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Generalized Watson's summation formula for ${}_3F_2(1)$

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Abstract

A summation formula is given for ${}_3F_2(a,b,c;\frac{1}{2}(a+b+i+1),2c+j;1)$ with fixed j and arbitrary i ($i,j\in\mathbb{Z}$). This result generalizes the classical Watson's theorem which deals with the case $i=j=0$.

Extensions to the cases of ${}_3F_2(a,1+i+j-a,c;e,1+i+2c-e;1)$, and ${}_3F_2(a,b,c;1+i+a-b,1+i+j+a-c;1)$ are given. Notice that the case $i=j=0$ corresponds to the classical theorems due to Whipple and Dixon, respectively.

Keywords: Generalized hypergeometric functions; Watson's summation theorem

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1. Introduction

The classical Watson's summation theorem may be written as

$$ {}_3F_2\left(\begin{matrix}a,b,c\\ d,2c\end{matrix}\middle|\,1\right)=2^{a+b-2}\frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b)\Gamma(d)\Gamma(c-d+1)\Gamma(c+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(a)\Gamma(b)\Gamma(c-\frac{1}{2}a+\frac{1}{2})\Gamma(c-\frac{1}{2}b+\frac{1}{2})} $$

$$ (d=\tfrac{1}{2}(a+b+1)), \tag{1.1} $$

provided $\Re(2c-a-b)>-1$ (see, e.g., [3, Section 4.4], or [9, Section 5.2.4], where the duplication formula for the Γ function should be used). Wimp [14] has shown that Watson's formula *cannot* be generalized in the sense that ${}_3F_2(a,b,c;d,2c;1)$ for unrestricted a,b,c,d cannot be expressed as a general ratio of Γ functions. (Later, Zeilberger [15] gave a short proof of Wimp's theorem.)

In several recent papers (see [4–6, 12]) functions of the type

$$ f_{i,j}(a,b,c):={}_3F_2\left(\begin{matrix}a,b,c\\ d+\frac{1}{2}i,2c+j\end{matrix}\middle|\,1\right)\quad(\Re(2c-a-b)>-i-2j-1) \tag{1.2} $$

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were studied for various values of the parameters i and j; [6] contains extensive tables of components of 25 formulae obtained for $i,j\in\{-2,-1,0,1,2\}$, by repeated use of relations linking contiguous hypergeometric functions (see [1]; or [11, pp. 80–85]).

In this paper we show that an analytical formula can be given for $f_{i,j}$ with fixed j and arbitrary $i\in\mathbb{Z}$ (cf. Theorem 2.4).

Also, we show that the recently studied problem of evaluation of

$$g_{i,j}(a,c,e):={}_3F_2\left(\begin{matrix}a, 1+i+j-a, c\\ e, 1+i+2c-e\end{matrix}\,\middle|\,1\right) \tag{1.3}$$

and

$$h_{i,j}(a,b,c):={}_3F_2\left(\begin{matrix}a, b, c\\ 1+i+a-b, 1+i+j+a-c\end{matrix}\,\middle|\,1\right) \tag{1.4}$$

(see [7, 8]) may be reduced to the evaluation of a function of the type (1.2) (see Sections 3 and 4, respectively). Notice that g_{00} and h_{00} may be evaluated by the classical Whipple's and Dixon's theorems, respectively.

2. Main result

Lemma 2.1. *For any $i,j\in\mathbb{Z}$ we have*

$$f_{ij}\equiv f_{i,j}(a,b,c)=\sum_{k=0}^{|i|}\gamma_{i,k}f_{0j}(a+k,b+|i|-k,c), \tag{2.1}$$

where the coefficients $\gamma_{i,k}\equiv\gamma_{i,k}(a,b)$ $(k=0,1,\ldots,|i|)$ are either given by

$$\gamma_{i,k}:=(-1)^k\binom{i}{k}\frac{(a)_k(b)_{i-k}(b-a+i-2k)}{(b-a-k)_{i+1}}\qquad(i\geqslant 0), \tag{2.2}$$

or are defined recursively by

$$\gamma_{i,k}(a,b)=C_i(a,b)\gamma_{i+1,k-1}(a+1,b)+C_i(b,a)\gamma_{i+1,k}(a,b+1)\quad(i\leqslant 0;\ \gamma_{0,0}\equiv 1) \tag{2.3}$$

with

$$C_i(a,b):=\frac{a(a-b+i+1)}{(a-b)(a+b+i+1)}.$$

(We adopt the convention that $\gamma_{i,q}\equiv 0$ for $q<0$ or $q>|i|$). Here

$$(\alpha)_m:=\Gamma(\alpha+m)/\Gamma(\alpha)=\prod_{k=0}^{m-1}(\alpha+k)\quad(m\geqslant 0).$$

Proof. The first part of the result (for $i\geqslant 0$) is obtained by iterating the formula

$$(a-b)f_{i,j}(a,b,c)=af_{i-1,j}(a+1,b,c)-bf_{i-1,j}(a,b+1,c)\quad(i=0,1,\ldots;\ j\in\mathbb{Z}),$$

which is a disguised form of a contiguity relation for ${}_3F_2$ given in [11, Section 48, Eq. (14)].

The second part (for $i \leqslant 0$) is obtained by iterating the contiguity relation

$$f_{i,j}(a,b,c) = C_i(a,b)f_{i+1,j}(a+1,b,c) + C_i(b,a)f_{i+1,j}(a,b+1,c)$$
$$(i=0,-1,-2,\ldots;\ j\in\mathbb{Z}),$$

which follows from another result of Rainville's monograph (see [11, Section 48, Eq. (15)]). $\square$

Lemma 2.2. *For any $i\in\mathbb{Z}$ we have*

$$f_{i0}(a,b,c) \equiv {}_3F_2\left(\left.\begin{matrix} a,b,c \\ d+\frac{1}{2}i, 2c \end{matrix}\right| 1\right) = P_i(X_\mu^{(0)}Q_i^{(0)} - X_\mu^{(1)}Q_i^{(1)}) \quad (\Re(2c-a-b) > -i-1), \tag{2.4}$$

where $\mu := |i| \bmod 2$, and

$$P_i \equiv P_i(a,b,c) := 2^{a+b+|i|-2}(-1)^{\lfloor |i|/2\rfloor}\frac{\Gamma(d+\frac{1}{2}|i|)\Gamma(c+\frac{1}{2})\Gamma(c-d-\frac{1}{2}|i|+1)}{\Gamma(\frac{1}{2})\Gamma(a)\Gamma(b)}, \tag{2.5}$$

$$X_\mu^{(l)} \equiv X_\mu^{(l)}(a,b,c) := \frac{\Gamma(\frac{1}{2}a+\frac{1}{2}l)\Gamma(\frac{1}{2}b+\frac{1}{2}\mu+\frac{1}{2}l)}{\Gamma(c-\frac{1}{2}a+\frac{1}{2}l+\frac{1}{2})\Gamma(c-\frac{1}{2}b-\frac{1}{2}\mu+\frac{1}{2}l+\frac{1}{2})} \quad (l=0,1), \tag{2.6}$$

$$Q_i^{(l)} := \sum_{m=0}^{\lfloor (|i|-l)/2\rfloor} \frac{(\frac{1}{2}a-c+m)_l}{(\frac{1}{2}b+\frac{1}{2}|i|-\frac{1}{2}-m)_l}\alpha_{i,2m+l}\beta_{i,m}^{(l)} \quad (l=0,1). \tag{2.7}$$

Here

$$\alpha_{ik} := \frac{\gamma_{i,k}}{(a)_k(b)_{|i|-k}}, \tag{2.8}$$

$$\beta_{im}^{(l)} := (\tfrac{1}{2}a+\tfrac{1}{2}l)_m(\tfrac{1}{2}a-\tfrac{1}{2}l+\tfrac{1}{2}-c)_m$$
$$\times(\tfrac{1}{2}b+\tfrac{1}{2}\mu+\tfrac{1}{2}l)_{\lfloor |i|/2\rfloor-m}(\tfrac{1}{2}b+\tfrac{1}{2}\mu-\tfrac{1}{2}l+\tfrac{1}{2}-c)_{\lfloor |i|/2\rfloor-m}. \tag{2.9}$$

Proof. Using notations (2.5) and (2.6), we may write Watson's formula (1.1) as

$$f_{00}(a,b,c) = P_0(a,b,c)X_0^{(0)}(a,b,c).$$

Now, it can be checked that

$$P_0(a+k,b+|i|-k,c) = P_i(a,b,c)\frac{(-1)^{\lfloor |i|/2\rfloor}}{(a)_k(b)_{|i|-k}}, \tag{2.10}$$

$$X_0^{(0)}(a+k,b+|i|-k,c) = (-1)^{\lfloor |i|/2\rfloor+l}\frac{(\frac{1}{2}a-c+m)_l}{(\frac{1}{2}b+\frac{1}{2}i-\frac{1}{2}-m)_l}\beta_{i,m}^{(l)}X_\mu^{(l)}(a,b,c), \tag{2.11}$$

where $\mu := |i| \bmod 2$, $m := \lfloor k/2\rfloor$, $l := k \bmod 2$. Using the above forms in the r.h.s. of (2.1) with $j=0$, we obtain the result. $\square$

Lemma 2.3. *For any $i\in\mathbb{Z}$, we have*

$$f_{i1}(a,b,c) \equiv {}_3F_2\left(\left.\begin{matrix} a,b,c \\ d+\frac{1}{2}i, 2c+1 \end{matrix}\right| 1\right) = P_i\left(X_\mu^{(0)}S_i^{(0)} - X_\mu^{(1)}S_i^{(1)}\right) \quad (\Re(2c-a-b) > -i-3) \tag{2.12}$$

with

$$S_i^{(l)} := Q_i^{(l)} + R_i^{(1-l)} \quad (l=0,1), \tag{2.13}$$

where we use the notation (2.5)–(2.7) *and*

$$R_i^{(l)} := \sum_{m=0}^{\lfloor(|i|-l)/2\rfloor} \frac{(\frac{1}{2}a+m)_l}{(\frac{1}{2}b+\frac{1}{2}|i|-\frac{1}{2}-c-m)_l}\alpha_{i,2m+l}\beta_{i,m}^{(1-l)} \quad (l=0,1). \tag{2.14}$$

Proof. A result in [4] may be written as

$$f_{01}(a,b,c) = P_0(X_0^{(0)} - X_0^{(1)}).$$

Observe that

$$X_0^{(1)}(a+k, b+|i|-k, c) = (-1)^{\lfloor |i|/2\rfloor + l}\frac{(\frac{1}{2}a+m)_l}{(\frac{1}{2}b+\frac{1}{2}|i|-\frac{1}{2}-c-m)_l}\beta_{i,m}^{(1-l)}X_\mu^{(l)}(a,b,c),$$

where $\mu := |i| \bmod 2$, $m := \lfloor k/2\rfloor$, $l := k \bmod 2$. Using this and (2.10), (2.11) in the r.h.s. of (2.1) with $j=1$, we obtain the result. □

Now, we are able to prove the main result of this paper. We have the following:

Theorem 2.4. *For any $i,j\in\mathbb{Z}$ we have*

$$f_{i,j}(a,b,c) \equiv {}_3F_2\left(\left.\begin{matrix} a,b,c \\ d+\frac{1}{2}i, 2c+j \end{matrix}\right| 1\right) = P_i\ (X_\mu^{(0)}T_{i,j}^{(0)} - X_\mu^{(1)}T_{i,j}^{(1)})\ (\Re(2c-a-b) > -i-2j-1), \tag{2.15}$$

where $\mu = |i| \bmod 2$,

$$T_{i,j}^{(l)} = A_{i,j}Q_i^{(l)} + B_{i,j}R_i^{(1-l)} \quad (l=0,1) \tag{2.16}$$

and where A_{ij} and B_{ij} are particular solutions of the difference equation in j (i being a parameter)

$$\begin{aligned} &(j+2c-a)(j+2c-b)(j+c)E_{j+1} \\ &\quad = (j+2c)[(j+2c-d)(2j+2c-d) + \tfrac{1}{2}i(j+2c-1) - (a-d)_2]E_j \\ &\qquad -(j+2c-1)_2(j+c-d+\tfrac{1}{2}i)E_{j-1} \quad (j\in\mathbb{Z}), \end{aligned} \tag{2.17}$$

obtained for the initial values $E_0=1$, $E_1=1$, and $E_0=0$, $E_1=1$, respectively.

Proof. The main tool used in the proof is the recurrence relation in j

$$\begin{aligned}(j+2c-a)(j+2c-b)(j+c)f_{i,j+1} \\ &= (j+2c)[(j+2c-d)(2j+2c-d)+\tfrac{1}{2}i(j+2c-1)-(a-d)_2]f_{i,j} \\ &\quad -(j+2c-1)_2(j+c-d+\tfrac{1}{2}i)f_{i,j-1} \quad (j\in\mathbb{Z}),\end{aligned} \tag{2.18}$$

which follows from a result given by Bailey [1] for contiguous functions of the type ${}_3F_2(1)$. Putting (2.15) into the above equation it is easy to observe that $T_{i,j}^{(l)}$ $(l=0,1)$ are also solutions of Eq. (2.17) with the initial values (cf. Lemmata 2.2 and 2.3)

$$T_{i,0}^{(l)} = Q_i^{(l)}, \qquad T_{i,1}^{(l)} = S_i^{(l)}.$$

Also, it is easy to observe that we can write

$$(-1)^l T_{i,j}^{(l)} = A_{i,j}^{(l)} Q_i^{(l)} + B_{i,j}^{(l)} R_i^{(1-l)} \quad (l=0,1), \tag{2.19}$$

where the rational coefficients $A_{i,j}^{(l)}$ and $B_{i,j}^{(l)}$ are solutions of the recurrence relation (2.17) with the initial conditions

$$A_{i,0}^{(0)} = A_{i,1}^{(0)} = 1, \qquad A_{i,0}^{(1)} = A_{i,1}^{(1)} = -1,$$

and

$$B_{i,0}^{(0)} = 0, \quad B_{i,1}^{(0)} = 1, \qquad B_{i,0}^{(1)} = 0, \quad B_{i,1}^{(1)} = -1,$$

respectively. Now, it is a simple observation that solutions $A_{i,j}^{(0)}$ and $B_{i,j}^{(0)}$ are linearly independent and that

$$A_{i,j}^{(1)} = -A_{i,j}^{(0)}, \quad B_{i,j}^{(1)} = -B_{i,j}^{(0)} \ (j\in\mathbb{Z}).$$

Thus, the formula (2.19) can be written as (2.16) with $A_{i,j} = A_{i,j}^{(0)}$, and $B_{i,j} = B_{i,j}^{(0)}$. □

Remark 1. A collection of forms for $A_{i,j}$ and $B_{i,j}$ with $i\in\mathbb{Z}$ and $-2 \leqslant j \leqslant 3$ is given in the appendix.

Remark 2. The recurrence relation (2.18) can also be obtained using Zeilberger algorithm (see, e.g., [10, Ch. 6]).

Remark 3. Note that the Eq. (2.18) may be used to compute $f_{i,j}$ recursively. However, in the present approach we are using this recursion only to produce *rational* expressions for $A_{i,j}$ and $B_{i,j}$, which results in a remarkable efficiency of the algorithm.

Using Theorem 2.4 with $j=2$, and the formulae (A.6) given in the appendix, we obtain the following:

Corollary 2.5. *For any* $i\in\mathbb{Z}$, *we have*

$$f_{i,2}(a,b,c) \equiv {}_3F_2\left(\begin{matrix} a,b,c \\ d+\frac{1}{2}i, 2c+2\end{matrix}\middle| 1\right) = P_i\left(X_\mu^{(0)}T_{i,2}^{(0)} + X_\mu^{(1)}T_{i,2}^{(1)}\right) (\Re(2c-a-b) > -i-5), \tag{2.20}$$

where $\mu = |i| \bmod 2$, *and*

$$T_{i,2}^{(l)} = \frac{2c+1}{c+1}\{[1-2(c-d+1)e]Q_i^{(l)} + [1+ie]R_i^{(1-l)}\} \qquad (l=0,1). \tag{2.21}$$

Here $e := c/[(2c-a+1)(2c-b+1)]$, *and the notation used is that of* (2.5)–(2.7) *and* (2.14).

Example 2.6. In particular, Theorem 2.4 and the formulae of the appendix imply the following results:

$$f_{5,0} = {}_3F_2\left(\begin{matrix} a,b,c \\ \frac{1}{2}a+\frac{1}{2}b+3, 2c\end{matrix}\middle| 1\right) = 2^{a+b+3}\frac{\Gamma(\frac{1}{2}a+\frac{1}{2}b+3)\Gamma(c+\frac{1}{2})\Gamma(c-\frac{1}{2}a-\frac{1}{2}b-2)}{(a-b-4)_9\Gamma(\frac{1}{2})\Gamma(a)\Gamma(b)}$$

$$\times\left\{w_{5,0}(a,b)\frac{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b)}{\Gamma(c-\frac{1}{2}b+\frac{1}{2})\Gamma(c-\frac{1}{2}a)} - w_{5,0}(b,a)\frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b+\frac{1}{2})}{\Gamma(c-\frac{1}{2}a+\frac{1}{2})\Gamma(c-\frac{1}{2}b)}\right\}, \tag{2.22}$$

where

$$\begin{aligned} w_{5,0}(a,b) := {} & 5\left(\tfrac{1}{2}b\right)_2\left(c-\tfrac{1}{2}b-\tfrac{3}{2}\right)_2(a-b-3)(a-b+2)_3 + \left(\tfrac{1}{2}a+\tfrac{1}{2}\right)_2 \\ & \times\left(c-\tfrac{1}{2}a-2\right)_2(a-b-4)_4 + 5b\left(\tfrac{1}{2}a+\tfrac{1}{2}\right)\left(c-\tfrac{1}{2}b-\tfrac{1}{2}\right)\left(c-\tfrac{1}{2}a-1\right) \\ & \times(a-b-4)_2(a-b+1)(a-b+4). \end{aligned}$$

$$f_{3,1} = {}_3F_2\left(\begin{matrix} a,b,c \\ \frac{1}{2}a+\frac{1}{2}b+2, 2c+1\end{matrix}\middle| 1\right) = 2^{a+b-2}\frac{\Gamma(\frac{1}{2}a+\frac{1}{2}b+2)\Gamma(c+\frac{1}{2})\Gamma(c-\frac{1}{2}a-\frac{1}{2}b-1)}{(a-b-2)_5\Gamma(\frac{1}{2})\Gamma(a)\Gamma(b)}$$

$$\times\left\{w_{3,1}(a,b)\frac{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}a+\frac{1}{2})}{\Gamma(c-\frac{1}{2}a+1)\Gamma(c-\frac{1}{2}b+\frac{1}{2})} - w_{3,1}(b,a)\frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b+\frac{1}{2})}{\Gamma(c-\frac{1}{2}a+\frac{1}{2})\Gamma(c-\frac{1}{2}b+1)}\right\}, \tag{2.23}$$

where

$$\begin{aligned} w_{3,1}(a,b) := {} & (a+1)(2c-a)(a-b-2)[(a-b-1)(2c-a-2)+3b(a-b+1)] \\ & +b(a-b+2)(2c-b-1)[(b+2)(a-b+1)+3(a-b-1)(2c-a)], \end{aligned}$$

$$f_{1,3} = {}_3F_2\left(\begin{matrix} a,b,c \\ \frac{1}{2}a+\frac{1}{2}b+1, 2c+3\end{matrix}\middle| 1\right) = 2^{a+b-3}\frac{\Gamma(\frac{1}{2}a+\frac{1}{2}b+1)\Gamma(c+\frac{3}{2})\Gamma(c-\frac{1}{2}a-\frac{1}{2}b)}{(a-b)(c+2)\Gamma(\frac{1}{2})\Gamma(a)\Gamma(b)}$$

$$\times\left\{w_{1,3}(a,b)\frac{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b)}{\Gamma(c-\frac{1}{2}a+2)\Gamma(c-\frac{1}{2}b+\frac{3}{2})} - w_{1,3}(b,a)\frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b+\frac{1}{2})}{\Gamma(c-\frac{1}{2}a+\frac{3}{2})\Gamma(c-\frac{1}{2}b+2)}\right\}, \tag{2.24}$$

where

$$w_{1,3}(a,b) := 2(2c^2+ab)(c-a+b+1)-2a^2$$
$$+2(b-a)(5c-2a-2b+2)+c(8c+a^2-3b^2+8).$$

$$f_{-3,1} = {}_3F_2\left(\begin{matrix} a,b,c \\ \frac{1}{2}a+\frac{1}{2}b-1, 2c+1\end{matrix}\,\middle|\, 1\right) = 2^{a+b-4}\frac{\Gamma(\frac{1}{2}a+\frac{1}{2}b-1)\Gamma(c+\frac{1}{2})\Gamma(c-\frac{1}{2}a-\frac{1}{2}b)}{\Gamma(\frac{1}{2})\Gamma(a)\Gamma(b)}$$
$$\times\left\{\frac{(4ac-\omega)\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b+\frac{1}{2})}{\Gamma(c-\frac{1}{2}a+\frac{1}{2})\Gamma(c-\frac{1}{2}b+1)}+\frac{(4bc-\omega)\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b)}{\Gamma(c-\frac{1}{2}a+1)\Gamma(c-\frac{1}{2}b+\frac{1}{2})}\right\}, \tag{2.25}$$

where $\omega := (a+b-2)(a+b-2c)$.

$$f_{-4,3} = {}_3F_2\left(\begin{matrix} a,b,c \\ \frac{1}{2}a+\frac{1}{2}b-\frac{3}{2}, 2c+3\end{matrix}\,\middle|\, 1\right) = 2^{a+b-4}\frac{\Gamma(\frac{1}{2}a+\frac{1}{2}b-\frac{3}{2})\Gamma(c+\frac{3}{2})\Gamma(c-\frac{1}{2}a-\frac{1}{2}b+\frac{3}{2})}{(c+2)\Gamma(\frac{1}{2})\Gamma(a)\Gamma(b)}$$
$$\times\left\{w_{-4,3}\frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b)}{\Gamma(c-\frac{1}{2}a+\frac{3}{2})\Gamma(c-\frac{1}{2}b+\frac{3}{2})}+z_{-4,3}\frac{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})}{\Gamma(c-\frac{1}{2}a+2)\Gamma(c-\frac{1}{2}b+2)}\right\}, \tag{2.26}$$

where

$$w_{-4,3} := (c+2)[(a+b)(a+b-4)+3]-2ab(a+b-2c-3),$$

$$z_{-4,3} := 8c^2(a+b-1)-c[3(a+b)(a+b-8)+4ab+29]$$
$$-(4a+4b-2ab)(a+b-5)+4(ab-6).$$

Note that the above results are obtained with the aid of a program written in Maple [2].

2.1. Special case

Corollary 2.7. *For $n=0,1,\ldots;\ p=0,1,$ and $i,j\in\mathbb{Z}$, we have*

$${}_3F_2\left(\begin{matrix} -2n-p, \alpha+2n+p, c \\ \frac{1}{2}\alpha+\frac{1}{2}i+\frac{1}{2}, 2c+j\end{matrix}\,\middle|\, 1\right)$$
$$= 2^{|i|}(\tfrac{1}{2}\alpha)_{\mu(1-p)}(\tfrac{1}{2}\alpha+\tfrac{1}{2}p+n)_{\mu p}$$
$$\times\frac{(\frac{1}{2}\alpha+\frac{1}{2}\mu+\frac{1}{2})_{\lfloor |i|/2\rfloor}(\frac{1}{2})_{n+p}(\frac{1}{2}\alpha+\frac{1}{2}\mu+\frac{1}{2}-c)_n}{(\frac{1}{2}\alpha+\frac{1}{2}\mu+\frac{1}{2}-c)_{\lfloor |i|/2\rfloor}(c+\frac{1}{2})_{n+p}(\frac{1}{2}\alpha+\frac{1}{2}p+\frac{1}{2}-\frac{1}{2}|\mu-p|)_n}T^{(p)}_{i,j}, \tag{2.27}$$

where $\mu=|i| \bmod 2$, and $T^{(p)}_{i,j}$ is defined as in Theorem 2.4, for $a:=-2n-p$, and $b:=\alpha+2n+p$.

Proof. First observe that using the duplication formula for the Γ function we can write (cf. (2.5) and (2.6))

$$P_i X_\mu^{(l)} = 2^{|i|}(-1)^{\lfloor |i|/2 \rfloor} \frac{(\frac{1}{2}b)_{\mu l}\Gamma(\frac{1}{2})\Gamma(c+\frac{1}{2})}{\Gamma(\frac{1}{2}a-\frac{1}{2}l+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2}-\frac{1}{2}|\mu-l|)} \times \frac{\Gamma(d+\frac{1}{2}|i|)\Gamma(c-d-\frac{1}{2}|i|+1)}{\Gamma(c-\frac{1}{2}a+\frac{1}{2}l+\frac{1}{2})\Gamma(c-\frac{1}{2}b-\frac{1}{2}\mu+\frac{1}{2}l+\frac{1}{2})} \qquad (l=0,1).$$

Letting $a := -2n-p$, and $b := \alpha + 2n + p$, where p equals 0 or 1, and n is non-negative integer, we observe that $P_i X_\mu^{(1-p)} = 0$, and $(-1)^p P_i X_\mu^{(p)}$ can be written in the form given in the right-hand side of (2.27) for the coefficient of $T_{i,j}^{(p)}$. Now, the result follows from Theorem 2.4. □

3. Generalization of Whipple's theorem

The classical Whipple's theorem is (see [3, Section 4.4]; or [13, Eq. (2.3.3.14)])

$$ {}_3F_2\left(\left.\begin{matrix} a, 1-a, c \\ e, 1+2c-e \end{matrix}\right| 1\right) = \frac{\pi\Gamma(e)\Gamma(1+2c-e)}{2^{2c-1}\Gamma(\frac{1}{2}e+\frac{1}{2}a)\Gamma(\frac{1}{2}+\frac{1}{2}e-\frac{1}{2}a)\Gamma(c-\frac{1}{2}e+\frac{1}{2}a+\frac{1}{2})\Gamma(1+c-\frac{1}{2}e-\frac{1}{2}a)}, \qquad (3.1)$$

where $\Re(c) > 0$.

In [7], a collection of 39 analytic expressions is given for

$$g_{i,j}(a,c,e) := {}_3F_2\left(\left.\begin{matrix} a, 1+i+j-a, c \\ e, 1+i+2c-e \end{matrix}\right| 1\right), \qquad (3.2)$$

where $\Re(c) > j$, for a selection of i, j such that $|i|, |j| \leqslant 3$, using a connection between the above functions and the functions (4.2).

The following theorem shows that the problem of computing (3.2) for any integer i and j can be reduced to the evaluation of $f_{i,j}$, discussed in Section 2.

Theorem 3.1. *For any $i, j \in \mathbb{Z}$, we have*

$$g_{i,j}(a,c,e) = \frac{\Gamma(e)\Gamma(1+i+2c-e)\Gamma(c)}{\Gamma(a)\Gamma(1+i-a+c)\Gamma(2c-j)} f_{i,j}(e-a, 1+i+2c-a-e, c-j), \qquad (3.3)$$

where $f_{i,j}$ is defined by (1.2).

Proof. The result is a simple consequence of a familiar transformation ([13, Eq. (2.3.3.7)])

$$ {}_3F_2\left(\left.\begin{matrix} a, b, c \\ e, f \end{matrix}\right| 1\right) = \frac{\Gamma(e)\Gamma(f)\Gamma(s)}{\Gamma(a)\Gamma(s+b)\Gamma(s+c)} {}_3F_2\left(\left.\begin{matrix} e-a, f-a, s \\ s+b, s+c \end{matrix}\right| 1\right), \qquad (3.4)$$

where $s := e+f-a-b-c$. □

Example 3.2. In particular, Theorems 2.4 and 3.1 imply the following formula:

$$
\begin{aligned}
g_{-4,1} &= {}_3F_2\left(\begin{matrix} a, -a-2, c \\ e, 2c-e-3 \end{matrix}\,\middle|\, 1\right) \\
&= 2^{-3-2a}\frac{\Gamma(e)\Gamma(2c-e-3)}{(c-1)\Gamma(e-a)\Gamma(2c-a-e-3)} \\
&\quad\times\left\{u_{-4,1}\frac{\Gamma(\frac12 e-\frac12 a)\Gamma(c-\frac12 a-\frac12 e-\frac32)}{\Gamma(\frac12 a+\frac12 e+1)\Gamma(c+\frac12 a-\frac12 e-\frac12)}\right. \\
&\quad\left.-v_{-4,1}\frac{\Gamma(\frac12 e-\frac12 a+\frac12)\Gamma(c-\frac12 a-\frac12 e-1)}{\Gamma(\frac12 a+\frac12 e+\frac32)\Gamma(c+\frac12 a-\frac12 e)}\right\},
\end{aligned}
$$

where

$$u_{-4,1} := a(a+1)(a+3) - ac(a+c) + ce(2c-e-5) + e(e+3),$$

$$v_{-4,1} := a(a^2+2a-1) + ac(a-c+4) - ce(2c-e-5) - 2c(c-2) - (e+1)(e+2).$$

4. Generalization of Dixon's theorem

The classical Dixon's theorem is (see [3, Section 4.4]; or [13, Eq. (2.3.3.5)])

$$ {}_3F_2\left(\begin{matrix} a, b, c \\ 1+a-b, 1+a-c \end{matrix}\,\middle|\, 1\right) = \frac{\Gamma(1+\frac12 a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac12 a-b-c)}{\Gamma(1+a)\Gamma(1+\frac12 a-b)\Gamma(1+\frac12 a-c)\Gamma(1+a-b-c)}, \tag{4.1}$$

where $\Re(a-2b-2c) > -2$.

In [8], a collection of 39 analytic expressions is given for

$$ h_{i,j}(a,b,c) := {}_3F_2\left(\begin{matrix} a, b, c \\ 1+i+a-b, 1+i+j+a-c \end{matrix}\,\middle|\, 1\right), \tag{4.2}$$

where $\Re(a-2b-2c) > -2i-j-2$, for a selection of i, j such that $|i|, |j| \leqslant 3$.

The following theorem shows that the problem of computing (4.2) for any integer i and j can be reduced to the evaluation of $f_{i,j}$, discussed in Section 2.

Theorem 4.1. *For any $i, j \in \mathbb{Z}$, we have*

$$ h_{i,j}(a,b,c) = \frac{\Gamma(1+i+j+a-c)\Gamma(\sigma)}{\Gamma(1+i+j-c)\Gamma(\sigma+a)} f_{i,j}(a, 1+i+a-2b, 1+i+a-b-c), \tag{4.3}$$

where $\sigma := 2+2i+j+a-2b-2c$, and $f_{i,j}$ is defined by (1.2).

Proof. Consider the following well-known transformation, obtained by reapplying formula (3.4) on itself,

$$ {}_3F_2\left(\begin{matrix} a, b, c \\ e, f \end{matrix}\,\middle|\, 1\right) = \frac{\Gamma(f)\Gamma(s)}{\Gamma(f-a)\Gamma(s+a)} {}_3F_2\left(\begin{matrix} a, e-b, e-c \\ e, s+a \end{matrix}\,\middle|\, 1\right),$$

where $s := e + f - a - b - c$. Hence, with $e = 1 + i + a - b$, and $f = 1 + i + j + a - c$, follows (4.3). $\square$

Example 4.2. Theorem 4.1 and (2.23) imply

$$h_{3,1} = {}_3F_2\left(\begin{matrix} a,b,c \\ 4+a-b, 5+a-c \end{matrix}\middle| 1\right)$$

$$= 2^{\alpha+\beta+3}\frac{\Gamma(a-b+4)\Gamma(a-c+5)\Gamma(\alpha+9)\Gamma(a-b-c+\frac{9}{2})}{(2b-6)_5(c-4)_4\Gamma(\frac{1}{2})\Gamma(a)\Gamma(\beta+4)\Gamma(a+\alpha+9)}$$

$$\times\left\{y_{3,1}\frac{\Gamma(\frac{1}{2}\beta+2)\Gamma(\frac{1}{2}a+\frac{1}{2})}{\Gamma(\frac{1}{2}\alpha+5)\Gamma(\frac{1}{2}\gamma+\frac{5}{2})} - z_{3,1}\frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}\beta+\frac{5}{2})}{\Gamma(\frac{1}{2}\alpha+\frac{9}{2})\Gamma(\frac{1}{2}\gamma+3)}\right\},$$

where

$$\begin{aligned} y_{3,1} :=& (a+1)(b-3)(\alpha+8)[(2b-5)(\alpha+6)+3(2b-3)(\beta+4)] \\ &+(b-1)(\beta+4)(\gamma+3)[(2b-3)(\beta+6)+3(2b-5)(\alpha+8)], \\ z_{3,1} :=& (b-1)(\beta+5)(\gamma+4)[(2b-3)(\gamma+2)+3a(2b-5)] \\ &+a(b-3)(\alpha+7)[(a+2)(2b-5)+3(2b-3)(\gamma+4)], \end{aligned}$$

and where $\alpha := a - 2b - 2c$, $\beta := a - 2b$, $\gamma := a - 2c$.

5. Concluding remarks

A general method is proposed for producing the analytical form of the generalized hypergeometric series of unit argument (1.2) for any $i, j \in \mathbb{Z}$, including classical Watson's result.

In contrast to the earlier approach of Lavoie et al. (see [4–8]), the obtained formula is *natural*, and does not require storing many coefficients.

The new method can be easily implemented in a computer algebra system programming language, like Maple or MATHEMATICA.

Appendix

In this section, we give a collection of forms for $A_{i,j}$ and $B_{i,j}$ ($i \in \mathbb{Z}$, and $-2 \leqslant j \leqslant 3$) which are ingredients of the formula (2.16).

We have

$$T_{i,j}^{(l)} = A_{i,j}Q_i^{(l)} + B_{i,j}R_i^{(1-l)} \quad (l = 0, 1;\ i \in \mathbb{Z},\ -2 \leqslant j \leqslant 3), \tag{A.1}$$

where

$$A_{i,-2} = \frac{[G(-2)+(c-1)i][G(-1)+(2c-1)i]}{2(2c-2)_2(c-d+\frac{1}{2}i-1)_2} - \frac{G(-1)}{(2c-1)(c-d+\frac{1}{2}i-1)},$$
$$B_{i,-2} = -\frac{G(0)[G(-2)+(c-1)i]}{2(2c-2)_2(c-d+\frac{1}{2}i-1)_2}, \tag{A.2}$$

$$A_{i,-1} = \frac{G(-1)+(2c-1)i}{(2c-1)(2c-2d+i)}, \qquad B_{i,-1} = -\frac{G(0)}{(2c-1)(2c-2d+i)}, \tag{A.3}$$

$$A_{i,0} = 1, \qquad B_{i,0} = 0, \tag{A.4}$$

$$A_{i,1} = B_{i,1} = 1, \tag{A.5}$$

$$A_{i,2} = H_2(2d-2c-2), \qquad B_{i,2} = H_2(i), \tag{A.6}$$

$$A_{i,3} = H_3(2d-2c-2), \qquad B_{i,3} = H_3(i). \tag{A.7}$$

Here $G(m) := (2c-a+m)(2c-b+m)$, and

$$H_2(W) := \frac{2c+1}{c+1}\,[1+cW/G(1)],$$

$$H_3(W) := \frac{(2c+1)_2}{(c+1)_2G(2)}\,\{[1+cW/G(1)][4(c-\tfrac{1}{2}d+1)_2-(a-d)_2+i(c+\tfrac{1}{2})]$$
$$-(c+1)(c-d+\tfrac{1}{2}i+2)\}.$$

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