, \left[ \frac{\sum m_i + n - 2}{n} \right] \right\},
\]

the case \( n = 2 \) being Segre’s bound [7].

In order to conjecture an upper bound for the regularity index of any set of fat points we introduce the following integers.

**Definition 2.1.** Let \( X = \{(P_1, m_1), \ldots, (P_s, m_s)\} \) be a set of fat points of \( \mathbb{P}^n \). We set

\[
h_X(i) = \max \left\{ \left[ \frac{\sum m_i}{i} \right] \right\},
\]

where \( A_i \) runs over all the linear subspaces of \( \mathbb{P}^n \) of dimension \( i \), and \( [q] \) denotes the greatest integer less than or equal to \( q \). We set, moreover

\[
h_X = \max \{ h_X(i) \mid i = 1, \ldots, n \}.
\]
We also need to define $h_0 = -1$.

The integers $h_X(i)$ can be computed by means of a package (which runs under CoCoA [1]), recently developed by Moncelli in her thesis [6], which does not involve the computation of the Hilbert function.

We conjecture that $h_X$ is a (sharp) bound for $\tau(X)$, i.e.,

**Conjecture.** $\tau(X) \leq h_X$.

For $n = 2$ this is exactly the bound given in [4,9].

If we assume the points of $\text{Supp}(X)$ are in general position and the multiplicities are non-increasing, we get

$$h_X(1) = m_1 + m_2 - 1$$

and

$$h_X(i) \leq h_X(1), \quad \forall i = 1, \ldots, n - 1.$$ 

In other words,

$$h_X = \max \left\{ m_1 + m_2 - 1, \left[ \frac{\sum m_i + n - 2}{n} \right] \right\},$$

i.e., we obtain the bound given in [2], which in turn, gives Segre’s bound for $n = 2$.

If the conjecture is true, then $h_X$ is a sharp bound for $\tau(X)$, in fact

**Corollary 2.2.** Let $X = \{(P_1,m_1),\ldots,(P_s,m_s)\}$ be a set of fat points of $\mathbb{P}^n$ which satisfies the conjecture. If there exists $X' \subset X$ such that $\tau(X') = h_X$, then $\tau(X) = h_X$.

In particular, if $h_X = h_X(1)$, then $\tau(X) = h_X = h_X(1)$.

**Proof.** The first statement follows immediately from [3, Proposition 2.1].

If $h_X = h_X(1)$, further observe that, in this case, there exists a subset $X'$ whose support consists of collinear points, hence such that $\tau(X') = h_X$. 

Another case in which $h_X$ is attained is when there exists a subset of points lying on the rational normal curve of some $\mathbb{P}^i$. To see this, one has to look at a set of fat points of $\mathbb{P}^n$ lying in a smaller $\mathbb{P}^i$ and to compare the relative regularity indices. This is done in [5]; in particular it is shown in Proposition 4.7 that, in the case of points lying on the rational normal curve of $\mathbb{P}^i$, the two indices coincide. This allows us to prove the following:

**Corollary 2.3.** Let $X = \{(P_1,m_1),\ldots,(P_s,m_s)\} \subset \mathbb{P}^n$ be a set of fat points satisfying the conjecture. Assume $h_X = h_X(i)$ and suppose there exists $X' \subset X$ with support lying on the rational normal curve of $\Lambda_i \cong \mathbb{P}^i$. Then

$$\tau(X) = h_X = h_X(i).$$

**Proof.** By Catalisano et al. [2] and Franceschini and Lorenzini [5, Proposition 4.7] we have that $\tau(X') = h_X$, hence $\tau(X) = h_X$ by the corollary above. 

3. A hyperplane criterion

If $H$ is a hyperplane through $P_1, \ldots, P_r$, then, after a linear change of variables, we may assume $H$ is the hyperplane with equation \( \{X_n = 0\} \), so that, for $l = 1, \ldots, r$, we have $P_l = [a_{l,0} : \ldots : a_{l,n-1} : 0]$.

Denote $\tilde{P}_l = [a_{l,0} : \ldots : a_{l,n-1}] \in \mathbb{P}^{n-1}$ and consider

\[
\tilde{Y} = \{(\tilde{P}_1, m_1), \ldots, (\tilde{P}_r, m_r)\} \subset \mathbb{P}^{n-1},
\]

\[Z = \{(P_1, m_1 - 1), \ldots, (P_r, m_r - 1), (P_{r+1}, m_{r+1}), \ldots, (P_s, m_s)\} \subset \mathbb{P}^n.
\]

Observe that

\[\delta(Z) + \delta(\tilde{Y}) = \delta(X).
\]

From now on we shall assume $m_i \geq 1$, for all $i = 1, \ldots, s$.

**Theorem 3.1.** Let $H$ be a hyperplane through $P_1, \ldots, P_r \in \text{Supp}(X)$ and

\[Z = \{(P_1, m_1 - 1), \ldots, (P_r, m_r - 1), (P_{r+1}, m_{r+1}), \ldots, (P_s, m_s)\} \subset \mathbb{P}^n.
\]

If there exists $t$ such that $\tau(\tilde{Y}) \leq t$ and $\tau(Z) \leq t - 1$, then

\[\tau(X) \leq t.
\]

**Proof.** Let $I = I_X$ and $J = I_\tilde{Y}$ and write $I_f = X_n I_{J_{l,1}} \oplus V$, where $V$ is a vector subspace of $R_l$, no form of which is divisible by $X_n$. Now write each $F \in V$ as $F = X_n F_1 + F_2$, with $F_2$ not involving $X_n$.

Consider the linear map $\phi : V \rightarrow (I_f)_l$ defined by

\[\phi(F) = \phi(X_n F_1 + F_2) = F_2,
\]

where $F_2$ is considered as a polynomial in $\tilde{R} = k[X_0, \ldots, X_{n-1}]$.

Clearly $\phi$ is injective, hence $\text{dim}_k V \leq \text{dim}_k (I_f)_l$. It follows from the hypotheses and direct computation that

\[
H(X, t) = \text{dim}_k R_l - \text{dim}_k J_{l-1} - \text{dim}_l V \geq \text{dim}_k R_l - \text{dim}_k J_{l-1} - \text{dim}_k (I_f)_l = \text{dim}_k R_l - \text{dim}_k J_{l-1} - \text{dim}_k \tilde{R}_l + \delta(\tilde{Y}) = \text{dim}_k \tilde{R}_l - \text{dim}_k J_{l-1} + \delta(\tilde{Y}).
\]

By using Theorem 3.1, we would be able to prove the conjecture, provided we were able to find a hyperplane which suitably lowers $h_X$.

**Condition H.** Given any set $X = \{(P_1, m_1), \ldots, (P_s, m_s)\} \subset \mathbb{P}^n$, there exists a hyperplane $H$ through $P_1, \ldots, P_s \in \text{Supp}(X)$ such that $h_Z \leq h_X - 1$, where $Z = \{(P_1, m_1 - 1), \ldots, (P_r, m_r - 1), (P_{r+1}, m_{r+1}), \ldots, (P_s, m_s)\}$.

It is not hard to prove, by induction on $\sum m_i$, that condition H implies the conjecture: when $\sum m_i > 1$, use Condition H to obtain $h_Z \leq h_X - 1$, hence, by induction, $\tau(Z) \leq t$. 

Consider $Y = \{(P_1, m_1), \ldots, (P_r, m_r)\} \subset \mathbb{P}^n$, corresponding to the set $\tilde{Y} \subset \mathbb{P}^{n-1}$, constructed above: by induction, $\tau(Y) \leq h_Y \leq h_X$; on the other hand, by Franceschini and Lorenzini [5, Theorem 4.1], we have $\tau(\tilde{Y}) \leq \tau(Y)$, hence the conclusion follows from Theorem 3.1, for $t = h_X$.

Unfortunately, Condition H does not always hold, already in $\mathbb{P}^3$, as the following example shows. However, we believe that it should hold generically.

**Example.** Let $X = \{(P_1, m), \ldots, (P_5, m)\} \subset \mathbb{P}^3$, with

- $P_1 = [1 : 0 : 0 : 0]$,
- $P_2 = [1 : 1 : 0 : 0]$,
- $P_3 = [1 : 0 : 1 : 0]$,
- $P_4 = [1 : 0 : 0 : 1]$,

In this case $h_X = h_X(1) = 2m - 1$ (which is reached by every line drawn in the picture) and the points are in general position, hence, by Catalisano et al. [2], $\tau(X) = h_X$. Nevertheless, a hyperplane lowering $h_X$ cannot be found, since the plane through any three of these five points does not intersect the line through the remaining two in points of $\text{Supp}(X)$.

When Condition H cannot be proved to hold, we shall replace it by the following result:

**Theorem 3.2.** Let $X = \{(P_1, m_1), \ldots, (P_s, m_s)\}$ and suppose there exists a form $G$ of degree $d$ passing through $P_1, \ldots, P_r$ and avoiding a point (which we may assume is $P_s$) of $\text{Supp}(X)$. Denote

$$X' = \{(P_1, m_1), \ldots, (P_{s-1}, m_{s-1}), (P_s, m_s - 1)\};$$

$$Z = \{(P_1, m_1 - 1), \ldots, (P_r, m_r - 1), (P_{r+1}, m_{r+1}), \ldots, (P_s, m_s)\}.$$

If there exists $t$ such that $\tau(Z) \leq t - d$ and $\tau(X') \leq t$, then

$$\tau(X) \leq t.$$
Proof. Denote
\[Z' = \{(P_1, m_1 - 1), \ldots, (P_r, m_r - 1), (P_{r+1}, m_{r+1}), \ldots, (P_{s-1}, m_{s-1}), (P_s, m_s - 1)\}\].
By hypothesis and [3, Proposition 2.1], we have that \(\tau(Z') \leq \tau(Z) \leq t - d\), thus we get that
\[\dim_k (I_Z')_{t-d} = \delta(Z) - \delta(Z') = u = \delta(X) - \delta(X').\]
Thus, we can find \(F_1, \ldots, F_u \in (I_Z')_{t-d}\) such that \(F_1, \ldots, F_u\) are linearly independent modulo \((I_Z)_{t-d}\). It is not hard to see, by using \(G(P_i) \neq 0\), that \(GF_1, \ldots, GF_u\) are in \((I_{X'})_{t} \setminus (I_X)_{t}\) and that they are linearly independent modulo \((I_X)_{t}\); in other words, that \(\dim_k (I_{X'})_{t}/(I_X)_{t} \geq u\). Therefore,
\[H(X, t) = \dim_k R_t - \dim_k (I_{X'})_{t} - \dim_k (I_X)_{t} \geq \delta(X') + u = \delta(X),\]
where the inequality follows also from the hypothesis \(\tau(X') \leq t\).

In order to find the hyperplane we need, we now introduce notions and results which we shall mainly use in \(\mathbb{P}^3\).

If \(S\) is any subset of \(\mathbb{P}^n\), we shall denote \(m(S) = \sum m_{ij}\), where \(P_{ij} \in S\) and \((P_{ij}, m_{ij}) \in X\).

Definition 3.3. We call a linear subspace \(A_i \cong \mathbb{P}^i\) maximal with respect to \(X\) (or simply maximal) if
\[\left[\frac{m(A_i) + i - 2}{i}\right] = h_X.\]

Remark 3.4. If \(A_i\) is maximal, then it easily follows from the definitions that
\[ih_X - i + 2 \leq m(A_i) \leq ih_X + 1 \leq nh_X + 1, \quad \forall i = 1, \ldots, n.\]
In particular,
\[\sum_{i=1}^s m_i \leq nh_X + 1.\]

Lemma 3.5. Let \(A_1, A_2 \cong \mathbb{P}^i\) be maximal linear subspaces, then
\[m(A_1 \cap A_2) \geq (2i - n)h_X - 2i + 3.\]

Proof. From Remark 3.4 it follows that
\[2(ih_X - i + 2) - m(A_1 \cap A_2) \leq m(A_1 \cup A_2) \leq \sum_{i=1}^s m_i \leq nh_X + 1.\]

It should be pointed out that Lemma 3.5 does not give much information about \(A_1 \cap A_2 \cap \text{Supp}(X)\), when the integer on the right-hand side is non-positive.
Definition 3.6. We say that $A_i \cong \mathbb{P}^i$ and $A_j \cong \mathbb{P}^j$ are skew with respect to $X$ if $A_i \cap A_j \cap \text{Supp}(X) = \emptyset$.

Obviously, if two linear subspaces are skew, then they are, a fortiori, skew with respect to $X$. The converse is not true in general, but this is the case for maximal lines (Lemma 3.7) and nearly so for maximal planes (Lemma 3.9).

Lemma 3.7. Let $L_1$ and $L_2$ be two maximal lines, then $L_1$ and $L_2$ are skew with respect to $X$ if and only if they are skew.

Proof. To see that the condition is necessary, suppose that $L_1 \cap L_2 \cap \text{Supp}(X) = \emptyset$, with $L_1 \cap L_2 \neq \emptyset$. Then there is a plane, $\pi$, containing $L_1$ and $L_2$, for which $h_X(2) \geq h_X \cap \pi \geq \left\lceil \frac{2h_X + 2}{2} \right\rceil = h_X + 1$,

thus contradicting the definition of $h_X$. □

As a consequence, we get that Condition H holds in $\mathbb{P}^2$: although we do not need this to prove the conjecture in this case, we explicitly state it, for future reference, and because it gives a proof of the conjecture for $n = 2$ easier than the ones given in [4,9].

Corollary 3.8. Let $X = \{(P_1, m_1), \ldots, (P_s, m_s)\} \subset \mathbb{P}^2$. Then Condition H holds for $X$.

Proof. We can choose any maximal line, if there is one, otherwise any line intersecting $\text{Supp}(X)$ in at least two points, thus lowering $h_X(2)$, but also $h_X(1)$, because of Lemma 3.7. □

Lemma 3.9. Let $\pi_1$ and $\pi_2$ be two maximal planes. If $\pi_1$ and $\pi_2$ are skew with respect to $X$, then they have at most one point in common.

Proof. If $\pi_1 \cap \pi_2$ is a line, then they generate a linear subspace $A_3 \cong \mathbb{P}^3$. On the other hand, $\pi_1 \cap \pi_2 \cap \text{Supp}(X) = \emptyset$ gives $m(\pi_1 \cup \pi_2) = m(\pi_1) + m(\pi_2)$, which, by Remark 3.4, is greater than or equal to $4h_X$. Thus,

$$h_X(3) \geq \left\lceil \frac{4h_X + 1}{3} \right\rceil > h_X,$$

contradicting the definition of $h_X$. □

Lemma 3.10. Let $L_1, \ldots, L_k$ be pairwise skew maximal lines. Then $k \leq n - 1$.

Proof. It follows from Remark 3.4 that

$$m(L_1 \cup \cdots \cup L_k) = k(h_X + 1) \leq \sum_{i=1}^{s} m_i \leq nh_X + 1,$$
which yields
\[ k \leq \frac{nh_X + 1}{h_X + 1} < n, \]
since \( n > 1. \)

4. The conjecture in \( \mathbb{P}^3 \)

We first try and see when Condition H holds in \( \mathbb{P}^3 \).

First observe that, if \( H \) is a plane through \( P_1, \ldots, P_r \in \text{Supp}(X) \), with \( r \geq 3 \), and if \( Z = \{(P_1, m_1 - 1), \ldots, (P_r, m_r - 1), (P_{r+1}, m_{r+1}), \ldots, (P_s, m_s)\} \), then \( h_Z(3) \leq h_X(3) - 1 \); thus, if \( H \) lowers \( h_X(1) \) or \( h_X(2) \), it necessarily lowers also \( h_X(3) \).

Lemma 4.1. Let \( X = \{(P, m), (P_1, 1), \ldots, (P_s, 1)\} \subset \mathbb{P}^3 \), with \( s > 0 \) and \( m = h_X \). Then \( X \) satisfies Condition H.

Proof. Observe that \( m = h_X \) necessarily yields \( h_X = h_X(1) \) (by considering the line through \( P \) and \( P_1 \)), hence also \( h_X(1) \geq h_X(2), h_X(3) \).

Furthermore, it follows from Remark 3.4, that \( s \leq 2h_X + 1. \)

We may assume \( h_X \geq 2 \), for otherwise we have at most four simple points, for which the statement is obvious.

When \( h_X = h_X(1) > h_X(2) \), there are no maximal lines avoiding \( P \) (otherwise the plane through \( P \) and such a line would be maximal), and we can choose a plane through \( P \) and any two other points of \( \text{Supp}(X) \).

Now assume \( h_X = h_X(1) = h_X(2) \). If there is no maximal plane through \( P \) then, as before, there cannot be maximal lines avoiding \( P \), and there is a unique maximal plane \( \pi \). In fact, we have
\[ m(\mathbb{P}^3 \setminus \{\pi \cup \{P\}\}) \leq 3h_X + 1 - (2h_X + h_X) = 1, \]
and so, for any other plane \( \pi' \) not through \( P \), we have
\[ m(\pi') \leq m(\pi \cap \pi') + m(\mathbb{P}^3 \setminus \{\pi \cup \{P\}\}) \leq h_X + 1 < 2h_X. \]

In this case, we can choose the plane through \( P \) and any two points of \( \pi \cap \text{Supp}(X) \).

If there are maximal planes through \( P \), we choose \( H = \pi \), with \( m(\pi) \) maximal among the maximal planes through \( P \).

Notice that, in both cases, we have also lowered \( h_X(3) \), as we have chosen a plane which contains at least three points of \( \text{Supp}(X) \).

Lemma 4.2. Let \( X = \{(P, m), (P_1, 2), \ldots, (P_r, 2), (P_{r+1}, 1), \ldots (P_s, 1)\} \subset \mathbb{P}^3 \) with \( r + s > 0 \) and \( m = h_X - 1. \) Then \( X \) satisfies Condition H.

Proof. From Remark 3.4, \( r + s \leq 2h_X + 2. \) Then, there are at most two skew maximal lines avoiding \( P \).
If $h_X = h_X(3) > h_X(1), h_X(2)$, then a plane through any three points of $\text{Supp}(X)$ lowers $h_X$.

If $h_X = h_X(1) > h_X(2)$, there is no maximal line avoiding $P$ (otherwise the plane through such a line and $P$ would give $h_X = h_X(2)$), and so we are done by choosing the plane through $P$ and any two other points of $\text{Supp}(X)$.

If $h_X = h_X(2) > h_X(1)$, then we are done by choosing a plane, $\pi$, with $m(\pi)$ maximal among all maximal planes.

Now suppose $h_X = h_X(1) = h_X(2) > 2$.

First assume there is no maximal plane through $P$, then there cannot be maximal lines avoiding $P$ (otherwise the plane through $P$ and such a line would be maximal). Moreover, there is a unique maximal plane, $\pi$. In fact,

$$m(\mathbb{P}^3 \setminus \{\pi \cup \{P\}\}) \leq 3h_X + 1 - (2h_X + h_X - 1) = 2,$$

and so, for any other plane $\pi'$ not through $P$, we have

$$m(\pi') \leq m(\pi \cap \pi') + m(\mathbb{P}^3 \setminus \{\pi \cup \{P\}\}) \leq h_X + 2 < 2h_X.$$

Thus, in this case, we can choose the plane through $P$ and any two points of $\pi \cap \text{Supp}(X)$.

If there is a maximal plane, $\pi$, through $P$, then $\pi$ must contain one of the two skew maximal lines, if they exist. In this case, we choose a plane through $P$, another point of $\pi \cap \text{Supp}(X)$ and any other point of $\text{Supp}(X)$ (the last two on the skew maximal lines, if they exist).

Finally, suppose $h_X = h_X(1) = h_X(2) = 2$.

If $r > 0$, then $r = 1$ and $X = \{(P, 1), (P_1, 2), (P_2, 1), \ldots, (P_1, 1)\}$, so we can apply Lemma 4.1.

If $r = 0$, then $X = \{(P, 1), (P_1, 1), \ldots, (P_1, 1)\}$ and, because $h_X(3) \leq h_X = 2$, with $s \leq 6$. Because $h_X(1) = h_X(2) = 2$, there is a maximal plane, $\pi$, containing at least four points of $X$ (three of which may be on a maximal line $L'$) and at most one maximal line, $L$, skew with $\pi$ with respect to $X$. Then, choose $H = \pi$ if $L$ does not exist, otherwise as the plane through $L$ and a point of $L' \cap X \setminus \{\pi \cap X, \text{if } L' \cap X \}$ (or any point of $\pi \cap X$, if $L' \cap X$ does not exist).

Notice that, again, in all the cases, we have also lowered $h_X(3)$.

Now we introduce a sort of ‘basic’ configuration which will lead us to the one we have to exclude in order to have Condition H satisfied.

**Configuration B.** Let $X = \{(P_1, m_1), \ldots, (P_s, m_s)\} \subset \mathbb{P}^3$, with $s \geq 5$ and $h_X = h_X(1)$. Call $L_1$, the line passing through $P_1$ and $P_2$, $L_3$ the line through $P_1$ and $P_3$, and $L_2$ the one through $P_4$ and $P_3$. Assume $L_1, L_2, L_3$ are all maximal. Call also $\pi_1$ the plane through $P_1, P_2$ and $P_3$, and assume $L_2 \cap \pi_1 \cap \text{Supp}(X) = \emptyset$.

**Remark 4.3.** If $X$ is as in Configuration B, then $s \geq 5$ yields $h_X(3) \geq 2$, while the existence of $L_2$ (together with Remark 3.4) gives

$$m(\mathbb{P}^3 \setminus L_2) \leq 3h_X + 1 - (h_X + 1) = 2h_X.$$
In other words
(1) \( h_X \geq 2; \)
(2) \( m(\pi_1) \leq 2h_X. \)

**Remark 4.4.** If \( X \) is as in Configuration B, we must also have \( m_1 \geq 2, \) for otherwise we would have
\[
m(L_1 \cup L_2 \cup L_3) = 3(h_X + 1) - 1,\
\]
thus contradicting Remark 3.4.

In particular, if \( X \) is a set of \( s \geq 5 \) simple points, then \( X \) cannot be as in Configuration B.

**Lemma 4.5.** Let \( X \) be a set of fat points. Then
(1) if \( L, L', L_2 \) are distinct maximal lines, with \( L \cap L_2 = \emptyset = L' \cap L_2, \) then
\[
m(\mathbb{P}^3 \setminus \{L \cup L_2 \cup L'\}) \leq m(L \cap L') - 2.
\]
Further, assume that \( X \) is as in Configuration B. Then
(2) if the line, \( L_4, \) through \( P_2 \) and \( P_3 \) is maximal, then
\[
m(\mathbb{P}^3 \setminus \{L_1 \cup L_2 \cup L_3 \cup L_4\}) \leq m - 2,
\]
where \( m = \min\{m_1, m_2, m_3\}; \)
(3) if \( L \) is a maximal line not in \( \pi_1 \) such that \( L \cap L_2 = \emptyset, \) then
\[
L \cap \{L_1 \cup L_3\} = \{P_1\};
\]
(4) if the line, \( L_4, \) through \( P_2 \) and \( P_3 \) is maximal and \( L \neq L_2 \) is a maximal line not in \( \pi_1, \) then \( L \cap \{L_1 \cup L_3\} \neq \emptyset \neq L \cap L_2; \)
(5) if \( L \) is a line with \( m(L) = h_X, \) then either \( L \cap \pi_1 \cap \text{Supp}(X) \neq \emptyset \) or \( L \cap L_2 \cap \text{Supp}(X) \neq \emptyset. \)

**Proof.** By Lemma 3.10, necessarily \( L \cap L' \neq \emptyset. \) Moreover, it follows from Remark 3.4 that
\[
3h_X + 1 \geq m(L \cup L_2 \cup L') = 3(h_X + 1) - m(L \cap L'),
\]
since \( L \cap L' \) is the only point of mutual intersection of \( L, L_2, L'. \) Then
\[
m(\mathbb{P}^3 \setminus \{L \cup L_2 \cup L'\}) \leq 3h_X + 1 - m(L \cup L_2 \cup L') \leq m(L \cap L') - 2,
\]
thus proving the first assertion.

To see the second statement, apply (1). to get
\[
m(\mathbb{P}^3 \setminus \{L_1 \cup L_2 \cup L_3\}) \leq m_1 - 2,
\]
\[
m(\mathbb{P}^3 \setminus \{L_1 \cup L_2 \cup L_4\}) \leq m_2 - 2,
\]
\[
m(\mathbb{P}^3 \setminus \{L_3 \cup L_2 \cup L_4\}) \leq m_3 - 2.
\]
To prove (3), suppose $L \cap \{ L_1 \cup L_3 \} \neq \{ P_i \}$, then it follows from (1) that

$$m(L) \leq m(L \cap \{ L_1 \cup L_3 \}) + m_1 - 2 < h_X + 1.$$

To show (4), suppose that either $L \cap \{ L_1 \cup L_3 \} = \emptyset$ or $L \cap L_2 = \emptyset$, and, in both cases, call $P \in \text{Supp}(X)$ the only possible point of intersection. Then, from (1), we would have, respectively,

$$m(L) \leq m(L \cap L_2) + m - 2 = m(P) + m - 2 < h_X + 1,$$

or

$$m(L) \leq m(L \cap \{ L_1 \cup L_3 \}) + m - 2 = m(P) + m - 2 < h_X + 1,$$

where $m = \min\{m_1, m_2, m_3\}$.

To prove the last statement, suppose

$$L \cap \pi_1 \cap \text{Supp}(X) = \emptyset = L \cap L_2 \cap \text{Supp}(X).$$

Then

$$m(L) \leq m_1 - 2 < h_X.$$

\[ \square \]

**Lemma 4.6.** Let $X$ be as in Configuration B, and suppose, moreover, $h_X(2) = h_X$. Let $\pi \neq \pi_1$ be a maximal plane.

1. If $m(\pi) = 2h_X + 1$, then $L_2 \subset \pi$ and $\pi \cap L \cap \text{Supp}(X) \neq \emptyset$, for any maximal line $L \subset \pi_1$.

2. If $m(\pi) = 2h_X$ and $L_i \not\subset \pi$ ($\forall i = 1, 2, 3$), then $\pi \cap L_2 \cap \text{Supp}(X) \neq \emptyset$. Moreover, $\pi \cap L \cap \text{Supp}(X) \neq \emptyset$, for any maximal line $L \subset \pi_1$.

3. If the line, $L_4$, through $P_2$ and $P_3$ is maximal and $2h_X - 2 \leq m(\pi) \leq 2h_X - 1$, then either $\pi \supset L_2$ or $|\pi \cap \{ \pi_1 \cup L_2 \} \cap \text{Supp}(X)| \geq 2$.

**Proof.** In order to prove (1), suppose $L_2 \not\subset \pi$. Then, because of (1) of Lemma 4.5

$$2h_X + 1 = m(\pi) \leq m(\pi \cap (L_1 \cup L_3)) + [m(\pi \cap L_2) + m_1] - 2 \leq 2(h_X + 1) - 2 = 2h_X.$$

Now consider $L = L_1$ and suppose that $\pi \cap L_1 \cap \text{Supp}(X) = \emptyset$. Then, as $L_3 \cap \pi \cap \text{Supp}(X)$ has at most one point (otherwise $L_3 \subset \pi \supset L_2$), we have

$$2h_X + 1 = m(\pi) \leq m(L_2) + [m(\pi \cap L_3) + m_1] - 2 \leq 2(h_X + 1) - 2 = 2h_X.$$

Similarly for $L = L_3$ and any other maximal line $L \subset \pi_1$.

In order to show (2), suppose there exists $i \in \{1,2,3\}$ such that $\pi \cap L_i \cap \text{Supp}(X) = \emptyset$, and let $j, k \in \{1,2,3\}$, with $j \neq k$ and both different from $i$. Since it follows from the hypotheses that both $\pi \cap L_j \cap \text{Supp}(X)$ and $\pi \cap L_k \cap \text{Supp}(X)$ have at most one point, we would then have

$$m(\pi \cap L_j) + m(\pi \cap L_k) \leq h_X + 1,$$

and so

$$2h_X = m(\pi) \leq m(\pi \cap L_j) + m(\pi \cap L_k) + m_1 - 2 \leq 2h_X - 1.$$
Similarly for \( L \) (the statement being trivial when \( L \subseteq \pi \)).

To show (3), suppose \( \pi \not\ni L_2 \) and \( |\pi \cap \{\pi_1 \cup L_2\} \cap \text{Supp}(X)| < 2 \). Then, by (2) of Lemma 4.5, if \( m = \min\{m_1, m_2, m_3\} \), then
\[
m(\pi) \leq m(\pi \cap \{L_1 \cup \cdots \cup L_4\}) + m - 2 \leq h_X - 1 < 2h_X - 2,
\]
where the last inequality follows from (1) of Remark 4.3. \( \square \)

**Definition 4.7.** For \( X \) as in Configuration B, define
\[
\mathcal{L} = \{L \subseteq \mathbb{P}^3|m(L) = h_X + 1, L \not\subseteq \pi_1, L \not\ni L_2\}.
\]

**Lemma 4.8.** Let \( X \) be as in Configuration B and suppose the line, \( L_4 \), through \( P_2 \) and \( P_3 \) is maximal. If \( \mathcal{L} \neq \emptyset \) and \( \{\bigcup_{L \in \mathcal{L}} L\} \cap \{\bigcup_{L \in \mathcal{L}} \{L_1 \cup L_3\} \not\ni \{P_1, P_2, P_3\} \), then all the lines of \( \mathcal{L} \) meet \( L_2 \) in the same point.

**Proof.** By (4) of Lemma 4.5, each line \( L \in \mathcal{L} \) meets \( L_2 \). Suppose there are \( L, L' \in \mathcal{L} \) such that \( L' \cap L_2 \neq L \cap L_2 \), with \( L \) such that \( L \cap \{L_1 \cup L_3\} \not\ni \{P_1, P_2, P_3\} \). Because \( L_1, \ldots, L_4, L, L' \) are all maximal, after recalling Remark 3.4, we get
\[
\begin{align*}
6(h_X + 1) - m(L_1 \cap L_3) - m(L_1 \cap L_4) - m(L \cap \{L_1 \cup L_3\}) \\
-m(L' \cap \{L_1 \cup L_3\}) - m(L \cap L') - m(L' \cap L_2) \\
-m(L_4 \cap L_3) - m(L \cap L_2) \leq 3h_X + 1,
\end{align*}
\]
i.e.,
\[
3h_X + 5 \leq m(L' \cap \{L_1 \cup L_3\}) + m(L' \cap L_2) + m(L \cap L') \\
+ m(L_1 \cap L_3) + m(L_1 \cap L_4) + m(L \cap \{L_1 \cup L_3\}) \\
+ m(L_3 \cap L_4) + m(L \cap L_2),
\]
assuming, for a moment, that the intersections which appear above are all distinct.

Now, clearly
\[
m(L' \cap \{L_1 \cup L_3\}) + m(L' \cap L_2) + m(L \cap L') \leq m(L') = h_X + 1 \quad (2)
\]
and
\[
m(L_3 \cap L_4) + m(L \cap L_2) \leq h_X + 1. \quad (3)
\]

We may assume \( L \cap \{L_1 \cup L_3\} \subset L_1 \) (the other case being similar), so that we also have
\[
m(L_1 \cap L_3) + m(L_1 \cap L_4) + m(L \cap \{L_1 \cup L_3\}) \leq m(L_1) = h_X + 1. \quad (4)
\]
Thus (1) would yield
\[
3h_X + 5 \leq 3h_X + 3. \quad (5)
\]

If the intersections appearing in (1) are all not distinct, then the only possible equalities are either
\[
L' \cap \{L_1 \cup L_3\} = \begin{cases} L_1 \cap L_3, \\ L_1 \cap L_4, \\ L_3 \cap L_4 \end{cases}
\]
(with the three points above necessarily distinct), or
\[ L' \cap \{L_1 \cup L_3\} = L \cap L' = L \cap \{L_1 \cup L_3\} \]
(and the two cases are mutually exclusive).

Observe that inequalities (3) and (4) are not affected by either of the cases above. Similarly, the first case does not affect (2). When the second case occurs, we may replace (2) by
\[ m(L' \cap \{L_1 \cup L_3\}) + m(L' \cap L_2) \leq m(L') = h_X + 1. \]
Also, (1) becomes
\[ 3h_X + 5 \leq m(L_1 \cap L_3) + m(L_1 \cap L_4) + m(L \cap \{L_1 \cup L_3\}) + m(L' \cap \{L_1 \cup L_3\}) + m(L' \cap L_2) + m(L_4 \cap L_3) + m(L \cap L_2). \]
The last two inequalities, together with (3) and (4), yield (5) again. □

**Corollary 4.9.** Let \( X \) be as in Configuration B and suppose the line, \( L_4 \), through \( P_2 \) and \( P_3 \) is maximal. If \(|\{\cup_{L \in \mathcal{L}} L\} \cap L_2| \geq 2\) then, for every \( L \in \mathcal{L}, \)
\[ L \cap \{L_1 \cup L_3\} \subset \{P_1, P_2, P_3\}. \]
If, furthermore, \(|\{\cup_{L \in \mathcal{L}} L\} \cap \{L_1 \cup L_3\}| \geq 2\), then \(|\{\cup_{L \in \mathcal{L}} L\} \cap L_2| = 2. \)

**Proof.** The first statement is an immediate consequence of the previous lemma.
As for the second statement, suppose \(|\{\cup_{L \in \mathcal{L}} L\} \cap L_2| \geq 3\). Then there exist at least three lines in \( \mathcal{L} \) intersecting \( L_2 \) in three distinct points of \( \text{Supp}(X) \) and intersecting \( \pi_1 \) on \( L_1 \cup L_3 \), by (4) of Lemma 4.5, hence in a subset of \( \{P_1, P_2, P_3\} \), by the first part of the statement. The (possible) mutual intersections of these three lines of \( \mathcal{L} \) must lie in \( \pi_1 \), otherwise there would be a plane containing \( L_2 \) and either \( L_1 \) or \( L_3 \) or \( L_4 \), thus contradicting \( L_2 \cap \pi_1 \cap \text{Supp}(X) = \emptyset \), because of Lemma 3.7. Label these three lines of \( \mathcal{L} \) as \( L_5, L_6 \) and \( L_7 \), with \( L_7 \not\subset P_1, L_5 \not\subset P_2, \) and \( L_6 \not\subset P_3 \). Then, by Remark 3.4, we must have
\[ 3h_X + 1 \geq 7(h_X + 1) - m(L_1 \cap L_3) - m(L_1 \cap L_4) - m(L_3 \cap L_4) \]
\[ - \sum_{i=5}^{7} m(L_i \cap L_2) - \sum_{i=5}^{7} m(L_i \cap \{L_1 \cup L_3\}), \]
with \( \sum_{i=5}^{7} m(L_i \cap L_2) \leq m(L_2) = h_X + 1 \), and
\[ m(L_1 \cap L_3) + m(L_7 \cap \{L_1 \cup L_3\}) \leq h_X + 1, \]
\[ m(L_1 \cap L_4) + m(L_5 \cap \{L_1 \cup L_3\}) \leq h_X + 1, \]
\[ m(L_3 \cap L_4) + m(L_6 \cap \{L_1 \cup L_3\}) \leq h_X + 1, \]
as in each of the three cases above we have two distinct points of \( \text{Supp}(X) \).
Therefore, we would obtain
\[ 3h_X + 1 \geq 7(h_X + 1) - 4(h_X + 1) = 3h_X + 3, \]
which is a contradiction. \( \Box \)

Now we introduce the only obstruction to Condition H in \( \mathbb{P}^3 \).

**Configuration A.** Let \( X \) be as in Configuration B, with \( \{ \bigcup_{i \in \mathcal{I}} L \} \cap L_2 = \{ P_4, P_5 \} \). Suppose the line, \( L_4 \), through \( P_2, P_3 \) is maximal and assume that, for all \( i = 1, 2, 3 \), the lines through \( P_i \) and \( P_4 \) and through \( P_i \) and \( P_5 \) are all maximal.

**Remark 4.10.** If \( X \) is as in Configuration A, then, because of (4) of Lemma 4.5, for any line \( L \in \mathcal{L} \) we have
\[ L \cap \{ L_1 \cup L_3 \} \neq \emptyset \neq L \cap L_2. \]

In order to prove that Configuration A is indeed the only obstruction to Condition H, we need the following:

**Lemma 4.11.** Let \( X = \{(P_i, m_i) : i = 1, \ldots, s\} \subset \mathbb{P}^3 \), with \( m_i \leq h_X - 2, \forall i = 1, \ldots, s \), and \( h_X = h_X(2) \). Let \( \Pi_1 \) and \( \Pi_2 \) be two distinct maximal planes. Then

1. \( m(\mathbb{P}^3 \setminus \{ \Pi_1 \cup \Pi_2 \}) \leq 2; \)
2. if \( L \) is a maximal line, with \( L \nsubseteq \Pi_1 \cup \Pi_2 \), then
   \[ L \cap \Pi_1 \cap \text{Supp}(X) \neq \emptyset \neq L \cap \Pi_2 \cap \text{Supp}(X); \]
3. if \( \Pi \neq \Pi_1, \Pi_2 \) is a maximal plane, then
   \[ |\Pi \cap \Pi_1 \cap \text{Supp}(X)| \geq 2 \leq |\Pi \cap \Pi_2 \cap \text{Supp}(X)|. \]

**Proof.** By Remark 3.4, we have
\[ 3h_X + 1 \geq m(\Pi_1 \cup \Pi_2) \geq 2h_X + 2 - m(\Pi_1 \cap \Pi_2), \]
on the other hand,
\[ m(\Pi_1 \cap \Pi_2) \leq h_X + 1, \]
whence
\[ r = m(\mathbb{P}^3 \setminus \{ \Pi_1 \cup \Pi_2 \}) \leq 2, \]
which proves the first assertion.

To prove (2), suppose \( L \cap \Pi_1 \cap \text{Supp}(X) = \emptyset \), then
\[ m(L) \leq m(\Pi_2 \cap L) + r \leq h_X - 2 + r \leq h_X, \]
hence \( L \) would not be maximal. Similarly if \( L \cap \Pi_2 \cap \text{Supp}(X) = \emptyset \).

To show (3), observe that, by Lemma 3.5, \( m(\Pi \cap \Pi_j) \geq h_X - 1 \) \((j = 1, 2)\), but \( m_i \leq h_X - 2 \), for all \( i = 1, \ldots, s \). \( \Box \)
Theorem 4.12. Let $X = \{(P_1,m_1),\ldots,(P_s,m_s)\} \subset \mathbb{P}^3$. If $X$ is not as in Configuration A, then Condition H holds for $X$.

Proof. Case 1: $h_X = h_X(1) > h_X(2)$. Clearly Condition H holds if there are no skew maximal lines, or there are at most three maximal lines.

So let us suppose that there are $k \geq 4$ maximal lines, two of which are skew (which is the maximum allowed by Lemma 3.10), $L_1 \cap L_2 = \emptyset$.

If any other maximal line meets one of $L_1, L_2$, say $L_2$, then the plane containing $L_2$ and any point of $L_1 \cap \text{Supp}(X)$ suffices.

Otherwise, there exists a maximal line $L_3$ such that $L_3 \cap L_2 = \emptyset$; but then, by Lemma 3.10, we necessarily have $L_3 \cap L_1 \neq \emptyset$. Call $\pi_1$ the plane containing $L_1 \cup L_3$.

If $L_2 \cap \pi_1 \cap \text{Supp}(X) \neq \emptyset$, then, because of the conditions so far assumed, $L_2 \cap \{L_1 \cup L_3\} \neq \emptyset$. In this case Condition H is satisfied by choosing $H = \pi_1$, for there cannot be maximal lines not intersecting $\pi_1$ in points of $\text{Supp}(X)$ (in fact, by (1) of Lemma 4.5, if $L$ is a line such that $L \cap \pi_1 \cap \text{Supp}(X) = \emptyset$, then $m(L) \leq m_1 - 2 + m(L \cap L_2) < h_X + 1$).

So suppose $L_2 \cap \pi_1 \cap \text{Supp}(X) = \emptyset$, i.e., $X$ is as in Configuration B.

If $\{\bigcup_{L \in \mathcal{L}} L\} \cap L_2 = \emptyset$, then either $\mathcal{L} = \emptyset$ or (by (3) of Lemma 4.5) any line $L \in \mathcal{L}$ contains $P_1$, so we are done by choosing $H$ as the plane through $L_1$ (or $L_3$) and any point of $L_2$.

If $L \cap L_2 = \{P\}$, for every $L \in \mathcal{L}$, then we are done by choosing $H$ as the plane through $L_1$ (or $L_3$) and $P$.

Now assume $|\{\bigcup_{L \in \mathcal{L}} L\} \cap L_2| \geq 2$. If every maximal line of $\pi_1$ contains $P_1 = L_1 \cap L_3$, by (3) of Lemma 4.5, we are done by choosing $H$ as the plane through $P_1$ and $L_2$.

So, suppose there exists a maximal line $L_4 \subset \pi_1$ such that $P_1 \not\subset L_4$. We may relabel the points, if necessary, so as to assume $P_2 = L_1 \cap L_4, P_3 = L_3 \cap L_4$, and $P_4, P_3 \in L_2$. Then, by Corollary 4.9 and (4) of Lemma 4.5, for each $L \in \mathcal{L}$, we have that $\emptyset \neq L \cap \{L_1 \cup L_3\} \subset \{P_1, P_2, P_3\}$. Condition H is then satisfied if there exists a $P_i$, for $i \in \{1,2,3\}$, which is contained in only one line, $L$, of $\mathcal{L}$, for we can choose $H$ to be the plane through the remaining two points and $L \cap L_2$. If each $P_i$ is contained in at least two lines of $\mathcal{L}$, then $X$ is as in Configuration A.

Case 2: $h_X = h_X(2)$. First assume there is only one maximal plane $\pi_1$. In view of Remark 3.4, because $2h_X \leq m(\pi_1) \leq 2h_X + 1$, we can have at most one maximal line, $L_2$, skew with $\pi_1$ with respect to $X$. If there is none, we are clearly done, by taking $H = \pi_1$ ($H$ contains at least three points of $\text{Supp}(X)$, thus lowering $h_X(3)$ as well).

Otherwise, by Remark 3.4, we necessarily have $m(\pi_1) = 2h_X$ (hence any plane through one point of $\pi_1 \cap \text{Supp}(X)$ will lower $h_X(2)$) and $\text{Supp}(X) \setminus \{\pi_1 \cup L_2\} = \emptyset$. Thus, if $X$ is not as in Configuration A, we are done by proceeding as in Case 1, as in each of the cases above we have chosen $H$ to contain at least one point of $\pi_1 \cap \text{Supp}(X)$ and at least three points of $\text{Supp}(X)$.

Now suppose there are at least two maximal planes, $\pi_1, \pi_2$, which, by Lemma 3.9, necessarily meet in points of $\text{Supp}(X)$. Because of Lemmas 4.1 and 4.2, we may assume that $m_i \leq h_X - 2$ for all $i = 1,\ldots,s$. Then, by (2) of Lemma 4.11, any maximal line $L \not\subset \pi_1 \cup \pi_2$ must intersect both $\pi_1$ and $\pi_2$ in points of $\text{Supp}(X)$, and, by (3) of
Lemma 4.11, any other maximal plane, \( \pi \), must intersect each of \( \pi_1 \) and \( \pi_2 \) at least in two distinct points of \( \text{Supp}(X) \).

Thus we are done by choosing \( H = \pi_1 \) or \( H = \pi_2 \), except if there exist a maximal line \( L_1 \subset \pi_1 \) and a maximal line \( L_2 \subset \pi_2 \) such that \( L_1 \cap L_0 \cap \text{Supp}(X) = \emptyset = L_2 \cap L_0 \cap \text{Supp}(X) \), where \( L_0 = \pi_1 \cap \pi_2 \).

In this case, it necessarily follows from Lemma 3.5, that \( m(L_0) = h_X - 1 \) and so (by Remark 3.4), that \( \text{Supp}(X) \subset L_0 \cup L_1 \cup L_2 \), and \( m(\pi_1) = m(\pi_2) = 2h_X \). If there is no other maximal line in \( \pi_1 \cup \pi_2 \), then any maximal line must intersect both \( L_1 \) and \( L_2 \) in points of \( \text{Supp}(X) \), so we are done by choosing a plane through \( L_1 \) (or \( L_2 \)) and any point of \( L_2 \cap \text{Supp}(X) \) (resp., of \( L_1 \cap \text{Supp}(X) \)). If there is at least another maximal line in \( \pi_1 \cup \pi_2 \), then \( X \) is as in Configuration B.

Notice that in each step of Case 1 relative to Configuration B, we have chosen a plane containing either \( L_2 \) and a point of \( L_1 \cup L_3 \), or a maximal line of \( \pi_1 \) and a point of \( L_2 \). Because of (1) and (2) of Lemma 4.6 and (2) of Remark 4.3, this is enough to lower \( h_X(2) \).

**Case 3:** \( h_X = h_X(3) \). We may assume \( h_X(3) > h_X(1), h_X(2) \): but then, as we have already observed, a plane through any three points of \( \text{Supp}(X) \) lowers \( h_X \).

**Corollary 4.13.** If \( X = \{ P_1, \ldots, P_s \} \subset \mathbb{P}^3 \) is a set of simple points, then Condition H holds for \( X \).

**Proof.** The result is trivial for \( s = 1, \ldots, 4 \); while, for \( s \geq 5 \), follows from Theorem 4.12, after recalling that, because of Remark 4.4, \( X \) cannot be as in Configuration A. ⊐

**Theorem 4.14.** If \( X = \{ P_1, \ldots, P_s \} \subset \mathbb{P}^3 \) is a set of simple points, then

\[
\tau(X) \leq h_X.
\]

**Proof.** By induction on \( s \), the result being trivial for \( s = 1 \). If \( s > 1 \), by Corollary 4.13, we can find a plane, \( H \), such that, if \( H \cap X = \{ P_1, \ldots, P_s \} \), and \( Z = \{ P_{s+1}, \ldots, P_t \} \), then \( h_Z \leq h_X - 1 \). By induction, we have that \( \tau(Z) \leq h_X - 1 \) and, by Theorem 3.1 (for \( t = h_X \)), this implies \( \tau(X) \leq h_X \). ⊐

Now we look at sets of fat points in Configuration A.

**Lemma 4.15.** Let \( X \) be as in Configuration A and assume

\[
\text{Supp}(X) \subset \pi_1 \cup L_2,
\]

then

1. \( m_1 = \cdots = m_5 \);
2. for any \( i, j = 1, \ldots, 5, i \neq j \), the line joining \( P_i \) and \( P_j \) intersects \( \text{Supp}(X) \) exactly in \( P_i \) and \( P_j \).
Proof. Observe that, because \( \text{Supp}(X) \subset \pi_1 \cup L_2 \), by Remark 4.10, there are exactly two points of \( \text{Supp}(X) \) on each line of \( \mathcal{L} \). In particular, \( h_X(1) = m_i + m_j - 1 \), with \( i = 1, 2, 3 \) and \( j = 4, 5 \). From this we get \( m_1 = m_2 = m_3 \) and \( m_4 = m_5 \). If \( m_1 < m_4 \), then
\[
h_X + 1 = m(L_2) \geq 2m_4 > m_1 + m_4 = h_X + 1
\]
(and similarly if \( m_1 > m_4 \)), and this proves the first assertion. The second statement follows from 1. and the initial observation. \( \square \)

Lemma 4.16. Let \( X \) be as in Configuration A and assume
\[
\text{Supp}(X) \cap \pi_1 = \{P_1, P_2, P_3\}
\]
and
\[
\text{Supp}(X) \cap L_2 = \{P_4, P_5\}.
\]
Then
(1) \( m_1 = \ldots = m_5 \);
(2) for any \( i, j = 1, \ldots, 5, i \neq j \), the line joining \( P_i \) and \( P_j \) intersects \( \text{Supp}(X) \) exactly in \( P_i \) and \( P_j \).

Proof. Because \( L_1, L_3 \) and \( L_4 \) are maximal and contain exactly two points, we have \( m_i + m_j = h_X + 1 \), with \( i, j = 1, 2, 3 \) and \( i \neq j \), which implies \( m_1 = m_2 = m_3 \). Since \( m_4 + m_5 = h_X + 1 \) (as \( L_2 \) contains no other point of \( \text{Supp}(X) \)), if \( m_4 < m_1 \), then \( m_5 > m_1 \), whence \( m_1 + m_5 > h_X + 1 \), which is a contradiction. Similarly if \( m_5 < m_1 \) and if \( m_4 \) or \( m_5 > m_1 \). This gives the first assertion and implies that there are exactly two points of \( \text{Supp}(X) \) on the lines of \( \mathcal{L} \), thus proving the second statement. \( \square \)

From now on, if \( X \) is as in Configuration A and \( \text{Supp}(X) \) contains at least six points, we shall denote
\[
\mathcal{S} = \{P_6, \ldots, P_5\}.
\]

Lemma 4.17. Let \( X \) be as in Configuration A and assume
\[
\text{Supp}(X) \subset \pi_1 \cup L_2.
\]
Let \( \pi \) be the plane through \( P_1, P_4 \) and \( P_6 \), and \( \pi_2 \) the plane through \( P_2, P_4 \) and \( P_6 \). If \( \mathcal{S} \subset \pi \), then \( \pi_2 \cap \mathcal{S} = \{P_6\} \).

Proof. If \( \mathcal{S} = \{P_6\} \), there is nothing to prove.

Observe that, by (2) of Lemma 4.15, \( \mathcal{S} \subset \pi_1 \) and so \( L = \pi \cap \pi_1 \supset \{P_1\} \) \( \cup \mathcal{S} \).

Now, if \( \mathcal{S} \) has more than one point, and we suppose there is \( P_l \in \pi_2 \), with \( l \geq 7 \), then \( \pi_1 \cap \pi_2 \ni P_6, P_l \), and so \( \pi_1 \cap \pi_2 = L_1 \). On the other hand, \( P_2 \in \pi_1 \cap \pi_2 \ni P_1 \), whence \( L = L_1 \supset \mathcal{S} \), contrary to (2) of Lemma 4.15. \( \square \)

Lemma 4.18. Let \( X \) be as in Configuration A and assume
\[
\text{Supp}(X) \cap \pi_1 = \{P_1, P_2, P_3\}
\]
and

\[ \text{Supp}(X) \cap L_2 = \{P_4, P_3\}. \]

Let \( \pi \) be the plane through \( L_2 \) and \( P_6 \), and \( \pi_2 \) the plane through \( P_2, P_3 \) and \( P_5 \). Assume both \( \pi \) and \( \pi_2 \) contain \( \mathcal{S} \). Call \( H_1 \) the plane through \( P_2, P_4 \) and \( P_6 \), and \( H_2 \) the plane through \( P_1, P_3 \) and \( P_5 \). Then \( H_1 \cap \mathcal{S} = \{P_6\} \) and \( H_2 \cap \mathcal{S} = \emptyset \).

**Proof.** If \( |\mathcal{S}| = 1 \), the first assertion is obvious. If \( |\mathcal{S}| > 1 \) and there is \( P_4 \in H_1 \), with \( l \geq 7 \), then, because \( L = \pi \cap \pi_2 \supset \mathcal{S} \), we would have \( H_1 \cap \pi \ni P_6, P_4 \); which implies \( H_1 \cap \pi = L \). But then \( P_3 \in \pi \cap \pi_2 = L = H_1 \cap \pi \ni P_4 \); i.e. \( L_2 = L \supset \mathcal{S} \), contrary to the assumption \( \text{Supp}(X) \cap L_2 = \{P_4, P_3\} \).

Now suppose \( P_1 \in H_2 \cap \mathcal{S} \), with \( l \geq 6 \); then \( P_1 \in H_2 \cap \pi_2 \), which is the line joining \( P_3 \) and \( P_5 \); thus contradicting (2) of Lemma 4.16. \( \square \)

**Lemma 4.19.** Let \( X \) be as in Configuration A and assume

\[ \text{Supp}(X) \cap \pi_1 = \{P_1, P_2, P_3\} \]

and

\[ \text{Supp}(X) \cap L_2 = \{P_4, P_5\}. \]

Let \( \pi \) be the plane through \( L_2 \) and \( P_6 \), \( \pi_2 \) the plane through \( P_2, P_3 \) and \( P_5 \), and \( \pi_3 \) the plane through \( P_1, P_4 \) and \( P_6 \). Assume that both \( \pi \) and \( \pi_3 \) contain \( \mathcal{S} \) and \( \pi_2 \cap \mathcal{S} = \emptyset \). Call \( H_1 \) the plane through \( P_1, P_2 \) and \( P_6 \), and \( H_2 \) the plane through \( P_3, P_4 \) and \( P_5 \). Then \( H_1 \cap \mathcal{S} = \{P_6\} \) and \( H_2 \cap \mathcal{S} = \emptyset \).

**Proof.** The first assertion is obvious if \( \mathcal{S} = \{P_6\} \). So, assume \( |\mathcal{S}| > 1 \) and \( P_4 \in H_1 \), with \( l \geq 7 \). Because \( L = \pi \cap \pi_3 \supset \mathcal{S} \), we would have \( H_1 \cap \pi_3 \ni P_6, P_4 \), which implies \( H_1 \cap \pi_3 = L \). But then \( P_4 \in \pi \cap \pi_3 = L = H_1 \cap \pi_3 \ni P_1 \); hence the line through \( P_1 \) and \( P_3 \) is \( L \supset \mathcal{S} \), contrary to (2) of Lemma 4.16.

Now suppose \( P_1 \in H_2 \cap \mathcal{S} \), with \( l \geq 6 \); then \( P_1 \in H_2 \cap \pi = L_2 \), contrary to (2) of Lemma 4.16. \( \square \)

**Theorem 4.20.** Let \( X \) be as in Configuration A, with \( s \geq 7 \). Then there exist planes \( H_1, H_2 \) such that \( \{H_1 \cup H_2\} \cap \text{Supp}(X) = \{P_1, \ldots, P_r\} \), with \( r \geq 6 \), and \( \text{Supp}(X) \setminus \{H_1 \cup H_2\} \neq \emptyset \).

**Proof.** First suppose that \( \text{Supp}(X) \subset \pi_1 \cup L_2 \). Then, from (2) of Lemma 4.15, we know that \( \mathcal{S} \subset \pi_1 \). Thus no point of \( \mathcal{S} \) can be on the plane, \( \pi_2 \), through \( P_2, P_3 \) and \( P_4 \), otherwise it would be on \( L_4 = \pi_1 \cap \pi_2 \), which is impossible by (2) of Lemma 4.15.

Now consider the plane, \( \pi \), through \( P_1, P_3 \) and \( P_6 \).

If there exists \( P_1 \in \mathcal{S} \) such that \( P_1 \notin \pi \), then we can choose \( H_1 = \pi \) and \( H_2 = \pi_2 \).

If \( \pi \) contains \( \mathcal{S} \), then, by Lemma 4.17, the plane \( \pi_2 \) through \( P_2, P_5 \) and \( P_6 \) does not contain any other point of \( \mathcal{S} \). In this case, we choose \( H_1 = \pi_2 \) and \( H_2 \) as the
plane through \(P_1, P_3\) and \(P_5\), which cannot contain any other point of \(\mathcal{F}\) (otherwise, as above, the further point would be on \(L_1 = H_2 \cap \pi_1\), contrary to (2) of Lemma 4.15).

Now suppose \(\text{Supp}(X) \not\subset \pi_1 \cup L_2\), and temporarily fix \(H_1 = \pi_1\). If either \(\pi_1\) contains at least four points of \(\text{Supp}(X)\), or \(L_2\) contains at least three points of \(\text{Supp}(X)\), then we are done by picking any plane through \(L_2\) which avoids one point of \(\mathcal{F}\). Thus we may assume \(\text{Supp}(X) \cap \pi_1 = \{P_1, P_2, P_3\}\) and \(\text{Supp}(X) \cap L_2 = \{P_4, P_5\}\).

If there is a plane through \(L_2\) and one point of \(\mathcal{F}\), say \(P_6\) (after relabeling the points, if necessary), which avoids at least another point of \(\mathcal{F}\), choose that as \(H_2\).

If not, drop the assumption \(H_1 = \pi_1\) and call \(\pi\) the plane through \(L_2\) and \(P_6\), and \(\pi_2\) the plane through \(P_2, P_3\) and \(P_5\). It may happen that either \(\emptyset \neq \pi_2 \cap \mathcal{F} \neq \mathcal{F}\), or \(\pi_2 \cap \mathcal{F} = \emptyset\), or \(\pi_2 \cap \mathcal{F} = \mathcal{F}\).

In the first case, choose \(H_2 = \pi_2\) and \(H_1\) as any plane through \(P_1, P_4\) avoiding any point of \(\mathcal{F}\).

In the second case, by Lemma 4.18, we can choose \(H_1\) as the plane through \(P_2, P_4\) and \(P_6\), and \(H_2\) as the plane through \(P_1, P_3\) and \(P_5\).

In the last case, we can choose \(H_2 = \pi_2\) and \(H_1\) as the plane through \(P_1, P_4\) and \(P_6\), provided the latter avoids one point of \(\mathcal{F}\). If not, use Lemma 4.19 to choose \(H_1\) as the plane through \(P_1, P_2\) and \(P_6\) and \(H_2\) as the plane through \(P_3, P_4\) and \(P_5\).

**Lemma 4.21.** Let \(X\) be as in Configuration A and let \(L \neq L_1 (\forall i = 1, \ldots, 4)\) be a line such that \(L \cap \{L_1 \cup \cdots \cup L_4\} \cap \text{Supp}(X) \subset \{P_1, \ldots, P_3\}\).

1. If \(L \cap \{P_1, \ldots, P_5\} = \emptyset\), then \(m(L) \leq h_X - 2\);
2. If \(L \cap \{P_1, \ldots, P_5\} = \{P_i\}\), then \(m(L) \leq h_X - 1\).

**Proof.** After setting \(m = \min\{m_1, m_2, m_3\}\), it follows from (2) of Lemma 4.5 that, in the first case,

\[ m(L) \leq m - 2 \leq h_X - 2, \]

thus proving (1); while, in the second case,

\[ m(L) \leq m(P_i) + m - 2 \leq h_X - 1, \]

which proves the second assertion. \(\square\)

**Lemma 4.22.** Let \(X\) be as in Configuration A and let \(\pi \neq \pi_1\) be a plane such that \(\pi \cap \{L_1 \cup \cdots \cup L_4\} \cap \text{Supp}(X) \subset \{P_1, \ldots, P_5\}\).

1. If \(m(\pi) = 2h_X + 1\), then \(|\pi \cap \{P_1, \ldots, P_5\}| \geq 4\);
2. If \(m(\pi) = 2h_X\), then \(|\pi \cap \{P_1, \ldots, P_5\}| \geq 3\);
3. If \(2h_X - 2 \leq m(\pi) \leq 2h_X - 1\), then \(|\pi \cap \{P_1, \ldots, P_5\}| \geq 2\).

**Proof.** Let \(m = \min\{m_1, m_2, m_3\}\), and recall ((2) of Lemma 4.5) that

\[ m(\mathbb{P}^3 \setminus \{L_1 \cup L_2 \cup L_3 \cup L_4\}) \leq m - 2, \]

whence

\[ m(\pi) \leq m(\pi \cap \{P_1, \ldots, P_5\}) + m - 2. \]
First suppose $\pi \cap \{P_1, \ldots, P_5\} \subset \{P_i, P_{i2}, P_{i3}\}$. If $m \neq m_i, m_{i2}, m_{i3}$, then we have

$$m(\pi) \leq (m_i + m_{i2} - 1) + (m_i + m - 1) \leq 2h_X.$$ 

Otherwise, if, say, $m = m_i$, we still have

$$m(\pi) \leq (m + m_i - 1) + (m_i + m - 1) \leq 2h_X.$$ 

If $\pi \cap \{P_1, \ldots, P_5\} \subset \{P_i, P_{i2}\}$, then

$$m(\pi) \leq (m_i + m_{i2} - 1) + m - 1 \leq 2h_X - 1.$$ 

If $\pi \cap \{P_1, \ldots, P_5\} \subset \{P_i\}$, then

$$m(\pi) \leq (m_i + m - 1) - 1 \leq h_X - 1,$$

provided $m \neq m_i$. If $m = m_i$, but there is an $m_i \in \{m_1, m_2, m_3\}$ such that $m = m_i \neq m_i$, then we still have

$$m(\pi) \leq (m_i + m - 1) - 1 \leq (m_i + m_i - 1) - 1 \leq h_X - 1,$$

Finally, if $m_i = m = m_1 = m_2 = m_3$, then (by considering the line $L_t$) we get again

$$m(\pi) \leq (m_i + m - 1) - 1 = (m_1 + m_2 - 1) - 1 \leq h_X - 1.$$ 

Now we get (3) by observing that $h_X - 1 < 2h_X - 2$ whenever $h_X > 1$, which is the case when $X$ is as in Configuration A, because of (1) of Remark 4.3.

**Theorem 4.23.** Let $X = \{(P_1, m_1), \ldots, (P_s, m_s)\} \subset \mathbb{P}^3, s \geq 7$. Then

$$\tau(X) \leq h_X.$$

**Proof.** We shall proceed by induction on $\sum_{i=1}^s m_i \geq 7$. If $\sum_{i=1}^s m_i = 7$, then the statement follows from Theorem 4.14, for $X$ consists of seven simple points, so suppose $\sum_{i=1}^s m_i > 7$.

If $X$ is not as in Configuration A, then, by Theorem 4.12, we can find a plane $H$ such that, if $H \cap \text{Supp}(X) = \{(P_1, m_1), \ldots, (P_r, m_r)\}$, and $Z = \{(P_1, m_1 - 1), \ldots, (P_r, m_r - 1), (P_{r+1}, m_{r+1}), \ldots, (P_s, m_s)\}$, then $h_Z \leq h_X - 1$. By induction we have that $\tau(Z) \leq h_X - 1$ and, by Theorem 3.1 (for $t = h_X$), this implies $\tau(X) \leq h_X$.

If $X$ is as in Configuration A, then, by Theorem 4.20, we have that there exist two planes $H_1$ and $H_2$ such that $H_1 \cup H_2$ avoids at least one point of $\text{Supp}(X)$, say $P_s$, and $\{H_1 \cup H_2\} \cap \text{Supp}(X) = \{P_1, \ldots, P_r\}$, with $r \geq 6$. Denote

$$X' = \{(P_1, m_1), \ldots, (P_{s-1}, m_{s-1}), (P_s, m_s - 1)\},$$

$$Z = \{(P_1, m_1 - 1), \ldots, (P_r, m_r - 1), (P_{r+1}, m_{r+1}), \ldots, (P_s, m_s)\}.$$ 

Obviously, $h_{X'} \leq h_X$, thus, by induction, $\tau(X') \leq h_X$.

On the other hand, since $r \geq 6$, clearly $h_Z(3) \leq h_X - 2$.

Furthermore, observe that in the proof of Theorem 4.20, we either chose $H_1 = \pi_1$ and $H_2 \supset L_3$ or we had $L \cap \{L_1 \cup \cdots \cup L_4\} \cap \text{Supp}(X) \subset \{P_1, \ldots, P_3\}$ and $\pi \cap \{L_1 \cup \cdots \cup L_4\} \cap \text{Supp}(X) \subset \{P_1, \ldots, P_3\}$, for any line $L$ and any plane $\pi$. 

In the first case, by (4) and (5) of Lemma 4.5, we have \( h_Z(1) \leq h_X - 2 \) and, by Lemma 4.21 we get \( h_Z(2) \leq h_X - 2 \).

In the other case, by (4) of Lemma 4.5 and by Lemma 4.21 we get \( h_Z(1) \leq h_X - 2 \) and, by Lemmas 4.6 and 4.22, we get \( h_Z(2) \leq h_X - 2 \).

Thus \( h_Z \leq h_X - 2 \), hence by induction \( \tau(Z) \leq h_X - 2 \).

Therefore, by taking \( G = H_1H_2 \) and \( t = h_X \) in Theorem 3.2, we get \( \tau_X \leq h_X \), as we wished. \( \Box \)

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References