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Computations of Critical Groups at a Degenerate Critical Point for Strongly Indefinite Functionals

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INTRODUCTION

It is well known that Morse theory is a powerful tool for studying the multiple solutions of differential equations which arise in the calculus of variations. In applications, in order to apply Morse theory, the computations of the critical groups are essential. There are some results in the computations of the critical groups: in the nondegenerate case, the critical groups are determined by the Morse index completely; in the degenerate case, we have the Spliting Lemma and Shifting Theorem, which reduce the computing of the critical groups for an isolated critical point to a functional which is defined on the kernel of the Hessian (a finite dimensional space in many cases). More information about Morse theory may be found in Chang [1]. Here, we would like to mention that a different approach to this theory (based on the Conley index (see [2])) has been developed by Benci [3]. On the other hand, some results are concerned with the critical groups of a critical point determined by standard minimax methods (see [4, 5]).

Suppose that E is a real Hilbert space; for a functional $f \in C^1(E, R)$ recall that (see [1]) the critical groups of an isolated critical point p of f are defined by

$$C_q(f, p) = H_q(f_a \cap U, f_a \cap U - \{p\}),$$

where $a = f(p)$, $f_a = \{x \in E : f(x) \le a\}$, U is a closed neighborhood of p, and $H_q(\cdot, \cdot)$ is the qth (singular) homology group with coefficients in a field F ($C_q(f, p)$ is independent of U by the excision property of homology). However, if f is a strongly indefinite functional, then $C_q(f, p) = 0$ for all q;

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this implies that the definition of the critical groups is of little use for strongly indefinite functionals. Here, we use a different notion of critical groups developed in [6], which is valuable for strongly indefinite functionals.

Recently, another kind of critical groups, the so-called critical groups at infinity, was introduced; see, for example, [7] for the strong resonant problems, [8] for the resonant problems, [9] for the strongly indefinite problems, and [10] for the Landesman–Lazer problems. Combined with the critical groups at a real critical point, the critical groups at infinity play an important role in dealing with the resonant problem of differential equations, especially in the studying of the asymptotically linear problem. So, accurate computations of the critical groups both at isolated critical points and at infinity are very important.

In this paper, using a different Morse theory developed in [6], we investigate the critical groups at degenerate critical points for a strongly indefinite functional, and then a different notion for the critical groups at infinity is given. By new computations of the critical groups at infinity and at the origin, we obtain some abstract critical point theorems, which will be very useful for studying the existence of nontrivial solutions to asymptotically linear Hamiltonian systems and the boundary value problems of elliptic systems.

Our paper is organized as follows. In Section 1, we recall some basic results and concepts in [6, 12]; in Section 2, we deal with critical groups at the origin; in Section 3, we deal with critical groups at infinity. Some existence results for nontrivial critical points of asymptotically quadratic strongly indefinite functions with Landsman–Lazer resonant problems are presented in Section 4.

1. MORSE THEORY FOR STRONGLY INDEFINITE FUNCTIONALS

Let H be a real Hilbert space with inner product $\langle , \rangle$ and norm$\|\cdot\|$, and $H = \oplus_{i=1}^{\infty} H_i$ with all subspaces H_i being mutually orthogonal and finite dimensional. Set $H^n = \oplus_{i=1}^{n} H_i$ and assume

(1) $f \in C^2(H, R)$ with the form $f(x) = \frac{1}{2}\langle Ax, x\rangle + G(x)$,

(2) A is a bounded, linear, self-adjoint operator with a finite dimensional kernel, and its zero eigenvalue is isolated in the spectrum of A,

(3) $G'(x) := K(x)$ is compact and is globally Lipschitz continuous on a bounded set.

The following key concepts are due to [10, 12].

Definition 1.1. Let $\Gamma = \{P_n \mid n = 1, 2, \ldots\}$ be a sequence of orthogonal projections. We call Γ an approximation scheme w.r.t. A if the following properties hold:

(1) $H_n = P_n H$ is finite dimensional for $\forall n$,

(2) $P_n \to I$ as $n \to \infty$ (strongly),

(3) $[P_n, A] = P_n A - AP_n \to 0$ (in the operator norm).

Definition 1.2 (Gromoll–Meyer Pair). Let f be a C^1-functional on a C^1-Finsler manifold M and S be a subset of the critical set K for f. A pair of subsets (W, W_-) is said to be a Gromoll–Meyer pair (for short G-M pair) for S associated with a pseudo gradient field X for f if considering the flow η generated by X the following conditions hold:

(1) W is a closed neighborhood of S, satisfying $W \cap K = S$ and $W \cap f_\alpha = \varnothing$ for some α,

(2) W_- is an exit set of W, i.e., $\forall x_0 \in W$, if $\forall t_1 > 0$ such that $\eta(x_0, t_1) \notin W$, there exists $t_0 \in [0, t_1)$ such that $\eta(x_0, [0, t_0]) \subset W$ and $\eta(x_0, t_0) \in W_-$,

(3) W_- is closed and is a union of a finite number of submanifolds that are transversal to the flow η.

Definition 1.3 (Dynamically Isolated Critical Sets). A subset S of the critical set K for f is said to be a dynamically isolated critical set if there exists a closed neighborhood O of S and regular values $\alpha < \beta$ of f such that

$$O \subset f^{-1}[\alpha, \beta] \qquad \text{and} \qquad cl(\widetilde{O}) \cap K \cap f^{-1}[\alpha, \beta] = S;$$

we say that (O, α, β) is an isolating triplet for S, where $\widetilde{O} = \cup_{t \in R} \eta(O, t)$, η is the flow associated with f.

For an isolated critical set S of f, we define the critical groups $C_*(f, S)$ (see [6]) by

$$C_*(f, S) := H^{*+m(P_n(A+P)P_n)}(W_n, W_{n-}),$$

where (W_n, W_{n-}) is a G-M pair for S_n, and S_n is the critical set of the restriction functional $f_n = f|_{H^n}$, $\{P_n | n = 1, 2 \ldots\}$ is an approximation scheme w.r.t. A, P is the orthogonal projection onto the kernel space of A, and $m(\cdot)$ is the Morse index of the operator $(\cdot)$. It has been shown in [6] that the critical groups are independent of the choice of isolating triplet, G-M pair, and approximation scheme.

In order to deal with the infinity of Morse indices of a strongly indefinite functional at its isolated critical points, Chang *et al.* [12] proved the following theorem and introduced the notion of an abstract Maslov index.

THEOREM 1.1. *Let B be a linear symmetric compact operator and suppose that $A+B$ has a bounded inverse. Then the difference of Morse indices*

$$m(P_n(A+B)P_n) - m(P_n(A+P)P_n)$$

eventually becomes a constant independent of n, where A is a bounded self-adjoint operator with a finite dimensional kernel N, the restriction $A|_{N^\perp}$ is invertible, P is the orthogonal projection from H to N, and $\{P_n\}_1^\infty$ is an approximation scheme w.r.t. A.

For the given invertible operator $A+B$ with compact symmetric B, we define an index

$$I(B) = \lim_{n\to\infty}(m(P_n(A+B)P_n) - m(P_n(A+P)P_n)).$$

DEFINITION 1.4. For a given compact linear symmetric operator B, let P_B be the orthogonal projection onto the kernel space ker $(A+B)$. We define

$$N(B) = \dim\ker(A+B) \qquad \text{and} \qquad I_-(B) = I(B+P_B)$$

and call the pair $(I_-(B), N(B))$ the abstract Maslov index of B w.r.t. A.

Remark 1.1. Let H_n be a sequence of invariant subsapces of $(A+B)$, and P_n be a sequence of orthogonal projection operators onto H_n. According to the positive, zero, and negative eigenvalue of $A+B$, we have $H_n = H_{n+}\oplus H_0 \oplus H_{n-}$. Then, by direct computation, we have $m(P_n(A+B+P_B)P_n) = m(P_n(A+B)P_n)$; thus $I_-(B) = m(P_n(A+B)P_n) - m(P_n(A+P)P_n)$ (for large n).

2. CRITICAL GROUPS AT THE ORIGIN

For the strongly indefinite functional $f(x) = \frac{1}{2}\langle Ax, x\rangle + G(x)$, we assume that

(H_1) A is a bounded self-adjoint operator defined on H.

(H_2) θ is an isolated critical point of f and 0 is the isolated critical value. $G'(x)$ is compact and globally Lipschitz continuous on any bounded set.

$(H_3^\pm)$ There exists a linear symmetric compact operator B_0, such that

$$\|G'(x) - B_0(x)\| \le c(\|x_0\|^\alpha + \|x_+ + x_-\|^\beta)$$

$$\frac{G(x_0) - \frac{1}{2}\langle B_0 x_0, x_0\rangle}{\|x_0\|^{2\alpha}} \longrightarrow \pm\infty, \qquad \text{as } \|x_0\| \to 0,$$

where $x = x_+ + x_0 + x_- \in H_+ \oplus H_0 \oplus H_-$ is the orthogonal decomposition corresponding to the spectrum of the operator $(A+B_0)$.

Lemma 2.1. *Let B be a compact linear operator. If $\Gamma = \{P_n \mid n = 1, 2, \ldots\}$ is an approximation scheme w.r.t. $(A + B)$, then it is also an approximation scheme w.r.t. A.*

Proof. Indeed, we need only to prove that

$$[P_n, A] = P_n A - AP_n \to 0 \qquad (n \to \infty) \text{ (in the operator norm).}$$

Since $[P_n, A + B] \to 0(n \to \infty)$ (in the operator norm) and B is compact, by [12, Lemma 2.1], we have $\|(P_n - I)B\| \to 0$ $(n \to \infty)$ and $\|B(P_n - I)\| \to 0$ $(n \to \infty)$. Thus

$$[P_n, A] = [P_n, A + B] - [(P_n - I), B] \to 0 \qquad (n \to \infty).$$

Theorem 2.1. *Under the assumptions (H_1)–$(H_3^{\pm})$, we have*

$$C_q(f, \theta) = \begin{cases} F, & \text{if } q = I_-(B_0) + N(B_0) \\ 0, & \text{otherwise} \end{cases} \text{ if } (H_3^-) \text{ holds}$$

$$C_q(f, \theta) = \begin{cases} F, & \text{if } q = I_-(B_0) \\ 0, & \text{otherwise} \end{cases} \text{ if } (H_3^+) \text{ holds},$$

where here and in the following, F denotes the coefficient field of the critical groups.

Proof. (0) For the compact operator B_0, let $\cdots \lambda_{-2} \le \lambda_{-1} < 0 < \lambda_1 \le \lambda_2 \cdots$ be the eigenvalues of $A + B_0$, and $\{e_j \mid j = \pm 1, \pm 2, \ldots\}$ be the eigenvectors of $A + B_0$ corresponding to $\{\lambda_j \mid j = \pm 1, \pm 2, \ldots\}$. For any $m \ge 0$, set $H_0 = \ker(A + B_0)$, $H_m = H_0 \oplus \operatorname{span}\{e_1, \ldots, e_m\} \oplus \operatorname{span}\{e_{-1}, \ldots, e_{-m}\}$, and let $\Gamma = \{P_m \mid m = 1, 2, \ldots\}$ be the orthogonal projections from H to H_m. Then $(A + B_0)P_m = P_m(A + B_0)$ and $P_m \to I, (m \to \infty)$(strongly). Consequently, $\Gamma = \{P_m \mid m = 1, 2, \ldots\}$ is an approximation scheme w.r.t. $(A + B_0)$, so it is w.r.t. A by Lemma 2.1.

(1) *Case* (H_3^-). Set $C := A + B_0$, $g(x) := G(x) - \frac{1}{2}\langle B_0 x, x\rangle$, and $m = \inf\{|\langle Cx_{\pm}, x_{\pm}\rangle| \; \|x_{\pm}\| = 1, x_{\pm} \in H_{\pm}\}$. Then we have

$$\begin{aligned} f(x) &= \tfrac{1}{2}\langle (A + B_0)x, x\rangle + G(x) - \tfrac{1}{2}\langle B_0 x, x\rangle, \\ &= \tfrac{1}{2}\langle Cx, x\rangle + g(x), \end{aligned}$$

satisfying

(i) $g(\theta) = 0$, and g' is compact and globally Lipschitz continuous on a bounded set,

(ii) $g'(\theta) = \theta$ and $\|g'(x)\| \le c(\|x_-\|^{\alpha} + \|x_+ + x_0\|^{\beta})$, for $\|x_0\|$ small enough,

(iii) $g(x_0)\|x_0\|^{-2\alpha} \to -\infty$ as $\|x_0\| \to 0$.

Since θ is an isolated critical point and 0 is an isolated critical value, it is easy to check that $\{\theta\}$ is a dynamically isolated critical set. Recall the critical group

$$C_*(f, \theta) = H^{*+m(P_n(A+P)P_n)}(W_n, W_{n-}),$$

where (W_n, W_{n-}) is a G-M pair for the critical set S_n of the restriction functional $f_n = f\,|_{H_n}$ in H_n, which is associated with the flow generated by $df_n = P_n(A + G')P_n$. Let (O, α, β) be an isolating triplet for $\{\theta\}$ satisfying that $(O \cap H_n, \alpha, \beta)$ is an isolating triplet for S_n (the existence of such a triplet is proved in [6]). For large n, we take a neighborhood N of S_n in $O \cap H_n$ of the form

$$N = \{x \mid \|x_+\|^2 - d\|x_-\|^2 - k\|x_0\|^{2\alpha} \le \epsilon r_0^2,\ \|x_-\|^2 + \|x_0\|^2 \le r_0^2\},$$

where d, k, ϵ, r_0 are to be determined later, $x = x_+ + x_0 + x_- \in H_n = H_n^+ \oplus H_0 \oplus H_n^-$. The boundary of N consists of two parts, namely

$$\Gamma_1 = \{x \mid \|x_+\|^2 - d\|x_-\|^2 - k\|x_0\|^{2\alpha} = \epsilon r_0^2,\ \ \|x_-\|^2 + \|x_0\|^2 \le r_0^2\},$$
$$\Gamma_2 = \{x \mid \|x_+\|^2 - d\|x_-\|^2 - k\|x_0\|^{2\alpha} \le \epsilon r_0^2,\ \ \|x_-\|^2 + \|x_0\|^2 = r_0^2\}.$$

Since the normal vector on Γ_1 is $n = x_+ - dx_- - k\alpha\|x_0\|^{2\alpha-2}x_0$, we have

$$\begin{aligned}
(df_n(x), n) &= (P_n(C + g')P_n(x), n) \\
&= (P_nCP_nx_+, x_+) - d(P_nCP_nx_-, x_-) + (P_ng'P_n(x), n) \\
&\ge m\|x_+\|^2 + dm\|x_-\|^2 - c(\|x_0\|^\alpha + \|x_+ + x_-\|^\beta) \\
&\quad \cdot (\|x_+\|^2 + d\|x_-\| + k\alpha\|x_0\|^{2\alpha-1}) \\
&\ge m\|x_+\|^2 + dm\|x_-\|^2 - c\|x_0\|^\alpha(\|x_+\| + d\|x_-\|) - L(x) \\
&\ge \tfrac{m}{2}(\|x_+\|^2 - d\|x_-\|^2 - k\|x_0\|^{2\alpha}) \\
&= \tfrac{1}{2}m\epsilon r_0^2 > 0, \qquad (2.1)
\end{aligned}$$

as k is large and r_0 small, where $L(x)$ consists of some higher terms w.r.t. $\|x_+\|^2$, $\|x_-\|^2$, and $\|x_0\|^\alpha$. Now we study the behavior of f_n near the boundary Γ_2:

$$\begin{aligned}
f_n(x) &= \tfrac{1}{2}\langle Cx_+, x_+\rangle + \tfrac{1}{2}\langle Cx_-, x_-\rangle + g(x) \\
&\le \tfrac{1}{2}\|C\|\|x_+\|^2 - \tfrac{1}{2}m\|x_-\|^2 + g(x_0) \\
&\quad + c(\|x_0\|^\alpha + \|x_+ + x_-\|^\beta)(\|x_+\| + \|x_-\|) \\
&\le \|C\|\|x_+\|^2 - \tfrac{1}{4}m\|x_-\|^2 + \tfrac{1}{2}g(x_0) \\
&\le \|C\|\epsilon r_0^2 + (\|C\|d - \tfrac{1}{4}m)\|x_-\|^2 + \tfrac{1}{4}g(x_0). \qquad (2.2)
\end{aligned}$$

Take d satisfying $\|C\|d - \frac{1}{4}m < 0$. For given r_0 and ϵ small enough, we choose constants r_1, r_2, and $\delta > 0, r_1 < r_2 < r_0$ such that

$$\begin{array}{lll} f_n(x) \geq -\frac{\delta}{2} & \text{if } x \in N, & \|x_0 + x_-\| \leq r_1 \\ f_n(x) < 0 & \text{if } x \in N, & \|x_0 + x_-\| > r_1 \\ f_n(x) \geq -\frac{3}{4}\delta & \text{if } x \in N, & \|x_0 + x_-\| \leq r_2 \\ f_n(x) < -\frac{\delta}{2} & \text{if } x \in N, & \|x_0 + x_-\| = r_2 \\ f_n(x) < -\delta & \text{if } x \in N, & \|x_0 + x_-\| = r_0. \end{array}$$

Let

$$N_1 = \{x \in N \mid \|x_0 + x_-\| \leq r_1\}, \qquad N_2 = \{x \in N \mid \|x_0 + x_-\| \leq r_2\},$$
$$\Gamma_{r_1} = \{x \in N \mid \|x_0 + x_-\| = r_1\}, \qquad \Gamma_{r_2} = \{x \in N \mid \|x_0 + x_-\| = r_2\}.$$

Set

$$W_n = \left\{\eta_n(t, u) \mid t \geq 0,\ u \in N_2,\ f_n(\eta_n(t, u)) \geq -\frac{3}{4}\delta\right\}$$
$$W_{n-} = W_n \cap f_n^{-1}\left(-\frac{3}{4}\delta\right),$$

where η_n is the negative gradient flow generated by df_n in H_n. Then (W_n, W_{n-}) is a G-M pair for $N \cap K_{f_n}$ associated with the flow η_n. Set

$$A_1 = \left\{\eta_n(t, u) \mid t \geq 0,\ u \in \Gamma_{r_2},\ f(\eta_n(t, u)) \geq -\frac{3}{4}\delta\right\}.$$

Then $W_n = N_2 \cup (A_1 \cup W_{n-})$. Since $\eta_n(t, u)$ can not enter N_1, whenever $u \in \Gamma_{r_2}$, it follows that if $\eta_n(t, u) \in W_n$, then there exists a unique t_1 such that $\eta_n(t_1, u) \in W_{n-}$. Let t_2 be the time of reaching Γ_1, $t = \min\{t_1, t_2\}$. Then the mapping

$$\sigma(s, u) = \begin{cases} \eta_n(st, u), & u \in \Gamma_{r_2} \\ u, & u \in W_{n-} \end{cases}$$

is a deformation retraction of $\Gamma_{r_2} \cup W_{n-}$ onto W_{n-}.

Similarly, we can deform $A_1 \cup W_{n-}$ to W_{n-}. So

$$\begin{aligned} H^*(W_n, W_{n-}) &= H^*(N_2 \cup (A_1 \cup W_{n-}), W_{n-}) \\ &\cong H^*(N_2 \cup W_{n-}, W_{n-}) \\ &\cong H^*(N_2 \cup W_{n-}, \Gamma_{r_2} \cup W_{n-}) \\ &= H^*(N_2, \Gamma_{r_2}) \\ &= \begin{cases} F, & \text{if } * = m(P_n(A + B_0)P_n) + \dim(\ker(A + B_0)) \\ 0, & \text{otherwise;} \end{cases} \end{aligned}$$

therefore (for large n),

$$C_*(f,\theta) = H^{*+m(P_n(A+P)P_n)}(W_n, W_{n-}) = \begin{cases} F, & \text{if } * = I_-(B_0)+N(B_0) \\ 0, & \text{otherwise.} \end{cases}$$

(2) *Case* (H_3^+). In this case, we define the neighborhood of S_n as

$$N = \{x \mid -d\|x_+\|^2 + \|x_-\|^2 - k\|x_0\|^{2\alpha} \le \epsilon r_0^2,\ \|x_+\|^2 + \|x_0\|^2 \le r_0^2\}.$$

Then the boundary of N consists of the following two parts:

$$\Gamma_{r0} = \{x \mid -d\|x_+\|^2 + \|x_-\|^2 - k\|x_0\|^{2\alpha} = \epsilon r_0^2,\ \|x_+\|^2 + \|x_0\|^2 \le r_0^2\},$$
$$\widetilde{\Gamma}_{r0} = \{x \mid -d\|x_+\|^2 + \|x_-\|^2 - k\|x_0\|^{2\alpha} \le \epsilon r_0^2,\ \|x_+\|^2 + \|x_0\|^2 = r_0^2\}.$$

We define

$$\Gamma_r = \{x \mid -d\|x_+\|^2 + \|x_-\|^2 - k\|x_0\|^{2\alpha} = \epsilon r_0^2,\ \|x_+\|^2 + \|x_0\|^2 \le r^2\}.$$

Then the normal vector on Γ_{r_0} is $n = -dx_+ + x_- - k\alpha\|x_0\|^{2\alpha-2}x_0$. Similarly to (1), we have

$$\begin{aligned}(df_n(x), n) &= (P_n(C+g')P_n(x), n) \\ &\le \tfrac{m}{2}(d\|x_+\|^2 - \|x_-\|^2 + k\|x_0\|^{2\alpha}) \\ &= -\tfrac{1}{2}m\epsilon r_0^2 < 0.\end{aligned} \tag{2.3}$$

This implies that the negative gradient of f_n is outward on Γ_{r_0}.

Since

$$\begin{aligned} f_n(x) &= \tfrac12\langle Cx_+, x_+\rangle + \tfrac12\langle Cx_-, x_-\rangle + g(x) \\ &\ge \tfrac12 m\|x_+\|^2 - \tfrac12\|C\|\|x_-\|^2 + g(x_0) \\ &\quad - c(\|x_0\|^\alpha + \|x_+ + x_-\|^\beta)(\|x_+\| + \|x_-\|) \\ &\ge -\|C\|\|x_0\|^2 + \tfrac14 m\|x_+\|^2 + \tfrac12 g(x_0) \\ &\ge -\|C\|\epsilon r_0^2 - (\|C\|d - \tfrac14 m)\|x_+\|^2 + \tfrac14 g(x_0), \end{aligned} \tag{2.4}$$

choose d satisfying $\|C\|d - \frac14 m < 0$, for given r_0 and small enough ϵ, and take positive constants $r_1, r_2, \delta > 0, r_1 < r_2 < r_0$ such that

$$\begin{aligned} f_n(x) &\le \tfrac{\delta}{2} && \text{if } x\in N, \quad \|x_0 + x_+\| \le r_1 \\ f_n(x) &> 0 && \text{if } x\in N, \quad \|x_0 + x_+\| \ge r_1 \\ f_n(x) &\ge \delta && \text{if } x\in N, \quad \|x_0 + x_+\| \ge r_2. \end{aligned}$$

First, we deform N to $\Gamma_{r_0} \cup N_2 = \{x \in N \mid \|x_0 + x_+\| \le r_2\}$ by a geometric deformation σ_2. Let

$$W_n = N_2 \cap f_{n(\frac{\delta}{2})} \cup \Gamma_{r_0}, \qquad W_{n-} = \Gamma_{r_0} \cap f_{n(\frac{\delta}{2})}.$$

Then it is easy to check that (W_n, W_{n-}) is a G-M pair we need.

Second, we use the negative gradient flow η_n generated by df_n to make a deformation. Let t_1 be the time of reaching the level set $f_{n(\delta/2)}$, let t_2 be the time of reaching the boundary Γ_{r_0}, take $t = \min\{t_1, t_2\}$, and define

$$\sigma_1(s,u) = \begin{cases} \eta_n(st, u), & u \in N_2 \cup \Gamma_{r_0},\ \ t > 0 \\ u, & u \in N_2 \cup \Gamma_{r_0},\ \ t = 0. \end{cases}$$

Then $\sigma = \sigma_2 \circ \sigma_1$ is a deformation retraction of $N_2 \cup \Gamma_{r_0}$ onto W_n. Hence

$$H^*(W_n, W_{n-}) \cong H^*(N_2 \cup \Gamma_{r_0}, \Gamma_{r_0}) = \begin{cases} F, & \text{if } * = m(P_n(A+B_0)P_n) \\ 0, & \text{otherwise.} \end{cases}$$

So (for large n),

$$\begin{aligned} C_*(f,\theta) &= H^{*+m(P_n(A+P)P_n)}(W_n, W_{n-}) \\ &= \begin{cases} F, & \text{if } * = m(P_n(A+B_0)P_n) - m(P_n(A+P)P_n) \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} F, & \text{if } * = I_-(B_0) \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

The theorem is proved.

3. CRITICAL GROUPS AT INFINITY

In this section, without loss of generality, we assume that f has finite critical values. That is, if S is the set consisting of all the critical points of f, then $S \subset f^{-1}[a,b]$ for some $a < b$. Clearly, S is a dynamically isolated set. We define the critical groups of infinity by $C_*(f,\infty) := C_*(f,S)$.

Suppose that (I_1), (I_2) are the same as (H_1), (H_2) in Section 2.

$(I_3^{\pm})$ There exists a linear compact operator B_∞ such that

$$\|G'(x) - B_\infty\| \le c(\|x\|^\alpha), \qquad \alpha \in (0,1) \qquad c > 0$$

$$\frac{G(P_0x) - (1/2)\langle B_\infty(P_0x), P_0x\rangle}{\|P_0x\|^{2\alpha}} \longrightarrow \pm\infty, \qquad \text{as } \|P_0x\| \to \infty,$$

where P_0 is the orthogonal projection from H to the kernel space of $(A + B_\infty)$.

THEOREM 3.1. *Under the assumptions* (I_1)–$(I_3^{\pm})$, *we have*

$$C_q(f,\infty) = \begin{cases} F, & \text{if } q = I_-(B_\infty) + N(B_\infty) \\ 0, & \text{otherwise,} \end{cases} \text{ if } (I_3^-) \text{ holds}$$

$$C_q(f,\infty) = \begin{cases} F, & \text{if } q = I_-(B_\infty) \\ 0, & \text{otherwise} \end{cases} \text{ if } (I_3^+) \text{ holds.}$$

Proof. (0) For the compact operator B_∞, let $\cdots \gamma_{-2} \le \gamma_{-1} < 0 < \gamma_1 \le \gamma_2 \cdots$ be the eigenvalues of $A + B_\infty$, and $\{q_j \mid j = \pm 1, \pm 2, \ldots\}$ be the eigenvectors of $A + B_\infty$ corresponding to $\{\gamma_j \mid j = \pm 1, \pm 2, \ldots\}$. For any $n \ge 0$, set $H_0 = \ker(A + B_\infty)$, $H_n = H_0 \oplus \operatorname{span}\{q_1, \ldots, q_n\} \oplus \operatorname{span}\{q_{-1}, \ldots, q_{-n}\}$, and let $\Gamma = \{P_n \mid n = 1, 2, \ldots\}$ be the orthogonal projection from H to H_n. Then $\Gamma = \{P_m \mid m = 1, 2, \ldots\}$ is an approximation scheme w.r.t. $(A + B_0)$, so it is w.r.t. A by Lemma 2.1.

(1) We deal with the case (I_3^-). Note that

$$\begin{aligned} f(x) &= \tfrac{1}{2}\langle (A + B_\infty)x, x\rangle + G(x) - \tfrac{1}{2}\langle B_\infty x, x\rangle \\ &= \tfrac{1}{2}\langle Cx, x\rangle + g(x), \end{aligned}$$

where $C := A + B_\infty$.
Let $m = \inf\{|\langle Cx_\pm, x_\pm\rangle| \; \|x_\pm\| = 1, x_\pm \in H_\pm\}$. Then, we have

(i) g' is compact and globally Lipschitz continuous on a bounded set,

(ii) $\|g'(x)\| \le c(\|x\|^\alpha + 1), \quad \alpha \in (0,1), \quad c > 0,$

(iii) $g(P_0 x)\|P_0 x\|^{-2\alpha} \to +\infty$ as $\|P_0 x\| \to \infty$.

In order to compute $C_*(f,\infty)$, we need to compute $H^{*+m(P_n(A+P)P_n)}(W_n, W_{n-})$ only (for large n), where $(W_n.W_{n-})$ is a G-M pair for S_n associated with the flow generated by df_n, and S_n is the set consisting of all the critical points of the restriction function $f_n = f|_{H_n}$. In the following, we define a "cylinder" in H_n (for n large enough) by

$$C_0 = \{x \mid \|x_+\|^2 - d\|x_-\|^2 - kh(\|x_0\|) \le M\},$$

where $d, k, M > 0$ will be determined later and

$$h(t) = \begin{cases} |t|^{2\alpha}, & \alpha > \frac{1}{2} \\ |t|^{2\alpha}, & \alpha \le \frac{1}{2}, \quad |t| \ge 2 \\ |t|^2, & \alpha \le \frac{1}{2}, \quad |t| \le 1 \\ \text{smooth}, & \alpha \le \frac{1}{2}, \quad 1 \le |t| \le 2. \end{cases} \tag{3.1}$$

Then the normal vector on ∂C_0 is $n = x_+ - dx_- - kh'(\|x_0\|)(x_0/\|x_0\|)$. For n large enough, we have

$$\begin{aligned}(df_n(x), n) &= (P_nCP_nx_+, x_+) - d(P_nCP_nx_-, x_-) + (P_ng'P_n(x), n)\\ &\geq m\|x_+\|^2 - dm\|x_-\|^2 - c(\|x\|^\alpha + 1)\\ &\quad\cdot\left(\|x_+\| + d\|x_-\| + kh'(\|x_0\|)\frac{x_0}{\|x_0\|}\right)\\ &\geq m\|x_+\|^2 + dm\|x_-\|^2 - c\|x_0\|^\alpha(\|x_+\| + d\|x_-\|) - L(x) - c,\end{aligned}$$

where $L(x)$ consists of lower terms w.r.t. $\|x_+\|^2, \|x_-\|^2$, and $\|x_0\|^{2\alpha}$. Choosing k large enough, we have

$$\begin{aligned}(df_n(x), n) &\geq \tfrac{m}{2}(\|x_+\|^2 - d\|x_-\|^2 - k\|x_0\|^{2\alpha}) - c\\ &= \tfrac{m}{2}M - c > 0, \qquad \text{if } M > \tfrac{2c}{m}.\end{aligned}$$

So the negative gradient $-df_n(x)$ points inward to C_0 on ∂C_0 and f_n has no critical points outside C_0.

Now we prove that $\forall x \in C_0$

$$f_n(x) \to -\infty \Longleftrightarrow \|x_- + P_0x\| \to \infty, \qquad \text{uniformly in } x_+. \tag{3.2}$$

In fact, for $\forall x \in C_0$

$$\begin{aligned}f_n(x) &= \tfrac{1}{2}(Cx_+, x_+) + \tfrac{1}{2}(Cx_-, x_-) + g(x)\\ &\leq \tfrac{1}{2}\|C\|\|x_+\|^2 - \tfrac{1}{2}m\|x_-\|^2 + g(x_0)\\ &\quad + c(\|x\|^\alpha + 1)(\|x_+ + x_-\|)\\ &\leq \|C\|\|x_+\|^2 - \tfrac{1}{4}m\|x_-\|^2 + \tfrac{1}{2}g(x_0) + c\\ &\leq (-\tfrac{1}{4}m + \|C\|d)\|x_-\|^2 + \tfrac{1}{4}g(x_0) + c.\end{aligned} \tag{3.3}$$

Choose d satisfying $-\frac{1}{4}m + \|c\|d < 0$, then

$$f_n(x) \to -\infty, \qquad \text{as } \|x_- + P_0x\| \to \infty \qquad \text{uniformly in } x_+.$$

On the other hand, by (I_3^-)

$$\begin{aligned}f_n(x) &= \tfrac{1}{2}(Cx_+, x_+) + \tfrac{1}{2}(Cx_-, x_-) + g(x)\\ &\geq -\tfrac{1}{2}\|C\|\|x_-\|^2 + g(x_0) - c(\|x\|^\alpha + 1)(\|x_+ + x_-\|)\\ &\geq -\tfrac{1}{2}\|C\|\|x_-\|^2 + g(x_0) - c\|x_0\|^{2\alpha} - L(x) - \bar{c}\\ &\geq -\|C\|\|x_-\|^2 + 2g(x_0) - \bar{c},\end{aligned} \tag{3.4}$$

where $L(x)$ consists of some lower terms with respect to $\|x_+\|^2$, $\|x_-\|^2$, and $\|x_0\|^{2\alpha}$, c, $\bar{c}$ are constants. Combining by (3.3) and (3.4), (3.2) is proved.

Take $T > 0$ large enough such that there are no critical points in f_{-T}. By (3.2), there exist $\alpha_1 < \alpha_2 < -T, R_1 > R_2 > 0$ such that

$$\begin{aligned} f_n(x) &\geq \alpha_2, \quad && \text{if } x \in C_0 \quad \|x_- + x_0\| < R_2 \\ f_n(x) &< \alpha_2, && \text{if } x \in C_0 \quad \|x_- + x_0\| > R_1 \\ f_n(x) &\geq \alpha_1, && \text{if } x \in C_0 \quad \|x_- + x_0\| < R_1. \end{aligned}$$

Set

$$W_n = \{x \in C_0 \mid \quad f_n(x) \geq \alpha_1\}, \qquad W_{n-} = \{x \in C_0 \mid \quad f_n(x) = \alpha_1\}.$$

Then it is easy to check that (W_n, W_{n-}) is a G-M pair for the critical set S_n which includes all the critical points of f_n.

Let

$$A_1 = \{x \in C_0 \mid f_n(x) \geq \alpha_1 \quad \text{and } \|x_- + x_0\| \geq R_2\}.$$

We define

$$\mu(t,x)=\begin{cases} x, & \text{if } \|x_-+P_0x\| \geq R_1 \\ x_+ + \frac{x_-+x_0}{\|x_-+x_0\|}(tR_1+(1-t)\|x_-+x_0\|) & \text{if } \|x_-+x_0\| \leq R_1. \end{cases}$$

Then $\sigma_1 = \mu(1, \cdot)$ is a deformation retraction of $C_2 := \{x \in C_0 \mid \|x_- + x_0\| \geq R_2\}$ onto $C_1 := \{x \in C_0 \mid \|x_- + x_0\| \geq R_1\}$.

For $x \in A_1$ we use the negative gradient flow η_n generated by df_n to make a deformation. Let t be the time of reaching the level set f_{α_1}. We define a deformation retraction σ_2 by

$$\sigma_2(s, x) = \begin{cases} \eta_n(st, x), & t > 0 \\ x, & t = 0. \end{cases}$$

Then $\sigma = \sigma_2 \circ \sigma_1$ is a deformation retraction of A_1 onto W_{n-}

Set

$$\begin{aligned} D_1 &:= \{x \in C_0 \mid \|x_- + x_0\| \leq R_1\}, \\ D_{12} &:= \{x \in C_0 \mid R_2 \leq \|x_- + x_0\| \leq R_1\}. \end{aligned}$$

Then

$$\begin{aligned} H^*(W_n, W_{n-}) &\cong H^*(W_n, A_1) \cong H^*(D_1, D_{12}) \\ &= \begin{cases} F, & \text{if } * = I_-(B_\infty) + N(B_\infty) \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

(2) *Case* (I_3^+). In this case, we define

$$C_0 = \{x \mid \quad \|x_-\|^2 - d\|x_+\|^2 - kh(\|x_0\|) \leq M\}.$$

Then the normal vector on ∂C_0 is $n = x_- - dx_+ - kh'(\|x_0\|)(x_0/\|x_0\|)$. Similarly to (1), we have

$$(df_n(x), n) \le \tfrac{m}{2}(d\|x_+\|^2 - \|x_-\|^2 + k\|x_0\|^{2\alpha}) + c$$
$$\le -\tfrac{m}{2}M + c < 0.$$

Here we choose k, M large enough.

This implies that f_n has no critical points outside C_0, and the negative gradient $-df_n$ points outside on ∂C_0.

Now we prove that $\forall x \in C_0$

$$f_n(x) \to +\infty \iff \|x_+ + x_0\| \to \infty, \qquad \text{uniformly in } x_-. \tag{3.5}$$

In fact, for $\forall x \in C_0$, by similar arguments as in (3.3) and (3.4), we have

$$\begin{aligned} f_n(x) &= \tfrac{1}{2}(Cx_+, x_+) + \tfrac{1}{2}(Cx_-, x_-) + g(x) \\ &\ge \tfrac{1}{2}m\|x_+\|^2 - \tfrac{1}{2}\|C\|\|x_-\|^2 + g(x_0) \\ &\quad - c(\|x\|^\alpha + 1)(\|x_+\| + \|x_-\|) \\ &\ge \tfrac{1}{4}m\|x_+\|^2 - \|C\|\|x_-\|^2 + \tfrac{1}{2}g(x_0) - c \\ &\ge (\tfrac{1}{4}m - \|C\|d)\|x_+\|^2 + \tfrac{1}{4}g(x_0) - c. \end{aligned} \tag{3.6}$$

On the other hand,

$$\begin{aligned} f_n(x) &\le \tfrac{1}{2}\|C\|\|x_+\|^2 + g(x_0) + c(\|x\|^\alpha + 1)(\|x_+\| + \|x_-\|) \\ &\le \|C\|\|x_+\|^2 + 2g(x_0) + c, \end{aligned} \tag{3.7}$$

and (3.6) and (3.7) imply (3.5).

So for $\forall T > 0$, $\exists M_2 > M_1 > T$ and $R_2 > R_1 > 0$ such that

$$\begin{aligned} &f_n(x) \le M_1, &&\text{if } x \in C_0 \quad \|x_0 + x_+\| < R_1 \\ &f_n(x) > M_1, &&\text{if } x \in C_0 \quad \|x_0 + x_-\| > R_2 \\ &f_n(x) \le M_2, &&\text{if } x \in C_0 \quad \|x_0 + x_+\| \le R_2. \end{aligned}$$

Set

$$W_n = \{x \in C_0 \mid f_n(x) \le M_1\}, \quad W_{n-} = \{x \in C_0 \mid f_n(x) \le M_1 \text{ and } x \in \partial C_0\}.$$

Let

$$N_i = \{x \in C_0 \mid \|x_0 + x_+\| \le R_i\}, \quad \Gamma_i = \{x \in \partial C_0 \mid \|x_0 + x_+\| \le R_i\}, \quad i = 1, 2.$$

Similarly to (1), first we deform N_2 to $N_1 \cup \Gamma_2$ by a geometric deformation σ_1. Second, let s_1 be the time of reaching ∂C_0 along the negative gradient flow, and let s_2 be the time of reaching the level set $f_n^{-1}(M_1)$. Take $s = \min(s_1, s_2)$ and define

$$\sigma_2(t, x) = \begin{cases} \eta(ts, x), & s > 0 \\ x, & s = 0. \end{cases}$$

Then, $\sigma = \sigma_2 \circ \sigma_1$ is a deformation retraction of W_n onto $N_1 \cup \Gamma_2$. Thus

$$\begin{aligned} H^*(W_n, W_{n-}) &\cong H^*(N_1 \cup \Gamma_2, W_{n-}) \\ &\cong H^*(N_1, \Gamma_1) \\ &= \begin{cases} F, & \text{if } * = m(P_n(A + B_\infty)P_n) \\ 0, & \text{otherwise;} \end{cases} \end{aligned}$$

hence

$$\begin{aligned} &H^{*+m(P_n(A+P)P_n)}(W_n, W_{n-}) \\ &\quad = \begin{cases} F, & \text{if } * + m(P_n(A+P)P_n) = m(P_n(A + B_\infty)P_n) \\ 0, & \text{otherwise} \end{cases} \\ &\quad = \begin{cases} F, & \text{if } * = m(P_n(A + B_\infty)P_n) - m(P_n(A+P)P_n) \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

So (for large n)

$$C_*(f, \infty) := C_*(f, S) = \begin{cases} F, & \text{if } * = I_-(B_\infty) \\ 0, & \text{otherwise.} \end{cases}$$

The theorem is proved.

4. CRITICAL POINT THEOREM FOR ASYMPTOTICALLY QUADRATIC FUNCTIONALS

Asymptotically quadratic functionals have been studied by many authors; see [1, 13] and the references therein. Applications can be found in semilinear elliptic boundary value problems, Hamiltonian systems, and the nonlinear wave equation. The corresponding functional is asymptotically quadratic provided the equation is asymptotically linear.

In this section, we use the same notation as in Sections 2 and 3. Consider the functional with the form $f(x) = \frac{1}{2}\langle Ax.x\rangle + G(x)$. Set $G'(x) =: K(x)$. Recall that the functional f is said to be asymptotically quadratic at infinity and at the origin, respectively, if there exist self-adjoint bounded operators B_0 and B_∞ such that

$$\frac{|B_\infty - K(x)|}{|x|} \to 0 \quad \text{as } |x| \to \infty \tag{4.1}$$

$$\frac{|B_0 - K(x)|}{|x|} \to 0 \quad \text{as } |x| \to 0. \tag{4.2}$$

Then it is easy to check that under the assumptions of $(H_3^{\pm})$ and $(I_3^{\pm})$ in Sections 2 and 3, the functional f is asymptotically quadratic.

THEOREM 4.1. *Let $f \in C^2(H, R)$, (H_1), and (H_2) hold, and $N(B_0) \neq 0$, $N(B_\infty) \neq 0$. If one of the following conditions holds,*

(1) (H_3^+), (I_3^+) *hold*, $I_-(B_0) \neq I_-(B_\infty)$

(2) (H_3^-), (I_3^+) *hold*, $N(B_0) + I_-(B_0) \neq I_-(B_\infty)$

(3) (H_3^+), (I_3^-) *hold*, $I_-(B_0) \neq I_-(B_\infty) + N(B_\infty)$

(4) (H_3^-), (I_3^-) *hold*, $N(B_0) + I(B_0) \neq I(B_\infty) + N(B_\infty)$,

then f has at least one nontrivial critical point.

Proof. We prove case (1) only; the others are similar. By Theorem 2.1 (H_3^+) and Theorem 3.1 (I_3^+), we have

$$C_q(f, \theta) = \begin{cases} F, & \text{if } q = I_-(B_0) \\ 0, & \text{otherwise} \end{cases}$$

$$C_q(f, \infty) = \begin{cases} F, & \text{if } q = I_-(B_\infty) \\ 0, & \text{otherwise.} \end{cases}$$

By (1), we have $C_q(f, \theta) \neq C_q(f, \infty)$. By the $I_-(B_\infty)$th Morse inequality, f has at least one nontrivial critical point x_1 satisfying $C_{I_-(B_\infty)}(f, x_1) \neq 0$. The proof is complete.

THEOREM 4.2. *In addition to the assumptions of Theorem* 4.1, *suppose that $x_0 \neq \theta$ is a nondegenerate critical point. Then f has another critical point $x_1 \neq \theta, x_0$, if one of the cases of Theorem* 4.1 *holds.*

Proof. If f only has the critical point θ, x_0, then

$$C_q(f, x_0) = \begin{cases} F, & \text{if } q = \mu \\ 0, & \text{otherwise,} \end{cases}$$

where μ is the Morse index of x_0, and

$$C_q(f, \theta) = \begin{cases} F, & \text{if } q = r_0 \\ 0, & \text{otherwise,} \end{cases}$$

where

$$r_0 = \begin{cases} I_-(B_0), & \text{if } (H_3^+) \text{ holds} \\ I_-(B_0) + N(B_0), & \text{if } (H_3^-) \text{ holds.} \end{cases}$$

By the last Morse equality

$$(-1)^{\bar{r}} = (-1)^{r_0} + (-1)^{\mu}, \qquad \text{where } \bar{r} = \begin{cases} I_-(B_\infty), & \text{if } (I_3^+) \text{ holds} \\ I_-(B_\infty) + N(B_\infty), & \text{if } (I_3^-) \text{ holds,} \end{cases}$$

a contradiction! The proof is finished.

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