Stable *LU* Factorization of *H*-Matrices

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ABSTRACT

Results are given concerning the *LU* factorization of *H*-matrices, and Gaussian elimination with column-diagonal-dominant pivoting is shown to be applicable to *H*-matrices. This algorithm, which uses a symmetric permutation to exchange the most diagonally dominant column of the unreduced submatrix into the pivotal position, is shown to be numerically stable by deriving an upper bound on the growth factor associated with the backward error analysis for Gaussian elimination.

1. INTRODUCTION

An $n \times n$ real matrix $A \equiv (a_{ij})$ is an *M-matrix* if $a_{ij} \leqslant 0$ for all $i \neq j$ and if $\operatorname{Re}\lambda \geqslant 0$ for all $\lambda \in \sigma(A)$, the spectrum of A. There are numerous

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equivalent conditions for a real matrix A having the M-matrix sign pattern to be either a singular or a nonsingular M-matrix (see e.g. Berman and Plemmons [2]). The *comparison matrix* $\mathscr{M}(A) \equiv (m_{ij})$ of an arbitrary $n \times n$ complex matrix A is defined by

$$m_{ij} = \begin{cases} |a_{ii}| & \text{if} \quad i = j, \\ -|a_{ij}| & \text{if} \quad i \neq j. \end{cases}$$

With the exception of M-matrices and comparison matrices, we assume that any matrix in this paper may have complex-valued entries. If A and B are real matrices of the same size, then $A \leqslant B$ means that $a_{ij} \leqslant b_{ij}$ for all i, j. Relative to any M-matrix B, we define the set Ω_B of complex matrices by

$$\Omega_B = \{ A \mid B \leqslant \mathscr{M}(A) \}.$$

An $n \times n$ matrix A is (column) *generalized diagonally dominant* if there exist scalars $d_i > 0$ $(1 \leqslant i \leqslant n)$ such that

$$d_j |a_{jj}| \geqslant \sum_{i \neq j} d_i |a_{ij}|, \qquad 1 \leqslant j \leqslant n.$$

We let $N = \{1, 2, \dots, n\}$, and if $\varnothing \subset \alpha \subset N$, the principal submatrix of A from the rows and columns specified by α is denoted by $A[\alpha]$.

We define an $n \times n$ (complex) matrix A to be an *H-matrix* if $\mathscr{M}(A)$ is an M-matrix. This is the definition given by Berman and Plemmons [2], although other definitions have been used. For example, Funderlic, Neumann, and Plemmons [5] call A an H-matrix if A is generalized diagonally dominant, while other definitions (see e.g. Varga [12], Carlson and Markham [3], and Neumann and Plemmons [10]) require that $\mathscr{M}(A)$ be a nonsingular M-matrix. It is well known that a matrix is generalized diagonally dominant if its comparison matrix is a nonsingular, or a singular and irreducible, M-matrix, but with our definition clearly there exist H-matrices that are not generalized diagonally dominant; for example,

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

After k steps $(1 \leqslant k \leqslant n-1)$ of the forward elimination of Gaussian elimination (without pivoting) applied to an $n \times n$ matrix A, the resultant *reduced matrix* is denoted by

$$A^{(k)} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,k} & a_{1,k+1} & \cdots & a_{1,n} \\ & a_{2,2}^{(1)} & \cdots & a_{2,k}^{(1)} & a_{2,k+1}^{(1)} & \cdots & a_{2,n}^{(1)} \\ & & \ddots & \vdots & \vdots & & \vdots \\ & & & a_{k,k}^{(k-1)} & a_{k,k+1}^{(k-1)} & \cdots & a_{k,n}^{(k-1)} \\ & & & & a_{k+1,k+1}^{(k)} & \cdots & a_{k+1,n}^{(k)} \\ & 0 & & & a_{k+2,k+1}^{(k)} & \cdots & a_{k+2,n}^{(k)} \\ & & & & \vdots & & \vdots \\ & & & & a_{n,k+1}^{(k)} & \cdots & a_{n,n}^{(k)} \end{bmatrix}$$

$$= \begin{bmatrix} \hat{U}^{(k)} & \hat{C}^{(k)} \\ 0 & \hat{A}^{(k)} \end{bmatrix}, \tag{1}$$

where $\hat{U}^{(k)}$ is a $k \times k$ upper triangular matrix, $\hat{C}^{(k)}$ is $k \times (n-k)$, and $\hat{A}^{(k)}$ is $(n-k)\times(n-k)$. Letting $A^{(0)} = A$, $A^{(k)}$ exists iff $\{A^{(i)} \mid 1 \leqslant i < k\}$ exists, and if $a_{k,k}^{(k-1)} = 0$ then $a_{j,k}^{(k-1)} = 0$ for $k+1 \leqslant j \leqslant n$. We note that

$$A^{(k+1)} = T_k A^{(k)}, \qquad 0 \leqslant k \leqslant n-2,$$

where T_k is an $n \times n$ elementary lower triangular matrix of order n and index $k+1$ (see Stewart [11, §3.2]), and contains the Gaussian-elimination "multipliers." If $a_{j,k+1}^{(k)} = 0$ for $k+1 \leqslant j \leqslant n$, then T_k is not uniquely defined and may, for example, be set equal to the $n \times n$ identity matrix (as all diagonal entries of T_k are 1), thus uniquely determining the elimination. The matrix $A^{(n-1)}$ is upper triangular, and $\hat{A}^{(k)}$ is often called the *Schur complement* of $A[1,2,\ldots,k]$ in A (when $A[1,2,\ldots,k]$ is nonsingular). Numerical stability of the Gaussian elimination algorithm (see e.g. Stewart [11]) requires controlling the size of the "growth factor" γ defined by

$$\gamma = \frac{\max_{i,j,k} |a_{ij}^{(k)}|}{\max_{i,j} |a_{ij}^{(0)}|}, \qquad 0 \leqslant k \leqslant n-1, \tag{2}$$

where $A^{(k)} \equiv (a_{ij}^{(k)})$ and $a_{ij}^{(0)} = a_{ij}$.

An $n \times n$ matrix A is said to *admit an LU factorization* if it can be written as $A = LU$, where L is an $n \times n$ *nonsingular* lower triangular matrix and U is an $n \times n$ upper triangular matrix. The following result is an immediate consequence of [8, Theorem 2]: a matrix A admits an LU factorization iff $(a_{k+1,1}, a_{k+1,2}, \dots, a_{k+1,k})$ lies in the space spanned by the row vectors of $A[1,2,\dots,k]$, for $k = 1,2,\dots, n-1$. Equivalently, it can be seen that A admits an LU factorization iff Gaussian elimination without pivoting may be successfully applied to reduce A to upper triangular form (i.e., iff the matrices $A^{(k)}$ exist, $1 \leqslant k \leqslant n-1$).

Necessary and sufficient conditions for an M-matrix to admit an LU factorization into M-matrices (i.e., such that L and U are M-matrices) are given by Varga and Cai [13]. We note [6] that there exist (singular, reducible) M-matrices which may be LU factored, but not into a product of M-matrices, and there exist M-matrices (again singular and reducible) which have no LU factorization. However, Kuo [7] proved that for any M-matrix A there exists a permutation matrix P such that PAP^T admits an LU factorization into M-matrices. A numerically stable algorithm which determines such a matrix P, and the LU factorization of PAP^T, is given by Ahac and Olesky [1].

The purpose of this paper is to show that the algorithm of Ahac and Olesky [1], Gaussian elimination with column-diagonal-dominant pivoting, is applicable to H-matrices (see Section 3). Our main results are contained in Section 4, where the stability of this algorithm is shown by deriving an upper bound on the growth factor γ. Our proofs of these results depend on extending known results on the LU factorization of an H-matrix whose comparison matrix is a nonsingular M-matrix to our broader class of H-matrices; this we do first, in Section 2. We conclude with two examples in Section 5.

2. *LU* FACTORIZATION OF *H*-MATRICES

We begin with a generalization of a result of Neumann [9, Proposition 5], who proved the following when $\mathcal{M}(A)$ is a nonsingular M-matrix.

LEMMA 1. *Let A be an $n \times n$ H-matrix, let $\mathcal{M}(A) \equiv (m_{ij})$ denote its comparison matrix, and suppose that $\mathcal{M}(A)$ admits an LU factorization into M-matrices. Then A admits an LU factorization, and if $A^{(k)}$ and $M^{(k)}$, respectively, denote the reduced matrices of A and $\mathcal{M}(A)$, then*

$$M^{(k)} \leqslant \mathcal{M}(A^{(k)}), \qquad 1 \leqslant k \leqslant n-1.$$

Proof. We note that all $M^{(k)}$ are M-matrices (see Varga and Cai [13, p. 186] and Fan [4]) and proceed by induction on k. Firstly, if $a_{11} = 0$ then $a_{i1} = 0$ for $2 \leqslant i \leqslant n$ [since $\mathcal{M}(A)$ admits an LU factorization], so in this case

$A^{(1)} = M^{(1)} = A$. If $a_{11} \neq 0$, then $A^{(1)}$ exists and for $2 \leqslant i \leqslant n$

$$|a_{ii}^{(1)}| = \left| a_{ii} - \frac{a_{i1}a_{1i}}{a_{11}} \right| \geqslant |a_{ii}| - \left| \frac{a_{i1}a_{1i}}{a_{11}} \right| = m_{ii} - \frac{m_{i1}m_{1i}}{m_{11}} = m_{ii}^{(1)}.$$

Similarly, for $2 \leqslant i,\ j \leqslant n$ and $i \neq j$,

$$|a_{ij}^{(1)}| = \left| a_{ij} - \frac{a_{i1}a_{1j}}{a_{11}} \right| \leqslant |a_{ij}| + \left| \frac{a_{i1}a_{1j}}{a_{11}} \right|$$

$$= -m_{ij} + \frac{(-m_{i1})(-m_{1j})}{m_{11}} = -m_{ij}^{(1)},$$

so that $m_{ij}^{(1)} \leqslant -|a_{ij}^{(1)}|$. Thus $M^{(1)} \leqslant \mathscr{M}(A^{(1)})$.

For given k $(1 \leqslant k \leqslant n-1)$ and for all p $(1 \leqslant p \leqslant k)$, assume that $A^{(p)}$ exists and $M^{(p)} \leqslant \mathscr{M}(A^{(p)})$. In particular, $M^{(k)} \leqslant \mathscr{M}(A^{(k)})$ implies that

$$m_{ii}^{(k)} \leqslant |a_{ii}^{(k)}|, \qquad k+1 \leqslant i \leqslant n,$$

and

$$m_{ij}^{(k)} \leqslant -|a_{ij}^{(k)}|, \qquad k+1 \leqslant i,\ j \leqslant n \text{ and } i \neq j.$$

Since $M^{(k+1)}$ exists, either $m_{k+1,k+1}^{(k)} \neq 0$ or $m_{i,k+1}^{(k)} = 0$ for $k+1 \leqslant i \leqslant n$. Thus either $a_{k+1,k+1}^{(k)} \neq 0$ or $a_{i,k+1}^{(k)} = 0$ for $k+1 \leqslant i \leqslant n$, implying that $A^{(k+1)}$ exists. In the latter case, $A^{(k+1)} = A^{(k)}$ and $M^{(k+1)} = M^{(k)}$, so the inductive hypothesis implies that $M^{(k+1)} \leqslant \mathscr{M}(A^{(k+1)})$. On the other hand, if $a_{k+1,k+1}^{(k)} \neq 0$, then for $k+2 \leqslant i \leqslant n$

$$|a_{ii}^{(k+1)}| = \left| a_{ii}^{(k)} - \frac{a_{i,k+1}^{(k)} a_{k+1,i}^{(k)}}{a_{k+1,k+1}^{(k)}} \right|$$

$$\geqslant |a_{ii}^{(k)}| - \left| \frac{a_{i,k+1}^{(k)} a_{k+1,i}^{(k)}}{a_{k+1,k+1}^{(k)}} \right|$$

$$\geqslant m_{ii}^{(k)} - \frac{(-m_{i,k+1}^{(k)})(-m_{k+1,i}^{(k)})}{m_{k+1,k+1}^{(k)}} \qquad \text{by the inductive hypothesis}$$

$$= m_{ii}^{(k+1)};$$

for $k+2 \leqslant i,\ j \leqslant n$ and $i \neq j$,

$$|a_{ij}^{(k+1)}| = \left| a_{ij}^{(k)} - \frac{a_{i,k+1}^{(k)} a_{k+1,j}^{(k)}}{a_{k+1,k+1}^{(k)}} \right|$$

$$\leqslant |a_{ij}^{(k)}| + \left| \frac{a_{i,k+1}^{(k)} a_{k+1,j}^{(k)}}{a_{k+1,k+1}^{(k)}} \right|$$

$$\leqslant -m_{ij}^{(k)} + \frac{m_{i,k+1}^{(k)} m_{k+1,j}^{(k)}}{m_{k+1,k+1}^{(k)}} \qquad \text{by the inductive hypothesis}$$

$$= -m_{ij}^{(k+1)}.$$

Thus $M^{(k+1)} \leqslant \mathscr{M}(A^{(k+1)})$, which completes the induction. ∎

We note that the converse of Lemma 1 is false: if A is an H-matrix which admits an LU factorization, then $\mathscr{M}(A)$ doesn't necessarily admit an LU factorization, as shown, for example, by

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Conditions which are necessary and sufficient for an H-matrix to admit an LU factorization do not seem to be known.

The proof of our next lemma follows immediately from Theorem 1 of Varga and Cai [13].

LEMMA 2. *Let A be an M-matrix which admits an LU factorization into M-matrices. If $A \leqslant B$ and $b_{ij} \leqslant 0$ for all $i \neq j$, then B is an M-matrix which admits an LU factorization into M-matrices.*

The above lemmas give the following sufficient condition for the LU factorization of matrices of the class Ω_B of H-matrices, defined relative to an M-matrix B.

COROLLARY 1. *Let B be an M-matrix which admits an LU factorization into M-matrices. If $A \in \Omega_B$, then A admits an LU factorization.*

Proof. By Lemma 2, $\mathscr{M}(A)$ admits an LU factorization into M-matrices for all $A \in \Omega_B$, and therefore Lemma 1 implies that A admits an LU factorization. ∎

In the remainder of this paper, we are concerned with the LU factorization of a symmetric permutation PAP^T of an H-matrix A.

3. COLUMN-DIAGONAL-DOMINANT PIVOTING ON H-MATRICES

The column-diagonal-dominant (cdd) pivoting strategy (see Ahac and Olesky [1]) for Gaussian elimination requires that the submatrix $\hat{A}^{(k)}$ of the reduced matrix $A^{(k)}$ [see (1)] have a column which is diagonally dominant. In this section we show that an H-matrix has a diagonally dominant column, and that Gaussian elimination with cdd pivoting may be applied to any H-matrix, producing an LU factorization of some symmetric permutation PAP^T of A. We first state a straightforward generalization of a known result for M-matrices.

THEOREM 1. *Given any $n \times n$ H-matrix A, there exists at least one index j such that*

$$|a_{jj}| \geqslant \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}|.$$

Proof. As $\mathcal{M}(A)$ is an M-matrix, this follows from the analogous result for M-matrices (see [1, Theorem 1]). ∎

If A is a nonsingular M-matrix, then there exists at least one index j such that strict inequality holds in Theorem 1 (see [1]). However, this does not extend to nonsingular H-matrices, as illustrated by

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Referring to (1), suppose that $\hat{A}^{(k)}$ is an H-matrix and that column j_k ($k < j_k \leqslant n$) has the *maximal* column sum in $\mathcal{M}(\hat{A}^{(k)})$; by Theorem 1, this sum is nonnegative. *Cdd pivoting* is the process of interchanging the $(k+1)$th and j_kth columns and rows of $A^{(k)}$ prior to the $(k+1)$th step of Gaussian elimination ($0 \leqslant k \leqslant n-2$). This interchanging is equivalent to forming $P_{k+1,j_k} A^{(k)} P^T_{k+1,j_k}$, where P_{k+1,j_k} is an elementary permutation matrix, and implies that the diagonal entry of the most diagonally dominant column of $\hat{A}^{(k)}$ is the pivotal element for this step of the elimination. The diagonal dominance of the column gives the important property that the sum of the absolute values of the Gaussian elimination multipliers is $\leqslant 1$ at each step.

Provided that $\hat{A}^{(k)}$ is an H-matrix for $1 \leqslant k \leqslant n-1$, Theorem 1 insures that the cdd pivoting strategy may be used at every step of the elimination. (We note that if, at any step, the pivotal element $a_{k+1,k+1}^{(k)}$ is zero, then Theorem 1 implies that $a_{l,k+1}^{(k)} = 0$ for $k+2 \leqslant l \leqslant n$, so that the LU factorization may continue.) The resultant LU factorization will be an LU factorization of PAP^T for some permutation matrix P. Our next result formalizes the application of Gaussian elimination with cdd pivoting to H-matrices.

THEOREM 2. *Let A be an $n \times n$ H-matrix. There exists a permutation matrix P such that PAP^T admits an LU factorization using Gaussian elimination with* cdd *pivoting.*

Proof. By Theorem 1, A has at least one diagonally dominant column. Let j be such that

$$|a_{jj}| - \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}| \geqslant |a_{kk}| - \sum_{\substack{i=1 \\ i \neq k}}^{n} |a_{ik}|, \qquad 1 \leqslant k \leqslant n,$$

and define $\tilde{A}^{(1)} = P_{1j} A P_{1j}^T$, where P_{1j} is the $n \times n$ permutation matrix with rows 1 and j interchanged. Then either the $(1,1)$ entry of $\tilde{A}^{(1)}$ is nonzero, or else the entire first column of $\tilde{A}^{(1)}$ is zero. In either case, one step of Gaussian elimination may also be applied to $\tilde{A}^{(1)}$, giving the reduced matrix $A^{(1)}$, say. Clearly $P_{1j}\mathscr{M}(A)P_{1j}^T \equiv \mathscr{M}(P_{1j}AP_{1j}^T)$ is an M-matrix to which one step of Gaussian elimination may be applied, giving a reduced matrix which we denote by $M^{(1)}$. By the first step of the induction argument in the proof of Lemma 1, $M^{(1)} \leqslant \mathscr{M}(A^{(1)})$. As $M^{(1)}$ is an M-matrix, $\mathscr{M}(A^{(1)})$ is also an M-matrix, and so $A^{(1)}$ is an H-matrix. This completes the first step of the Gaussian elimination, and as $A^{(1)}$ is an H-matrix, it is clear that the process may be continued until some symmetric permutation of A has been reduced to upper triangular form. ■

Given an H-matrix A, we note that there may exist more than one permutation matrix P such that PAP^T has an LU factorization with the property that the sum of the absolute values of the Gaussian elimination multipliers is $\leqslant 1$ at each step. For example, Gaussian elimination with cdd pivoting may be applied to $\mathscr{M}(A)$, producing an LU factorization of, say, $P\mathscr{M}(A)P^T$. Thus PAP^T is an H-matrix such that $\mathscr{M}(PAP^T)$ has an LU factorization, so that Lemma 1 implies that PAP^T also has an LU factorization and, moreover, that the comparison matrix of the kth reduced matrix of PAP^T is greater than or equal to the kth reduced matrix of $\mathscr{M}(PAP^T)$. As the LU factorization of $P\mathscr{M}(A)P^T$ uses cdd pivoting, the sum of the absolute values of the multipliers of the LU factorization of PAP^T must also be $\leqslant 1$. However, such an LU factorization does not necessarily (strictly-speaking) correspond to the use of cdd pivoting, as it may not use the pivots based upon the *maximal* column sums. We illustrate this in the next example.

EXAMPLE 1. Consider the following H-matrix A and its comparison matrix M:

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & 0 \\ -2 & 0 & 2 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 6 & -2 & -2 \\ -2 & 3 & 0 \\ -2 & 0 & 2 \end{bmatrix}.$$

The LU factorization of M using cdd pivoting (no interchanges are necessary) gives the reduced matrices

$$M^{(1)} = \begin{bmatrix} 6 & -2 & -2 \\ 0 & \frac{7}{3} & -\frac{2}{3} \\ 0 & -\frac{2}{3} & \frac{4}{3} \end{bmatrix} \quad \text{and} \quad M^{(2)} = \begin{bmatrix} 6 & -2 & -2 \\ 0 & \frac{7}{3} & -\frac{2}{3} \\ 0 & 0 & \frac{8}{7} \end{bmatrix}.$$

However, Gaussian elimination with cdd pivoting for A gives

$$A^{(1)} = \begin{bmatrix} 6 & -2 & 2 \\ 0 & \frac{7}{3} & \frac{2}{3} \\ 0 & -\frac{2}{3} & \frac{8}{3} \end{bmatrix},$$

and thus rows and columns 2 and 3 of $A^{(1)}$ must be interchanged, giving

$$A^{(2)} = \begin{bmatrix} 6 & 2 & -2 \\ 0 & \frac{8}{3} & -\frac{2}{3} \\ 0 & 0 & \frac{5}{2} \end{bmatrix}.$$

Thus, the permutation matrices P_1 and P_2, say, of the "cdd factorizations" of an H-matrix A and its comparison matrix $\mathscr{M}(A)$, respectively, are in general different. However, the results of the next section show that the computed LU factorization of each of $P_1AP_1^T$ and $P_2AP_2^T$ is numerically stable, since in either case the growth factor γ is guaranteed to be small (as this depends only on the abovementioned property of the multipliers).

Before discussing stability, consider an arbitrary M-matrix B, and suppose P is a permutation matrix such that the LU factorization of PBP^T corresponds to the application of Gaussian elimination with cdd pivoting to B. Then for any $A \in \Omega_B$, $\mathscr{M}(PAP^T)$ admits an LU factorization into M-matrices (by Lemma 2), and thus (by Lemma 1) the H-matrix PAP^T admits an LU factorization such that the sum of the absolute values of the multipliers is $\leqslant 1$ at each step of the elimination. That is, the permutation matrix P permits an LU factorization of PAP^T for all $A \in \Omega_B$, and the multipliers have the above property.

4. NUMERICAL STABILITY OF THE ALGORITHM

To examine the stability of Gaussian elimination using cdd pivoting, we must bound the growth factor γ defined in (2). The row and column interchanges of the pivoting strategy complicate expressions for the entries of the reduced matrices $A^{(k)}$, making the analysis tedious. However, as usual (see e.g. Theorem 2.9 of Stewart [11, p. 125]), we can assume without loss of generality that all of the interchanges required for the particular pivoting strategy have been done prior to doing the elimination.

In the following lemma we derive an upper bound for the magnitude of any entry $a_{ij}^{(k)}$ of $\hat{A}^{(k)}$ [see (1)] in terms of certain entries in the jth column of $\hat{A}^{(k-t)}$, where $1 \leqslant t \leqslant k$. Our stability result is a consequence of the case $k = t$ in this lemma, as this gives an upper bound for $|a_{ij}^{(k)}|$ in terms of certain entries in the given matrix $A^{(0)} \equiv A$.

LEMMA 3. *Let A' be any $n \times n$ H-matrix, and let P denote a permutation matrix such that Gaussian elimination without pivoting applied to $A = PA'P^T$ is identical to Gaussian elimination with* cdd *pivoting applied to A'. Let $1 \leqslant k \leqslant n-1$ and $k+1 \leqslant i, j \leqslant n$. Then for any t such that $1 \leqslant t \leqslant k$,*

$$|a_{ij}^{(k)}| \leqslant |a_{ij}^{(k-t)}| + \sum_{l=k+1-t}^{k} |a_{lj}^{(k-t)}|.$$

Proof. We proceed by induction on t. If $a_{kk}^{(k-1)} = 0$, then $a_{lk}^{(k-1)} = 0$ for $k+1 \leqslant l \leqslant n$; so $a_{ij}^{(k)} = a_{ij}^{(k-1)}$ and clearly $|a_{ij}^{(k)}| \leqslant |a_{ij}^{(k-1)}| + |a_{kj}^{(k-1)}|$. Otherwise, if $a_{kk}^{(k-1)} \neq 0$, then

$$|a_{ij}^{(k)}| = \left| a_{ij}^{(k-1)} - \frac{a_{ik}^{(k-1)} a_{kj}^{(k-1)}}{a_{kk}^{(k-1)}} \right| \leqslant |a_{ij}^{(k-1)}| + |a_{kj}^{(k-1)}|,$$

as $|a_{ik}^{(k-1)}/a_{kk}^{(k-1)}| \leqslant 1$ using cdd pivoting. Thus the lemma is true for $t = 1$.

Assume that the lemma is true for $t = p$, where $1 \leqslant p \leqslant k-1$; that is,

$$|a_{ij}^{(k)}| \leqslant |a_{ij}^{(k-p)}| + \sum_{l=k+1-p}^{k} |a_{lj}^{(k-p)}|.$$

If $a_{k-p,k-p}^{(k-p-1)} = 0$, then $a_{ij}^{(k-p)} = a_{ij}^{(k-p-1)}$; thus

$$|a_{ij}^{(k)}| \leqslant |a_{ij}^{(k-p-1)}| + \sum_{l=k+1-p}^{k} |a_{lj}^{(k-p-1)}|,$$

so clearly

$$|a_{ij}^{(k)}| \leqslant |a_{ij}^{(k-p-1)}| + \sum_{l=k-p}^{k} |a_{lj}^{(k-p-1)}|.$$

On the other hand, if $a_{k-p,k-p}^{(k-p-1)} \neq 0$ then

$$\begin{aligned}
|a_{ij}^{(k)}| &\leqslant \left| a_{ij}^{(k-p-1)} - \frac{a_{i,k-p}^{(k-p-1)} a_{k-p,j}^{(k-p-1)}}{a_{k-p,k-p}^{(k-p-1)}} \right| \\
&\quad + \sum_{l=k+1-p}^{k} \left| a_{lj}^{(k-p-1)} - \frac{a_{l,k-p}^{(k-p-1)} a_{k-p,j}^{(k-p-1)}}{a_{k-p,k-p}^{(k-p-1)}} \right| \\
&\leqslant |a_{ij}^{(k-p-1)}| + \sum_{l=k+1-p}^{k} |a_{lj}^{(k-p-1)}| + |a_{k-p,j}^{(k-p-1)}| \\
&\quad \times \left[\frac{|a_{i,k-p}^{(k-p-1)}| + \sum_{l=k+1-p}^{k} |a_{l,k-p}^{(k-p-1)}|}{a_{k-p,k-p}^{(k-p-1)}} \right] \\
&\leqslant |a_{ij}^{(k-p-1)}| + \sum_{l=k-p}^{k} |a_{lj}^{(k-p-1)}|,
\end{aligned}$$

since the factor multipling $|a_{k-p,j}^{(k-p-1)}|$ above is a sum of absolute values of multipliers at the $(k-p)$th step of the cdd factorization, and thus is bounded by one. The above inequalities imply that the lemma is true for $t = p+1$, completing the induction. ■

Letting $t = k$ in Lemma 3, we obtain a bound for $|a_{ij}^{(k)}|$ in terms of entries of A.

COROLLARY 2. *With A', A, i, j, and k defined as in Lemma 3,*

$$|a_{ij}^{(k)}| \leqslant |a_{ij}| + \sum_{l=1}^{k} |a_{lj}|.$$

This corollary clearly implies that

$$|a_{ij}^{(k)}| \leqslant (k+1) \max_{i,j} |a_{ij}|,$$

so on letting $k = n-1$ we obtain a bound on the maximal growth possible in any entry of any reduced matrix $A^{(k)}$ in terms of the maximum entry of $|A|$. We thus have the following stability result.

THEOREM 3. *Let A be an $n \times n$ H-matrix. The growth factor γ resulting from application of Gaussian elimination with* cdd *pivoting to A is bounded as follows*:

$$\gamma = \frac{\max_{i,j,k} |a_{ij}^{(k)}|}{\max_{i,j} |a_{ij}|} \leqslant n.$$

5. EXAMPLES

EXAMPLE 2. To illustrate that a large growth factor γ is possible in H-matrices, consider the matrix

$$M = \begin{bmatrix} 2 & 0 & -x \\ -x & x & -1 \\ 0 & -1 & x \end{bmatrix},$$

which is a nonsingular M-matrix for all $x > \sqrt{2}$. Gaussian elimination without pivoting applied to M yields an unbounded growth factor γ of $x/2 + 1/x$, whereas with cdd pivoting $\gamma = 1$.

EXAMPLE 3. In [13], Varga and Cai give the following example of a singular, reducible M-matrix which does not admit an LU factorization into M-matrices. Let

$$M=\begin{bmatrix} 6 & -1 & 0 & 0 & 0 & 0 \\ -1 & 6 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & -1 \\ -1 & 0 & 0 & 0 & -1 & 6 \end{bmatrix}.$$

If A is an H-matrix with $\mathscr{M}(A)=M$, then A may or may not admit an LU factorization. However, by Theorem 2 there exists a permutation matrix P such that PAP^T admits an LU factorization (using cdd pivoting), and by Theorem 3 this computation is numerically stable. For example, if

$$A=\begin{bmatrix} 6 & -1 & 0 & 0 & 0 & 0 \\ -1 & 6 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & -1 \\ -1 & 0 & 0 & 0 & -1 & 6 \end{bmatrix},$$

Gaussian elimination with cdd pivoting gives the LU factorization

$$PAP^T=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{6} & 1 & 0 & 0 & 0 \\ -\frac{1}{6} & 0 & -\frac{1}{35} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\times\begin{bmatrix} 6 & 0 & -1 & -1 & -1 & 0 \\ 0 & 6 & -1 & 0 & 0 & 0 \\ 0 & 0 & \frac{35}{6} & -1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1219}{210} & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix},$$

where

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

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