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Complete intersection lattice ideals

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Abstract

In this paper we completely characterize lattice ideals that are complete intersections or equivalently complete intersections finitely generated semigroups of $\mathbb{Z}^n \oplus T$ with no invertible elements, where T is a finite abelian group. We also characterize the lattice ideals that are set-theoretic complete intersections on binomials.

Keywords: Lattice ideals; Semigroups; Complete intersections; Semigroup gluing

1. Introduction

Let S be a finitely generated, cancellative, abelian semigroup with no invertible elements. S can be considered as a subsemigroup of a finitely generated abelian group $\mathbb{Z}^n \oplus T$ such that $S \cap (-S) = \{\mathbf{0}\}$, where T is a torsion group. In the case that the torsion group is trivial the semigroup S is called *affine semigroup*. Let $A = \{\mathbf{a}_i \mid i \in \{1, \ldots, m\}\}$ be a set of generators for the semigroup S, thus $S = \mathbb{N}A$, where $\mathbb{N}$ is the set of nonnegative integers. Let L denote the kernel of the group homomorphism from $\mathbb{Z}^m$ to $\mathbb{Z}^n \oplus T$ which sends $\mathbf{e}_i$

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to $\mathbf{a}_i$, where $\{\mathbf{e}_i \mid i \in \{1, \dots, m\}\}$ is the canonical basis of $\mathbb{Z}^m$. L is a sublattice of $\mathbb{Z}^m$, the *lattice ideal* associated to L is the binomial ideal

$$I_L = \left(\left\{\mathbf{x}^{\alpha^+} - \mathbf{x}^{\alpha^-} \;\middle|\; \alpha = \alpha^+ - \alpha^- \in L\right\}\right) \subset K[x_1, \dots, x_m],$$

where K is a field of any characteristic. The semigroup S is a complete intersection if and only if $I_L \subset \mathbb{Q}[x_1, \dots, x_m]$ is a complete intersection, which means that the minimal number of generators of I_L is equal to the height of I_L.

The problem of determining complete intersection semigroups or equivalently complete intersection lattice ideals has a long history. It was solved for affine semigroups gradually in a series of papers by J. Herzog [12], Ch. Delorme [5], R.P. Stanley [19], M.N. Ishida [13], K. Watanabe [21], H. Nakajima [14], U. Schäfer [17], J.C. Rosales, and P.A. Garcia-Sanchez [16]. Finally, in 1997 K.G. Fischer, W. Morris, and J. Shapiro [11] characterized all complete intersections affine semigroups of $\mathbb{Z}^n$ using mixed dominating matrices and the notion of semigroup gluing introduced by J.C. Rosales [15]. Recently D. Dais and M. Henk [4] used Nakajima's classification to describe the precise form of the binomial equations which determine toric locally complete intersection singularities.

Another related problem that drew the attention of a number of authors over the last years was the generation of a lattice ideal by binomials up to radical [1–3,6–9]. In 2002 K. Eto [8] characterized complete intersection finitely generated, abelian semigroups with no invertible elements or equivalently complete intersection lattice ideals as those that are set-theoretic complete intersection on binomials in characteristic zero. A generalization of the corresponding result for affine semigroups or equivalently toric varieties, which was provided by M. Barile, M. Morales, and A. Thoma [2]. Note that a binomial ideal I is set-theoretic complete intersection on binomials if there exist $r = \mathrm{height}(I)$ binomials $F_1, \dots, F_r$ such that $\mathrm{rad}(I) = \mathrm{rad}(F_1, \dots, F_r)$. Recently M. Barile and G. Lyubeznik [1] used p-gluing of affine semigroups and étale cohomology to give a class of toric varieties which are set-theoretic complete intersections only over fields of one positive characteristic p.

The aim of this article is twofold. On the one hand, we give a complete characterization of all finitely generated, cancellative, abelian semigroups with no invertible elements or equivalently lattice ideals that are complete intersections by introducing the notion of gluing lattices and extending the notion of semigroup gluing. On the other hand we characterize all lattice ideals that are set-theoretic complete intersection on binomials in any characteristic by extending the notion of p-gluing. The characterization depends on the characteristic.

2. Semigroup and lattice gluing

A *lattice* is a finitely generated free abelian group. A *partial character* (L, ρ) on $\mathbb{Z}^m$ is a homomorphism ρ from a sublattice L of $\mathbb{Z}^m$ to the multiplicative group $K^* = K - \{0\}$. Given a partial character (L, ρ) on $\mathbb{Z}^m$, we define the ideal

$$I_{L,\rho} := \left(\left\{\mathbf{x}^{\alpha^+} - \rho(\alpha)\mathbf{x}^{\alpha^-} \;\middle|\; \alpha = \alpha^+ - \alpha^- \in L\right\}\right) \subset K[x_1, \dots, x_m]$$

called *lattice ideal*. Here $\alpha^+ \in \mathbb{N}^m$ and $\alpha^- \in \mathbb{N}^m$ denote the positive and negative part of α, respectively, and $\mathbf{x}^\beta = x_1^{b_1} \cdots x_m^{b_m}$ for $\beta = (b_1, \ldots, b_m) \in \mathbb{N}^m$. We will denote by $F(\alpha)$ the binomial $\mathbf{x}^{\alpha^+} - \mathbf{x}^{\alpha^-}$ and by $F_\rho(\alpha)$ the binomial $\mathbf{x}^{\alpha^+} - \rho(\alpha)\mathbf{x}^{\alpha^-}$. Lattice ideals are binomial ideals. The theory of binomial ideals was developed by Eisenbud and Sturmfels in [6]. A prime lattice ideal is called a *toric ideal*, while the set of zeroes in K^m is an *affine toric variety* in the sense of [20], since we do not require normality.

Let $A = \{\mathbf{a}_i \mid 1 \leqslant i \leqslant m\} \subset \mathbb{Z}^n \oplus T$ be such that the semigroup $\mathbb{N}A$ has no invertible element. That means that although the group $\mathbb{Z}^n \oplus T$ has torsion elements, no nonzero element in the semigroup $\mathbb{N}A$ is a torsion element. This remark will be very useful in the sequel.

Let $\psi : \mathbb{Z}^m \to \mathbb{Z}^n \oplus T$ be a group homomorphism such that $\psi(\mathbf{e}_i) = \mathbf{a}_i \in \mathbb{Z}^n \oplus T$, where $\mathbf{e}_1, \ldots, \mathbf{e}_m$ is the canonical basis of $\mathbb{Z}^m$. We will denote by L the lattice $\ker(\psi)$. The fact that the semigroup $\mathbb{N}A$ has no invertible element is equivalent with the fact that the lattice L is *positive*, that is $L \cap \mathbb{N}^m = \{\mathbf{0}\}$. This means that the lattice ideal $I_{L,\rho}$ is homogeneous with respect to some positive grading. In this case by the graded Nakayama's lemma all minimal systems of generators of the ideal $I_{L,\rho}$ have the same cardinality.

For a lattice L and a prime number p, let $(L : p^\infty)$ be the lattice

$$\{\mathbf{u} \in \mathbb{Z}^m \mid p^k\mathbf{u} \in L \text{ for some } k \in \mathbb{N}\}.$$

For a semigroup S, $(S : p^\infty)$ denotes the semigroup

$$\{\mathbf{b} \in \mathbb{Z}^n \oplus T \mid p^k\mathbf{b} \in S \text{ for some } k \in \mathbb{N}\}.$$

Let $E \subset \{1, \ldots, m\}$, for a set $P \subset \mathbb{Z}$ we denote by

$$P^E := \{(p_1, \ldots, p_m) \in \mathbb{Z}^m \mid p_i \in P \text{ for } i \in E,\ p_i = 0 \text{ for } i \notin E\}.$$

L_E denotes the lattice $L \cap \mathbb{Z}^E$ and $\mathbb{N}A^E$ the semigroup generated by $A^E = \{\mathbf{a}_i \mid i \in E\}$. For a single element $\mathbf{u} \in \mathbb{Z}^m$ we denote

$$\mathbf{u}^E = \{(\mathbf{u}'_i) \in \mathbb{Z}^m \mid \mathbf{u}'_i = \mathbf{u}_i \text{ for } i \in E,\ \mathbf{u}'_i = 0 \text{ for } i \notin E\}.$$

Lemma 2.1. *Let $U \subset L \subset \mathbb{Z}^m$ be two lattices. Then $p^kL \subset U$ for some $k \in \mathbb{N}$ if and only if*

$$(L : p^\infty) = (U : p^\infty).$$

Proof. Suppose that $p^kL \subset U$ for some $k \in \mathbb{N}$. From $U \subset L$ we have $(U : p^\infty) \subset (L : p^\infty)$. Let $\mathbf{u} \in (L : p^\infty)$. Then there exists $n \in \mathbb{N}$ such that $p^n\mathbf{u} \in L$, and the hypothesis implies that $p^{n+k}\mathbf{u} \in U$. Therefore $\mathbf{u} \in (U : p^\infty)$. For the converse, suppose that $L = \sum_{i=1}^r \mathbb{Z}\mathbf{u}_i$. Then, since $\mathbf{u}_i \in L$, we have

$$\mathbf{u}_i \in (L : p^\infty) = (U : p^\infty).$$

Which means that there exists $k_i \in \mathbb{N}$ such that $p^{k_i}\mathbf{u}_i \in U$, $1 \leqslant i \leqslant r$. By choosing k the maximum of all k_i, we have $p^k L \subset U$. $\square$

We give the definitions of semigroup gluing (respectively p-gluing) for subsemigroups of $\mathbb{Z}^n \oplus T$ and gluing (respectively p-gluing) of lattices.

Definition 2.2. Let E_1, E_2 be two nonempty subsets of $\{1,\dots,m\}$ such that $E_1 \cup E_2 = \{1,\dots,m\}$ and $E_1 \cap E_2 = \emptyset$. The semigroup $\mathbb{N}A$ is called the gluing (respectively the p-gluing) of the semigroups $\mathbb{N}A^{E_1}$ and $\mathbb{N}A^{E_2}$ if there is a nonzero $\mathbf{a} \in \mathbb{N}A^{E_1} \cap \mathbb{N}A^{E_2}$ (respectively $\mathbf{a} \in ((\mathbb{N}A^{E_1} \cap \mathbb{N}A^{E_2}) : p^\infty)$) such that $\mathbb{Z}\mathbf{a} = \mathbb{Z}A^{E_1} \cap \mathbb{Z}A^{E_2}$.

Definition 2.3. Let E_1, E_2 be two nonempty subsets of $\{1,\dots,m\}$ such that $E_1 \cup E_2 = \{1,\dots,m\}$ and $E_1 \cap E_2 = \emptyset$. The lattice L is called the gluing (respectively p-gluing) of the lattices L_{E_1} and L_{E_2} if there is a nonzero $\mathbf{u} \in L$ with $\mathbf{u}^+ = \mathbf{u}^{E_1}$ and $\mathbf{u}^- = -\mathbf{u}^{E_2}$, such that $L = L_{E_1} + L_{E_2} + \langle \mathbf{u} \rangle$ (respectively

$$\bigl(L : p^\infty\bigr) = \bigl(\bigl(L_{E_1} + L_{E_2} + \langle \mathbf{u} \rangle\bigr) : p^\infty\bigr).$$

A set of elements $\mathbf{a}_1,\dots,\mathbf{a}_s$ of $\mathbb{Z}^n \oplus T$ is called linearly independent if the space of relations is $\{\mathbf{0}\}$, that means the relation $\sum_{i=1}^s n_i\mathbf{a}_i = 0$ in $\mathbb{Z}^n \oplus T$, with $n_i \in \mathbb{Z}$, implies $n_1 = \cdots = n_s = 0$.

Definition 2.4. We call a semigroup completely glued (respectively p-glued) if it belongs to C (respectively P), which is the smallest class of finitely generated, cancellative, abelian semigroups with no invertible elements that includes all semigroups generated by linearly independent elements and is closed under gluing (respectively p-gluing).

In the sequel we prove some general results that relate the gluing of semigroups with the gluing of lattices. We remind the reader that L denotes the kernel of the group homomorphism from $\mathbb{Z}^m$ to $\mathbb{Z}^n \oplus T$ which sends $\mathbf{e}_i$ to $\mathbf{a}_i$, where $\{\mathbf{e}_i \mid i \in \{1,\dots,m\}\}$ is the canonical basis of $\mathbb{Z}^m$. Thus with every semigroup $\mathbb{N}A \subset \mathbb{Z}^n \oplus T$ we associate a lattice $L \subset \mathbb{Z}^m$. Also with every lattice $L \subset \mathbb{Z}^m$ we associate the semigroup generated by $\mathbf{e}_i + L$ in $\mathbb{Z}^m/L$, where $i \in \{1,\dots,m\}$. We define a lattice to be *completely glued* (respectively p-glued) if and only if the associated semigroup is completely glued (respectively p-glued).

Theorem 2.5. *The semigroup $\mathbb{N}A$ is the p-gluing* (*respectively gluing*) *of the semigroups $\mathbb{N}A^{E_1}$ and $\mathbb{N}A^{E_2}$ if and only if the lattice L is the p-gluing* (*respectively gluing*) *of the lattices L_{E_1} and L_{E_2}.*

Proof. Suppose that $\mathbb{N}A$ is the p-gluing of $\mathbb{N}A^{E_1}$ and $\mathbb{N}A^{E_2}$. Let

$$\mathbf{a} \in \bigl(\bigl(\mathbb{N}A^{E_1} \cap \mathbb{N}A^{E_2}\bigr) : p^\infty\bigr)$$

such that $\mathbb{Z}\mathbf{a} = \mathbb{Z}A^{E_1} \cap \mathbb{Z}A^{E_2}$. Then $p^k\mathbf{a} = \sum_{i\in E_1} u_i\mathbf{a}_i = \sum_{i\in E_2}(-u_i\mathbf{a}_i)$, for some $k \in \mathbb{N}$. Then $\mathbf{u} = (u_i) \in L$ with $\mathbf{u}^+ = \mathbf{u}^{E_1}$ and $\mathbf{u}^- = -\mathbf{u}^{E_2}$. Let $\mathbf{l} = (l_i) \in (L : p^\infty)$. Then $p^s\mathbf{l} \in L$ for some $s \in \mathbb{N}$, which implies $\sum_{i\in\{1,\dots,m\}} p^s l_i\mathbf{a}_i = \mathbf{0}$. Consider the element

$$\mathbf{b} = \sum_{i\in E_1} p^s l_i\mathbf{a}_i = \sum_{i\in E_2}\bigl(-p^s l_i\mathbf{a}_i\bigr) \in \mathbb{Z}A^{E_1} \cap \mathbb{Z}A^{E_2} = \mathbb{Z}\mathbf{a}.$$

There exists a $\mu \in \mathbb{Z}$ such that $\mathbf{b} = \mu\mathbf{a}$, which means

$$\sum_{i\in E_1} p^{k+s} l_i\mathbf{a}_i = \mu\sum_{i\in E_1} u_i\mathbf{a}_i \quad \text{and} \quad \sum_{i\in E_2}(-p^{k+s} l_i\mathbf{a}_i) = \mu\sum_{i\in E_2}\bigl(-u_i\mathbf{a}_i\bigr).$$

Therefore $\mathbf{l}_1 = (p^{k+s}l_i - \mu u_i)^{E_1} \in L_{E_1}$ and $\mathbf{l}_2 = (p^{k+s}l_i - \mu u_i)^{E_2} \in L_{E_2}$, and $p^{k+s}\mathbf{l} = \mathbf{l}_1 + \mathbf{l}_2 + \mu\mathbf{u}$. Therefore

$$\bigl(L : p^\infty\bigr) \subset \bigl(\bigl(L_{E_1} + L_{E_2} + \langle\mathbf{u}\rangle\bigr) : p^\infty\bigr).$$

The other inclusion is obvious.

Suppose that

$$\bigl(L : p^\infty\bigr) = \bigl(\bigl(L_{E_1} + L_{E_2} + \langle\mathbf{u}\rangle\bigr) : p^\infty\bigr)$$

with $\mathbf{u}^+ = \mathbf{u}^{E_1}$ and $\mathbf{u}^- = -\mathbf{u}^{E_2}$. By virtue of Lemma 2.1, there exists an $s \in \mathbb{N}$ such that $p^sL \subset (L_{E_1} + L_{E_2} + \langle\mathbf{u}\rangle)$. Set $\mathbf{c} = \sum_{i\in E_1} u_i\mathbf{a}_i = \sum_{i\in E_2} -u_i\mathbf{a}_i$. Then $\mathbf{c} \in \mathbb{N}A^{E_1} \cap \mathbb{N}A^{E_2}$. Let $\mathbf{b} \in \mathbb{Z}A^{E_1} \cap \mathbb{Z}A^{E_2}$, then $\mathbf{b} = \sum_{i\in E_1} l_i\mathbf{a}_i = \sum_{i\in E_2} -l_i\mathbf{a}_i$. This implies that $\mathbf{l} = (l_i) \in L$, therefore $p^s\mathbf{l} = \mathbf{l}_1 + \mathbf{l}_2 + \mu\mathbf{u}$ for some $\mathbf{l}_1 \in L_{E_1}$, $\mathbf{l}_2 \in L_{E_2}$ and $\mu \in \mathbb{Z}$. But then

$$p^s\mathbf{b} = \sum_{i\in E_1} p^s l_i\mathbf{a}_i = \sum_{i\in E_1}\bigl(\mathbf{l}_1 + \mu\mathbf{u}^+\bigr)_i\mathbf{a}_i = \mu\mathbf{c}.$$

Among the elements of $\mathbb{Z}A^{E_1} \cap \mathbb{Z}A^{E_2}$ choose $\mathbf{a}$ such that μ is positive and the smallest possible, set $\mu = \mu_{\mathbf{a}}$. Then it follows that $\mathbb{Z}\mathbf{a} = \mathbb{Z}A^{E_1} \cap \mathbb{Z}A^{E_2}$. Now $\mathbf{c} \in \mathbb{Z}A^{E_1} \cap \mathbb{Z}A^{E_2}$, therefore there exists a natural number λ such that $\mathbf{c} = \lambda\mathbf{a}$. Then from $p^s\mathbf{a} = \mu_{\mathbf{a}}\mathbf{c}$ we have $p^s\mathbf{a} = \mu_{\mathbf{a}}\lambda\mathbf{a}$. Which implies that $\lambda = p^k$ for some $k \in \mathbb{N}$, since the order of $\mathbf{a}$ is not finite, as for every nonzero element in $\mathbb{N}A$. Therefore

$$\mathbf{a} \in \bigl(\bigl(\mathbb{N}A^{E_1} \cap \mathbb{N}A^{E_2}\bigr) : p^\infty\bigr).$$

The proof of the gluing part of the theorem follows from the proof of the p-gluing part by setting $p = 1$. Actually the second part of the proof is much simpler. □

The next theorem shows how the gluing (respectively p-gluing) of lattices reflects on the (respectively radical of the) lattice ideal. The first part of the theorem is a generalization of the corresponding result by J.C. Rosales [15] for toric ideals.

Theorem 2.6. *Let E_1, E_2 be two nonempty subsets of $\{1,\dots,m\}$ such that $E_1 \cup E_2 = \{1,\dots,m\}$ and $E_1 \cap E_2 = \emptyset$. The lattice L is the gluing of the lattices L_{E_1} and L_{E_2} if and only if*

$$I_L = I_{L_{E_1}} + I_{L_{E_2}} + \langle F(\mathbf{u})\rangle,$$

where $\mathbf{u}$ is a nonzero element in L such that $\mathbf{u}^+ = \mathbf{u}^{E_1}$ and $\mathbf{u}^- = -\mathbf{u}^{E_2}$. The lattice L is the p-gluing of the lattices L_{E_1} and L_{E_2} if and only if

$$\operatorname{rad}(I_L) = \operatorname{rad}\bigl(I_{L_{E_1}} + I_{L_{E_2}} + \langle F(\mathbf{u})\rangle\bigr),$$

in characteristic $p > 0$, where $\mathbf{u}$ is a nonzero element in L such that $\mathbf{u}^+ = \mathbf{u}^{E_1}$ and $\mathbf{u}^- = -\mathbf{u}^{E_2}$.

Proof. We prove only the second claim, since the proof of the first is simpler and follows from the proof of the second by putting $p = 1$, even in positive characteristic, and taking out the radicals. Suppose that the lattice L is the p-gluing of the lattices L_{E_1} and L_{E_2}. Then

$$\bigl(L : p^\infty\bigr) = \bigl(\bigl(L_{E_1} + L_{E_2} + \langle\mathbf{u}\rangle\bigr) : p^\infty\bigr).$$

By Theorem 2.5, the semigroup $\mathbb{N}A$ is the p-gluing of the semigroups $\mathbb{N}A^{E_1}$ and $\mathbb{N}A^{E_2}$. Then we know that $\mathbb{Z}\mathbf{a} = \mathbb{Z}A^{E_1} \cap \mathbb{Z}A^{E_2}$, where $p^k\mathbf{a} = \sum_{i\in E_1} u_i\mathbf{a}_i = \sum_{i\in E_2}(-u_i\mathbf{a}_i)$. Let $F(\mathbf{v}) \in I_L$. Then $\mathbf{v} \in L$ and so $\sum_{i=1}^m v_i\mathbf{a}_i = 0$. Then

$$\sum_{i\in E_1} v_i^+\mathbf{a}_i + \sum_{i\in E_2} v_i^+\mathbf{a}_i = \sum_{i\in E_1} v_i^-\mathbf{a}_i + \sum_{i\in E_2} v_i^-\mathbf{a}_i.$$

Therefore

$$\gamma := \sum_{i\in E_1} v_i^+\mathbf{a}_i - \sum_{i\in E_1} v_i^-\mathbf{a}_i = \sum_{i\in E_2} v_i^-\mathbf{a}_i - \sum_{i\in E_2} v_i^+\mathbf{a}_i \in \mathbb{Z}A^{E_1} \cap \mathbb{Z}A^{E_2} = \mathbb{Z}\mathbf{a}.$$

That means that $\gamma = \tau\sum_{i\in E_1} u_i\mathbf{a}_i = \tau\sum_{i\in E_2}(-u_i\mathbf{a}_i)$, for some $\tau \in \mathbb{Z}$, which without loss of generality we can suppose to be positive. Then, since the characteristic is $p > 0$, we have

$$\begin{aligned}\bigl(F(\mathbf{v})\bigr)^{p^k} = F\bigl(p^k\mathbf{v}\bigr) = \mathbf{x}^{p^k\mathbf{v}^+} - \mathbf{x}^{p^k\mathbf{v}^-} &= \bigl(\mathbf{x}^{p^k(\mathbf{v}^+)^{E_1}} - \mathbf{x}^{p^k(\mathbf{v}^-)^{E_1}+\tau\mathbf{u}^{E_1}}\bigr)\mathbf{x}^{p^k(\mathbf{v}^+)^{E_2}} \\ &\quad - \bigl(\mathbf{x}^{p^k(\mathbf{v}^-)^{E_2}} - \mathbf{x}^{p^k(\mathbf{v}^+)^{E_2}+\tau\mathbf{u}^{E_2}}\bigr)\mathbf{x}^{p^k(\mathbf{v}^-)^{E_1}} \\ &\quad + \mathbf{x}^{p^k(\mathbf{v}^-)^{E_1}+p^k(\mathbf{v}^+)^{E_2}}\bigl(\mathbf{x}^{\tau\mathbf{u}^{E_1}} - \mathbf{x}^{\tau\mathbf{u}^{E_2}}\bigr).\end{aligned}$$

From which it is easy to see that

$$\bigl(F(\mathbf{v})\bigr) \in \operatorname{rad}\bigl(I_{L_{E_1}} + I_{L_{E_2}} + \langle F(\mathbf{u})\rangle\bigr).$$

The reverse inclusion is obvious.

Suppose that $\mathrm{rad}(I_L) = \mathrm{rad}(I_{L_{E_1}} + I_{L_{E_2}} + \langle F(\mathbf{u})\rangle)$. Let U be the lattice $L_{E_1} + L_{E_2} + \langle \mathbf{u}\rangle$; then $U \subset L$ and thus also $I_U \subset I_L$. Also note that $I_{L_{E_1}} \subset I_U$, $I_{L_{E_2}} \subset I_U$ and $\langle F(\mathbf{u})\rangle \subset I_U$. Therefore $I_{L_{E_1}} + I_{L_{E_2}} + \langle F(\mathbf{u})\rangle \subset I_U$, which implies

$$\mathrm{rad}\big(I_{L_{E_1}} + I_{L_{E_2}} + \big\langle F(\mathbf{u})\big\rangle\big) \subset \mathrm{rad}(I_U).$$

Then from the hypothesis we have $\mathrm{rad}(I_U) = \mathrm{rad}(I_L)$. It follows from [6, Corollary 2.2], that in characteristic zero $I_U = I_L$ and so $U = L$, and in characteristic $p > 0$ that $I_{(U:p^\infty)} = I_{(L:p^\infty)}$ and so $(U : p^\infty) = (L : p^\infty)$. Note that in [6] $(L : p^\infty)$ is denoted by $\mathrm{Sat}_p(L)$. □

3. Complete intersections

In this section we will give a series of results that will characterize complete intersection lattice ideals and complete intersection semigroups. We also characterize lattice ideals that are set-theoretic complete intersections on binomials.

Let L be a nonzero positive sublattice of $\mathbb{Z}^m$ of rank r, and (L, ρ) be a partial character on $\mathbb{Z}^m$. The height of the lattice ideal $I_{L,\rho}$ is equal to r, the rank of the lattice L, see [6, Corollary 2.2].

Remark 3.1. Any variable x_i is a nonzero divisor for $I_{L,\rho}$.

We grade $K[x_1, \ldots, x_m]$ by setting $\deg_{(\mathbb{Z}^m/L)}(x_i) = \mathbf{a}_i$, for $i \in \{1, \ldots, m\}$. Then the $\mathbb{Z}^m/L$-degree of the monomial $x^{\mathbf{u}}$ is

$$\deg_{(\mathbb{Z}^m/L)}\big(x^{\mathbf{u}}\big) = u_1\mathbf{a}_1 + \cdots + u_m\mathbf{a}_m \in \mathbb{N}A,$$

where $\mathbb{N}A$ is the semigroup generated by A. The lattice ideal $I_{L,\rho} \subset K[x_1, \ldots, x_m]$ is $\mathbb{Z}^m/L$-homogeneous, since all generators are $\mathbb{Z}^m/L$-homogeneous. In particular, let $\mathbf{v} \in \mathbb{Z}^m$, $A, B \in K^*$, and $G(\mathbf{v}) = A\mathbf{x}^{\mathbf{v}^+} - B\mathbf{x}^{\mathbf{v}^-}$, then $G(\mathbf{v}) \in I_{L,\rho}$ implies $\mathbf{v} \in L$. Since, if $\mathbf{v} \notin L$, then $G(\mathbf{v})$ is not $\mathbb{Z}^m/L$-homogeneous. Then the monomial $\mathbf{x}^{\mathbf{v}_+}$ must be in $I_{L,\rho}$ since $I_{L,\rho}$ is $\mathbb{Z}^m/L$-homogeneous. This is impossible since any variable x_i is a nonzero divisor for $I_{L,\rho}$.

Lemma 3.2. *Let $I, J, K \subset R$ be three ideals in a noetherian ring R such that $J \subset I$ and* $\mathrm{rad}(I) = \mathrm{rad}(J)$, *then*

$$\mathrm{rad}(I + K) = \mathrm{rad}(J + K).$$

Proof. The inclusion $\mathrm{rad}(J + K) \subset \mathrm{rad}(I + K)$ is clear. Now let $g \in \mathrm{rad}(I + K)$. Then $g^q \in I + K$ and we can write $g^q = h_1 + h_2$, with $h_1 \in I, h_2 \in K$. Hence there exists l such that $h_1^l \in J$, so $g^{ql} = h_1^l + h_2'$ with $h_2' \in K$, which proves the assertion. □

Lemma 3.3. *Consider r vectors $\mathbf{u}_1,\dots,\mathbf{u}_r \in \mathbb{Z}^m$, let $L=\sum_{i=1}^r \mathbb{Z}u_i$ be the lattice generated by them. The following are equivalent*:

(1) $I_L=(F(\mathbf{u}_1),\dots,F(\mathbf{u}_r))$ *and* $F(\mathbf{u}_1),\dots,F(\mathbf{u}_r)$ *is a regular sequence*;
(2) $I_{L,\rho}=(F_\rho(\mathbf{u}_1),\dots,F_\rho(\mathbf{u}_r))$ *and* $F_\rho(\mathbf{u}_1),\dots,F_\rho(\mathbf{u}_r)$ *is a regular sequence for any partial character* (L,ρ) *on* $\mathbb{Z}^m$.

Proof. First we remark that any variable x_i is a nonzero divisor of I_L, this implies that the sequence $F(\mathbf{u}_1),\dots,F(\mathbf{u}_r),x_1\dots x_m$, is a regular sequence. Let (L,ρ) be a partial character on $\mathbb{Z}^m$. Then $\rho(\mathbf{u})$ is a unit for every $\mathbf{u}\in L$. Thus by [18, Theorem 2.7] the sequence $F_\rho(\mathbf{u}_1),\dots,F_\rho(\mathbf{u}_r),x_1\dots x_m$, is regular. Let $\mathbf{u}\in L$ any nonzero vector, we can write $\mathbf{u}=n_1\mathbf{u}_1+\dots+n_r\mathbf{u}_r$. From the identity

$$\frac{\mathbf{x}^{\mathbf{u}^+}}{\mathbf{x}^{\mathbf{u}^-}}-\rho(\mathbf{u})=\prod_{i=1}^r\left(\frac{\mathbf{x}^{\mathbf{u}_i^+}}{\mathbf{x}^{\mathbf{u}_i^-}}\right)^{n_i}-\prod_{i=1}^r\left(\frac{\rho(\mathbf{u}_i^+)}{\rho(\mathbf{u}_i^-)}\right)^{n_i}$$

by clearing denominators we get an identity in $K[x_1,\dots,x_m]$ which shows that there exists a monomial P such that $PF_\rho(\mathbf{u})$ belongs to $(F_\rho(\mathbf{u}_1),\dots,F_\rho(\mathbf{u}_r))$. But $F_\rho(\mathbf{u}_1),\dots,F_\rho(\mathbf{u}_r),x_1\dots x_m$, is a regular sequence which implies that $F_\rho(\mathbf{u})\in(F_\rho(\mathbf{u}_1),\dots,F_\rho(\mathbf{u}_r))$, therefore $I_{L,\rho}=(F_\rho(\mathbf{u}_1),\dots,F_\rho(\mathbf{u}_r))$.

The proof of the other implication follows from applying (2) to the trivial character. □

Corollary 3.4. *For any lattice ideal $I_{L,\rho}$ the fact that $I_{L,\rho}$ is a complete intersection is independent from the character ρ.*

Definition 3.5 [10]. A matrix M with coefficients in $\mathbb{Z}$ is called mixed if every row has a positive and a negative entry. M is called dominating if it does not contain any square mixed submatrix.

We also define the empty matrix $(0\times d)$ to be mixed dominating.

We denote by $M(\mathbf{u}_1,\dots,\mathbf{u}_r)$ the $r\times m$ matrix whose rows are the vectors $\mathbf{u}_1,\dots,\mathbf{u}_r$ of $\mathbb{Z}^m$.

Theorem 3.6. *Let L be a nonzero positive sublattice of $\mathbb{Z}^m$ of rank r, and (L,ρ) be a partial character on $\mathbb{Z}^m$. Consider r vectors $\mathbf{u}_1,\dots,\mathbf{u}_r\in L$. The following are equivalent*:

(1) $\mathrm{rad}(I_{L,\rho})=\mathrm{rad}(F_\rho(\mathbf{u}_1),\dots,F_\rho(\mathbf{u}_r))$;
(2) • *the matrix $M(\mathbf{u}_1,\dots,\mathbf{u}_r)$ is mixed dominating,*
• *in characteristic* 0 *we have that $L=\sum_{i=1}^r\mathbb{Z}\mathbf{u}_i$ and in characteristic $p>0$,*

$$(L:p^\infty)=\left(\sum_{i=1}^r\mathbb{Z}\mathbf{u}_i:p^\infty\right).$$

Proof. (1) $\Rightarrow$ (2) Since $L \subset \mathbb{Z}^m$ is a positive sublattice, the matrix M is mixed. Now we prove that M is dominating, i.e., no square submatrix of M is mixed. Assume that N is a mixed $s \times s$ submatrix of M, with $s \geqslant 1$ and suppose that s is maximal with respect to this property. Then up to permutations of the rows and of the variables we may assume that N consists of the first s lines and the first s columns, so that we can write

$$M = \begin{pmatrix} N & B \\ C & D \end{pmatrix}.$$

From Lemma 3.2 we have

$$\operatorname{rad}\bigl(I_L + (x_1, \dots, x_s)\bigr) = \operatorname{rad}\bigl(F(\mathbf{u}_1), \dots, F(\mathbf{u}_r), x_1, \dots, x_s\bigr).$$

Since N is mixed,

$$\bigl(F(\mathbf{u}_1), \dots, F(\mathbf{u}_s)\bigr) \subset (x_1, \dots, x_s)$$

so in fact we have

$$\operatorname{rad}\bigl(I_L + (x_1, \dots, x_s)\bigr) = \operatorname{rad}\bigl(F(\mathbf{u}_{s+1}), \dots, F(\mathbf{u}_r), x_1, \dots, x_s\bigr).$$

On the other hand, x_1 is not a zero divisor of I_L, therefore height(rad($I_L + (x_1, \dots, x_s)$)) $\geqslant r + 1$, but the height of rad($F(\mathbf{u}_{s+1}), \dots, F(\mathbf{u}_r), x_1, \dots, x_s$) is at most r. This is a contradiction, therefore M is mixed dominating.

Since $M(\mathbf{u}_1, \dots, \mathbf{u}_r)$ is mixed dominating, by Fischer–Shapiro [10, Theorem 2.9], we get that the ideal $(F(\mathbf{u}_1), \dots, F(\mathbf{u}_r))$ is equal to the lattice ideal I_U, where $U = \sum_{i=1}^{r} \mathbb{Z}\mathbf{u}_i$. By Lemma 3.3, this implies $(F_\rho(\mathbf{u}_1), \dots, F_\rho(\mathbf{u}_r)) = I_{U,\rho}$. Now by hypothesis there exists k such that

$$F_\rho(\mathbf{v})^{p^k} \in \bigl(F_\rho(\mathbf{u}_1), \dots, F_\rho(\mathbf{u}_r)\bigr)$$

for any $\mathbf{v} \in L$. (L, ρ) is a partial character on $\mathbb{Z}^m$ therefore $\rho(p^k\mathbf{v}) = (\rho(\mathbf{v}))^{p^k}$. If the characteristic of K is equal to p, this implies

$$F_\rho(\mathbf{v})^{p^k} = F_\rho\bigl(p^k\mathbf{v}\bigr) \in \bigl(F_\rho(\mathbf{u}_1), \dots, F_\rho(\mathbf{u}_r)\bigr) = I_{U,\rho}$$

and then $p^k\mathbf{v} \in U$, since $I_{U,\rho}$ is $\mathbb{Z}^m/U$-homogeneous. Therefore $(L : p^\infty) = (U : p^\infty)$. If the characteristic of K is zero, $I_{U,\rho}$ is a radical ideal, see Eisenbud–Sturmfels [6, Corollary 2.2], then

$$F_\rho(\mathbf{v})^{p^k} \in \bigl(F_\rho(\mathbf{u}_1), \dots, F_\rho(\mathbf{u}_r)\bigr) = I_{U,\rho}$$

implies $F_\rho(\mathbf{v}) \in I_{U,\rho}$, therefore $\mathbf{v} \in U$ and $L = U$.

(2) $\Rightarrow$ (1) Since $M(\mathbf{u}_1, \dots, \mathbf{u}_r)$ is mixed dominating, by Fischer–Shapiro [10, Theorem 2.9], we get $(F(\mathbf{u}_1), \dots, F(\mathbf{u}_r)) = I_U$ and by Lemma 3.3 this implies $(F_\rho(\mathbf{u}_1), \dots,$

$F_\rho(\mathbf{u}_r)) = I_{U,\rho}$. If the characteristic of K is zero we have $U = L$, so $I_{L,\rho} = I_{U,\rho} = (F_\rho(\mathbf{u}_1), \ldots, F_\rho(\mathbf{u}_r))$. If the characteristic of K is p positive, for any $\mathbf{v} \in L$, we have

$$F_\rho(\mathbf{v})^{p^k} = F_\rho\big(p^k\mathbf{v}\big) \in I_{U,\rho} = \big(F(\mathbf{u}_1)_\rho, \ldots, F(\mathbf{u}_r)_\rho\big)$$

and then $I_{L,\rho} \subset \mathrm{rad}(I_{U,\rho}) = \mathrm{rad}(F_\rho(\mathbf{u}_1), \ldots, F_\rho(\mathbf{u}_r))$. This completes the proof. □

Remark 3.7. By Fischer–Shapiro [10, Corollary 2.8], if the matrix $M(\mathbf{u}_1, \ldots, \mathbf{u}_r)$ is mixed dominating, then the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_r$ are linearly independent.

Corollary 3.8. *For any lattice ideal $I_{L,\rho}$ the fact that $I_{L,\rho}$ is a set-theoretical complete intersection on binomials is independent from the character ρ. Moreover, if* $\mathrm{rad}(I_L) = \mathrm{rad}(F(\mathbf{u}_1), \ldots, F(\mathbf{u}_r))$, *then for any character ρ,*

$$\mathrm{rad}(I_{L,\rho}) = \mathrm{rad}\big(F_\rho(\mathbf{u}_1), \ldots, F_\rho(\mathbf{u}_r)\big).$$

The proof follows from Theorem 3.6, since condition (2) is independent of the character.

Theorem 3.9. *Let L be a nonzero positive sublattice of $\mathbb{Z}^m$ of rank r, and (L, ρ) be a partial character on $\mathbb{Z}^m$. Consider r vectors $\mathbf{u}_1, \ldots, \mathbf{u}_r \in L$, the following are equivalent*:

(1) $I_{L,\rho} = (F_\rho(\mathbf{u}_1), \ldots, F_\rho(\mathbf{u}_r))$.
(2) • *The matrix $M(\mathbf{u}_1, \ldots, \mathbf{u}_r)$ is mixed dominating,*
  • $L = \sum_{i=1}^r \mathbb{Z}\mathbf{u}_i$.

The proof follows from the proof of Theorem 3.6 by taking out the radicals and putting $p = 1$ even in positive characteristic. Theorem 3.9 characterizes complete intersection lattice ideals: a lattice ideal $I_{L,\rho}$ is a complete intersection if and only if the lattice L has a basis $\mathbf{u}_1, \ldots, \mathbf{u}_r$ such that the matrix $M(\mathbf{u}_1, \ldots, \mathbf{u}_r)$ is mixed dominating.

Corollary 3.10. *Let L be a nonzero positive sublattice of $\mathbb{Z}^m$ of rank r, and (L, ρ) be a partial character on $\mathbb{Z}^m$. If the characteristic of K is zero, we have* $\mathrm{rad}(I_{L,\rho}) = \mathrm{rad}(F_\rho(\mathbf{u}_1), \ldots, F_\rho(\mathbf{u}_r))$ *if and only if* $I_{L,\rho} = (F_\rho(\mathbf{u}_1), \ldots, F_\rho(\mathbf{u}_r))$.

The proof of the corollary follows from the proof of Theorem 3.6. Corollary 3.10 states that in zero characteristic a lattice ideal is a set-theoretic complete intersection on binomials if and only if it is a complete intersection, see also [8, Theorem 2.1].

The aim of the next theorems is to prove Theorems 3.15 and 3.16, which give an exact characterization of complete intersection lattice ideals and complete intersection semigroups. Lattices that correspond to lattice ideals that are set-theoretic complete intersection on binomials are also characterized.

We recall the following decomposition theorem of K. Fischer, W. Morris, and J. Shapiro, for mixed dominating matrices (see [11, Theorem 2.2]) whose claim we adjust to our notation.

Theorem 3.11. *Let $M(\mathbf{u}_1,\ldots,\mathbf{u}_r)$ be a mixed dominating $r\times m$ matrix with $m\geqslant r>0$. Then there exist E_1, E_2 disjoint nonempty subsets of $\{1,\ldots,m\}$ with $E_1\cup E_2=\{1,\ldots,m\}$, and disjoint subsets S_1, S_2 of $\{1,\ldots,r\}$ with $S_1\cup S_2=\{1,\ldots,r\}-\{q\}$ for some q, such that the matrices $M(\{\mathbf{u}_i\mid i\in S_1\})$, $M(\{\mathbf{u}_i\mid i\in S_2\})$ are mixed dominating, where $(\mathbf{u}_i)^{E_j}=\mathbf{u}_i$ for every $i\in S_j$, $j\in\{1,2\}$ and $(\mathbf{u}_q)^{E_1}=\mathbf{u}_q^+$, $(\mathbf{u}_q)^{E_2}=-\mathbf{u}_q^-$.*

Lemma 3.12. *The notation being that of Theorem* 3.11 *we have for $j\in\{1,2\}$,*

$$\left(\sum_{i=1}^{r}\mathbb{Z}\mathbf{u}_i\right)_{E_j}=\sum_{i\in S_j}\mathbb{Z}\mathbf{u}_i$$

and the lattice $U=\sum_{i=1}^r\mathbb{Z}\mathbf{u}_i$ is the gluing of the lattices U_{E_1}, U_{E_2}.

Proof. Without loss of generality we take $j=1$. Recall that $L_{E_1}=L\cap\mathbb{Z}^{E_1}$, and since $(\mathbf{u}_i)^{E_1}=\mathbf{u}_i$ for every $i\in S_1$, we conclude that $\sum_{i\in S_1}\mathbb{Z}\mathbf{u}_i\subset(\sum_{i=1}^r\mathbb{Z}\mathbf{u}_i)_{E_1}$. Let $\mathbf{u}\in(\sum_{i=1}^r\mathbb{Z}\mathbf{u}_i)_{E_1}\subset\sum_{i=1}^r\mathbb{Z}\mathbf{u}_i$. Then $\mathbf{u}=\mathbf{u}^{E_1}$ and

$$\mathbf{u}=\sum_{i\in S_1}\lambda_i\mathbf{u}_i+\sum_{i\in S_2}\lambda_i\mathbf{u}_i+\lambda_q\mathbf{u}_q.$$

From which we have that

$$\mathbf{u}^{E_1}=\sum_{i\in S_1}\lambda_i\mathbf{u}_i^{E_1}+\sum_{i\in S_2}\lambda_i\mathbf{u}_i^{E_1}+\lambda_q\mathbf{u}_q^{E_1}.$$

But then

$$\mathbf{u}=\sum_{i\in S_1}\lambda_i\mathbf{u}_i+\lambda_q\mathbf{u}_q^+.$$

The last equality implies that the vector $\lambda_q\mathbf{u}_q^+$ belongs to the positive lattice $\sum_{i=1}^r\mathbb{Z}\mathbf{u}_i$, which is impossible except if $\lambda_q=0$. So $\mathbf{u}=\sum_{i\in S_1}\lambda_i\mathbf{u}_i$. We conclude that $(\sum_{i=1}^r\mathbb{Z}\mathbf{u}_i)_{E_1}$ $=\sum_{i\in S_1}\mathbb{Z}\mathbf{u}_i$. Therefore, $U=U_{E_1}+U_{E_2}+\langle\mathbf{u}_q\rangle$, where $(\mathbf{u}_q)^{E_1}=\mathbf{u}_q^+$, $(\mathbf{u}_q)^{E_2}=-\mathbf{u}_q^-$. □

Theorem 3.13. *Let K be a field of positive characteristic p. The lattice ideal $I_{L,\rho}\subset K[x_1,\ldots,x_m]$ is set-theoretic complete intersection on binomials if and only if the lattice L is the p-gluing of the two lattices L_{E_1} and L_{E_2} and both lattice ideals $I_{L_{E_1},\rho}$, $I_{L_{E_2},\rho}$ are set-theoretic complete intersections on binomials.*

Proof. Suppose that $\operatorname{rad}(I_{L,\rho})=\operatorname{rad}(F(\mathbf{u}_1),\ldots,F(\mathbf{u}_r))$. Then Theorem 3.6 gives us that the matrix $M(\mathbf{u}_1,\ldots,\mathbf{u}_r)$ is mixed dominating. Therefore there exist E_1, E_2, S_1, S_2 as provided in Theorem 3.11. By virtue of Lemma 3.12 the lattice $U = \sum_{i=1}^{r} \mathbb{Z}\mathbf{u}_i$ is the gluing of the lattices U_{E_1}, U_{E_2}. Now $U \subset L$ and from Theorem 3.6 we have

$$\left(L : p^\infty\right) = \left(\sum_{i=1}^{r} \mathbb{Z}\mathbf{u}_i : p^\infty\right),$$

therefore by Lemma 2.1 there exists a positive integer k such that $p^k L \subset U = U_{E_1} + U_{E_2} + \langle \mathbf{u}_q \rangle$. But $U_{E_1} \subset L_{E_1}$ and $U_{E_2} \subset L_{E_2}$ so that $p^k L \subset L_{E_1} + L_{E_2} + \langle \mathbf{u}_q \rangle$. Note also that $L_{E_1} + L_{E_2} + \langle \mathbf{u}_q \rangle \subset L$. Therefore, by Lemma 2.1, we have that

$$\left(L : p^\infty\right) = \left(L_{E_1} + L_{E_2} + \langle \mathbf{u}_q \rangle : p^\infty\right).$$

Which means that L is the p-gluing of L_{E_1} and L_{E_2}.

Note also that

$$\left(L_{E_j} : p^\infty\right) = \left(L : p^\infty\right)_{E_j} = \left(U : p^\infty\right)_{E_j} = \left(U_{E_j} : p^\infty\right),$$

for $j \in \{1, 2\}$. By Remark 3.7, the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_r$ are linearly independent and by Theorem 3.11 the matrices $M(\{\mathbf{u}_i \mid i \in S_j\})$ are mixed dominating, for $j \in \{1, 2\}$. Therefore, by Theorem 3.6 again, we conclude that $\operatorname{rad}(I_{L_{E_j}}) = \operatorname{rad}(F(\mathbf{u}_i) \mid i \in S_j)$, for $j \in \{1, 2\}$. Recall that $\operatorname{height}(I_{L_{E_j},\rho}) = \operatorname{rank}(\sum_{i \in S_j} \mathbb{Z}\mathbf{u}_i) = |S_j|$, for $j \in \{1, 2\}$, therefore both $I_{L_{E_1},\rho}, I_{L_{E_2},\rho}$ are set-theoretic complete intersections on binomials.

The proof of the converse implication follows from Theorem 2.6 and the remark that for the lattice (p-) gluing for positive lattices we have $\operatorname{rank}(L) = \operatorname{rank}(L_{E_1}) + \operatorname{rank}(L_{E_2}) + 1$. □

Notice that by Corollary 3.10, in the zero characteristic case, lattice ideals that are binomial set theoretic complete intersection are complete intersections. Therefore they are characterized also by the next theorem.

Theorem 3.14. *The lattice ideal $I_{L,\rho} \subset K[x_1, \ldots, x_m]$ is a complete intersection if and only if the lattice L is the gluing of the two lattices L_{E_1} and L_{E_2} and both lattice ideals $I_{L_{E_1},\rho}, I_{L_{E_2},\rho}$ are complete intersections.*

Proof. The proof follows the lines of the proof of Theorem 3.13 by taking out the radicals and putting $p = 1$ even in positive characteristic. □

The next theorem is the main result of the article and characterizes all lattice ideals that are complete intersections and also all lattice ideals that are set-theoretic complete intersections on binomials, in all characteristics.

Theorem 3.15. *Let K be a field of any characteristic. The lattice ideal $I_{L,\rho} \subset K[x_1, \ldots, x_m]$ is a complete intersection if and only if the lattice L is completely glued.*

In the characteristic zero case (*respectively positive characteristic p case*), *the lattice ideal $I_{L,\rho}$ is a set-theoretic complete intersection on binomials if and only if the lattice L is completely glued* (*respectively completely p-glued*).

Proof. The proof follows by induction on the rank r and is based on Theorems 3.13, 3.14. Note that if a lattice has rank zero then the elements of the associated semigroup are linearly independent and therefore the lattice is completely (respectively p-) glued and of course a complete intersection. □

The property for a lattice ideal to be a complete intersection does not depend on the field, but only on the lattice $L \subset \mathbb{Z}^m$. Therefore, translating Theorem 3.15 for semigroups, we have:

Theorem 3.16. *A finitely generated, cancellative, abelian semigroup with no invertible elements is a complete intersection if and only if it is completely glued.*

Theorem 3.16 restricted to affine semigroups gives an exact characterization of complete intersection affine semigroups: *an affine semigroup is a complete intersection if and only if it is completely glued*. An affine semigroup is *completely glued* if it belongs to the smallest class of affine semigroups that includes all free affine semigroups and is closed under gluing.

Example 3.17. The results of this section help us to provide examples of lattice ideals that are complete intersections or set-theoretic complete intersections on binomials. Any mixed dominating integer matrix $M(\mathbf{u}_1, \ldots, \mathbf{u}_r)$ gives a completely glued lattice, the $L = \sum_{i=1}^{r} \mathbb{Z}\mathbf{u}_i$, and a complete intersection lattice ideal, the $I_{L,\rho}$ in $K[x_1, \ldots, x_m]$, where K is any field and (L, ρ) a partial character on $\mathbb{Z}^m$. Also the semigroup $\langle \mathbf{e}_i + L \mid i \in \{1, \ldots, m\}\rangle \subset \mathbb{Z}^m/L$ is completely glued. Considering a lattice L' such that $(L' : p^\infty) = (L : p^\infty)$ for some prime number p, the lattice ideal $I_{L',\rho}$ in $K[x_1, \ldots, x_m]$ is set-theoretic complete intersection on binomials, where K is a field of characteristic p.

Mixed dominating matrices can be constructed easily. Let M_1 and M_2 be mixed dominating matrices of sizes $m_1 \times n_1$ and $m_2 \times n_2$ with $m_1 \geqslant 0$ and $m_2 \geqslant 0$. Let $\mathbf{u}^+ \in \mathbb{N}^{n_1}$ and $\mathbf{u}^- \in \mathbb{N}^{n_2}$ be any two vectors. Then the matrix

$$\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \\ \mathbf{u}^+ & -\mathbf{u}^- \end{pmatrix}$$

is mixed dominating. To start with, we can consider both matrices M_1, M_2 to be empty. Subsequently we use already constructed mixed dominating matrices to construct new ones. Actually the decomposition theorem, see Theorem 3.11 or [11, Theorem 2.2], of mixed dominating matrices says that all mixed dominating matrices can be taken in this way.

For example, take as M_1 the 1×3 mixed dominating matrix $(1 \quad 3 \quad -4)$, M_2 the empty 0×1 matrix, $\mathbf{u}^+ = (3, 1, 0)$ and $\mathbf{u}^- = (4)$. Then the matrix

$$\begin{pmatrix} 1 & 3 & -4 & 0 \\ 3 & 1 & 0 & -4 \end{pmatrix}$$

is mixed dominating. Therefore the lattice $L = \mathbb{Z}(1, 3, -4, 0) + \mathbb{Z}(3, 1, 0, -4)$ is completely glued and the lattice ideal $I_{L,\rho}$ is a complete intersection for any character ρ. The associated semigroup of the lattice L is isomorphic to the semigroup generated by $(4, 0, 0)$, $(0, 4, 0)$, $(1, 3, 0)$ and $(3, 1, 1)$ in $\mathbb{Z}^2 \oplus \mathbb{Z}_4$. Which is completely glued.

Let $L' = \mathbb{Z}(1, 3, -4, 0) + \mathbb{Z}(0, 2, -3, 1)$. L' is the associated lattice of the affine semigroup generated by $(4, 0)$, $(0, 4)$, $(1, 3)$ and $(3, 1)$ in $\mathbb{Z}^2$. Which is not a complete intersection affine semigroup. Therefore there is no basis $\mathbf{u}_1, \mathbf{u}_2$ of L' such that the matrix $M(\mathbf{u}_1, \mathbf{u}_2)$ is mixed dominating. Notice that $(L' : 2^\infty) = (L : 2^\infty)$, since $4L' \subset L \subset L'$. This implies that in characteristic 2 the two ideals $I_{L',\rho}$, $I_{L,\rho}$ have the same radical. Therefore $I_{L',\rho}$ is set-theoretic complete intersection on binomials in characteristic 2.

4. Extreme rays of a complete intersection semigroup cone

Let ϕ be the projection homomorphism from $\mathbb{Z}^n \oplus T$ to $\mathbb{Z}^n$ and denote $\phi(\mathbf{b}) = \bar{\mathbf{b}}$ for $\mathbf{b} \in \mathbb{Z}^n \oplus T$. Let $\overline{A} = \{\bar{\mathbf{a}}_i \mid 1 \leqslant i \leqslant m\}$. We associate with the semigroup $\mathbb{N}A$ (or with the lattice ideal $I_{L,\rho}$) the rational polyhedral cone $\sigma = \mathrm{pos}_{\mathbb{Q}}(\overline{A}) := \{l_1\bar{\mathbf{a}}_1 + \cdots + l_m\bar{\mathbf{a}}_m \mid l_i \in \mathbb{Q}$ and $l_i \geqslant 0\}$. A cone σ is *strongly convex* if $\sigma \cap -\sigma = \{\mathbf{0}\}$. The condition that the lattice L is positive is equivalent with the condition that the cone σ is strongly convex.

A ray R in the cone of $\overline{A}$ is an *extreme ray* of the cone of $\overline{A}$, if given any vector $\mathbf{u} \in R$, positive integers $\mu, c_1, \ldots, c_t$ and elements $\mathbf{w}_1, \ldots, \mathbf{w}_t$ of $\mathbb{N}\overline{A}$ such that

$$\mu\mathbf{u} = c_1\mathbf{w}_1 + \cdots + c_t\mathbf{w}_t,$$

then $\mathbf{w}_j \in R$ for all $j = 1, \ldots, t$. In [11] it was shown that for an n-dimensional complete intersection affine semigroup with $n \geqslant 2$, its cone contains no more than $2n - 2$ extreme rays. The corresponding statement is true for semigroups of $\mathbb{Z}^n \oplus T$ or equivalently lattice ideals which are complete intersections. But also for lattice ideals that are set theoretic complete intersections on binomials.

Theorem 4.1. *Let $\mathbb{N}A$ be an n-dimensional semigroup of $\mathbb{Z}^n \oplus T$ which is completely glued or completely p-glued, $n \geqslant 2$. Then the cone of $\overline{A}$ contains no more than $2n - 2$ extreme rays.*

Proof. The proof almost follows the lines of the proof of [11, Corollary 2.4]. Let $\mathbb{N}A$ be a semigroup of $\mathbb{Z}^n \oplus T$ which is completely glued or completely p-glued. Let $\psi : \mathbb{Z}^m \to \mathbb{Z}^n \oplus T$ be the group homomorphism such that $\psi(\mathbf{e}_i) = \mathbf{a}_i \in \mathbb{Z}^n \oplus T$, where $\mathbf{e}_1, \ldots, \mathbf{e}_m$ is the canonical basis of $\mathbb{Z}^m$. Let L be the lattice $\ker(\psi)$ of rank $r = m - n$. We will use induction on r. If $r = 0$, then $m = n$. Hence the vectors in $\overline{A}$ are linearly independent and the cone has exactly n extreme rays. Since $n \geqslant 2$, we have $n \leqslant 2n - 2$.

If $r \geqslant 1$, we can write A as the disjoint union of A^{E_1}, A^{E_2} such that $\mathbb{Z}\mathbf{a} = \mathbb{Z}A^{E_1} \cap \mathbb{Z}A^{E_2}$ and there is a multiple of $\mathbf{a}$ in $\mathbb{N}A^{E_1} \cap \mathbb{N}A^{E_2}$, for some disjoint subsets E_1, E_2 of $\{1, \ldots, m\}$. Then we have $\bar{\mathbf{a}} \in \mathbb{Z}\overline{A}^{E_1} \cap \mathbb{Z}\overline{A}^{E_2}$. Let $\bar{\mathbf{b}} \in \mathbb{Z}\overline{A}^{E_1} \cap \mathbb{Z}\overline{A}^{E_2}$. Then $g\mathbf{b} \in \mathbb{Z}A^{E_1} \cap \mathbb{Z}A^{E_2} = \mathbb{Z}\mathbf{a}$, where g is the order of the finite group T. Therefore $g\mathbf{b} = \lambda\mathbf{a}$ and so $g\bar{\mathbf{b}} = \lambda\bar{\mathbf{a}}$. Thus $\mathbb{Z}\overline{A}^{E_1} \cap \mathbb{Z}\overline{A}^{E_2}$ is one-dimensional and if $\bar{\mathbf{c}}$ is any generator, then $\bar{\mathbf{a}} = \mu\bar{\mathbf{c}}$. We conclude that a multiple of $\bar{\mathbf{c}}$ belongs to $\mathbb{N}\overline{A}^{E_1} \cap \mathbb{N}\overline{A}^{E_2}$.

Let n_1, n_2 be the dimensions of $\mathbb{N}A^{E_1}, \mathbb{N}A^{E_2}$, respectively. Then $n_1 + n_2 = n + 1$. Let r_i be the rank of the lattice L_{E_i}, $i \in \{1, 2\}$. It follows from $n_1 + n_2 = n + 1$ that $r_1 + r_2 = r - 1$. Therefore each r_i is less than r. Each extreme ray of the cone of $\overline{A}$ is an extreme ray for either the cone of $\overline{A}^{E_1}$ or $\overline{A}^{E_2}$. Therefore, the number of extreme rays of the cone of $\overline{A}$ is bounded by the sum of the number of extreme rays in the cones of $\overline{A}^{E_1}$ and $\overline{A}^{E_2}$. Hence as long as $n_i \geqslant 2$, the inductive hypothesis gives that the number of extreme rays of the cone of $\overline{A}$ is bounded by $2n_1 - 2 + 2n_2 - 2 = 2n - 2$. But if $r_1 = 1$ say, then since the two cones of $\overline{A}^{E_1}$ and $\overline{A}^{E_2}$ intersect in a semiline, it follows that the cone of $\overline{A}^{E_1}$ is contained in the cone of $\overline{A}^{E_2}$. Therefore the cone of $\overline{A}$ is the same with the cone of $\overline{A}^{E_2}$. But r_2 is smaller than r, therefore the inductive hypothesis gives the result. □

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