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Triples on Functor Categories

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INTRODUCTION

The results in this paper were obtained by making certain observations about modules and generalizing to functor categories. The idea is to generalize the techniques of homological algebra to functor categories, in particular, to find a suitable definition for a tensor product of functors. This was done in [8] for additive functor categories, and recently the invention of ends and coends in [3], and the use to which they were put in the work of Dubuc [4], has allowed us to extend this definition of the tensor product to a much larger class of functors (this definition is given in Section 2 of this paper).

Many of our results parallel those of Bunge [1]. Her approach was inspired by certain observations about sets, whereas our approach stems from a more "homological" viewpoint. We feel that the latter approach not only simplifies the exposition of these results, but also allows us to be much more general in some cases.

We now give some definitions, state our observations on modules, and give a rough idea how these observations generalize.

Let $\mathbf{X}$ be a category. A *triple* $\mathbf{T}$ on $\mathbf{X}$ is given by $\mathbf{T} = (T, \eta, \mu)$, where $T: \mathbf{X} \to \mathbf{X}$ is a functor, $\eta: 1_{\mathbf{X}} \Rightarrow T$ is a natural transformation from the identity functor on $\mathbf{X}$ to T, and $\mu: TT \Rightarrow T$ is a natural transformation from T composed with itself to T, for which the identities $\mu \cdot \eta T = \mu \cdot T\eta = 1_T$ and $\mu \cdot \mu T = \mu \cdot T\mu$ hold. Dually, one defines a cotriple on $\mathbf{X}$.

Examples of a triple and of a cotriple which are pertinent to this paper are the following. Let R be a ring, and let ${}_R\mathbf{M}$ be the category of left R-modules (we shall always assume that our rings and modules are unitary). Suppose S is a ring and $f: R \to S$ is a ring homomorphism. Then f induces a left and

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right R-module structure on S, making S into an $R - R$ bimodule. We define a triple $\mathbf{T}_f = (T_f, \eta_f, \mu_f)$ on ${}_R\mathbf{M}$ by $T_f(M) = S \otimes_R M$,

$$\eta_f(M) = f \otimes 1_M : R \otimes_R M \to S \otimes_R M$$

(identifying $R \otimes_R -$ with the identity functor on ${}_R\mathbf{M}$ via the canonical isomorphism $R \otimes_R M \cong M$), and

$$\mu_f(M) = m \otimes 1_M : S \otimes_R S \otimes_R M \to S \otimes_R M,$$

where $m: S \otimes_R S \to S$ is the "multiplication" map given by $m(s \otimes s') = ss'$. The ring structure on S gives the required identities for $\mathbf{T}_f$ to be a triple. We refer to $\mathbf{T}_f$ as the *scalar extension triple* associated to f. Dually, we construct a cotriple $\mathbf{G}_f = (G_f, \epsilon_f, \delta_f)$ on ${}_R\mathbf{M}$ by letting $G_f(M) = \mathrm{Hom}_R(S, M)$,

$$\epsilon_f(M) = \mathrm{Hom}_R(f, M): \mathrm{Hom}_R(S, M) \to \mathrm{Hom}_R(R, M) \cong M,$$

and

$$\delta_f(M) = \mathrm{Hom}_R(m, M): \mathrm{Hom}_R(S, M) \to \mathrm{Hom}_R(S \otimes_R S, M).$$

The main theorem of this paper is a generalization of the following theorem for modules: if $\mathbf{T} = (T, \eta, \mu)$ is a triple on ${}_R\mathbf{M}$ for which T is cocontinuous (i.e., T preserves direct sums and cokernels), then there is a ring S and a ring homomorphism $f: R \to S$ for which one has an isomorphism of triples $\mathbf{T} \cong \mathbf{T}_f$, $\mathbf{T}_f$ the scalar extension triple associated to f.

The proof of the theorem for modules is an application of Watt's theorem [13], which may be stated as follows: if R and R' are rings and $U: {}_R\mathbf{M} \to {}_{R'}\mathbf{M}$ is a cocontinuous functor, then there is an $R' - R$ bimodule E for which $U \cong E \otimes_R -$. Consequently, given a triple $\mathbf{T} = (T, \eta, \mu)$ on ${}_R\mathbf{M}$ for which $T: {}_R\mathbf{M} \to {}_R\mathbf{M}$ is cocontinuous, by Watts' theorem there is a $R - R$ bimodule S for which $T \cong S \otimes_R -$, and then one shows that the triple structure on T induces a ring structure on S and a ring homomorphism $f: R \to S$.

We now describe how this theorem, and its proof, are generalized in this paper.

Let **Ab** denote the category of Abelian groups, let **C** be a small preadditive category (a category whose Hom sets are endowed with an Abelian group structure in such a way that composition in **C** is bilinear), and let [**C**, **Ab**] be the category whose objects are additive functors from **C** to **Ab** and whose morphisms are natural transformations between these functors. The functor category [**C**, **Ab**] generalizes a module category in the following way: if one associates to a ring R a preadditive category **R** with one object whose morphisms are elements of R, composition is multiplication in R, and with Abelian group structure on the (unique) Hom set the additive structure of R, then one has an isomorphism of categories $[\mathbf{R}, \mathbf{Ab}] \cong {}_R\mathbf{M}$. (One also has an isomorphism of categories $[\mathbf{R}^o, \mathbf{Ab}] \cong \mathbf{M}_R$, the category of right R-modules, where $\mathbf{C}^o$ denotes the dual or "opposite" category of $\mathbf{C}$ for any category $\mathbf{C}$.)

Let $\mathbf{X}$ and $\mathbf{A}$ be any two categories. An *adjoint pair* $(U, F)\colon \mathbf{A} \to \mathbf{X}$ is a pair of functors $U\colon \mathbf{A} \to \mathbf{X}$ and $F\colon \mathbf{X} \to \mathbf{A}$ for which F is adjoint to U, i.e., for every pair of objects X of $\mathbf{X}$ and A of $\mathbf{A}$ one has a bijection

$$\theta_{(X,A)}\colon \mathbf{A}(FX, A) \cong \mathbf{X}(X, UA)$$

with $\theta_{(X,A)}$ natural in both X and A. An adjoint pair $(U, F)\colon \mathbf{A} \to \mathbf{X}$ induces a triple $\mathbf{T}$ on $\mathbf{X}$ (see [7]) as follows. One defines natural transformations $\eta\colon 1_{\mathbf{X}} \Rightarrow UF$ and $\epsilon\colon FU \Rightarrow 1_{\mathbf{A}}$, called the *unit* and *counit* of the adjoint pair (U, F) respectively, by letting $\eta_X = \theta_{(X,FX)}(1_{FX})$ for all objects X of $\mathbf{X}$ and $\epsilon_A = \theta^{-1}_{(UA,A)}(1_{UA})$ for all objects A of $\mathbf{A}$. Then the triple $\mathbf{T}$ on $\mathbf{X}$ induced by (U, F) is given by $\mathbf{T} = (UF, \eta, U\epsilon F)$. Similarly, one has a cotriple $\mathbf{G}$ on $\mathbf{A}$ induced by (U, F) given by $\mathbf{G} = (FU, \epsilon, F\eta U)$.

As an example, let $f\colon R \to S$ be a ring homomorphism, let $U_f\colon {}_S\mathbf{M} \to {}_R\mathbf{M}$ be the functor which regards every S-module N as an R-module by defining the operation of R on N by $ry = f(r)y$ for every $r \in R$ and $y \in N$, and let $F_f\colon {}_R\mathbf{M} \to {}_S\mathbf{M}$ be given by $F_f(M) = S \otimes_R M$ for every R-module M. Then $(U_f, F_f)\colon {}_S\mathbf{M} \to {}_R\mathbf{M}$ is an adjoint pair, and the triple induced by (U_f, F_f) is the scalar extension triple $\mathbf{T}_f$ associated to f.

More generally, let $\mathbf{C}$ and $\mathbf{C}'$ be small preadditive categories, and let $x\colon \mathbf{C} \to \mathbf{C}'$ be an additive functor. Then x induces a functor

$$\mathbf{Ab}^x\colon [\mathbf{C}', \mathbf{Ab}] \to [\mathbf{C}, \mathbf{Ab}]$$

given by $\mathbf{Ab}^x(G) = G \cdot x$ for every object $G\colon \mathbf{C}' \to \mathbf{Ab}$ of $[\mathbf{C}', \mathbf{Ab}]$. It is well known that $\mathbf{Ab}^x$ has a left adjoint $(\mathbf{Ab}^x)^\ell\colon [\mathbf{C}, \mathbf{Ab}] \to [\mathbf{C}', \mathbf{Ab}]$. We let $\mathbf{T}_x$ denote the triple on $[\mathbf{C}, \mathbf{Ab}]$ induced by the adjoint pair

$$(\mathbf{Ab}^x, (\mathbf{Ab}^x)^\ell)\colon [\mathbf{C}', \mathbf{Ab}] \to [\mathbf{C}, \mathbf{Ab}].$$

$\mathbf{T}_x$ is a generalization of the scalar extension triple, for if $\mathbf{C}$ and $\mathbf{C}'$ both have exactly one object, then they are preadditive categories associated to rings R and S, the functor $x\colon \mathbf{C} \to \mathbf{C}'$ induces a ring homomorphism $f\colon R \to S$, and $\mathbf{T}_x \cong \mathbf{T}_f$.

Our main result for additive functor categories is the above theorem for modules translated to additive functor categories, namely: if $\mathbf{C}$ is a small preadditive category and if $\mathbf{T}$ is a triple on $[\mathbf{C}, \mathbf{Ab}]$ whose functor is cocontinuous, then there is a small preadditive category $\mathbf{C}'$ and an additive functor $x\colon \mathbf{C} \to \mathbf{C}'$ for which $\mathbf{T} \cong \mathbf{T}_x$. Using the tensor product of functors defined in [8] (which generalizes that of modules) we were able in [9] to extend Watts' theorem to additive functor categories, the role of bimodules being filled by biadditive bifunctors $u: \mathbf{C} \times \mathbf{C}' \to \mathbf{Ab}$. Our main theorem follows from this generalized version of Watt's theorem.

The work of Dubuc in [4] has allowed us to generalize from additive functor categories to a much larger class of functor categories, and it is at this level of generality that we state and prove our results in this paper. More specifically, let **V** be a category with sufficient structure so that one can do category theory relative to **V**, generalizing category theory relative to sets, by which one means that:

(i) there is a notion of a **V**-category **A**, which differs from the notion of an ordinary category in that, given two objects A and B of **A**, the Hom set $\mathbf{A}(A, B)$ becomes an object of **V** instead of a set;

(ii) given two **V**-categories **A** and **B** there is a notion of a **V**-functor $F: \mathbf{A} \to \mathbf{B}$, which differs from the notion of an ordinary functor in that, given two objects A and B of **A**, $F_{AB}: \mathbf{A}(A, B) \to \mathbf{B}(FA, FB)$ is a morphism of **V** instead of a map of sets;

(iii) **V** is itself a **V**-category.

For the sufficient structure on **V**, one assumes **V** is a closed category (i.e., what in [5] is called a symmetric monoidal closed category) which we define in Section 1. If **V** is bicomplete (both complete and cocomplete in the ordinary category theory sense) then for any small **V**-category **C** there is a **V**-category $\mathbf{V}^{\mathbf{C}}$ whose objects are the **V**-functors $F: \mathbf{C} \to \mathbf{V}$. All of **V**-category theory is done in such a way that **Ab** is an example of a bicomplete closed category and **Ab**-category theory is category theory restricted to preadditive categories and additive functors (in particular, for any small **Ab**-category **C**, $\mathbf{Ab}^{\mathbf{C}}$ is the functor category $[\mathbf{C}, \mathbf{Ab}]$ discussed above). **Ens**, the category of sets, is also a bicomplete closed category, and **Ens**-category theory is just ordinary category theory.

There are many other examples (e.g., **Ban**, the category of complex Banach spaces and linear maps of norm $\leqslant 1$, is also a bicomplete closed category, and **Ban**-category theory would deal with categories whose Hom-sets are complex Banach spaces).

In Section 1, we define a closed category **V** and outline **V**-category theory following the exposition given by Dubuc in [4], whose viewpoint we find most useful (in fact, essential) for our purposes.

In Section 2, we generalize our results of [8] and [9] to **V**-categories. In particular, Theorem 2.10 becomes the suitable **V**-functor category version of Watts' theorem.

In Section 3, we prove our main result. Recall that in applying Watts' theorem to obtain our result for modules, we observed that "the triple structure on T induces a ring structure on S and a ring homomorphism

$f: R \to S$." Most of Section 3 is devoted to making this observation more precise in the situation of **V**-functor categories. Once this is done, we may apply the Watts' theorem of Section 2 to obtain our main theorem, Theorem 3.12, which says that given any small **V**-category **C** and a **V**-triple **T** on $\mathbf{V}^{\mathbf{C}}$ whose functor is **V**-cocontinuous, then there is a small **V**-category **C**′ and a **V**-functor $x: \mathbf{C} \to \mathbf{C}'$ which is a surjection on objects so that $\mathbf{T} \cong \mathbf{T}_x$, where $\mathbf{T}_x$ is the **V**-triple associated to the induced functor $\mathbf{V}^x: \mathbf{V}^{\mathbf{C}'} \to \mathbf{V}^{\mathbf{C}}$ and its left adjoint.

In Section 4, we use this result to obtain some characterizations of functor categories. To describe these results, we first give some definitions.

Recall that above we described how every adjoint pair $(U, F): \mathbf{A} \to \mathbf{X}$ gives rise to a triple **T** on **X**. Conversely, in [7] it is shown that a triple $\mathbf{T} = (T, \eta, \mu)$ on **X** gives rise to an adjoint pair in the following way. We define a **T**-algebra to be a pair (A, α) with A an object of **X** and $\alpha: TA \to A$ a morphism of **X** for which $\alpha \cdot \eta_A = 1_A$ and $\alpha \cdot \mu_A = \alpha \cdot T\alpha$. If (A, α) and (B, β) are **T**-algebras, then a morphism of **T**-algebras $f: (A, \alpha) \to (B, \beta)$ is a morphism $f: A \to B$ in **X** for which $f \cdot \alpha = \beta \cdot Tf$. The **T**-algebras and their morphisms form a category $\mathbf{X}^{\mathbf{T}}$, the category of **T**-algebras (or the *Eilenberg–Moore category* associated to **T**). There are functors $U^{\mathbf{T}}: \mathbf{X}^{\mathbf{T}} \to \mathbf{X}$ and $F^{\mathbf{T}}: \mathbf{X} \to \mathbf{X}^{\mathbf{T}}$, given on objects by $U^{\mathbf{T}}(A, \alpha) = A$ for every **T**-algebra (A, α) and $F^{\mathbf{T}}X = (TX, \mu_X)$ for every object X of **X**, and one shows that $(U^{\mathbf{T}}, F^{\mathbf{T}}): \mathbf{X}^{\mathbf{T}} \to \mathbf{X}$ is an adjoint pair.

Given an adjoint pair $(U, F): \mathbf{A} \to \mathbf{X}$, let **T** be the triple on **X** induced by (U, F), and let $(U^{\mathbf{T}}, F^{\mathbf{T}}): \mathbf{X}^{\mathbf{T}} \to \mathbf{X}$ be the adjoint pair associated to **T** as described above, with $\mathbf{X}^{\mathbf{T}}$ the category of **T**-algebras. Then there is a canonical functor $\Phi: \mathbf{A} \to \mathbf{X}^{\mathbf{T}}$, called the comparison functor, for which $U^{\mathbf{T}} \cdot \Phi = U$ (Φ is given on objects A of **A** by $\Phi(A) = (UA, U\epsilon_A)$). One says that the adjoint pair (U, F) is *tripleable* if Φ is an equivalence of categories.

Dually, for a cotriple $\mathbf{G} = (G, \epsilon, \delta)$ on **A**, the category of **G**-coalgebras, ${}^{\mathbf{G}}\mathbf{A}$, is defined with objects pairs (C, γ), C an object of **A** and $\gamma: C \to GC$ a morphism of A such that $\epsilon_C \cdot \gamma = 1_C$ and $\delta_C \cdot \gamma = G\gamma \cdot \gamma$. There is an adjoint pair $({}^{\mathbf{G}}U, {}^{\mathbf{G}}F): \mathbf{A} \to {}^{\mathbf{G}}\mathbf{A}$ associated to **G**. If $(U, F): \mathbf{A} \to \mathbf{X}$ is an adjoint pair and **G** is the cotriple on **A** associated to (U, F), there is a canonical comparison functor $\Phi: \mathbf{X} \to {}^{\mathbf{G}}\mathbf{A}$ such that ${}^{\mathbf{G}}F \cdot \Phi = F$. One says that (U, F) is *cotripleable* if Φ is an equivalence of categories.

As an example, let $f: R \to S$ be a ring homomorphism, and consider the adjoint pair $(U_f, F_f): {}_S\mathbf{M} \to {}_R\mathbf{M}$ defined above. In fact, (U_f, F_f) is tripleable, so that the comparison functor gives an equivalence of categories ${}_S\mathbf{M} \cong ({}_R\mathbf{M})^{\mathbf{T}_f}$.

In particular, let R be any ring and let Z denote the ring of integers. Then there is a unique homomorphism of rings $f: Z \to R$ giving us a canonical adjoint pair $(U_f, F_f): {}_R\mathbf{M} \to {}_Z\mathbf{M} = \mathbf{Ab}$ (in this case, U_f is the functor that considers every R-module as an Abelian group). Then ${}_R\mathbf{M} \cong \mathbf{Ab}^{\mathbf{T}_f}$, where $\mathbf{T}_f$ is the triple associated to the adjoint pair (U_f, F_f), and $\mathbf{T}_f$ has a cocontinuous functor. Furthermore, our main result (for modules) shows that this property characterizes module categories, namely, given any triple $\mathbf{T}$ on $\mathbf{Ab}$ whose functor is cocontinuous, there is a ring R for which ${}_R\mathbf{M} \cong \mathbf{Ab}^{\mathbf{T}}$.

In Theorem 4.1, we generalize this characterization of module categories to show that $\mathbf{V}$-functor categories of the form $\mathbf{V}^{\mathbf{C}}$ are characterized by the property that they are equivalent to $\mathbf{V}$-categories of $\mathbf{T}$-algebras for some $\mathbf{V}$-cocontinuous $\mathbf{V}$-triple $\mathbf{T}$ on $\mathbf{V}^S$, where S is some index set (which may be taken to be the set of objects of $\mathbf{C}$) and $\mathbf{V}^S$ is the category obtained by taking the product of S copies of $\mathbf{V}$ and making it into a $\mathbf{V}$-category (see Section 4).

The rest of this section is devoted to showing that Bunge's first and second characterizations of functor categories [1(4.6, 4.16)], with somewhat weaker hypotheses, follow from the characterization given above.

In Section 5, we dualize the results of Section 3 to obtain a generalization of the following theorem for modules: if $\mathbf{G} = (G, \epsilon, \delta)$ is a cotriple on ${}_R\mathbf{M}$ for which G is continuous (i.e., G preserves products and kernels), then there is a ring S and a ring homomorphism $f: R \to S$ for which one has an isomorphism $\mathbf{G} \cong \mathbf{G}_f$ and for which ${}^{\mathbf{G}}({}_R\mathbf{M}) \cong {}_S\mathbf{M}$.

We have considered generalizations of triples on ${}_R\mathbf{M}$ with cocontinuous functors and cotriples on ${}_R\mathbf{M}$ with continuous functors. We next attack the generalizations of the following: Suppose R is a commutative ring and Λ is an R-coalgebra, i.e., Λ is an R-module and there are R-homomorphisms $\bar{\epsilon}: \Lambda \to R$ and $\bar{\delta}: \Lambda \to \Lambda \otimes_R \Lambda$ satisfying the usual identities. We can then define a cotriple $\mathbf{G} = (G = \Lambda \otimes_R -, \epsilon = \bar{\epsilon} \otimes_R -, \delta = \bar{\delta} \otimes_R -)$ on ${}_R\mathbf{M}$ and a triple $\mathbf{T} = (T = \mathrm{Hom}_R(\Lambda, -), \eta = \mathrm{Hom}_R(\bar{\epsilon}, -), \mu = \mathrm{Hom}_R(\bar{\delta}, -))$ on ${}_R\mathbf{M}$ with G cocontinuous and T continuous. Then ${}^{\mathbf{G}}({}_R\mathbf{M})$ is the category of comodules over the coalgebra Λ and $({}_R\mathbf{M})^{\mathbf{T}}$ is the category of contramodules over Λ (see [6, Chapter III, Section 5]). Generally, these categories are not equivalent to categories of modules. However, when Λ is a finitely generated projective R-module, i.e., G and T are both continuous and cocontinuous, ${}^{\mathbf{G}}({}_R\mathbf{M}) \cong ({}_R\mathbf{M})^{\mathbf{T}} \cong {}_{\Lambda^*}\mathbf{M}$ where $\Lambda^* = \mathrm{Hom}_R(\Lambda, R)$. The equivalence of these categories follows from the fact that for Λ a finitely generated projective R-module the isomorphisms $\mathrm{Hom}_R(\Lambda, -) \cong \Lambda^* \otimes_R$ and $\Lambda \otimes_R - \cong \mathrm{Hom}_R(\Lambda^*, -)$ give rise to equivalences of triples and cotriples $\mathbf{T} \cong \mathbf{T}_f$ and $\mathbf{G} \cong \mathbf{G}_f$ where $f: R \to \Lambda^*$ is the ring homomorphism

$$R \cong \mathrm{Hom}_R(R, R) \xrightarrow{\mathrm{Hom}(\bar{\epsilon}, R)} \Lambda^*.$$

The key to generalizing this situation is the dualization functor $*: (\mathbf{V}^{\mathbf{C}})^o \to \mathbf{V}^{\mathbf{C}^o}$ which is the usual dualization functor $*: {}_R\mathbf{M}^o \to \mathbf{M}_R$ given by $M^* = \mathrm{Hom}_R(M, R)$ when $\mathbf{C} = \mathbf{R}$ and $\mathbf{V} = \mathbf{Ab}$. Using this dualization functor we prove (5.10) that a $\mathbf{V}$-triple $\mathbf{T}$ on $\mathbf{V}^{\mathbf{C}}$ whose functor is $\mathbf{V}$-continuous and $\mathbf{V}$-cocontinuous is equivalent to a $\mathbf{V}$-triple $\mathbf{T}_x$ where $x: \mathbf{C} \to \mathbf{C}'$ is a $\mathbf{V}$-cocontinuous functor and is surjective on objects and $\mathbf{V}^x: \mathbf{V}^{\mathbf{C}'} \to \mathbf{V}^{\mathbf{C}}$ and its left adjoint induce the V-triple $\mathbf{T}_x$. The dual of this theorem (5.11) for a $\mathbf{V}$-cotriple on $\mathbf{V}^{\mathbf{C}}$ with a $\mathbf{V}$-continuous and $\mathbf{V}$-cocontinuous functor is proved for $\mathbf{V} = \mathbf{Ens}$ or ${}_R\mathbf{M}$ where R is a commutative ring.

1. Review of Relative Category Theory

In this section, we review the definition of a closed category and outline the development of category theory relative to a closed category, following [4].

(1.1) Definition. A *monoidal category* $\mathbf{V}$ consists of (i) a category $\mathbf{V}_o$; (ii) a bifunctor $- \otimes -: \mathbf{V}_o \times \mathbf{V}_o \to \mathbf{V}_o$, called the *tensor product* of $\mathbf{V}$ which we assume is associative, i.e., for any three objects U, V, and W there is an isomorphism in $\mathbf{V}_o$, $a: (U \otimes V) \otimes W \to U \otimes (V \otimes W)$, natural in U, V, and W; (iii) an object I of $\mathbf{V}$ which is the *unit* of $\mathbf{V}$, i.e., for each object V of $\mathbf{V}$ there are isomorphisms in $\mathbf{V}_o$, $r: V \otimes I \to V$ and $l: I \otimes V \to V$, natural in V. The isomorphisms satisfy coherence conditions which may be found in [11]. A monoidal category $\mathbf{V}$ satisfying (iv) for each pair of objects U and V of $\mathbf{V}_0$ there is an isomorphism $\gamma: U \otimes V \to V \otimes U$ natural in U and V (and coherence conditions) is called *symmetric*. A symmetric monoidal category satisfying (v) there is a bifunctor $\mathbf{V}(-, -): (\mathbf{V}_o)^o \times \mathbf{V}_o \to \mathbf{V}_o$, called the *internal Hom* of $\mathbf{V}$, such that for any three objects U, V, and W of $\mathbf{V}$, there is a bijection $\omega: \mathbf{V}_o(U \otimes V, W) \to \mathbf{V}_o(U, \mathbf{V}(V, W))$ natural in U, V, and W; is called a *closed* category. We refer the reader to [1] and [11] for a more detailed discussion of these concepts.

Ab is a closed category, with tensor product the usual tensor product of Abelian groups, with unit the group of integers, and with internal Hom given by, for all Abelian groups A and B, $\mathbf{Ab}(A, B) = \mathrm{Hom}(A, B)$ with Abelian group structure the usual one of "pointwise" addition. **Ens** is also a closed category, with tensor product the Cartesian product of sets, with unit any set with exactly one element, and with internal Hom given by $\mathbf{Ens}(X, Y) = Y^X$ (the set of all functions from X to Y) for any two sets X and Y. We refer the reader to [1] for more examples.

(1.2) Definition. Let $\mathbf{V}$ be a closed category. A $\mathbf{V}$-*category* $\mathbf{A}$ consists of (i) a class of objects $|\mathbf{A}|$; (ii) for any two objects A and B of $\mathbf{A}$, an object $\mathbf{A}(A, B)$ of $\mathbf{V}$; (iii) for any three objects A, B, and C of $\mathbf{A}$, a morphism $c: \mathbf{A}(B, C) \otimes \mathbf{A}(A, B) \to \mathbf{A}(A, C)$ of $\mathbf{V}_o$, called composition, which is associative, i.e., the two morphisms from $(\mathbf{A}(C, D) \otimes \mathbf{A}(B, C)) \otimes \mathbf{A}(A, B)$ to $\mathbf{A}(A, D)$, $c(\mathbf{A}(C. D) \otimes c)a$ and $c(c \otimes \mathbf{A}(A, B))$, are equal for any four objects A, B, C and D of $\mathbf{A}$; (iv) for each object A of $\mathbf{A}$, a morphism $j_A: I \to \mathbf{A}(A, A)$, called the identity of A, with the property that for every object B in $\mathbf{A}$, $c(j_A \otimes \mathbf{A}(B, A)) = r$, and $c(\mathbf{A}(A, B) \otimes j_A) = l$.

We note that associated to every $\mathbf{V}$-category $\mathbf{A}$ is an ordinary category $\mathbf{A}_0$, called the *underlying category* of $\mathbf{A}$, whose objects are those of $\mathbf{A}$ and, for any two objects A and B of $\mathbf{A}_0$, $\mathbf{A}_0(A, B) = \mathbf{V}_0(I, \mathbf{A}(A, B))$.

We mention the following operations on $\mathbf{V}$-categories:

(i) to a $\mathbf{V}$-category $\mathbf{A}$ is associated its $\mathbf{V}$-*dual category* $\mathbf{A}^o$, the $\mathbf{V}$-category whose objects are those of $\mathbf{A}$ and, for every $A, B \in |\mathbf{A}|$, $\mathbf{A}^o(A, B) = \mathbf{A}(B, A)$ (note that $(\mathbf{A}^o)^o = \mathbf{A}$);

(ii) to two $\mathbf{V}$-categories $\mathbf{A}$ and $\mathbf{B}$ is associated their $\mathbf{V}$-*product category* $\mathbf{A} \otimes \mathbf{B}$, the $\mathbf{V}$-category whose objects are ordered pairs (A, B) with $A \in |\mathbf{A}|$ and $B \in |\mathbf{B}|$, and for every $A, A' \in |\mathbf{A}|$ and $B, B' \in |\mathbf{B}|$,

$$\mathbf{A} \otimes \mathbf{B}((A, B), (A', B')) = \mathbf{A}(A, A') \otimes \mathbf{B}(B, B').$$

(1.3) DEFINITION. Let $\mathbf{A}$ and $\mathbf{B}$ be $\mathbf{V}$-categories. A $\mathbf{V}$-*functor* $F: \mathbf{A} \to \mathbf{B}$ consists of (i) a function $F: |\mathbf{A}| \to |\mathbf{B}|$ assigning to each object A of $\mathbf{A}$ an object FA of $\mathbf{B}$; (ii) for each pair of objects A and B of $\mathbf{A}$, a morphism $F_{AB}: \mathbf{A}(A, B) \to \mathbf{B}(FA, FB)$ of $\mathbf{V}_0$ such that for any three objects A, B, and C of $\mathbf{A}$, $c(F_{BC} \otimes F_{AB}) = F_{AC}c$, and for any object A of $\mathbf{A}$, $F_{AA}j_A = j_{FA}$.

We note that (i) given two $\mathbf{V}$-categories $\mathbf{A}$ and $\mathbf{B}$, we may define the *contravariant* $\mathbf{V}$-functors from $\mathbf{A}$ to $\mathbf{B}$ to be the $\mathbf{V}$-functors from $\mathbf{A}^o$ to $\mathbf{B}$; and (ii) given $\mathbf{V}$-categories $\mathbf{A}$, $\mathbf{A}'$, and $\mathbf{B}$, we may define the $\mathbf{V}$-*bifunctors* from $\mathbf{A}$ and $\mathbf{A}'$ to $\mathbf{B}$ to be the $\mathbf{V}$-functors from $\mathbf{A} \otimes \mathbf{A}'$ to $\mathbf{B}$.

Given any $\mathbf{V}$-category $\mathbf{A}$ and any morphism $\beta: B \to B'$ of $\mathbf{A}_0$ (that is, a morphism $\beta: I \to A(B, B')$ of $\mathbf{V}_0$), then for any object A of $\mathbf{A}$, there is a morphism $\mathbf{A}(A, \beta): \mathbf{A}(A, B) \to \mathbf{A}(A, B')$ in $\mathbf{V}_0$ defined by $\mathbf{A}(A, \beta) = c(\beta \otimes \mathbf{A}(A, B))l^{-1}$. Similarly, one defines, for every object C of $\mathbf{A}$, a morphism $\mathbf{A}(\beta, C): \mathbf{A}(B', C) \to \mathbf{A}(B, C)$.

(1.4) DEFINITION. Let $F, G: \mathbf{A} \to \mathbf{B}$ be two $\mathbf{V}$-functors. A $\mathbf{V}$-*natural transformation* $\varphi: F \Rightarrow G$ is a collection $\{\varphi_A \in \mathbf{B}_0(FA, GA) \mid A \in |\mathbf{A}|\}$ such that, for any two objects A and B of $\mathbf{A}$, $\mathbf{B}(FA, \varphi_B)F_{AB} = \mathbf{B}(\varphi_A, GB)G_{AB}$.

For any object V of a closed category $\mathbf{V}$, we observe that the natural bijections ω of (v) of (1.1) imply that the functor $- \otimes V: \mathbf{V}_0 \to \mathbf{V}_0$ is adjoint to the functor $\mathbf{V}(V, -): \mathbf{V}_0 \to \mathbf{V}_0$ for all $V \in |\mathbf{V}|$. Consequently, the internal Hom of $\mathbf{V}$ makes $\mathbf{V}$ into a $\mathbf{V}$-category.

Furthermore, let $\mathbf{A}$ be a $\mathbf{V}$-category and let $A \in |\mathbf{A}|$. Then we may define a **V**-functor $\mathbf{A}(A, -): \mathbf{A} \to \mathbf{V}$ by letting $\mathbf{A}(A, -)(B) = \mathbf{A}(A, B)$ for all objects B of **A** and, for any pair of objects B and C of **A**, letting

$$(\mathbf{A}(A, -))_{BC}: \mathbf{A}(B, C) \to \mathbf{V}(\mathbf{A}(A, B), \mathbf{A}(A, C))$$

be the morphism corresponding to composition $c: \mathbf{A}(B, C) \otimes \mathbf{A}(A, B) \to \mathbf{A}(A, C)$ under ω. We shall say that a **V**-functor $G: \mathbf{A} \to \mathbf{V}$ is **V**-*representable* and is **V**-*represented* by an object $A \in |\mathbf{A}|$ if there is a **V**-natural isomorphism $G \cong \mathbf{A}(A, -)$. Similarly, we have the *contravariant* **V**-*representable* **V**-*functors*.

For the closed category of sets **Ens**, **Ens**-categories, **Ens**-functors, and **Ens**-natural transformations are just the ordinary categories, functors, and natural transformations. For the closed category of Abelian groups **Ab**, **Ab**-categories are the preadditive categories and **Ab**-functors are the additive functors (**Ab**-natural transformations and natural transformations of additive functors are the same).

We now wish to define the notion of a **V**-cocomplete **V**-category and **V**-cocontinuous **V**-functor. Dubuc [4] has shown that a straightforward generalization of the notion of a complete category is not desirable, and that the notions of "**V**-cocomplete" and **V**-cocontinuous" should incorporate "tensors" and "coends" as well as "colimits." We begin by defining these concepts.

(1.5) DEFINITION. Let **A** be a **V**-category, A be an object of **A**, and V an object of **V**. The *tensor* of V with A is an object $V \otimes_{\mathbf{A}} A$ of A which **V**-represents the **V**-functor $\mathbf{V}(V, \mathbf{A}(A, -)): \mathbf{A} \to \mathbf{V}$. **A** is said to be *tensored* if for every $A \in |\mathbf{A}|$ and $V \in |\mathbf{V}|$, $V \otimes_{\mathbf{A}} A$ exists in **A**.

Dually, the *cotensor* of V and A is an object $\bar{\mathbf{A}}(V, A)$ which **V**-represents the **V**-contravariant functor $\mathbf{V}(V, \mathbf{A}(-, A)): \mathbf{A}^o \to \mathbf{V}$, and **A** is said to be *cotensored* if, for every $A \in |\mathbf{A}|$ and $V \in |\mathbf{V}|$, $\bar{\mathbf{A}}(V, A)$ exists.

We note that **V** is always tensored and cotensored, with $\otimes_{\mathbf{V}} = \otimes$ and $\bar{\mathbf{V}}(-, -) = \mathbf{V}(-, -)$. If **A** is a cocomplete **Ab**-category, then **A** is tensored, and for all $A \in |\mathbf{A}|$, $- \otimes_{\mathbf{A}} A: \mathbf{Ab} \to \mathbf{A}$ is the unique cocontinuous functor for which $Z \otimes_{\mathbf{A}} A \cong A$ (Z being the group of integers). If **X** is a complete **Ens**-category then **X** is cotensored, and for all $A \in |\mathbf{X}|$ and $S \in |\mathbf{Ens}|$, $\bar{\mathbf{X}}(S, A) = \Pi_S A$, the product of S-copies of A.

(1.6) DEFINITION. Let **C** be a **V**-category and let $T: \mathbf{C}^o \otimes \mathbf{C} \to \mathbf{V}$ be a **V**-functor. The *coend* of T is an object of **V**, denoted by $\int^C T(C, C)$, and a **V**-natural family of morphisms $\{\lambda_C: T(C, C) \to \int^C T(C, C)\}_{C \in |\mathbf{C}|}$ of **V** satisfying the universal property that given any object V of **V** and a **V**-natural family $\{a_C: T(C, C) \to V\}_{C \in |\mathbf{C}|}$ of **V**, there is a unique morphism $a: \int^C T(C, C) \to V$ in **V** satisfying the equation $a\lambda_C = a_C$ for each $C \in |\mathbf{C}|$. The *end* of T, $\int_C T(C, C)$, is defined dually.

Suppose $\mathbf{A}$ is a $\mathbf{V}$-category and $S: \mathbf{C}^o \otimes \mathbf{C} \to \mathbf{A}$ is a $\mathbf{V}$-functor. Then the *coend* of S is an object $\int^C S(C, C)$ of $\mathbf{A}$, together with a $\mathbf{V}$-natural family of morphism $\{\lambda_C: S(C, C) \to \int^C S(C, C)\}_{C \in |\mathbf{C}|}$ of $\mathbf{A}$, such that for all $A \in |\mathbf{A}|$, the object $\mathbf{A}(\int^C S(C, C), A)$, together with the family $\{\mathbf{A}(\lambda_C, A)\}_{C \in |C|}$, is the end of the $\mathbf{V}$-functor $\mathbf{A}(S(-, -), A): \mathbf{C} \otimes \mathbf{C}^o \to \mathbf{V}$.

Let $\mathbf{C}$ be a small category and $F, G: \mathbf{C} \to \mathbf{Ens}$ be functors. Define a functor $T: \mathbf{C}^o \times \mathbf{C} \to \mathbf{Ens}$ by $T(C, C') = \mathrm{Mor}(FC, GC')$, the set of maps from FC to GC', for all objects C and C' of $\mathbf{C}$. Then the end of T, $\int_C T(C, C)$, is the set of natural transformations from F to G.

Let $\mathbf{C}$ be a small $\mathbf{Ab}$-category, $F: \mathbf{C}^o \to \mathbf{Ab}$ and $G: \mathbf{C} \to \mathbf{Ab}$ be additive functors. Define $T: \mathbf{C}^o \otimes \mathbf{C} \to \mathbf{Ab}$ by $T(C, C') = FC \otimes GC'$ for all C, $C' \in |\mathbf{C}|$. Then the coend of T, $\int^C T(C, C')$ exists, and is the tensor product of functors $F \otimes_C G$ as defined in [8]. In particular, if R is a ring and $\mathbf{C} = \mathbf{R}$, then F corresponds to a right R-module N, G corresponds to a left R-module M, and $\int^C T(C, C) \cong N \otimes_R M$.

The notions of end and coend were introduced in [3] and further discussed in [11] (the integral sign notation is due to Yoneda). They will be of fundamental importance for this paper for, as the examples indicate, they will allow us to define "$\mathbf{V}$-functor categories" at the end of this section and the "tensor product of $\mathbf{V}$-functors" in Section 2.

Recall that, if $F: \mathbf{\Gamma} \to \mathbf{A}$ is a functor ($\mathbf{\Gamma}$ and $\mathbf{A}$ ordinary categories), a *cone* of F is a family of morphisms $\{a_\lambda: A \to F(\lambda)\}_{\lambda \in |\mathbf{\Gamma}|}$ of $\mathbf{A}$ (for some object A of $\mathbf{A}$) such that for every morphism $j: \lambda \to \lambda'$ of $\mathbf{\Gamma}$, $F(j) \cdot a_\lambda = a_{\lambda'}$, and that the *limit* of F is an object $\lim F$ of $\mathbf{A}$ together with a cone $\{i_\lambda: \lim F \to F(\lambda)\}_{\lambda \in |\mathbf{\Gamma}|}$ satisfying the universal property that, given any object A of $\mathbf{A}$ and given any cone $\{a_\lambda: A \to F(\lambda)\}_{\lambda \in |\mathbf{\Gamma}|}$, there is a unique morphism $a: A \to \lim F$ in $\mathbf{A}$ for which $i_\lambda \cdot a = a_\lambda$ for all $\lambda \in |\mathbf{\Gamma}|$. Dually, one has "cocones" and "colimits."

(1.7) Definition. Let $\mathbf{A}$ be a $\mathbf{V}$-category, $\mathbf{\Gamma}$ a category, and $F: \mathbf{\Gamma} \to \mathbf{A}$ a functor. Then the $\mathbf{V}$-*colimit* of F is an object of $\mathbf{A}$, denoted by $\mathbf{V}$-colim F, together with a cocone $\{p_\lambda: F(\lambda) \to V\text{-colim} F\}_{\lambda \in |\mathbf{\Gamma}|}$, such that for every $A \in |\mathbf{A}|$, the cone $\{A(p_\lambda, A)\}_{\lambda \in |\mathbf{\Gamma}|}$ is a limit in $\mathbf{V}$. Dually, one defines $\mathbf{V}$-lim F.

We note that every $\mathbf{V}$-colimit in $\mathbf{A}$ is a colimit in $\mathbf{A}$ and that every colimit in $\mathbf{V}$ is a $\mathbf{V}$-colimit. If $\mathbf{A}$ is a tensored $\mathbf{V}$-category and $\mathbf{C}$ is any $\mathbf{V}$-category, the coend of a $\mathbf{V}$-functor $S: \mathbf{C}^o \otimes \mathbf{C} \to \mathbf{A}$ is a $\mathbf{V}$-colimit [4, p. 38].

(1.8) Definition. A $\mathbf{V}$-category $\mathbf{A}$ is called $\mathbf{V}$-*cocomplete* if it is tensored and, for all small categories $\mathbf{\Gamma}$ and for all functors $F: \mathbf{\Gamma} \to \mathbf{A}$, $\mathbf{V}$-colim F exists. Dually, we define $\mathbf{V}$-*complete*.

As examples, an $\mathbf{Ens}$-complete category is an ordinary complete category, while an $\mathbf{Ab}$-cocomplete category is an $\mathbf{Ab}$-category with coproducts and cokernels.

We note that if **V** is cocomplete, then **V** is **V**-cocomplete.

(1.9) DEFINITION. Let **A** and **B** be **V**-categories and let $f: \mathbf{A} \to \mathbf{B}$ be a **V**-functor. Then f is called **V**-*cocontinuous* if f preserves tensors, coends, and **V**-colimits. Dually, we define **V**-continuous.

We state below the duals of [4, Propositions III.1.1, 2, and 5, pp. 112–114].

(1.10) PROPOSITION. *For* **V**-*categories* **A** *and* **B**: (i) *for every* $A \in |\mathbf{A}|$, $\mathbf{A}(-, A): \mathbf{A}^o \to \mathbf{V}$ *is* **V**-*continuous*;

(ii) *a* **V**-*functor* $f: \mathbf{A} \to \mathbf{B}$ *is* **V**-*cocontinuous iff for every* $B \in |\mathbf{B}|$ *the* **V**-*functor* $\mathbf{B}(f-, B): \mathbf{A}^o \to \mathbf{V}$ *is* **V**-*continuous*;

(iii) *if* **B** *is tensored and* $V \in |\mathbf{V}|$, *then the* **V**-*functor* $V \otimes_{\mathbf{B}} -: \mathbf{B} \to \mathbf{B}$ *is* **V**-*cocontinuous.* ∎

We sketch briefly the notions of **V**-adjoint functors and **V**-triples, which are straightforward generalizations of the ordinary notions discussed in the Introduction.

(1.11) DEFINITION. Let **X** and **A** be **V**-categories. Then a **V**-*adjoint pair* $(U, F): \mathbf{A} \to \mathbf{X}$ is a pair of **V**-functors $U: \mathbf{A} \to \mathbf{X}$ and $F: \mathbf{X} \to \mathbf{A}$ for which F is **V**-adjoint to U, i.e., for every $X \in |\mathbf{X}|$ and for every $A \in |\mathbf{A}|$ there is an isomorphism in **V**, $\theta_{(X,A)}: \mathbf{A}(FX, A) \to \mathbf{X}(X, UA)$, which is **V**-natural in X and A.

As in the Introduction, one obtains **V**-natural transformations $\eta: 1_{\mathbf{X}} \Rightarrow UF$ and $\epsilon: FU \Rightarrow 1_{\mathbf{A}}$, called the *unit* and *counit* of (U, F), respectively. One also has the following

(1.12) PROPOSITION. [4, Proposition III.1.3 (and its dual), p. 114]. *If* $(U, F): \mathbf{A} \to \mathbf{X}$ *is a* **V**-*adjoint pair, then* U *is* **V**-*continuous and* F *is* **V**-*cocontinuous.* ∎

(1.13) DEFINITION. Let **X** be a **V**-category. Then a **V**-*triple* on **X** is a triple $\mathbf{T} = (T, \eta, \mu)$ with $T: \mathbf{X} \to \mathbf{X}$ a **V**-functor and $\eta: 1_{\mathbf{X}} \Rightarrow T$ and $\mu: TT \Rightarrow T$ **V**-natural transformations such that $\mu \cdot \eta T = \mu \cdot T\eta = 1_T$ and $\mu \cdot T\mu = \mu \cdot \mu T$. Dually, one defines a **V**-*cotriple* on $\mathbf{X}\,\mathbf{G} = (G, \epsilon, \delta)$.

As in the ordinary case, every **V**-adjoint pair $(U, F): \mathbf{A} \to \mathbf{X}$ induces a **V**-triple **T** on **X** by letting $\mathbf{T} = (UF, \eta, U\epsilon F)$, where η and ϵ are the unit and counit of (U, F), respectively.

Given a **V**-triple **T** on a **V**-category **X**, one may construct a **V**-category of **T**-algebras $\mathbf{X}^{\mathbf{T}}$ if one assumes **V** has equalizers. The objects of $\mathbf{X}^{\mathbf{T}}$ are **T**-algebras. $(A, a: TA \to A)$ as defined in the Introduction. If (A, a) and (B, b) are **T**-algebras, one lets $\mathbf{X}^{\mathbf{T}}((A, a), (B, b))$ be the object of **V** which is the equalizer of the two morphisms of **V** from $\mathbf{X}(A, B)$ to $\mathbf{X}(TA, B)$, $\mathbf{X}(a, B)$ and $\mathbf{X}(TA, b)T_{AB}$. One obtains a **V**-adjoint pair $(U^{\mathbf{T}}, F^{\mathbf{T}})$: $\mathbf{X}^{\mathbf{T}} \to \mathbf{X}$ by letting $U^{\mathbf{T}}(A, a) = A$ for every **T**-algebra (A, a) and by letting $F^{\mathbf{T}}(X) = (TX, \mu_X)$ for every $X \in |\mathbf{X}|$. Furthermore, the **V**-triple on **X** induced by $(U^{\mathbf{T}}, F^{\mathbf{T}})$ is **T**.

Given a **V**-adjoint pair (U, F): $\mathbf{A} \to \mathbf{X}$, let **T** be the triple on **X** induced by (U, F), and (assuming **V** has equalizers) let $(U^{\mathbf{T}}, F^{\mathbf{T}})$ be the **V**-adjoint pair defined above. Then there is a canonical **V**-functor Φ: $\mathbf{A} \to \mathbf{X}^{\mathbf{T}}$, called the *comparison functor*, for which $U^{\mathbf{T}} \cdot \Phi = U$ (Φ is given on objects by $\Phi(A) = (UA, U\epsilon_A)$ for all $A \in |\mathbf{A}|$).

(1.14) Definition. Suppose **V** has equalizers and suppose (U, F): $\mathbf{A} \to \mathbf{X}$ is a **V**-adjoint pair. Then (U, F) is said to be **V**-*tripleable* if the comparison functor Φ is a **V**-equivalence of **V**-categories.

As is done in the Introduction, one dualizes this definition to define **V**-*cotripleable.*

We now turn our attention to the subject of Kan extensions.

(1.15) Definition. Suppose S: $\mathbf{C} \to \mathbf{A}$ and T: $\mathbf{C} \to \mathbf{B}$ are **V**-functors. Then the **V**-*left Kan extension of T along S* is a **V**-functor $\mathrm{Lan}_S\, T$: $\mathbf{A} \to \mathbf{B}$, together with a **V**-natural transformation η: $T \Rightarrow (\mathrm{Lan}_S\, T) \cdot S$, satisfying the universal property that given any **V**-functor F: $\mathbf{A} \to \mathbf{B}$ and any **V**-natural transformation φ: $T \Rightarrow FS$, there is a unique **V**-natural transformation $\bar{\varphi}$: $\mathrm{Lan}_S\, T \Rightarrow F$ for which $\bar{\varphi}S \cdot \eta = \varphi$. Dually, one defines the **V**-*right Kan extension of T along S* $\mathrm{Ran}_S\, T$: $\mathbf{A} \to \mathbf{B}$.

The following is a special case of the dual of [4, Theorem I.4.2, p. 50].

(1.16) Proposition. *Let* **B** *be a* **V**-*cocomplete* **V**-*category,* **C** *be a small* **V**-*category, and* S: $\mathbf{C} \to \mathbf{A}$ *and* T: $\mathbf{C} \to \mathbf{B}$ *be* **V**-*functors. Then* $\mathrm{Lan}_S\, T$: $\mathbf{A} \to \mathbf{B}$ *exists and, for every* $A \in |\mathbf{A}|$, *one has*

$$\mathrm{Lan}_S\, T(A) = \int^{C} \mathbf{A}(SC, A) \otimes_{\mathbf{B}} TC.$$

Dually, if **B** *is* **V**-*complete and* **C** *is a small* **V**-*category,* $\mathrm{Ran}_S\, T$: $\mathbf{A} \to \mathbf{B}$ *exists and, for every* $A \in |\mathbf{A}|$, *one has*

$$\mathrm{Ran}_S\, T(A) = \int_{C} \bar{\mathbf{B}}(\mathbf{A}(A, SC), TC). \quad \blacksquare$$

Suppose S: $\mathbf{C} \to \mathbf{A}$ is a **V**-functor, with **C** a small **V**-category and **A** a **V**-cocomplete **V**-category. Then from (1.16) $\mathrm{Lan}_S\, S$ exists. Furthermore, the universal property of $\mathrm{Lan}_S\, S$ allows us to construct a **V**-cotriple structure on

$\text{Lan}_S\, S$, $\mathbf{G}_S = (\text{Lan}_S\, S, \epsilon_S, \delta_S)$, called the *density cotriple* of S (see [4, pp. 67–71] where Dubuc considers the dual situation and constructs a triple structure on $\text{Ran}_S\, S$, called the *codensity triple* of S).

(1.17) DEFINITION. Let $\mathbf{A}$ be a $\mathbf{V}$-cocomplete $\mathbf{V}$-category, let $S: \mathbf{C} \to \mathbf{A}$ be a $\mathbf{V}$-functor with $\mathbf{C}$ a small $\mathbf{V}$-category, and let $\mathbf{G}_S = (\text{Lan}_S\, S, \epsilon_S, \delta_S)$ be its density cotriple. Then we say S $\mathbf{V}$-*generates* if, for every $A \in |\mathbf{A}|$, $(\epsilon_S)_A$: $\text{Lan}_S S(A) \to A$ is a $\mathbf{V}$-epimorphism in $\mathbf{A}$. We say that S is $\mathbf{V}$-*dense* if ϵ_S: $\text{Lan}_S\, S \Rightarrow 1_{\mathbf{A}}$ is a $\mathbf{V}$-natural equivalence.

Dually, for $\mathbf{A}$ $\mathbf{V}$-complete, we define the notions of $\mathbf{V}$-*cogenerates* and $\mathbf{V}$-*codense* in terms of the unit of the codensity triple of S.

This definition is less general than that given by Dubuc in [4, Definitions II.3.1 and 2, p. 82], as we assume our $\mathbf{V}$-category is $\mathbf{V}$-cocomplete (or $\mathbf{V}$-complete). However, our definition is adequate for the purposes of this paper and avoids some technicalities.

Let R be a ring and let ${}_R\mathbf{M}$ be the category of left R-modules. Construct the $\mathbf{Ab}$-category $\mathbf{R}$ associated to R and let S: $\mathbf{R}^o \to {}_R\mathbf{M}$ be the (additive) functor that assigns to the one object of $\mathbf{R}^o$ the ring R considered as a left R-module over itself and assigns to every "morphism" a of $\mathbf{R}^o$ the left R-homomorphism $S(a)$: $R \to R$ given by $S(a)(x) = xa$ for every $x \in R$. Then S is $\mathbf{Ab}$-dense, since this statement is equivalent to the fact that every left R-module is the cokernel of a homomorphism between free R-modules.

In the manner of the Introduction this statement generalizes to functor categories and is given in 1.20.

We now develop the notion of $\mathbf{V}$-functor categories following [4, Chapter IV]. For this purpose, we shall assume from now on that $\mathbf{V}$ is a complete category.

Let $\mathbf{C}$ be a small $\mathbf{V}$-category and let $\mathbf{D}$ be any $\mathbf{V}$-category. We define a $\mathbf{V}$-category $\mathbf{D}^{\mathbf{C}}$ by letting the objects of $\mathbf{D}^{\mathbf{C}}$ be the $\mathbf{V}$-functors $F: \mathbf{C} \to \mathbf{D}$ and, for all $\mathbf{V}$-functors $F, G: \mathbf{C} \to \mathbf{D}$, letting $\mathbf{D}^{\mathbf{C}}(F, G) = \int_C \mathbf{D}(FC, GC)$, the end of the $\mathbf{V}$-functor $\mathbf{D}(F-, G-): \mathbf{C}^o \otimes \mathbf{C} \to \mathbf{V}$ (composition and identities of $\mathbf{D}^{\mathbf{C}}$ are naturally defined using the universal properties of ends). The morphisms of the underlying category of $\mathbf{D}^{\mathbf{C}}$ are the $\mathbf{V}$-natural transformations.

From the definition of end (1.6) we have, for every $F, G \in |\mathbf{D}^{\mathbf{C}}|$, a $\mathbf{V}$-natural family $\{(e_C)_{FG}: \mathbf{D}^{\mathbf{C}}(F, G) \to \mathbf{D}(FC, GC)\}_{C \in |\mathbf{C}|}$ of morphisms of $\mathbf{V}$, called the *evaluation morphisms*. Using these we may define, for every $C \in |\mathbf{C}|$, a $\mathbf{V}$-functor $e_C: \mathbf{D}^{\mathbf{C}} \to \mathbf{D}$ called the *evaluation functor* at C by letting $e_C(F) = FC$ for every $F \in |\mathbf{D}^{\mathbf{C}}|$.

The fact that limits in an ordinary functor category can be constructed "point-wise" generalizes to the statement that the family of evaluation functors $\{e_C: \mathbf{D}^{\mathbf{C}} \to \mathbf{D}\}$ "creates" $\mathbf{V}$-limits. More generally, one has

(1.18) PROPOSITION. [4, Proposition IV.1.4, p. 153]. *For all* **V**-*categories* **D** *and for all small* **V**-*categories* **C**, *the family of evaluation functors* $\{e_C: \mathbf{D}^{\mathbf{C}} \to \mathbf{D}\}_{C \in |\mathbf{C}|}$ *creates ends, coends, tensors, cotensors,* **V**-*limits and* **V**-*colimits. In particular, if* **D** *is* **V**-*complete* (**V**-*cocomplete*), *then* $\mathbf{D}^{\mathbf{C}}$ *is also, and for every* $C \in |\mathbf{C}|$, $e_C: \mathbf{D}^{\mathbf{C}} \to \mathbf{D}$ *is* **V**-*continuous* (**V**-*cocontinuous*). ▮

Now we let $\mathbf{D} = \mathbf{V}$ and obtain **V**-versions of the Yoneda lemma and properties of the Yoneda embeddings.

(1.19) THEOREM (**V**-Yoneda lemma, [4, Proposition IV.1.1, p. 152]. *Let* **C** *be a small* **V**-*category*, $C \in |\mathbf{C}|$, *and* $F \in |\mathbf{V}^{\mathbf{C}}|$. *Then*

$$\mathbf{V}^{\mathbf{C}}(\mathbf{C}(C, -), F) \cong FC. \quad ▮$$

For **C** a small **V**-category, define the *right Yoneda functor* $R_{\mathbf{C}} = R: \mathbf{C} \to \mathbf{V}^{\mathbf{C}^o}$ to be the **V**-functor defined by $R(C) = \mathbf{C}(-, C)$ and for all $D, D' \in |\mathbf{C}|$, $R_{DD'}: \mathbf{C}(D, D') \to \mathbf{V}^{\mathbf{C}^o}(RD, RD')$ is the unique morphism in V for which $e_C R_{DD'} = (\mathbf{C}(C, -))_{DD'}$ for every $C \in |\mathbf{C}|$. We define the *left Yoneda functor* $L_{\mathbf{C}}: \mathbf{C} \to (\mathbf{V}^{\mathbf{C}})^o$ by letting $L_{\mathbf{C}}{}^o: \mathbf{C}^o \to \mathbf{V}^{\mathbf{C}}$ be $R_{\mathbf{C}^o}$, the right Yoneda functor on $\mathbf{C}^o$ (for every $C \in |\mathbf{C}|$, $L_{\mathbf{C}}(C) = \mathbf{C}(C, -)$). It follows from 1.19 that for all $D, D' \in |\mathbf{C}|$, $R_{DD'}$ is an isomorphism in **V**, so that the right and left Yoneda functors are embeddings.

(1.20) THEOREM ([4, Proposition IV.1.2, p. 152], although we assume **V** is cocomplete so that we may use our (1.17)). *Suppose* **V** *is cocomplete. Then for all small* **V**-*categories* **C**, *the right Yoneda functor* $R_{\mathbf{C}}$ *is* **V**-*dense and the left Yoneda functor* $L_{\mathbf{C}}$ *is* **V**-*codense.* ▮

We remark that if **C** is a small **Ab**-category, then $R_{\mathbf{C}^o}: \mathbf{C}^o \to [\mathbf{C}, \mathbf{Ab}]$ being **Ab**-dense is equivalent to every additive functor $F: \mathbf{C} \to \mathbf{Ab}$ being the cokernel of a natural transformation between coproducts of representable functors. As the Yoneda lemma implies that the representable functors of $[\mathbf{C}, \mathbf{Ab}]$ are projective objects, this implies the "homological" statement that $[\mathbf{C}, \mathbf{Ab}]$ has enough projectives.

2. TENSOR PRODUCTS OF FUNCTORS AND ADJOINT PAIRS

In this section we define the tensor product of functors and prove our **V**-functor category version of Watts' theorem.

Throughout the rest of this paper, we shall assume that **V** is a closed category which is bicomplete, i.e., both complete and cocomplete (as are the closed categories **Ens** and **Ab**), that **D** is a **V**-bicomplete **V**-category, and that **C** is a small **V**-category.

Then for $f: \mathbf{C}^o \to \mathbf{V}$ and $g: \mathbf{C} \to \mathbf{D}$, let $f - \otimes_{\mathbf{D}} g -: \mathbf{C}^o \otimes \mathbf{C} \to \mathbf{D}$ denote the $\mathbf{V}$-functor defined by $(f - \otimes_{\mathbf{D}} g -)(C', C) = fC' \otimes_{\mathbf{D}} gC$ for all C', $C \in |\mathbf{C}|$. We define the tensor product of f and g by $f \,\overline{\otimes}_{\mathbf{C}}\, g = \int^C fC \otimes_{\mathbf{D}} gC$ (this generalizes the tensor product of modules, as discussed previously in the examples after (1.6)). This tensor product can be made into a functor, so that we have the following definition.

(2.1) Definition. We define the *tensor product functor*,

$$-\overline{\otimes}_{\mathbf{C}} -: \mathbf{V}^{\mathbf{C}^o} \otimes \mathbf{D}^{\mathbf{C}} \to \mathbf{D},$$

to be the $\mathbf{V}$-functor given by

$$f \,\overline{\otimes}_{\mathbf{C}}\, g = \int^C fC \otimes_{\mathbf{D}} gC,$$

for every $f \in |\mathbf{V}^{\mathbf{C}^o}|$ and $g \in |\mathbf{D}^{\mathbf{C}}|$.

Note that if $\mathbf{D} = \mathbf{V}$, then $f \,\overline{\otimes}_{\mathbf{C}}\, g \cong g \,\overline{\otimes}_{\mathbf{C}^o}\, f$, as $\otimes_{\mathbf{V}} = \otimes$ is symmetric. If g is in $\mathbf{D}^{\mathbf{C}}$ and $R_{\mathbf{C}}: \mathbf{C} \to \mathbf{V}^{\mathbf{C}^o}$ is the right Yoneda embedding, then

$$-\overline{\otimes}_{\mathbf{C}}\, g = \mathrm{Lan}_{R_{\mathbf{C}}}(g) : \mathbf{V}^{\mathbf{C}^o} \to \mathbf{D}$$

(the "realization" functor of g in [4, p. 158]) and $(-\overline{\otimes}_{\mathbf{C}}\, g)\, R_{\mathbf{C}} = g$ [4, pp. 52–59]. The functor $-\overline{\otimes}_{\mathbf{C}}\, g$ has a $\mathbf{V}$-right adjoint $\mathbf{D}(g -, -): \mathbf{D} \to \mathbf{V}^{\mathbf{C}^o}$ (called the "singular" functor of g in [4, p. 158]) defined on objects by $\mathbf{D}(g -, -)(D)(C) = \mathbf{D}(gC, D)$ for every object C of $\mathbf{C}^o$ and D of $\mathbf{D}$.

We recall the following proposition.

(2.2) Proposition. (Day and Kelly [3, (6.3), p. 188].) *For g in $\mathbf{D}^{\mathbf{C}}$, $-\overline{\otimes}_{\mathbf{C}} g: \mathbf{V}^{\mathbf{C}^o} \to \mathbf{D}$ is a $\mathbf{V}$-cocontinuous $\mathbf{V}$-functor for which $(-\overline{\otimes}_{\mathbf{C}}\, g)\, R_{\mathbf{C}} = g$. Furthermore, if $G: \mathbf{V}^{\mathbf{C}^o} \to \mathbf{D}$ is a $\mathbf{V}$-cocontinuous $\mathbf{V}$-functor for which $GR_{\mathbf{C}} = g$, there is a unique $\mathbf{V}$-natural equivalence $\Psi: -\overline{\otimes}_{\mathbf{C}}\, g \Rightarrow G$ such that $\Psi R_{\mathbf{C}} = 1_g$.*

Putting this information together, we obtain the following

(2.3) Theorem. *Let $F: \mathbf{V}^{\mathbf{C}^o} \to \mathbf{D}$ be a $\mathbf{V}$-functor. Then there is a $\mathbf{V}$-natural transformation $\epsilon_F: -\overline{\otimes}_{\mathbf{C}} FR_{\mathbf{C}} \Rightarrow F$ satisfying the universal property that if $\alpha: F' \Rightarrow F$ is a $\mathbf{V}$-natural transformation and $F': \mathbf{V}^{\mathbf{C}^o} \to \mathbf{D}$ is $\mathbf{V}$-cocontinuous, then there is a unique $\mathbf{V}$-natural transformation $\bar{\alpha}: F' \Rightarrow -\overline{\otimes}_{\mathbf{C}} FR$ for which $\epsilon_F \cdot \bar{\alpha} = \alpha$. Furthermore, ϵ_F is an equivalence iff F is $\mathbf{V}$-cocontinuous.*

Proof. Let $g: \mathbf{C} \to \mathbf{D}$ be a $\mathbf{V}$-functor. Then $\mathrm{Lan}_{R_{\mathbf{C}}} g = -\otimes_{\mathbf{C}}\, g$. Since $g = (-\overline{\otimes}_{\mathbf{C}}\, g) R_{\mathbf{C}}$, (1.15) in this case gives us that for every $\mathbf{V}$-functor $F: \mathbf{V}^{\mathbf{C}^o} \to \mathbf{D}$ and for every $\mathbf{V}$-natural transformation $\varphi: g \Rightarrow FR_{\mathbf{C}}$, there is a unique $\mathbf{V}$-natural transformation $\bar{\varphi}: -\overline{\otimes}_{\mathbf{C}}\, g \Rightarrow F$ for which $\bar{\varphi} R_{\mathbf{C}} = \varphi$.

For a given $F\colon \mathbf{V}^{\mathbf{C}^o} \to \mathbf{D}$, let $g = FR_{\mathbf{C}}$, let φ be the identity on FR_C, and let ϵ_F be the corresponding $\bar{\varphi}$. We note that $\epsilon_F R_C = 1_{FR_C}$, so that if F is $\mathbf{V}$-cocontinuous, ϵ_F is an equivalence by (2.2).

Suppose $F'\colon \mathbf{V}^{\mathbf{C}^o} \to \mathbf{D}$ is $\mathbf{V}$-cocontinuous and $\alpha\colon \mathbf{F}' \Rightarrow \mathbf{F}$ is a $\mathbf{V}$-natural transformation. One obtains a commutative diagram

$$\begin{array}{ccc} -\overline{\otimes}_{\mathbf{C}} F'R_{\mathbf{C}} & \xRightarrow{\;\epsilon_{F'}\;} & F' \\ {\scriptstyle -\overline{\otimes}_{\mathbf{C}}\alpha R_{\mathbf{C}}}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \alpha} \\ -\overline{\otimes}_{\mathbf{C}} FR_{\mathbf{C}} & \xRightarrow[\;\epsilon_F\;]{} & F \end{array}$$

Since F' is $\mathbf{V}$-cocontinuous, $\epsilon_{F'}$ is an equivalence, hence we may let $\bar{\alpha} = (-\overline{\otimes}_{\mathbf{C}}\, \alpha R_{\mathbf{C}})(\epsilon_{F'})^{-1}$ so that $\epsilon_F \cdot \bar{\alpha} = \alpha$, demonstrating the existence of $\bar{\alpha}$.

For any $\mathbf{V}$-natural transformation $\beta\colon F' \Rightarrow -\overline{\otimes}_{\mathbf{C}} FR_{\mathbf{C}}$ such that $\epsilon_F \cdot \beta = \alpha$, one has $\beta R_{\mathbf{C}} = \alpha R_{\mathbf{C}}$ since $\epsilon_F R_{\mathbf{C}}$ is the identity. It follows from the uniqueness of $-\overline{\otimes}_{\mathbf{C}}\, \alpha R_{\mathbf{C}}$ ($-\overline{\otimes}_{\mathbf{C}}\, \alpha R_{\mathbf{C}}$ is unique such that $(-\overline{\otimes}_{\mathbf{C}}\, \alpha R_{\mathbf{C}}) R_{\mathbf{C}} = \alpha R_{\mathbf{C}}$) that $\beta \cdot \epsilon_{F'} = -\overline{\otimes}_{\mathbf{C}}\, \alpha R_{\mathbf{C}}$. Hence $\beta = \bar{\alpha}$. ▮

Let $\mathbf{C}'$ be another small $\mathbf{V}$-category. Then for any $\mathbf{V}$-functor $g\colon \mathbf{C} \to \mathbf{D}$, let $\mathbf{D}(g\,-, -)_*\colon \mathbf{D}^{\mathbf{C}'} \to (\mathbf{V}^{\mathbf{C}^o})^{\mathbf{C}'} \cong \mathbf{V}^{\mathbf{C}^o \otimes \mathbf{C}'}$ be the functor induced by the singular functor of g, $\mathbf{D}(g\,-, -)\colon \mathbf{D} \to \mathbf{V}^{\mathbf{C}^o}$. Since $-\overline{\otimes}_{\mathbf{C}}\, g$ is $\mathbf{V}$-left adjoint to $\mathbf{D}(g\,-, -)$, we have a proposition:

(2.4) Proposition. *For any* $\mathbf{V}$*-functor* $g\colon \mathbf{C} \to \mathbf{D}$, *the* $\mathbf{V}$*-left adjoint to* $\mathbf{D}(g\,-, -)_*$ *is* $(-\overline{\otimes}_{\mathbf{C}}\, g)_*\colon \mathbf{V}^{\mathbf{C}^o \otimes \mathbf{C}'} \to \mathbf{D}^{\mathbf{C}'}$, *the functor induced by*

$$-\overline{\otimes}_C\, g : \mathbf{V}^{\mathbf{C}^o} \to \mathbf{D}. \quad ▮$$

For any $\mathbf{V}$-functor $u\colon \mathbf{C}^o \otimes \mathbf{C}' \to \mathbf{V}$, we shall denote $(-\overline{\otimes}_{\mathbf{C}}\, g)_*(u)$ by $u \,\overline{\otimes}_{\mathbf{C}}\, g$.

Dualizing 2.1, we have the following definition.

(2.5) Definition. The *symbolic Hom functor* is the $\mathbf{V}$-functor

$$[-, -]_{\mathbf{C}}\colon (\mathbf{V}^{\mathbf{C}})^o \otimes \mathbf{D}^{\mathbf{C}} \to \mathbf{D}$$

defined by $[f, g]_{\mathbf{C}} = \int_C \bar{\mathbf{D}}(fC, gC)$ for every $f \in |\mathbf{V}^{\mathbf{C}}|$ and $g \in |\mathbf{D}^{\mathbf{C}}|$.

We note that for $g\colon \mathbf{C} \to \mathbf{D}$ a $\mathbf{V}$-functor, $[-, g]_{\mathbf{C}}\colon (\mathbf{V}^{\mathbf{C}})^o \to \mathbf{D}$ is the "corealization" functor of [4], and that we have $[-, g]_{\mathbf{C}} = \mathrm{Ran}_{L_{\mathbf{C}}}(g)$, where $L_{\mathbf{C}}\colon \mathbf{C} \to (V^{\mathbf{C}})^o$ is the left Yoneda embedding. Also, if $\mathbf{D} = \mathbf{V}$, $[f, g]_{\mathbf{C}} = \mathbf{V}^{\mathbf{C}}(f, g)$.

Dualizing (2.2) and (2.3) gives us the following proposition.

(2.6) PROPOSITION. *For g in $\mathbf{D}^{\mathbf{C}}$, $[-,g]_{\mathbf{C}}\colon (\mathbf{V}^{\mathbf{C}})^{o} \to \mathbf{D}$ is a* $\mathbf{V}$*-continuous* $\mathbf{V}$*-functor for which $[-,g]_{\mathbf{C}} \cdot L_{\mathbf{C}} = g$. Furthermore, if $G\colon (\mathbf{V}^{\mathbf{C}})^{o} \to \mathbf{D}$ is a* $\mathbf{V}$*-continuous* $\mathbf{V}$*-functor for which $GL_{\mathbf{C}} = g$, there is a unique* $\mathbf{V}$*-natural equivalence $\Psi\colon G \Rightarrow [-,g]_{\mathbf{C}}$ such that $\Psi L_{\mathbf{C}} = 1_g$.* ∎

(2.7) THEOREM. *Let $F\colon (\mathbf{V}^{\mathbf{C}})^{o} \to \mathbf{D}$ be a* $\mathbf{V}$*-functor. Then there is a* $\mathbf{V}$*-natural transformation $\eta_F\colon F \Rightarrow [-,FL_{\mathbf{C}}]_{\mathbf{C}}$ satisfying the universal property that if $\beta\colon F \Rightarrow F'$ is a* $\mathbf{V}$*-natural transformation and $F'\colon (\mathbf{V}^{\mathbf{C}})^{o} \to \mathbf{D}$ is* $\mathbf{V}$*-continuous, then there is a unique* $\mathbf{V}$*-natural transformation $\bar{\beta}\colon [-,FL_{\mathbf{C}}]_{\mathbf{C}} \Rightarrow F'$ for which $\bar{\beta} \cdot \eta_F = \beta$. Furthermore, η_F is an equivalence iff F is* $\mathbf{V}$*-continuous.* ∎

To dualize (2.4), let $f \in |\mathbf{D}^{\mathbf{C}'}|$ and consider the $\mathbf{V}$-functor

$$\mathbf{D}(-,f-)\colon \mathbf{D} \to (\mathbf{V}^{\mathbf{C}'})^{o},$$

the "cosingular" functor of f of [4], defined on objects by $\mathbf{D}(-,f-)(D)(C') = \mathbf{D}(D,fC')$. $\mathbf{D}(-,f-)$ is $\mathbf{V}$-left adjoint to $[-,f]_{\mathbf{C}'}$, so that the induced functor,

$$\mathbf{D}(-,f-)_* : \mathbf{D}^{\mathbf{C}} \to ((\mathbf{V}^{\mathbf{C}'})^{o})^{\mathbf{C}} \cong (\mathbf{V}^{\mathbf{C}^{o}\otimes \mathbf{C}'})^{o},$$

is $\mathbf{V}$-left adjoint to $([-,f]_{\mathbf{C}'})_*$. For any $\mathbf{V}$-functor $u\colon \mathbf{C}^{o} \otimes \mathbf{C}' \to \mathbf{V}$, we shall denote $([-,f]_{\mathbf{C}'})_*(u)$ by $[u,f]_{\mathbf{C}'}$.

One observes that for all $f \in |\mathbf{D}^{\mathbf{C}'}|$ and $g \in |\mathbf{D}^{\mathbf{C}}|$, $\mathbf{D}(-,f)_*(g) = \mathbf{D}(g-,-)_*(f)$ as objects in $\mathbf{V}^{\mathbf{C}^{o}\otimes\mathbf{C}'}$. Thus, (2.4) and its dual give us this theorem:

(2.8) THEOREM. *Let* $\mathbf{C}$ *and* $\mathbf{C}'$ *be small* $\mathbf{V}$*-categories and $u\colon \mathbf{C}^{o} \otimes \mathbf{C}' \to \mathbf{V}$ be a* $\mathbf{V}$*-bifunctor. Then $u \mathbin{\overline{\otimes}}_{\mathbf{C}} -\colon \mathbf{D}^{\mathbf{C}} \to \mathbf{D}^{\mathbf{C}'}$ is* $\mathbf{V}$*-left adjoint to $[u,-]_{\mathbf{C}'}$.* ∎

(2.9) DEFINITION. We say that a $\mathbf{V}$-bifunctor $u\colon \mathbf{C}^{o} \otimes \mathbf{C} \to \mathbf{V}$ *represents* the $\mathbf{V}$-adjoint pair $([u,-]_{\mathbf{C}'}, u \mathbin{\overline{\otimes}}_{\mathbf{C}} -)\colon \mathbf{D}^{\mathbf{C}'} \to \mathbf{D}^{\mathbf{C}}$.

The following is the $\mathbf{V}$-functor category version of Watts' theorem.

(2.10) THEOREM. *Every* $\mathbf{V}$*-adjoint pair $(U,F)\colon \mathbf{V}^{\mathbf{C}'} \to \mathbf{V}^{\mathbf{C}}$ is represented uniquely by a* $\mathbf{V}$*-bifunctor $u\colon \mathbf{C}^{o} \otimes \mathbf{C}' \to \mathbf{V}$.*

Proof. Since $F\colon \mathbf{V}^{\mathbf{C}} \to \mathbf{V}^{\mathbf{C}'}$ is $\mathbf{V}$-cocontinuous, we have from (2.3 with $\mathbf{D} = \mathbf{V}^{\mathbf{C}'}$) that F is $\mathbf{V}$-equivalent to $- \mathbin{\overline{\otimes}}_{\mathbf{C}^{o}} FR \cong FR \mathbin{\overline{\otimes}}_{\mathbf{C}} -$, where $R\colon \mathbf{C}^{o} \to \mathbf{V}^{\mathbf{C}}$ is the right Yoneda embedding. Let u be the $\mathbf{V}$-functor corresponding to $FR\colon \mathbf{C}^{o} \to \mathbf{V}^{\mathbf{C}'}$ under the $\mathbf{V}$-equivalence $(\mathbf{V}^{\mathbf{C}'})^{\mathbf{C}^{o}} \cong \mathbf{V}^{\mathbf{C}^{o}\otimes\mathbf{C}'}$, so that $F \cong u \mathbin{\overline{\otimes}}_{\mathbf{C}} -$. Since U is $\mathbf{V}$-right adjoint to F and, by (2.8), $[u,-]_{\mathbf{C}'}$ is $\mathbf{V}$-right adjoint to $u \mathbin{\overline{\otimes}}_{\mathbf{C}} -$, we have $U \cong [u,-]_{\mathbf{C}'}$ by uniqueness of adjoints. ∎

Suppose $u\colon \mathbf{C}^o \otimes \mathbf{C}' \to \mathbf{V}$ and $v\colon \mathbf{C}'^o \otimes \mathbf{C}'' \to \mathbf{V}$ are $\mathbf{V}$-functors. Then we may define a $\mathbf{V}$-functor $v \mathbin{\overline{\otimes}_{\mathbf{C}'}} u\colon \mathbf{C}^o \otimes \mathbf{C}'' \to \mathbf{V}$ so that $(v \mathbin{\overline{\otimes}_{\mathbf{C}'}} u)(C, C'') = v(-, C'') \mathbin{\overline{\otimes}_{\mathbf{C}'}} u(C, -)$ for all $C \in |\mathbf{C}|$ and $C'' \in |\mathbf{C}''|$. Note that $u \mathbin{\overline{\otimes}_{\mathbf{C}}} -\colon \mathbf{V}^{\mathbf{C}} \to \mathbf{V}^{\mathbf{C}'}$ and $v \mathbin{\overline{\otimes}_{\mathbf{C}'}} -\colon \mathbf{V}^{\mathbf{C}'} \to \mathbf{V}^{\mathbf{C}''}$, while

$$(v \mathbin{\overline{\otimes}_{\mathbf{C}'}} u) \mathbin{\overline{\otimes}_{\mathbf{C}}} -\colon \mathbf{V}^{\mathbf{C}} \to \mathbf{V}^{\mathbf{C}''}.$$

The following is a consequence of (2.2).

(2.11) THEOREM. *For* $\mathbf{V}$-*functors* $u\colon \mathbf{C}^o \otimes \mathbf{C}' \to \mathbf{V}$ *and* $v\colon \mathbf{C}'^o \otimes \mathbf{C}'' \to \mathbf{V}$, *there is a canonical* $\mathbf{V}$-*equivalence*

$$(v \mathbin{\overline{\otimes}_{\mathbf{C}'}} -) \cdot (u \mathbin{\overline{\otimes}_{\mathbf{C}}} -) \simeq (v \mathbin{\overline{\otimes}_{\mathbf{C}'}} u) \mathbin{\overline{\otimes}_{\mathbf{C}}} -. \quad \blacksquare$$

A corollary follows immediately:

(2.12) COROLLARY. *If a* $\mathbf{V}$-*adjoint pair* $(U, F)\colon \mathbf{V}^{\mathbf{C}'} \to \mathbf{V}^{\mathbf{C}}$ *is represented by* $u\colon \mathbf{C}^o \otimes \mathbf{C}' \to \mathbf{V}$ *and a* $\mathbf{V}$-*adjoint pair* $(U', F')\colon \mathbf{V}^{\mathbf{C}''} \to \mathbf{V}^{\mathbf{C}'}$ *is represented by* $v\colon \mathbf{C}'^o \otimes \mathbf{C}'' \to \mathbf{V}$, *then the composition of adjoint pairs* $(UU', F'F)$ *is represented by* $v \mathbin{\overline{\otimes}_{\mathbf{C}'}} u\colon \mathbf{C}^o \otimes \mathbf{C}'' \to \mathbf{V}$. $\blacksquare$

We apply these results to the following situation. Let $x\colon \mathbf{C} \to \mathbf{C}'$ be a $\mathbf{V}$-functor, let $\mathbf{D}^x\colon \mathbf{D}^{\mathbf{C}'} \to \mathbf{D}^{\mathbf{C}}$ be the $\mathbf{V}$-functor induced by x ($\mathbf{D}^x(h) = hx$ for every $h \in |\mathbf{D}^{\mathbf{C}'}|$), and let $\mathbf{C}'(x-, -)\colon \mathbf{C}^o \otimes \mathbf{C}' \to \mathbf{V}$ and

$$\mathbf{C}'(-, x-)\colon \mathbf{C}'^o \otimes \mathbf{C} \to \mathbf{V}$$

be the $\mathbf{V}$-bifunctors defined by $\mathbf{C}'(x-, -)(C, C') = \mathbf{C}'(xC, C')$ and $\mathbf{C}'(-, x-)(C', C) = \mathbf{C}'(C', xC)$ for all $C \in |\mathbf{C}|$ and $C' \in |\mathbf{C}'|$. One checks that $[\mathbf{C}'(x-, -), -]_{\mathbf{C}'} \simeq \mathbf{D}^x \simeq \mathbf{C}'(-, x-) \mathbin{\overline{\otimes}_{\mathbf{C}'}} -$. It follows from (2.8) that

(2.13) THEOREM. *Let* $x\colon \mathbf{C} \to \mathbf{C}'$ *be a* $\mathbf{V}$-*functor and let* $\mathbf{D}^x\colon \mathbf{D}^{\mathbf{C}'} \to \mathbf{D}^{\mathbf{C}}$ *be the* $\mathbf{V}$-*functor induced by* x. *Then* $\mathbf{D}^x$ *is* $\mathbf{V}$-*equivalent to* $\mathbf{C}'(-, x-) \otimes_{\mathbf{C}'} -$ *and to* $[\mathbf{C}'(x-, -), -]_{\mathbf{C}'}$. *Furthermore,* $\mathbf{D}^x$ *has a* $\mathbf{V}$-*left adjoint* $(\mathbf{D}^x)^l \simeq \mathbf{C}'(x-, -) \mathbin{\overline{\otimes}_{\mathbf{C}}} -$ *and a* $\mathbf{V}$-*right adjoint* $(\mathbf{D}^x)^r \simeq [\mathbf{C}'(-, x-), -]_{\mathbf{C}}$.

3. V-COCONTINUOUS TRIPLES AND MONOIDS

We now consider the underlying category of the category of $\mathbf{V}$-bifunctors $\mathbf{V}^{\mathbf{C}^o \otimes \mathbf{C}}$, which we will denote by $[\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}]$ (the objects of this category are also known as Benabou's "profunctors" or "distributors" from $\mathbf{C}$ to $\mathbf{C}$). When $\mathbf{C} = \mathbf{R}$ and $\mathbf{V} = \mathbf{Ab}$, $[\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}]$ is the category of R-bimodules. The category $[\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}]$ is almost a closed category, for all it lacks is the symmetry of the tensor product. It is an example of the following.

(3.1) Definition (Lambek [9, p. 98].) A category $\mathbf{W}$ is called *biclosed* if it is a monoidal category with tensor product $\otimes: \mathbf{W} \times \mathbf{W} \to \mathbf{W}$ and furthermore, for all $W \in |\mathbf{W}|$, $W \otimes -: \mathbf{W} \to \mathbf{W}$ has a right adjoint $W\backslash -: \mathbf{W} \to \mathbf{W}$ and $- \otimes W$ has a right adjoint $-/W: \mathbf{W} \to \mathbf{W}$.

The following is mentioned in [2, pp. 36–37].

(3.2) Proposition. *The category* $[\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}]$ *is biclosed.*

Proof. We have defined a tensor product $u \mathbin{\overline{\otimes}_{\mathbf{C}}} v$ of two $\mathbf{V}$-bifunctors u and v in $[\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}]$ above (in the paragraph preceding (2.11)). It follows from (2.11) that one has a natural equivalence

$$a: (u \mathbin{\overline{\otimes}_{\mathbf{C}}} v) \mathbin{\overline{\otimes}_{\mathbf{C}}} w \Rightarrow u \mathbin{\overline{\otimes}_{\mathbf{C}}} (v \mathbin{\overline{\otimes}_{\mathbf{C}}} w).$$

The unit is the Hom bifunctor $\mathbf{C} = \mathbf{C}(-, -): \mathbf{C}^o \otimes \mathbf{C} \to \mathbf{V}$, and the left and right identity natural equivalences $l: \mathbf{C} \mathbin{\overline{\otimes}_{\mathbf{C}}} u \Rightarrow u$ and $r: u \mathbin{\overline{\otimes}_{\mathbf{C}}} \mathbf{C} \Rightarrow u$ follow from (2.2).

It follows from (2.8) that $u \mathbin{\overline{\otimes}_{\mathbf{C}}} -$ has a right adjoint $u\backslash -$. This functor is given by $(u\backslash v)(C', C) = \mathbf{V}^{\mathbf{C}}(u(C, -), v(C', -))$ for all bifunctors $v: \mathbf{C}^o \otimes \mathbf{C} \to \mathbf{V}$ and objects $C, C' \in |\mathbf{C}|$.

Since $- \mathbin{\overline{\otimes}_{\mathbf{C}}} u = u \mathbin{\overline{\otimes}_{\mathbf{C}^o}} -$, it follows from the above that $- \mathbin{\overline{\otimes}_{\mathbf{C}}} u$ has a right adjoint $-/u$. This functor is given by

$$(v/u)(C', C) = \mathbf{V}^{\mathbf{C}^o}(u(-, C'), v(-, C)).$$

for all bifunctors $v: \mathbf{C}^o \otimes \mathbf{C} \to \mathbf{V}$ and objects $C, C' \in |\mathbf{C}|$. ∎

The metacategory End($\mathbf{V}^{\mathbf{C}}$), with objects being $\mathbf{V}$-functors from $\mathbf{V}^{\mathbf{C}}$ to itself and with morphisms being $\mathbf{V}$-natural transformations, is a strict monoidal category with tensor product of two functors being their composition and the unit being the identity functor on $\mathbf{V}^{\mathbf{C}}$ ("strict" means that the "a," "l," and "r" of (1.1) are identity maps).

(3.3) Definition. Let $\mathbf{M} = (\mathbf{M}_o, \otimes, I, a, l, r)$ and $\mathbf{M}' = (\mathbf{M}_o', \otimes', I', a', l', r')$ be two monoidal categories. Then a *morphism of monoidal categories* $\Phi: \mathbf{M} \to \mathbf{M}'$ consists of (i) a functor $\Phi: \mathbf{M}_o \to \mathbf{M}_o'$;

(ii) for each $A, B \in |\mathbf{M}|$, a morphism $\varphi: \Phi A \otimes' \Phi B \to \Phi(A \otimes B)$ in $\mathbf{M}'$, natural in A and B, so that for all $A, B, C \in |\mathbf{M}|$ the diagram

$$\begin{array}{ccccc}
(\Phi A \otimes' \Phi B) \otimes' \Phi C & \xrightarrow{a'} & \Phi A \otimes' (\Phi B \otimes' \Phi C) & \xrightarrow{\Phi A \otimes' \varphi} & \Phi A \otimes' \Phi(B \otimes C) \\
\downarrow{\scriptstyle \varphi \otimes' \Phi C} & & & & \downarrow{\scriptstyle \varphi} \\
\Phi(A \otimes B) \otimes' \Phi C & \xrightarrow{\varphi} & \Phi((A \otimes B) \otimes C) & \xrightarrow{\Phi(a)} & \Phi(A \otimes (B \otimes C))
\end{array}$$

commutes;

(iii) a morphism $\varphi_o\colon I' \to \Phi I$ in $\mathbf{M}'$ so that, for every $A \in \mathbf{M}$, one has

$$(I' \otimes' \Phi A \xrightarrow{\varphi_o \otimes' \Phi A} \Phi I \otimes' \Phi A \xrightarrow{\varphi} \Phi(I \otimes A) \xrightarrow{\Phi(l)} \Phi A) = l',$$

and

$$(\Phi A \otimes' I' \xrightarrow{\Phi A \otimes' \varphi_o} \Phi A \otimes' \Phi I \xrightarrow{\varphi} \Phi(A \otimes I) \xrightarrow{\Phi(r)} \Phi A) = r'.$$

We say that Φ is a *strict* morphism of monoidal categories if φ_o and the φ's are isomorphisms in $\mathbf{M}_o'$.

The right Yoneda embedding $R_{\mathbf{C}_o}\colon \mathbf{C}^o \to \mathbf{V}^{\mathbf{C}}$ induces a metafunctor,

$$R^*\colon \mathrm{End}(\mathbf{V}^{\mathbf{C}}) \to [\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}],$$

given by R^*F is the bifunctor corresponding to $FR_{\mathbf{C}^o}\colon \mathbf{C}^o \to \mathbf{V}^{\mathbf{C}}$ under the equivalence $\mathbf{V}^{\mathbf{C}^o \otimes \mathbf{C}} \simeq (\mathbf{V}^{\mathbf{C}})^{\mathbf{C}^o}$ for all $F\colon \mathbf{V}^{\mathbf{C}} \to \mathbf{V}^{\mathbf{C}}$.

(3.4) THEOREM. *The metafunctor* $R^*\colon \mathrm{End}(\mathbf{V}^{\mathbf{C}}) \to [\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}]$ *is a morphism of monoidal metacategories. Furthermore,* R^* *has a left adjoint* R^l, *given by* $R^l(u) = u \mathbin{\overline{\otimes}}_{\mathbf{C}} -$ *for u in* $[\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}]$, *which is a strict morphism of monoidal metacategories. The restriction of* R^* *to the full submetacategory of* $\mathrm{End}(\mathbf{V}^{\mathbf{C}})$ *whose objects are the* $\mathbf{V}$*-cocontinuous* $\mathbf{V}$*-endofunctors on* $\mathbf{V}^{\mathbf{C}}$ *gives an equivalence of categories between this submetacategory and* $[\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}]$.

Proof. The adjointness follows from the universal property of Kan extensions, as $R^l(u) = u \mathbin{\overline{\otimes}}_{\mathbf{C}} - = - \mathbin{\overline{\otimes}}_{\mathbf{C}^o} u = \mathrm{Lan}_{R_{\mathbf{C}^o}} u$ for $u \in [\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}]$ (thought of as a $\mathbf{V}$-functor $u\colon \mathbf{C}^o \to \mathbf{V}^{\mathbf{C}}$ under the equivalence $\mathbf{V}^{\mathbf{C}^o \otimes \mathbf{C}} \simeq (\mathbf{V}^{\mathbf{C}})^{\mathbf{C}^o}$). The $\mathbf{V}$-natural transformations,

$$R^l(u) \cdot R^l(v) = (u \mathbin{\overline{\otimes}}_{\mathbf{C}} -) \cdot (v \mathbin{\overline{\otimes}}_{\mathbf{C}} -) \Rightarrow (u \mathbin{\overline{\otimes}}_{\mathbf{C}} v) \mathbin{\overline{\otimes}}_{\mathbf{C}} - = R^l(u \mathbin{\overline{\otimes}}_{\mathbf{C}} v),$$

of (2.11) and

$$1_{\mathbf{V}^{\mathbf{C}}} \Rightarrow R^l(\mathbf{C})$$

(2.10), with $F = 1_{\mathbf{V}^{\mathbf{C}}}$, are equivalences, so that R^l is a strict morphism of monoidal categories. It follows from Maranda [12, Corollary, p. 776] that R^* is a morphism of monoidal categories.

The unit of the metaadjoint pair,

$$(R^*, R^l)\colon \mathrm{End}(\mathbf{V}^{\mathbf{C}}) \to [\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}],$$

is an equivalence by (2.2). The counit for this metaadjoint pair is seen to be ϵ_F of (2.3) (replacing $\mathbf{C}^o$ by $\mathbf{C}$ and letting $\mathbf{D} = \mathbf{V}^{\mathbf{C}}$), which is an equivalence iff F is $\mathbf{V}$-cocontinuous. The rest of the theorem follows from these observations. ∎

We let $\mathscr{C}(\mathbf{V^C})$ denote the full submetacategory of End $(\mathbf{V^C})$ whose objects are the **V**-cocontinuous **V**-endofunctors on $\mathbf{V^C}$, and let $\gamma: \mathscr{C}(\mathbf{V^C}) \to [\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}]$ be the equivalence of metacategories obtained by restricting R^* to $\mathscr{C}(\mathbf{V^C})$. Then the inclusion metafunctor $i: \mathscr{C}(\mathbf{V^C}) \to \mathrm{End}(\mathbf{V^C})$ has a metaright adjoint, i.e., $\mathscr{C}(\mathbf{V^C})$ is a coreflective submetacategory of $\mathrm{End}(\mathbf{V^C})$. Furthermore, γ and i are strict morphisms of monoidal categories.

(3.5) DEFINITION. Let $\mathbf{M} = (\mathbf{M}_o, \otimes, I, a, l, r)$ be a monoidal category. A *monoid object* of $\mathbf{M}$ is a triple (M, η, μ), with M an object of $\mathbf{M}_o$ and $\eta: I \to M$ and $\mu: M \otimes M \to M$ morphisms of $\mathbf{M}_o$, for which the diagrams

$$\begin{array}{ccccc} I \otimes M & \xrightarrow{\eta \otimes M} & M \otimes M & \xrightarrow{M \otimes \eta} & M \otimes I \\ & {\scriptstyle l} \searrow & \downarrow {\scriptstyle \mu} & \swarrow {\scriptstyle r} & \\ & & M & & \end{array}$$

and

$$\begin{array}{ccccc} (M \otimes M) \otimes M & \xrightarrow{a} & M \otimes (M \otimes M) & \xrightarrow{M \otimes \mu} & M \otimes M \\ {\scriptstyle \mu \otimes M} \downarrow & & & & \downarrow {\scriptstyle \mu} \\ M \otimes M & & \xrightarrow{\quad\mu\quad} & & M \end{array}$$

commute in $\mathbf{M}_o$.

Dually, a *comonoid object* of $\mathbf{M}$ is a triple (N, ϵ, δ), with N an object of $\mathbf{M}_o$ and $\epsilon: N \to I$ and $\delta: N \to N \otimes N$ morphisms of $\mathbf{M}_o$ so that $(\epsilon \otimes N) \cdot \delta = l$, $(N \otimes \epsilon) \cdot \delta = r$, and $a \cdot (\delta \otimes N) \cdot \delta = (N \otimes \delta) \cdot \delta$.

If $\mathbf{M}$ and $\mathbf{M}'$ are monoidal categories and $\Phi: \mathbf{M} \to \mathbf{M}'$ is a morphism of monoidal categories, then Φ preserves the monoid objects of $\mathbf{M}$ in the sense that, if (M, η, μ) is a monoid object of $\mathbf{M}$, $(\Phi M, \Phi(\eta) \cdot \varphi_o, \Phi(\mu) \cdot \varphi)$ is a monoid object of $\mathbf{M}'$ (where φ and φ_o are the morphisms associated to Φ in (3.3)). In fact, if (M_1, η_1, μ_1) and (M_2, η_2, μ_2) are monoid objects of $\mathbf{M}$, define a *homomorphism of monoid objects* $t: (M_1, \eta_1, \mu_1) \to (M_2, \eta_2, \mu_2)$ to be a morphism $t: M_1 \to M_2$ in $\mathbf{M}_o$ for which $(t \cdot \eta_1) = \eta_2$ and $t \cdot \mu_1 = \mu_2 \cdot (t \otimes t)$. If we let Mon($\mathbf{M}$) denote the category of monoid objects of $\mathbf{M}$ and their homomorphisms, a morphism of monoidal categories $\Phi: \mathbf{M} \to \mathbf{M}'$ induces a functor Φ: Mon($\mathbf{M}$) $\to$ Mon($\mathbf{M}'$). We note that if Φ is strict, we may make similar observations about the comonoid objects of $\mathbf{M}$.

One sees that the monoid objects of $\mathrm{End}(\mathbf{V^C})$ are the **V**-triples on $\mathbf{V^C}$ and that the monoid objects of $\mathscr{C}(\mathbf{V^C})$ are the **V**-triples on $\mathbf{V^C}$ with **V**-cocontinuous functor (the comonoid objects of $\mathrm{End}(\mathbf{V^C})$ are the **V**-cotriples on $\mathbf{V^C}$).

(3.6) COROLLARY. *Let* $\mathbf{T}$ *be a* $\mathbf{V}$*-triple on* $\mathbf{V}^{\mathbf{C}}$. *Then there is a* V*-triple* $\bar{\mathbf{T}}$ *on* $\mathbf{V}^{\mathbf{C}}$ *with* $\mathbf{V}$*-cocontinuous functor and a homomorphism of* V*-triples* $\epsilon_{\mathbf{T}}: \bar{\mathbf{T}} \Rightarrow \mathbf{T}$ *satisfying the universal property that if* $\mathbf{T}'$ *is any* $\mathbf{V}$*-triple on* $\mathbf{V}^{\mathbf{C}}$ *with* $\mathbf{V}$*-cocontinuous functor and if* $\alpha: \mathbf{T}' \Rightarrow \mathbf{T}$ *is a homomorphism of* $\mathbf{V}$*-triples, then there is a unique homomorphism of* V*-triples* $\bar{\alpha}: \mathbf{T}' \Rightarrow \bar{\mathbf{T}}$ *for which* $\epsilon_{\mathbf{T}} \cdot \bar{\alpha} = \alpha$. *Furthermore,* $\epsilon_{\mathbf{T}}$ *is an equivalence iff the functor of* $\mathbf{T}$ *is* $\mathbf{V}$*-cocontinuous.*

Proof. $\mathscr{C}(\mathbf{V}^{\mathbf{C}})$ is a coreflective submetacategory of $\text{End}(\mathbf{V}^{\mathbf{C}})$ whose inclusion metafunctor i is a strict morphism of monoidal metacategories. It follows that $\text{Mon}(\mathscr{C}(\mathbf{V}^{\mathbf{C}}))$ is a coreflective submetacategory of $\text{Mon}(\text{End}(\mathbf{V}^{\mathbf{C}}))$, which is an equivalent statement of the corollary. ▮

Since $\gamma: \mathscr{C}(\mathbf{V}^{\mathbf{C}}) \to [\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}]$ is an equivalence of monoidal metacategories, we have the following

(3.7) COROLLARY. *The equivalence* $\gamma: \mathscr{C}(\mathbf{V}^{\mathbf{C}}) \to [\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}]$ *induces:* (a) *an equivalence between the metacategory of* $\mathbf{V}$*-triples on* $\mathbf{V}^{\mathbf{C}}$ *with* $\mathbf{V}$*-cocontinuous functors and the category of monoid objects in* $[\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}]$;

(b) *an equivalence between the metacategory of* $\mathbf{V}$*-cotriples on* $\mathbf{V}^{\mathbf{C}}$ *with* $\mathbf{V}$*-cocontinuous functors and the category of comonoids in* $[\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}]$. ▮

$\mathbf{V}$ is itself a monoidal category, and we may consider the monoid objects of $\mathbf{V}$. When $\mathbf{V} = \mathbf{Ens}$, $\text{Mon}(\mathbf{V})$ is the category of monoids (sets with an associative binary operation and a unit). When $\mathbf{V} = \mathbf{Ab}$, $\text{Mon}(\mathbf{V})$ is the category of rings and ring homomorphisms. As an immediate application of 3.7, we have the following

(3.8) COROLLARY. (a) *the category of monoids in* $\mathbf{V}$ *is equivalent to the metacategory of* $\mathbf{V}$*-triples on* $\mathbf{V}$ *with* $\mathbf{V}$*-cocontinuous functors.*

(b) *The category of comonoids of* $\mathbf{V}$ *is equivalent to the metacategory of* $\mathbf{V}$*-cotriples on* $\mathbf{V}$ *with* $\mathbf{V}$*-cocontinuous functor.*

Proof. One lets $\mathbf{I}$ denote the $\mathbf{V}$-category with one object $*$, with $\mathbf{I}(*, *) = I$, and with obvious composition and identity. One has an equivalence of monoidal metacategories $\gamma: \mathscr{C}(\mathbf{V}^{\mathbf{I}}) \to [\mathbf{I}^o \otimes \mathbf{I}, \mathbf{V}]$. But we observe that $\mathbf{V}^{\mathbf{I}} \simeq \mathbf{V}$ and $[\mathbf{I}^o \otimes \mathbf{I}, \mathbf{V}] \simeq \mathbf{V}$, and the corollary follows. ▮

The rest of this section is devoted to identifying $\text{Mon}([\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}])$ (Benabou's category of "protriples").

(3.9) DEFINITION. Given a monoid object,

$$(u, \eta: \mathbf{C} \Rightarrow u, \mu: u \mathbin{\bar{\otimes}_{\mathbf{C}}} u \Rightarrow u),$$

of $[\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}]$, we define a $\mathbf{V}$-category $\mathbf{C}_u$ and a $\mathbf{V}$-functor $x_u: \mathbf{C} \to \mathbf{C}_u$ as follows:

(C1) the objects of $\mathbf{C}_u$ are those of $\mathbf{C}$;

(C2) for objects C and C' of $\mathbf{C}_u$, $\mathbf{C}_u(C', C) = u(C', C)$;

(C3) for objects A, B, C of $\mathbf{C}_u$ composition in $\mathbf{C}_u$ is defined by the commutative diagram

$$\begin{array}{ccc} u(B, C) \otimes u(A, B) & \xrightarrow{\lambda_B} & \int^B u(B, C) \otimes u(A, B) = (u \mathbin{\overline{\otimes}_{\mathbf{C}}} u)(A, C) \\ & \searrow^{c} & \downarrow{\scriptstyle \mu(A,C)} \\ & & u(A, C); \end{array}$$

(C4) for C an object of $\mathbf{C}_u$, the identity of C in $\mathbf{C}_u$ is defined to be the composition of the identity $I \to \mathbf{C}(C, C)$ of C and $\eta(C, C): \mathbf{C}(C, C) \to u(C, C)$;

(C5) the $\mathbf{V}$-functor $x_u: \mathbf{C} \to \mathbf{C}_u$ is defined by $x_u(C) = C$ for every object C of $\mathbf{C}$ and by $x_u(A, B) = \eta(A, B)$ for every pair of objects A and B of $\mathbf{C}$.

Conversely, given a $\mathbf{V}$-category $\mathbf{C}'$ and a $\mathbf{V}$-functor $x: \mathbf{C} \to \mathbf{C}'$, we define a monoid object (u_x, η_x, μ_x) of $[\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}]$ as follows:

(M1) $u_x = \mathbf{C}'(x\,-, x\,-)$;

(M2) $\eta_x(A, B) = x_{AB}$ for every pair of objects A and B of $\mathbf{C}$;

(M3) $\mu_x(A, C): (u_x \mathbin{\overline{\otimes}_{\mathbf{C}}} u_x)(A, C) \to u_x(A, C)$ is the unique morphism for which the diagram

$$\begin{array}{ccc} \mathbf{C}'(xB, xC) \otimes \mathbf{C}'(xA, xB) & \xrightarrow{\lambda_{xB}} & \mathbf{C}'(x-, xC) \mathbin{\overline{\otimes}_{\mathbf{C}}} \mathbf{C}'(xA, x-) \\ & \searrow^{c} & \downarrow{\scriptstyle \mu_x(A,C)} \\ & & \mathbf{C}'(xA, xC) \end{array}$$

commutes for all objects B of C, where "c" denotes composition in $\mathbf{C}'$.

One verifies that, given a monoid object (u, η, μ) of $[\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}]$, the data (C1–5) satisfies the requirements for $\mathbf{C}_u$ to be a $\mathbf{V}$-category and $x_u: \mathbf{C} \to \mathbf{C}_u$ to be a $\mathbf{V}$-functor and, given a $\mathbf{V}$-functor $x: \mathbf{C} \to \mathbf{C}'$, the data (M1–3) does define a monoid object (u_x, η_x, μ_x) of $[\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}]$. Furthermore, one sees these constructions are almost inverses of each other. We have the following

(3.10) PROPOSITION. *There is an equivalence of categories between* Mon($[\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}]$) *and the category whose objects are pairs* $(x, \mathbf{C}')$, *where* $\mathbf{C}'$ *is a*

$\mathbf{V}$-category and $x: \mathbf{C} \to \mathbf{C}'$ is a $\mathbf{V}$-functor which is a surjection on objects, and whose morphisms $f: (x, \mathbf{C}') \to (y, \mathbf{C}'')$ are $\mathbf{V}$-functors $f: \mathbf{C}' \to \mathbf{C}''$ for which $fx = y$. This equivalence is given by the assignment $(u, \eta, \mu) \mapsto (x_u, \mathbf{C}_u)$ for all monoid objects (u, η, μ) of $[\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}]$, and its inverse is given by the assignment $(x, \mathbf{C}') \mapsto (u_x, \eta_x, \mu_x)$ for all $\mathbf{V}$-functors $x: \mathbf{C} \to \mathbf{C}'$ which are surjective on objects. ∎

Let $x: \mathbf{C} \to \mathbf{C}'$ be a $\mathbf{V}$-functor on small $\mathbf{V}$-categories $\mathbf{C}$ and $\mathbf{C}'$. Then x induces a $\mathbf{V}$-functor $\mathbf{V}^x: \mathbf{V}^{\mathbf{C}'} \to \mathbf{V}^{\mathbf{C}}$. By (2.13), $\mathbf{V}^x$ has a left adjoint $(\mathbf{V}^x)^l$, and both $\mathbf{V}^x$ and $(\mathbf{V}^x)^l$ are $\mathbf{V}$-cocontinuous. Therefore, the $\mathbf{V}$-triple $\mathbf{T}_x$ induced by the $\mathbf{V}$-adjoint pair $(\mathbf{V}^x, (\mathbf{V}^x)^l): \mathbf{V}^{\mathbf{C}'} \to \mathbf{V}^{\mathbf{C}}$ has $\mathbf{V}$-cocontinuous functor. Furthermore, we have the following

(3.11) Proposition. *If $x: \mathbf{C} \to \mathbf{C}'$ is a surjection on objects, the $\mathbf{V}$-adjoint pair $(\mathbf{V}^x, (\mathbf{V}^x)^l): \mathbf{V}^{\mathbf{C}'} \to \mathbf{V}^{\mathbf{C}}$ is $\mathbf{V}$-tripleable.*

Proof. We use [4, Theorem II.2.1, p. 78] and the terminology defined there.

Since $\mathbf{V}^{\mathbf{C}}$ is $\mathbf{V}$-cocomplete and $\mathbf{V}^x$ is $\mathbf{V}$-cocontinuous, $\mathbf{V}^x$ detects and preserves $\mathbf{V}$-coequalizers. Since x is a surjection on objects, it follows that $\mathbf{V}^x$ reflects isomorphisms and hence, since $\mathbf{V}^{\mathbf{C}}$ is $\mathbf{V}$-cocomplete, $\mathbf{V}^x$ reflects $\mathbf{V}$-coequalizers. Thus, by the theorem in [4], $(\mathbf{V}^x, (\mathbf{V}^x)^l)$ is $\mathbf{V}$-tripleable. ∎

We now put together the equivalences of (3.7) and (3.10), along with the previous proposition, to obtain the following (the "main theorem" of the Introduction).

(3.12) Theorem. *Let $\mathbf{T} = (T, \eta, \mu)$ be a $\mathbf{V}$-triple on $\mathbf{V}^{\mathbf{C}}$ with T a $\mathbf{V}$-cocontinuous functor. Then there is a $\mathbf{V}$-category $\mathbf{C}'$ and a $\mathbf{V}$-functor $x: \mathbf{C} \to \mathbf{C}'$ which is a surjection on objects so that $\mathbf{T}$ is isomorphic to the V-triple $\mathbf{T}_x$ induced by the $\mathbf{V}$-adjoint pair $(\mathbf{V}^x, (\mathbf{V}^x)^l)$. Furthermore, the $\mathbf{V}$-comparison functor gives an equivalence of $\mathbf{V}$-categories $(\mathbf{V}^{\mathbf{C}})^{\mathbf{T}} \simeq \mathbf{V}^{\mathbf{C}'}$.*

Proof. Let $\mathbf{C}' = \mathbf{C}_{\gamma T}$ and $x = x_{\gamma T}$. The equivalences of (3.7) and (3.10) give us that $T \simeq \mathbf{C}'(x-, x-) \overline{\otimes}_{\mathbf{C}} -$. But, by (2.13), $\mathbf{V}^x \simeq \mathbf{C}'(-, x-) \overline{\otimes}_{\mathbf{C}'} -$ and $(\mathbf{V}^x)^l \simeq \mathbf{C}'(x-, -) \overline{\otimes}_{\mathbf{C}} -$. Therefore, by (2.11), the functor of $\mathbf{T}_x$ is given by

$$\mathbf{V}^x \cdot (\mathbf{V}^x)^l \simeq (\mathbf{C}'(-, x-) \overline{\otimes}_{\mathbf{C}'} \mathbf{C}'(x-, -)) \overline{\otimes}_{\mathbf{C}} -,$$

and one sees that

$$\mathbf{C}'(-, x-) \overline{\otimes}_{\mathbf{C}'} \mathbf{C}'(x-, -) \simeq \mathbf{C}'(x-, x-).$$

Therefore, $\mathbf{T} \simeq \mathbf{T}_x$. The last assertion of the theorem follows from (3.11). ∎

4. Characterizations of V-Functor Categories

In this section, we apply (3.12) to obtain some characterizations of **V**-functor categories, including those given by Bunge in [1].

Let S be a set. We construct from S a corresponding "**V**-discrete" **V**-category **S** whose objects are the elements of S and, for any two elements i and j of S, $\mathbf{S}(i, j) = 0$, the initial object of **V**, if $i \neq j$, and $\mathbf{S}(i, i) = I$, the unit of **V**. We note the following:

(1) $\mathbf{V}^{\mathbf{S}}$ is **V**-equivalent to the category $\mathbf{V}^{s}$, the product category of S copies of **V**, made into a **V**-category by defining

$$\mathbf{V}^{S}((A_i)_S, (B_i)_S) = \Pi_S \mathbf{V}(A_i, B_i),$$

for all families $(A_i)_S$ and $(B_i)_S$ of objects of **V** indexed by S;

(2) for every small **V**-category **C** for which there is a bijection $\sigma: S \to |\mathbf{C}|$, there is a unique **V**-functor $\bar{\sigma}: \mathbf{S} \to \mathbf{C}$ for which $\bar{\sigma}(i) = \sigma(i)$ for every $i \in S$.

(4.1) Theorem. *Let* **B** *be a* **V**-*category. Then* **B** *is* **V**-*equivalent to* $\mathbf{V}^{\mathbf{C}}$ *for some small* **V**-*category* **C** *iff there is a set* S *and a* **V**-*triple* **T** *on* $\mathbf{V}^{\mathbf{S}}$ *with* **V**-*cocontinuous functor for which* **B** *is* **V**-*equivalent to the* **V**-*category of* **T**-*algebras* $(\mathbf{V}^{\mathbf{S}})^{\mathbf{T}}$.

Proof. Let **C** be a small category and let S be any set for which there is a bijection $\sigma: S \cong |\mathbf{C}|$. Then the V-functor $\bar{\sigma}: \mathbf{S} \to \mathbf{C}$ induces a V-functor $\mathbf{V}^{\bar{\sigma}}: \mathbf{V}^{\mathbf{C}} \to \mathbf{V}^{\mathbf{S}}$. If we let $\mathbf{T}_{\sigma}$ denote the **V**-triple on $\mathbf{V}^{\mathbf{S}}$ induced by the adjoint pair $(\mathbf{V}^{\bar{\sigma}}, (\mathbf{V}^{\bar{\sigma}})^{l})$, we have by (3.11) that $\mathbf{V}^{\mathbf{C}} \cong (\mathbf{V}^{\mathbf{S}})^{\mathbf{T}_{\sigma}}$, so that if **B** is **V**-equivalent to $\mathbf{V}^{\mathbf{C}}$, then **B** is **V**-equivalent to $(\mathbf{V}^{\mathbf{S}})^{\mathbf{T}_{\sigma}}$.

Conversely, if **B** is **V**-equivalent to $(\mathbf{V}^{\mathbf{S}})^{\mathbf{T}}$, **T** a **V**-triple with **V**-cocontinuous functor, then by (3.12) there is a **V**-category **C** for which $(\mathbf{V}^{\mathbf{S}})^{\mathbf{T}} \cong \mathbf{V}^{\mathbf{C}}$, so that $\mathbf{B} \cong \mathbf{V}^{\mathbf{C}}$. ∎

As a corollary, we obtain the "first characterization of functor categories" of Bunge [1 (4.6)].

(4.2) Theorem. *Let* **B** *be a* **V**-*category,* S *a set, and suppose there is a* **V**-*adjoint pair* $(U, F): \mathbf{B} \to \mathbf{V}^{\mathbf{S}}$ *for which* U *is* **V**-*cocontinuous. Then there is a* **V**-*category* **C**, *a bijection* $\sigma: S \cong |\mathbf{C}|$, *and a* **V**-*equivalence* $\Phi: \mathbf{B} \to \mathbf{V}^{\mathbf{C}}$ *for which* $U = \mathbf{V}^{\bar{\sigma}} \cdot \Phi$ *iff* **B** *has* **V**-*coequalizers and* U *reflects isomorphisms.*

Proof. For any small category **C**, $\mathbf{V}^{\mathbf{C}}$ is **V**-bicomplete, so that $\mathbf{V}^{\mathbf{C}}$ has **V**-coequalizers. If S is any set and $\sigma: S \cong |\mathbf{C}|$ is a bijection, $\mathbf{V}^{\bar{\sigma}}: \mathbf{V}^{\mathbf{C}} \to \mathbf{V}^{\mathbf{S}}$

reflects isomorphisms. Consequently, if there is an equivalence of categories $\Phi: \mathbf{B} \to \mathbf{V}^{\mathbf{C}}$ for which $U = \mathbf{V}^{\bar{\sigma}} \cdot \Phi$, **B** must have **V**-coequalizers and U must reflect isomorphisms.

Conversely, suppose **B** has **V**-coequalizers and U reflects isomorphisms. Let **T** be the **V**-triple on $\mathbf{V}^{\mathbf{S}}$ induced by the **V**-adjoint pair (U, F). Since **B** has **V**-coequalizers which are preserved by the **V**-cocontinuous functor U and U reflects isomorphisms, it follows from [4, Theorem II.2.1, p. 78] that (U, F) is **V**-tripleable, i.e., there is an equivalence of categories $\Phi: \mathbf{B} \to (\mathbf{V}^{\mathbf{S}})^{\mathbf{T}}$ for which $U = U^{\mathbf{T}} \cdot \Phi$. But F, being a **V**-adjoint, is **V**-cocontinuous and U is assumed **V**-cocontinuous so that the functor of **T**, UF, is **V**-cocontinuous. The rest follows from (4.1) and its proof. ▌

(4.3) DEFINITION. Let **B** be a **V**-category, and let $K \in |\mathbf{B}|$. Then K is said to be a **V**-*atom* of **B** if the **V**-functor $\mathbf{B}(K, -)$ is **V**-cocontinuous.

The **Ab**-atoms of the **Ab**-category $_R\mathbf{M}$, for any ring R, are the finitely generated projective R-modules. We also note that for any bicomplete closed category **V**, for any small **V**-category **C**, and for each $C \in |\mathbf{C}|$, $\mathbf{C}(C, -)$ is a **V**-atom of $\mathbf{V}^{\mathbf{C}}$, since the evaluation functor $e_{\mathbf{C}}: \mathbf{V}^{\mathbf{C}} \to \mathbf{V}$ is **V**-cocontinuous by (1.18) and is **V**-represented by $\mathbf{C}(C, -)$ by the **V**-Yoneda lemma (1.19).

(4.4) DEFINITION. A **V**-cocomplete category **B** is said to be **V**-*atomic* if there is a small category **C** and a **V**-functor $g: \mathbf{C} \to \mathbf{B}$ which is **V**-generating (1.17) and has the property that gC is a **V**-atom of **B** for every C in **C**.

We note that for any bicomplete closed category **V** and small **V**-category **C**, $\mathbf{V}^{\mathbf{C}}$ is **V**-atomic since the **V**-functor $R_{\mathbf{C}^o}: \mathbf{C}^o \to \mathbf{V}^{\mathbf{C}}$ is **V**-generating and $R_{\mathbf{C}^o}C = \mathbf{C}(C, -)$ is a **V**-atom for every C in **C**.

We say that a **V**-category **B** is **V**-*coregular* if every **V**-epimorphism of **B** is a **V**-coequalizer. If **V** is **V**-coregular, so is $\mathbf{V}^{\mathbf{C}}$ for every small **V**-category **C**. The proof of this statement is similar to the one for coregularity done in the proof of (4.9) in [1].

The following is a generalization of the "second characterization of functor categories" given by Bunge [1 (4.16)].

(4.5) THEOREM. *Let* **V** *be a bicomplete closed coregular category, and let* **B** *be a* **V**-*category. Then there is a small* **V**-*category* **C** *for which* **B** *is* **V**-*equivalent to* $\mathbf{V}^{\mathbf{C}}$ *iff* **B** *is a* **V**-*cocomplete* **V**-*atomic category which is* **V**-*coregular.*

Proof. The conditions on **B** are necessary since we have already noted above that a **V**-functor category satisfies them.

Suppose **B** is a **V**-cocomplete **V**-atomic category which is **V**-coregular. Then there exists a small category **C** and a **V**-functor $g: \mathbf{C} \to \mathbf{B}$ which is **V**-generating and has the property that gC is a **V**-atom of **B** for every object C

of $\mathbf{C}$. Since $\mathbf{B}$ is $\mathbf{V}$-cocomplete, the singular and realization $\mathbf{V}$-functors of g, $U = \mathbf{B}(g-, -)$ and $-\otimes_{\mathbf{C}} g$ exist and form an adjoint pair

$$(U, -\otimes_{\mathbf{C}} g) : \mathbf{B} \to \mathbf{V}^{\mathbf{C}^o}.$$

Since the family of evaluation $\mathbf{V}$-functors $\{e_C : \mathbf{V}^{\mathbf{C}^o} \to \mathbf{V}\}_{C\in|\mathbf{C}|}$ collectively creates $\mathbf{V}$-colimits and $e_C U = \mathbf{B}(gC, -)$ is $\mathbf{V}$-cocontinuous for every object C of $\mathbf{C}$, the functor U is $\mathbf{V}$-cocontinuous. Thus the triple $\mathbf{T}$ induced by the adjoint pair $(U, -\otimes_{\mathbf{C}} g)$ on $\mathbf{V}^{\mathbf{C}^o}$ is $\mathbf{V}$-cocontinuous. By (3.11) we have $(\mathbf{V}^{\mathbf{C}^o})^{\mathbf{T}}$ is $\mathbf{V}$-equivalent to $\mathbf{V}^{\mathbf{C}'}$ for some small $\mathbf{V}$-category $\mathbf{C}'$. If we show that U is $\mathbf{V}$-tripleable we will have that $\mathbf{B}$ is $\mathbf{V}$-equivalent to $(\mathbf{V}^{\mathbf{C}^o})^{\mathbf{T}}$ and hence to $\mathbf{V}^{\mathbf{C}'}$. Since $\mathbf{B}$ has $\mathbf{V}$-coequalizers and U preserves them, we need only show that U reflects isomorphisms by [4, Theorem II.2.1, p. 78].

Since the functor g is $\mathbf{V}$-generating we have $\epsilon_B : \mathrm{Lan}_g g(B) \to B$ is a $\mathbf{V}$-epimorphism for every B in $\mathbf{B}$. Thus,

$$\mathbf{B}(\epsilon_B, B') : \mathbf{B}(B, B') \to \mathbf{B}(\mathrm{Lan}_g g(B), B')$$

is a $\mathbf{V}$-monomorphism for every object B' of $\mathbf{B}$. Since the diagram

$$\begin{array}{ccc}
\mathbf{B}(B, B') & \xrightarrow{\mathbf{B}(\epsilon_B, B')} & \mathbf{B}(\mathrm{Lan}_g g(B), B') \\
 & \searrow & \downarrow \wr \\
 & & \mathbf{B}\left(\int^C \mathbf{B}(gC, B) \otimes_{\mathbf{B}} gC, B'\right) \\
 & U_{BB'} \searrow & \downarrow \wr \\
 & & \int_C \mathbf{V}(\mathbf{B}(gC, B), \mathbf{B}(gC, B')) \\
 & \searrow & \downarrow \wr \\
 & & \mathbf{V}^{\mathbf{C}^o}(UB, UB')
\end{array}$$

commutes, $U_{BB'}$ is a $\mathbf{V}$-monomorphism for all objects B and B' in $\mathbf{B}$. Thus U is $\mathbf{V}$-faithful. Let $f : B \to B'$ be a morphism in $\mathbf{B}$ for which $U(f)$ is an isomorphism in $\mathbf{V}^{\mathbf{C}^o}$. Since U is $\mathbf{V}$-faithful it follows that f is both a $\mathbf{V}$-monomorphism and a $\mathbf{V}$-epimorphism. Since $\mathbf{B}$ is $\mathbf{V}$-coregular, f is a $\mathbf{V}$-coequalizer of a pair of morphisms (α, β) of $\mathbf{B}$. But since f is also a $\mathbf{V}$-monomorphism we have $\alpha = \beta$, and hence f, being the $\mathbf{V}$-coequalizer of equal morphisms, is an isomorphism. ∎

It follows immediately from the above that

(4.6) COROLLARY. *Let* $\mathbf{V}$ *be a bicomplete closed coregular category. Then any* $\mathbf{V}$*-cocomplete* $\mathbf{V}$*-atomic category which is* $\mathbf{V}$*-coregular is* $\mathbf{V}$*-complete.* ∎

We note that if **V** is a bicomplete closed category with the property that the underlying functor $\mathbf{V}_0(I, -)$: $\mathbf{V}_0 \to \mathbf{Ens}$ is faithful and **B** is a **V**-cocomplete category, then **B** has the property that there exists a small **V**-category **C** and a **V**-generating functor $g: \mathbf{C} \to \mathbf{B}$ iff the set of objects $\{gC\}_{C \in \mathbf{C}}$ is a generating set for **B**. If, in addition, **V** has the property that $\mathbf{V}_0(I, -)$ reflects isomorphisms, then a **V**-category **B** is **V**-cocomplete iff it is cocomplete; and if **B** is cocomplete then a functor $h: \mathbf{B} \to \mathbf{V}$ is **V**-cocontinuous iff h preserves colimits, since in this case, **V**-colimits, tensors and coends in **B** can be constructed as colimits in **B** (see [4, Theorem A.2, p. 168].) If **V** has all these properties and **B** is a **V**-category which is cocomplete, then **B** is **V**-atomic iff **B** is atomic in the sense of Bunge [1, Definition 4.13]. Thus we sharpen Bunge's "second characterization of functor categories" [1 (4.16)]. (We note that if **V** is coregular and $\mathbf{V}_0(I, -)$ is faithful, then $\mathbf{V}_0(I, -)$ reflects isomorphisms.)

(4.7) Corollary (Bunge). *Let* **V** *be a bicomplete coregular closed category with the property that the underlying functor* $\mathbf{V}_0(I, -)$: $\mathbf{V} \to \mathbf{Ens}$ *is faithful. Then a* **V**-*category* **B** *is* **V**-*equivalent to* $\mathbf{V}^{\mathbf{C}}$ *for some small* **V**-*category* **C**, *iff* **B** *is a cocomplete coregular atomic category.* ▮

5. V-Continuous Cotriples and Monoids

We would like to dualize the results of Sections 2 and 3 to a **V**-cotriple $\mathbf{G} = (G, \epsilon, \delta)$ on $\mathbf{V}^{\mathbf{C}}$ for which G is **V**-continuous. To do this, we need to know that every **V**-continuous functor on $\mathbf{V}^{\mathbf{C}}$ has a **V**-left adjoint. This necessitates the following

(5.1) Definition. A **V**-bicomplete **V**-category **D** is said to *have a small cogenerating* functor if there exist a small **V**-category **B** and a **V**-functor $k: \mathbf{B} \to \mathbf{D}$ which is **V**-cogenerating (1.17), i.e.,

$$\eta_D : D \to (\mathrm{Ran}_k\, k)D = \int_B \bar{\mathbf{D}}(\mathbf{D}(D, kB), kB)$$

is a **V**-monomorphism for every D in **D**.

(5.2) Proposition. *If* **V** *has a small cogenerating functor, so does* $\mathbf{V}^{\mathbf{C}}$ *for every small* **V**-*category* **C**.

Proof. By hypothesis, we have a **V**-functor $k: \mathbf{B} \to \mathbf{V}$ with **B** small and with $\eta_V: V \to (\mathrm{Ran}_k\, k)V$ a **V**-monomorphism for every V in **V**. Define $u: \mathbf{C}^o \otimes \mathbf{B} \to \mathbf{V}^{\mathbf{C}}$ to be the **V**-functor defined on objects by $u(C', B)C =$

$\mathbf{V}(\mathbf{C}(C, C'), kB)$ for (C', B) in $|\mathbf{C}^o \otimes \mathbf{B}|$ and C in $|\mathbf{C}|$. We note that for every g in $|\mathbf{V}^{\mathbf{C}}|$ and (C', B) in $|\mathbf{C}^o \otimes \mathbf{B}|$, $\mathbf{V}^{\mathbf{C}}(g, u(C', B)) = \mathbf{V}(gC', kB)$. Now $\mathrm{Ran}_k(k)\colon \mathbf{V} \to \mathbf{V}$ is a $\mathbf{V}$-triple, and hence induces a V-triple

$$(\mathrm{Ran}_k\, k)^{\mathbf{C}}\colon \mathbf{V}^{\mathbf{C}} \to \mathbf{V}^{\mathbf{C}}$$

with unit $\eta^{\mathbf{C}}\colon 1_{\mathbf{V}^{\mathbf{C}}} = (1_{\mathbf{V}})^{\mathbf{C}} \Rightarrow (\mathrm{Ran}_k\, k)^{\mathbf{C}}$ which is pointwise a $\mathbf{V}$-monomorphism. Thus we are done if we show that $(\mathrm{Ran}_k\, k)^{\mathbf{C}} = \mathrm{Ran}_u\, u$ as triples.

Since $\mathrm{Ran}_k\, k = \int_B \mathbf{V}(\mathbf{V}(-, kB), kB)$ by (1.16) there are $\mathbf{V}$-natural transformations $\delta_B\colon \mathrm{Ran}_k\, k \Rightarrow \mathbf{V}(\mathbf{V}(-, kB), kB)$ for each object B of $\mathbf{B}$. If $F\colon \mathbf{V}^{\mathbf{C}} \to \mathbf{V}^{\mathbf{C}}$ is a $\mathbf{V}$-functor and $\varphi\colon F \Rightarrow (\mathrm{Ran}_k\, k)^{\mathbf{C}}$ is a $\mathbf{V}$-natural transformation, φ is equivalent to a $\mathbf{V}$-natural family $\{\varphi_{gC}\colon I \to \mathbf{V}((Fg)C, (\mathrm{Ran}_k\, k)\, gC)\}$. This family defines, by the following commutative diagram, a $\mathbf{V}$-natural family $\{\xi_{gCB}\}$ with g in $|\mathbf{V}^{\mathbf{C}}|$, C in $|\mathbf{C}|$, and $B \in |\mathbf{B}|$:

$$\begin{array}{lc}
I \xrightarrow{\ \varphi_{gC}\ } & \mathbf{V}((Fg)C, (\mathrm{Ran}_k k)(gC)) \\
 & \downarrow{\scriptstyle \mathbf{V}(FgC,\delta_B(gC))} \\
 & \mathbf{V}((Fg)C, \mathbf{V}(\mathbf{V}(gC, kB), kB)) \\
 & \downarrow{\scriptstyle \wr} \\
\quad\searrow{\scriptstyle \xi_{gCB}} & \mathbf{V}(\mathbf{V}(gC, kB), \mathbf{V}((Fg)C, kB)) \\
 & \downarrow{\scriptstyle \wr} \\
 & \mathbf{V}(\mathbf{V}^{\mathbf{C}}(g, u(C, B)), \mathbf{V}^{\mathbf{C}}(Fg, u, (C, B))).
\end{array}$$

Conversely, a $\mathbf{V}$-natural family $\{\xi_{gCB}\}$ defines uniquely a $\mathbf{V}$-natural family $\{\varphi_{gC}\}$ for which the above diagram commutes. By the representability theorem of Eilenberg–Kelly [5, Theorem 8.6], there is a one-to-one correspondence between $\mathbf{V}$-natural families $\{\xi_{gCB}\}$ and $\mathbf{V}$-natural families

$$\{\bar{\varphi}_{(C,B)}\colon I \to \mathbf{V}^{\mathbf{C}}(Fu(C, B), u(C, B))\},$$

i.e., a $\mathbf{V}$-natural transformation $\bar{\varphi}\colon Fu \Rightarrow u$. Thus, $(\mathrm{Ran}_k\, k)^{\mathbf{C}} = \mathrm{Ran}_u\, u$. ▮

A category is said to be *well powered* if the subobjects of any object in the category are in one-to-one correspondence with a set. (We note that **Ens** and **Ab** are well powered categories.)

(5.3) PROPOSITION. *Let* $\mathbf{V}$ *be a well powered bicomplete closed category which has a small cogenerating functor. Then every* $\mathbf{V}$*-continuous functor* $G\colon \mathbf{V}^{\mathbf{C}} \to \mathbf{A}$ *has a* $\mathbf{V}$*-left adjoint which is* $\mathrm{Ran}_G(id)$.

Proof. Since $\mathbf{V}$ is well powered $\mathbf{V}^{\mathbf{C}}$ is $\mathbf{V}$-well powered [4, Proposition IV.1.3]. This, together with (5.2) and [4, Theorem III.2.2] in which $\mathrm{Ran}_G(id)$ is shown to exist and be the $\mathbf{V}$-left adjoint of G, finishes the proof. ∎

We shall assume henceforth that $\mathbf{V}$ is well powered and has a small cogenerating functor.

Let $\mathscr{L}(\mathbf{V}^{\mathbf{C}})$ be the full submetamonoidal category of $\mathrm{End}(\mathbf{V}^{\mathbf{C}})$ whose objects are $\mathbf{V}$-continuous functors on $\mathbf{V}^{\mathbf{C}}$, with the inclusion functor $\iota: \mathscr{L}(\mathbf{V}^{\mathbf{C}}) \to \mathrm{End}(\mathbf{V}^{\mathbf{C}})$ a strict monoidal metafunctor. By the above, every object G of $\mathscr{L}(\mathbf{V}^{\mathbf{C}})$ has a $\mathbf{V}$-left adjoint $F = \mathrm{Ran}_G(id)$, and by (2.10) the adjoint pair (G, F) is represented by a $\mathbf{V}$-functor $u: \mathbf{C}^o \otimes \mathbf{C} \to \mathbf{V}$ with $G \cong [u, -]_{\mathbf{C}}$ and $F \cong u \otimes_{\mathbf{C}} -$.

For a monoidal category $\mathbf{M}$, let ${}^t\mathbf{M}$ be the transpose monoidal category with the same objects as $\mathbf{M}$ and with tensor, ${}^t\otimes$, defined by $M \,{}^t\!\otimes M' = M' \otimes M$ for all objects M and M' in $\mathbf{M}$. A monoid in $\mathbf{M}$ becomes a monoid in ${}^t\mathbf{M}$. Then we have by (2.10) and (5.3)

(5.4) THEOREM. *The functor* $N: {}^t[\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}]^o \to \mathscr{L}(\mathbf{V}^{\mathbf{C}})$ *defined by* $N(u) = [u, -]_{\mathbf{C}}$ *for* u *in* $[\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}]$ *is a strict morphism of monoidal metacategories which is an adjoint equivalence with right adjoint* N^r *defined by* $N^r(G)(C', C) = F((C', -))C$, *where* $F = \mathrm{Ran}_G(id)$, *the* $\mathbf{V}$*-left adjoint of* G. ∎

(5.5) COROLLARY.

(1) *The functor N induces an equivalence of monoidal categories between the category of comonoids in* $[\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}]$ *and the category of* $\mathbf{V}$*-triples on* $\mathbf{V}^{\mathbf{C}}$ *with* $\mathbf{V}$*-continuous functors.*

(2) *The functor N induces an equivalence of monoidal categories between the category of monoids in* $[\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}]$ *and the category of* $\mathbf{V}$*-cotriples on* $\mathbf{V}^{\mathbf{C}}$ *with* $\mathbf{V}$*-continuous functors.* ∎

(5.6) COROLLARY. *Let* $\mathbf{G} = (G, \epsilon, \delta)$ *be a* $\mathbf{V}$*-cotriple on* $\mathbf{V}^{\mathbf{C}}$ *with* G $\mathbf{V}$*-continuous. Then there is a* $\mathbf{V}$*-category* $\mathbf{C}'$ *and a* $\mathbf{V}$*-functor* $x: \mathbf{C} \to \mathbf{C}'$, *which is a surjection on objects, such that the* $\mathbf{V}$*-category of* $\mathbf{G}$*-coalgebras* ${}^{\mathbf{G}}(\mathbf{V}^{\mathbf{C}})$ *is* $\mathbf{V}$*-equivalent to the* $\mathbf{V}$*-functor category* $\mathbf{V}^{\mathbf{C}'}$. *Furthermore,* $\mathbf{V}^x: \mathbf{V}^{\mathbf{C}'} \to \mathbf{V}^{\mathbf{C}}$ *and its* $\mathbf{V}$*-right adjoint* $(\mathbf{V}^x)^r$ *induce the* $\mathbf{V}$*-cotriple* $\mathbf{G}$.

Proof. By 5.5(2) there is a monoid $u: \mathbf{C}^o \otimes \mathbf{C} \to \mathbf{V}$ with $N(u) = [u, -]_{\mathbf{C}} \cong G$. Since $u \,\overline{\otimes}_{\mathbf{C}} -$ is $\mathbf{V}$-left adjoint to G and is a $\mathbf{V}$-triple with its triple structure induced by the cotriple structure on G, we can apply (3.12), (2.13), and [1 (2.7 and 2.8)] to get the desired result. ∎

In the remainder of this paper we investigate $\mathbf{T}$-algebras ($\mathbf{G}$-coalgebras) when $\mathbf{T}$ ($\mathbf{G}$) is a bicontinuous triple (cotriple) on $\mathbf{V}^{\mathbf{C}}$. As mentioned in the introduction this generalizes the situation when $\mathbf{C} = \mathbf{R}$ (R a commutative ring) and $T = \mathrm{Hom}_R(\Lambda, -)(G = \Lambda \otimes_R -)$ where Λ is an R-coalgebra which is a finitely generated projective R-module.

Define the dualization functor D: ${}^t[\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}]^o \to [\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}]$ to be the composition of the monoidal functors N: ${}^t[\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}]^o \to \mathscr{L}(\mathbf{V}^{\mathbf{C}})$ and $R^*\iota$: $\mathscr{L}(\mathbf{V}^{\mathbf{C}}) \to [\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}]$. By direct calculation one sees that on objects u in $[\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}]$ and (A, B) in $\mathbf{C}^o \otimes \mathbf{C}$, $\mathrm{D}u(A, B) = (u\backslash \mathbf{C})(A, B) = \mathbf{V}^{\mathbf{C}}(u(B, -), \mathbf{C}(A, -))$. (When $\mathbf{V} = \mathbf{Ab}$ and $\mathbf{C} = \mathbf{R}$ (R any ring) then $[\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}]$ is the category of R-bimodules and for M and R-bimodule $DM =$ the group of left R-homomorphisms from M to R, regarded as an R-bimodule in the usual way.) Since D is a map of monoidal categories, it maps comonids of $[\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}]$ into monoids of $[\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}]$.

For each f in $\mathbf{V}^{\mathbf{C}}$ we can define f^* in $\mathbf{V}^{\mathbf{C}^o}$ by letting f^* be the composition of $R_{\mathbf{C}^o}$: $\mathbf{C}^o \to \mathbf{V}^{\mathbf{C}}$ and $\mathbf{V}^{\mathbf{C}}(f, -)$: $\mathbf{V}^{\mathbf{C}} \to \mathbf{V}$ (one sees that $f^*(C) = \mathbf{V}^{\mathbf{C}}(f, \mathbf{C}(C, -))$ for every object C of $\mathbf{C}^o$). The operation $*$ defines a contravariant $\mathbf{V}$-functor $-^*$: $(\mathbf{V}^{\mathbf{C}})^o \to \mathbf{V}^{\mathbf{C}^o}$. We note that f^* is always $\mathbf{V}$-continuous.

(5.7) PROPOSITION. *Let f be in $\mathbf{V}^{\mathbf{C}}$.*

(1) *There is a $\mathbf{V}$-natural transformation*

$$\epsilon_f: f^* \mathbin{\overline{\otimes}}_{\mathbf{C}} - \Rightarrow \mathbf{V}^{\mathbf{C}}(f, -),$$

which is an equivalence iff $\mathbf{V}^{\mathbf{C}}(f, -)$ is $\mathbf{V}$-cocontinuous.

(2) *There is a $\mathbf{V}$-natural transformation*

$$\eta_f: -^* \mathbin{\overline{\otimes}}_{\mathbf{C}} f \Rightarrow \mathbf{V}^{\mathbf{C}}(-, f),$$

which is an equivalence iff $-^* \mathbin{\overline{\otimes}}_{\mathbf{C}} f$ is $\mathbf{V}$-continuous ($-^* \otimes f$ is the composition of $-^*$: $(\mathbf{V}^{\mathbf{C}})^o \to \mathbf{V}^{\mathbf{C}^o}$ and $- \mathbin{\overline{\otimes}}_{\mathbf{C}} f$: $\mathbf{V}^{\mathbf{C}^o} \to \mathbf{V}$).

In addition, $\eta_f(f) = \epsilon_f(f)$.

Proof. Statement (1) is (2.3) with $F = \mathbf{V}^{\mathbf{C}}(f, -)$ and $\mathbf{C}$ replaced by $\mathbf{C}^o$. Statement (2) is (2.7) with $F = -^* \mathbin{\overline{\otimes}}_{\mathbf{C}} f$. ∎

Now if w: $\mathbf{C}^o \otimes \mathbf{C} \to \mathbf{V}$ is a $\mathbf{V}$-functor, $Dw(-, C) = w(C, -)^*$ for all C in $\mathbf{C}$. Thus there is a $\mathbf{V}$-natural transformation

$$\epsilon_w: Dw \mathbin{\overline{\otimes}}_{\mathbf{C}} - \Rightarrow [w, -]_{\mathbf{C}},$$

between endofunctors on $\mathbf{V}^{\mathbf{C}}$ and we have

(5.8) COROLLARY. *For a $\mathbf{V}$-functor w: $\mathbf{C}^o \otimes \mathbf{C} \to \mathbf{V}$ the following statements are equivalent:*

(a) $\epsilon_w: Dw \overline{\otimes}_{\mathbf{C}} - \Rightarrow [w, -]_{\mathbf{C}}$ *is an equivalence*;

(b) $[w, -]_{\mathbf{C}}$ *is* **V**-*cocontinuous*;

If $\mathbf{V} = \mathbf{Ens}$ *or* ${}_R\mathbf{M}$ *for* R *a commutative ring then* (b) *is equivalent to*

(c) $w(C, -)$ *is a retract of a representable functor for every* C *in* $\mathbf{C}$.

Proof. (a) is equivalent to (b) by (5.7). The equivalence of (b) and (c) follows from the following lemma. ▮

(5.9) Lemma. *Let* $\mathbf{V} = \mathbf{Ens}$ *or* ${}_R\mathbf{M}$ *where* R *is a commutative ring. Then the following are equivalent for* f *in* $\mathbf{V}^{\mathbf{C}}$:

(a) $-\overline{\otimes}_{\mathbf{C}} f: \mathbf{V}^{\mathbf{C}^o} \to \mathbf{V}$ *is* **V**-*continuous*;

(b) $-{}^* \overline{\otimes}_{\mathbf{C}} f: (\mathbf{V}^{\mathbf{C}})^o \to \mathbf{V}$ *is* **V**-*continuous*;

(c) $\mathbf{V}^{\mathbf{C}}(f, -)$ *is* **V**-*cocontinuous*;

(d) f *is a retract of a representable functor.*

Proof. (a) $\Rightarrow$ (b) is obvious since $-{}^* \overline{\otimes}_{\mathbf{C}} f$ is the composition of two **V**-continuous functors if $\overline{\otimes}_{\mathbf{C}} f$ is **V**-continuous. (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a) can be proved by using the arguments in [8 (4.4)] where β is the η of (5.7) and [9 (4.2)] where β is the ϵ of (5.7). ▮

(5.10) Theorem. *Let* $\mathbf{T} = (T, \eta, \mu)$ *be a triple on* $\mathbf{V}^{\mathbf{C}}$ *with* $\mathbf{T}$ *a bicontinuous* **V**-*functor. Then there is a* **V**-*category* $\mathbf{C}'$ *and a cocontinuous* **V**-*functor* $x: \mathbf{C} \to \mathbf{C}'$ *which is a surjection on objects, such that the category of* **T**-*algebras is equivalent to the category* $\mathbf{V}^{\mathbf{C}'}$. *Furthermore* $\mathbf{V}^x: \mathbf{V}^{\mathbf{C}'} \to \mathbf{V}^{\mathbf{C}}$ *and its left adjoint induce the triple* **T**.

Proof. We need only show that the functor $x: \mathbf{C} \to \mathbf{C}'$ of (3.12) is **V**-cocontinuous. By (5.5) (1), $T \cong [w, -]_{\mathbf{C}}$ for some comonoid w of $[\mathbf{C}^o \otimes \mathbf{C}, \mathbf{V}]$. By (5.8) $T \simeq Dw \overline{\otimes}_{\mathbf{C}} -$. Since $T \simeq \mathbf{C}'(x-, x-) \overline{\otimes}_{\mathbf{C}} -$ by (3.12) we have $Dw \simeq \mathbf{C}'(x-, x-)$. In particular $\mathbf{C}'(x-, xC) \simeq Dw(-, C) = w(C, -)^*$ is continuous for every C in $|\mathbf{C}|$. Since x is a surjection on objects this implies x is **V**-cocontinuous. ▮

(5.11) Theorem. *Let* $\mathbf{V} = \mathbf{Ens}$ *or* ${}_R\mathbf{M}$ *where* R *is a commutative ring and* $\mathbf{G} = (G, \epsilon, \delta)$ *be a cotriple on* $\mathbf{V}^{\mathbf{C}}$ *with* G *a* **V**-*bicontinuous functor. Then there is a* **V**-*category* $\mathbf{C}'$ *and a continuous* **V**-*functor* $x: \mathbf{C} \to \mathbf{C}'$, *which is a surjection on objects, such that the category of* **G**-*coalgebras is equivalent to the category* $\mathbf{V}^{\mathbf{C}'}$. *Furthermore* $\mathbf{V}^x$ *and its right adjoint induce the cotriple* **G**.

Proof. We need only show that the functor $x: \mathbf{C} \to \mathbf{C}'$ of (5.6) is continuous where $G \simeq [u, -]_{\mathbf{C}}$ for the monoid $u = \mathbf{C}'(x-, x-)$. Since G is cocontinuous, (5.8) implies $\mathbf{C}'(xC, x-)$ is a retract of a representable functor and therefore continuous for every C in $\mathbf{C}$. Thus, since x is surjective on objects, x is continuous. ∎

An unanswered question is the following: What is a nice description of the closed categories for which (5.9) holds and hence (5.11) holds?

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