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On an Operator Analogue of the Brenner–Newcomb–Ruehr Inequality

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1. INTRODUCTION

Let A and B be positive semi-definite operators on a Hilbert space H. If they commute with each other, then the mean σ_r, $0 \leqslant r \leqslant 1$, defined by

$$A\sigma_r B = (AB^{1-r} + BA^{1-r})(A^{1-r} + B^{1-r})^{-1} \tag{1}$$

satisfies the inequality

$$(A \nabla B)\, \sigma_r (C \nabla D) \geqslant (A\sigma_r C) \nabla (B\sigma_r D), \tag{2}$$

where $A \nabla B = (A+B)/2$ is the arithmetic mean of operators A and B.

This result in the case of scalars first appeared as the problem of J. L. Brenner, W. A. Newcomb, and O. G. Ruehr in [1]. The above operator analogue was given by J. I. Fujii, F. Kubo, and K. Kubo in [2]. They also noted that the dual inequality holds as well:

$$(A\,!\,B)\, \sigma_r (C\,!\,D) \leqslant (A\sigma_r C)\,!\,(B\sigma_r D), \tag{3}$$

where $A\,!\,B = \lim_{c \downarrow 0} (A_c^{-1} + B_c^{-1})^{-1}$, where $X_c = X + cI$.

In this paper, we give a generalization of (2).

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2. Preliminaries and Definitions

All operators in this article are positive semi-definite.

Let us consider the mean σ_r. Since A and B are permutable operators, we can easily see that

$$A\sigma_r B = \frac{A^r + B^r}{A^{r-1} + B^{r-1}}. \tag{4}$$

This mean is known as the rth-counter-harmonic mean (see [3, p. 185]).

From (4), it is clear that we can consider the case with several operators. Also, in the case of scalar inequalities, (2) is the well-known Beckenbach inequality [3, pp. 187–188; 4, pp. 87–88] and, in that case, if $1 \leqslant r \leqslant 2$, we have the reverse inequality in (2). (We show that the same is true in operators as well).

Denote by $S(J)$ the set of all self-adjoint operators on a Hilbert space whose spectra are contained in an interval $J = (\alpha, \beta)$.

Definition 1. Let $A = (A_1, ..., A_n)$ and $W = (W_1, ..., W_n)$ be the sequences of permutable operators from $S(0, \infty)$, $r \in R$. Then the rth-counter-harmonic mean of A (with weight W), $H_n^{[r]}(A; W)$ is defined by

$$H_n^{[r]}(A; W) = \frac{\sum_{i=1}^n A_i^r W_i}{\sum_{i=1}^n A_i^{r-1} W_i}. \tag{5}$$

More generally, we can introduce the following means (for scalars see [3, pp. 189–190]).

Definition 2. Let A and W be as in Definition 1, $p, q \in R$, $p > q$. Then

$$B_n^{p,q}(A; W) = \left[\frac{\sum_{i=1}^n W_i A_i^p}{\sum_{i=1}^n W_i A_i^q}\right]^{1/(p-q)}. \tag{6}$$

Note that for $q = 0$, we have the power mean $M_n^{[p]}(A; W)$ $(p \neq 0)$. Note also that the power means of operators are defined by

$$M_n^{[r]}(A; W) = \left[\frac{\sum_{i=1}^n W_i A_i^r}{\sum_{i=1}^n W_i}\right]^{1/r}, \qquad r \neq 0. \tag{7}$$

It is known that if A_i, W_i are permutable operators from $S(0, \infty)$, then for $r < s$ $(rs \neq 0)$, we have [5]

$$M_n^{[r]}(A; W) \leqslant M_n^{[s]}(A; W), \tag{8}$$

i.e., in this case, the fundamental means inequality is also valid. Note also that in [5], the following Hölder and Minkowski inequalities for permutable operators were proved:

Hölder's Inequality. Let $A_1, ..., A_n$, $B_1, ..., B_n$, $W_1, ..., W_n$ be permutable operators from $S(0, \infty)$, $p, q \in R$, $pq \neq 0$, $p^{-1} + q^{-1} = 1$. Then

(a) If $p > 1$,

$$\sum_{i=1}^{n} W_i A_i B_i \leqslant \left(\sum_{i=1}^{n} W_i A_i^p \right)^{1/p} \left(\sum_{i=1}^{n} W_i B_i^q \right)^{1/q}. \tag{9}$$

(b) If either $p < 0$ or $q < 0$, then inequality (9) is reversed.

Minkowski Inequality. Let $A_1, ..., A_n$, $B_1, ..., B_n$, $W_1, ..., W_n \in S(0, \infty)$ be permutable operators and let $p \in R$. If $p > 1$, then

$$\left[\sum_{i=1}^{n} W_i (A_i + B_i)^p \right]^{1/p} \leqslant \left[\sum_{i=1}^{n} W_i A_i^p \right]^{1/p} + \left[\sum_{i=1}^{n} W_i B_i^p \right]^{1/p}. \tag{10}$$

If $p < 1$, $p \neq 0$, then inequality (10) is reversed.

Setting, in (9), $p^{-1} = s$, $q^{-1} = 1 - s$, $A_i^p \to A_i^u$, $B_i^q \to A_i^v$, we get [5]

$$G(su + (1-s)v) \leqslant G(u)^s G(v)^{1-s} \qquad (0 \leqslant s \leqslant 1), \tag{11}$$

where

$$G(p) = \sum_{i=1}^{n} W_i A_i^p. \tag{12}$$

3. Results

THEOREM 1. *Let U_j, V_i, W_j, A_{ij}, $B_{ij} \in S(0, \infty)$, $i = 1, ..., n$, $j = 1, ..., m$ be permutable operators and let $p, r, \alpha, \beta \in R$ be such that $\alpha p - \beta r = 1$.*

(a) *If $p \geqslant 1$, $0 < r \leqslant 1$, $\alpha > 0$, $\beta > 0$, then*

$$\frac{[\sum_{j=1}^{m} U_j [\sum_{i=1}^{n} V_i A_{ij}]^p]^\alpha}{[\sum_{j=1}^{m} W_j [\sum_{i=1}^{n} V_i B_{ij}]^r]^\beta} \leqslant \sum_{i=1}^{n} V_i \frac{[\sum_{j=1}^{m} U_j A_{ij}^p]^\alpha}{[\sum_{j=1}^{m} W_j B_{ij}^r]^\beta}. \tag{13}$$

(b) *If $0 \leqslant p \leqslant 1$, $r < 0$, $\alpha > 0$, $\beta > 0$, the reverse inequality in* (13) *holds.*

Proof. By the extended Minkowski inequality [5], we have

$$\left(\sum_{j=1}^{m} U_j \left(\sum_{i=1}^{n} V_i A_{ij} \right)^p \right)^{1/p} \leqslant \sum_{i=1}^{n} V_i \left(\sum_{j=1}^{m} U_j A_{ij}^p \right)^{1/p} \tag{14}$$

if $p \geqslant 1$. The reverse inequality holds if $p < 1$ ($\neq 0$), i.e., for $r < 1$, we have

$$\left(\sum_{j=1}^{n} W_j\left(\sum_{i=1}^{n} V_i B_{ij}\right)^r\right)^{1/r} \geqslant \sum_{i=1}^{n} V_i\left(\sum_{j=1}^{m} W_j B_{ij}^r\right)^{1/r} \tag{15}$$

(a) We have $\alpha p > 0$, $\beta r > 0$, so (14) and (15) become

$$\left(\sum_{j=1}^{m} U_j\left(\sum_{i=1}^{n} V_i A_{ij}\right)^p\right)^{\alpha} \leqslant \left[\sum_{i=1}^{n} V_i\left(\sum_{j=1}^{m} U_j A_{ij}^p\right)^{1/p}\right]^{\alpha p} \tag{16}$$

and

$$\left(\sum_{j=1}^{m} W_j\left(\sum_{i=1}^{n} V_i B_{ij}\right)^r\right)^{\beta} \geqslant \left[\sum_{i=1}^{n} V_i\left(\sum_{j=1}^{m} W_j B_{ij}^r\right)^{1/r}\right]^{\beta r}. \tag{17}$$

Hence

$$\begin{aligned}
\frac{(\sum_{j=1}^{m} U_j(\sum_{i=1}^{n} V_i A_{ij})^p)^{\alpha}}{(\sum_{j=1}^{m} W_j(\sum_{i=1}^{n} V_i B_{ij})^r)^{\beta}} &\leqslant \frac{[\sum_{i=1}^{n} V_i(\sum_{j=1}^{m} U_j A_{ij}^p)^{1/p}]^{\alpha p}}{[\sum_{i=1}^{n} V_i(\sum_{j=1}^{m} W_j B_{ij}^r)^{1/r}]^{\beta r}} \\
&= \left[\sum_{i=1}^{n} V_i\left\{\left(\sum_{j=1}^{m} U_j A_{ij}^p\right)^{\alpha}\right\}^{1/\alpha p}\right]^{\alpha p} \\
&\quad \times\left[\sum_{i=1}^{n} V_i\left\{\left(\sum_{j=1}^{m} W_j B_{ij}^r\right)^{-\beta}\right\}^{-1/\beta r}\right]^{-\beta r} \\
&\leqslant \sum_{i=1}^{n} V_i\left\{\sum_{j=1}^{m} U_j A_{ij}^p\right\}^{\alpha}\left\{\sum_{j=1}^{m} W_j B_{ij}^r\right\}^{-\beta} \\
&= \sum_{i=1}^{n} V_i\frac{[\sum_{j=1}^{m} U_j A_{ij}^p]^{\alpha}}{[\sum_{j=1}^{m} W_j B_{ij}^r]^{\beta}}.
\end{aligned} \tag{18}$$

The last inequality is a consequence of the reverse inequality of (9).

(b) Now we have $\alpha p > 0$ and $\beta r < 0$. Note first, that (14) with the reverse inequality of (15) holds, and, therefore, the reverse inequalities hold in both (16) and (17). The reverse inequality also holds in (18) since, in this case, we apply (9).

COROLLARY 1. *Let* U_j, V_i, W_j, A_{ij}, B_{ij} *be defined as in Theorem* 1. *If* $p \geqslant 1 > r > 0$, $p, r \in R$ *then*

$$\left\{\frac{\sum_{j=1}^{m} U_j(\sum_{i=1}^{n} V_i A_{ij})^p}{\sum_{j=1}^{m} W_j(\sum_{i=1}^{n} V_i B_{ij})^r}\right\}^{1/(p-r)} \leqslant \sum_{i=1}^{n} V_i\left\{\frac{\sum_{j=1}^{m} U_j A_{ij}^p}{\sum_{j=1}^{m} W_j B_{ij}^r}\right\}^{1/(p-r)}. \tag{19}$$

If $1 \geqslant p > 0 > r$, *then the reverse inequality in* (19) *holds.*

This is the case $\alpha=\beta$ of Theorem 1. Setting in Corollary 1, $p\to r$, $r\to r-1$, we get

COROLLARY 2. *Let U_j, W_j, V_i, A_{ij}, B_{ij} be defined as in Theorem 1. If $1\leqslant r\leqslant 2$, then*

$$\frac{\sum_{j=1}^{m} U_j(\sum_{i=1}^{n} V_i A_{ij})^r}{\sum_{j=1}^{m} W_j(\sum_{i=1}^{n} V_i B_{ij})^{r-1}} \leqslant \sum_{i=1}^{n} V_i \frac{\sum_{j=1}^{m} U_j A_{ij}^r}{\sum_{j=1}^{m} W_j B_{ij}^{r-1}}. \tag{20}$$

If $0<r\leqslant 1$, then the reverse inequality in (20) holds.

COROLLARY 3. *Let $W_1, ..., W_n$, $A_1, ..., A_n$, $B_1, ..., B_n \in S(0,\infty)$ be permutable operators. If $p\geqslant 1>r>0$, $p, r\in R$, then*

$$B_n^{p,r}(A+B; W)\leqslant B_n^{p,r}(A; W)+B_n^{p,r}(B, W). \tag{21}$$

If $1\geqslant p>0>r$, then the reverse inequality in (21) holds.

Proof. In (19), set $n=2$, $m\to n$, $V_1=V_2=I$, $A_{1j}=B_{1j}=A_j$, $A_{2j}=B_{2j}=B_j$, $U_j=W_j$, $j=1, ..., n$.

COROLLARY 4. *Let W_i, A_i, B_i be defined as in Corollary 3. If $1\leqslant r\leqslant 2$, then*

$$H_n^{[r]}(A+B; W)\leqslant H_n^{[r]}(A; W)+H_n^{[r]}(B; W), \tag{22}$$

while if $0<r<1$, then the reverse inequality in (22) holds.

Remark. Corollary 4 gives a generalization of (2) as well as of the Beckenbach inequality, while Corollary 3 is a generalization of Dresher's inequality for sums (see [4, p. 28]) to the case of operators.

We now give further operator inequalities analogous to results given in [3].

THEOREM 2. *If A and W are sequences of permutable operators from $S(0,\infty)$ then*

(a) *If $1\leqslant r<\infty$, then*

$$H_n^{[r]}(A; W)\geqslant M_n^{[r]}(A; W) \tag{23}$$

while if $-\infty<r\leqslant 1$, $p\neq 0$, the inequality (23) is reversed.

(b) *If $-\infty<r<0$, then*

$$H_n^{[r]}(A; W)\leqslant M_n^{[r+1]}(A; W). \tag{24}$$

Proof. (a) If $1 \leqslant r < \infty$, then (8) and

$$H_n^{[r]}(A; W) = M_n^{[r]}(A; W) \left\{ \frac{M_n^{[r]}(A; W)}{M_n^{[r-1]}(A; W)} \right\}^{r-1}$$

imply (23); while if $-\infty < r \leqslant 1$, the reverse of (23) is implied.

(b) If $r < 0$, then

$$\begin{aligned} H_n^{[r]}(A; W) &= (H_n^{[-r+1]}(A^{-1}; W))^{-1} \\ &\leqslant (M^{[-r+1]}(A^{-1}; W))^{-1} \qquad \text{(by } (a)) \\ &= M_n^{[r+1]}(A; W). \end{aligned}$$

For our next results, let us consider the function G defined by (12). Inequality (11) and the version

$$G((1-s)\,u + sv) \leqslant G(u)^{1-s}\, G(v)^s \qquad (0 \leqslant s \leqslant 1)$$

give for $0 \leqslant s \leqslant 1$, the inequality

$$G(su + (1-s)\,v)\, G((1-s)\,u + sv) \leqslant G(u)\, G(v). \tag{25}$$

Setting in (25), $u \to a$, $v = b + h$, $s(v - u) \to h(>0)$, we get, for $b > a$,

$$\frac{G(b)}{G(a)} \leqslant \frac{G(b+h)}{G(a+h)}, \tag{26}$$

i.e., the following theorem holds.

THEOREM 3. *If A and W are sequences of permutable operators from $S(0, \infty)$ and $r < s$, $rs \neq 0$, then*

$$H_n^{[r]}(A; W) \leqslant H_n^{[s]}(A; W). \tag{27}$$

Proof. Set, in (26), $b = r$, $a = r - 1$, $h = -r$.

Now, setting $sv + (1-s)\,v = w$ $(u < w < v)$, in (11), gives Liapunov's inequality

$$G(w) \leqslant G(u)^{(v-w)/(v-u)}\, G(v)^{(w-u)/(v-u)} \tag{28}$$

i.e.,

$$\left(\frac{G(w)}{G(u)}\right)^{1/(w-u)} \leqslant \left(\frac{G(w)}{G(v)}\right)^{1/(w-v)}. \tag{29}$$

Moreover, this inequality holds for all u, w, v such that $u \leqslant v$, $u, v \neq w$, i.e.,

$$\left(\frac{G(w)}{G(u)}\right)^{1/(w-u)}$$

is an increasing function of u. Thus, we also have

$$\left(\frac{G(v)}{G(w)}\right)^{1/(v-w)} \leqslant \left(\frac{G(v)}{G(z)}\right)^{1/(v-z)} \qquad (w \leqslant z;\, w, z \neq v). \tag{30}$$

By combining (29) and (30), we obtain

$$\left(\frac{G(w)}{G(u)}\right)^{1/(w-u)} \leqslant \left(\frac{G(z)}{G(v)}\right)^{1/(z-v)} \tag{31}$$

if $u \leqslant v$, $w \leqslant z$, $u \neq w$, $z \neq v$.

A simple consequence of this inequality is the following:

THEOREM 4. *If* $u \leqslant v$, $w \leqslant z$, $u \neq w$, $z \neq v$, *then*

$$B_n^{u,w}(A; W) \leqslant B_n^{v,z}(A; W), \tag{32}$$

where A *and* W *are sequences of permutable operators from* $S(0, \infty)$.

Remark. Theorems 3 and 4, in the real case, are given in [3, pp. 187, 189–190].

Remark. In fact, known results in the real case can be used in proofs of similar results for permutable operators via spectral theory (see [6, pp. 209–291]). So, the Brenner–Newcomb–Ruehr inequality can be used in a proof of (1). Also, we can give our inequality (13) first in the real case and then, via spectral theory, obtain Theorem 1.

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