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Idempotents in Symmetric Semigroups

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We count the number of idempotent elements in a certain section of the symmetric semigroup S_n on n letters. As a corollary of our result we have that every maximal principal right ideal of S_n contains

$$\sum_{i=1}^{n-1} i^{n-i-1} \left(\binom{n-2}{i-1} + \binom{n-1}{i} \right)$$

idempotent elements. Let T_r ($1 \leqslant r \leqslant n - 1$) be the set of all elements of S_n of rank less than or equal to r, and let D_r denote the set of all elements of S_n of rank r. Then T_r is a semigroup generated by the idempotent elements of D_r. We shall obtain a maximal mutant of $T_{n-1} = S_n/D_n$.

1. INTRODUCTION

Harris and Schoenfeld [2] and Tainiter [10] have given different proofs of known formulas regarding the number of idempotents in subsets of the symmetric semigroup S_n on n letters. In Theorem 1, we shall get a counting formula for the number of idempotents in a certain section of the symmetric semigroup S_n. Any (known) counting formula for the number of idempotents in a subset of S_n can be obtained as a corollary of our result (Theorem 1). In Section 2, we shall explain the meaning of the term "section" which appeared in the abstract and the above. In Sections 3 and 4, we shall count the number of idempotents in a certain section of S_n and prove Theorem 1.

Howie [4] has proved that T_{n-1} is a semigroup generated by the idempotents of T_{n-1}. In Section 5, we shall state Theorem 2 and then observe that Howie's result [4, Theorem 1] is a particular case of Theorem 2. Kim [6] has proved that, if T is a topologic semigroup and $a \in T$ is not an idempotent, then there exists a maximal open mutant of T containing a. (This does not give any information about the actual form of a maximal mutant of a semigroup.) In Section 6, we shall give (in Theorem 3) an explicit form of a maximal mutant of the semigroup T_{n-1} ($n > 3$).

2. Notation

Let $S = S_n = S_X$ be the symmetric (or full transformation) semigroup on n letters $\{u_i : i = 1, 2,..., n\} = X$. The basic results on S can be found in [1, pp. 51–57]. If $X = \{u_1, u_2, ..., u_n\}$ and $\alpha \in S$, then we may use the (classical) notation

$$\alpha = \begin{pmatrix} u_1 & u_2 & \cdots & u_n \\ v_1 & v_2 & \cdots & v_n \end{pmatrix}$$

to mean that α is the mapping of X defined by $u_i\alpha = v_i$ ($i = 1, 2,..., n$).

Let f_α be the equivalence relation on X defined by $uf_\alpha v(u, v \in X)$ iff $u\alpha = v\alpha$. Then with each element α in S we associate two sets: (1) the range $N(\alpha) = X\alpha$ of α, and (2) the partition $N(\alpha) = X/f_\alpha$, the equivalence classes of X mod f_α. $\varphi(\alpha)$ denotes the rank of α, that is, $\varphi(\alpha) = |M(\alpha)|$. If $M(\alpha) = \{v_i : i = 1, 2,..., r\}$ and if we define

$$v_i\alpha^{-1} = V_i = \{u \in X : u\alpha = v_i\}$$

then we may write $N(\alpha) = \{V_1, V_2, ..., V_r\} = \{V_i\}$, the partition on X corresponding to α; we may use the notation

$$\alpha = \begin{pmatrix} V_1 & V_2 & \cdots & V_r \\ v_1 & v_2 & \cdots & v_r \end{pmatrix} = (V_1, V_2, ..., V_r; v_1, v_2, ..., v_r) = (V_i, v_i : i = 1, 2,..., r).$$

r is called the rank of the partition $N(\alpha)$.

Definition 1. $\pi(X)$ denotes the collection of all partitions on X and $\pi_r(X) = \{N \in \pi(X) :$ the rank of N is $r\}$.

(i) Let $N \in \pi_r(X)$ and $N = \{U_1, U_2, ..., U_r\}$. U_i is called a block of N (see Rota [9]).

(ii) Let $N_1 = \{U_1, U_2, ..., U_r\}$ and $N_2 = \{V_1, V_2, ..., V_s\}$ be elements of $\pi(X)$. If, for every block U_i of N_1, there is a block V_j in N_2 such that U_i is a subset of V_j, then we write $N_1 \subset N_2$.

(iii) $p(X)$ denotes the collection of all non-empty subsets of X, and $p_r(X)$ denotes the collection of all subsets of X with r elements.

(iv) Let $N \in \pi(X)$ and $M \in p(X)$. Then a pair $[N, M]$ is called a partition-range. $\pi_r x p_s(X) = \{[N, M] : N \in \pi_r(X)$ and $M \in p_s(X)\}$. D_r denotes the set of all elements $\alpha \in S_n$ of rank r.

(v) Let $[N, M] \in \pi_r x p_s(X)$. $[N, M](D_r) = \{\beta \in D_r : M(\beta) \subset M$ and $N(\beta) \supset N\}$ is called the NM section of D_r. (Now we can describe the meaning of the term "section" which appeared in the abstract.)

We shall count the number of idempotents in $[N, M](D_r)$, where $N \in \pi(X)$ and $M \in p(X)$.

3. Lemmas

Let $\alpha \in S$. By Lemmas 2.5, 2.6, and 2.7 in [1], if L_α, R_α, H_α, D_α denote, respectively, the L, R, H, D-class containing α, then we can write $N(\alpha) = N(H_\alpha) = N(R_\alpha)$ and $M(\alpha) = M(H_\alpha) = M(L_\alpha)$. We shall use the notation $NM(H_\alpha) = [N(\alpha), M(\alpha)]$ and $H_\alpha = (V_1, V_2, ..., V_r; v_1, v_2, ..., v_r)$, where $N(\alpha) = \{V_i : i = 1, 2, ..., r\}$ and $M(\alpha) = \{v_i : i = 1, 2, ..., r\}$. We rewrite Theorem 2.10-(i) of [1] in the following:

Lemma 1. *Let H be an H-class of rank r with $H = (V_1, V_2, ..., V_r; v_1, v_2, ..., v_r)$. Then H contains an idempotent iff $|V_i \cap M(H)| = 1$ for all blocks V_i $(i = 1, 2, ..., r)$.*

Definition 2. (i) Let H be as in Lemma 1. $N(H)$ is said to be a cross section of $N(H)$ if $|V_i \cap M(H)| = 1$ for every $i = 1, 2, ..., r$ (see [1, p. 54]). $N \# M$ means M is a cross section of N.

(ii) Let $N \in \pi_r(X)$ and $N = \{V_i\}$. Let $M \in p_s(X)$. Then an unordered arrangement $(m_1, m_2, ..., m_r)$ is called the section number of $[N, M]$, where $m_i = |V_i \cap M|$ $(i = 1, 2, ..., r)$.

(iii) Let $Y \in p_r(X)$. By the Y column of $[N, M](D_r)$ is meant the set of all elements β in $[N, M](D_r)$ such that $M(\beta) = Y$.

Lemma 2. *Let $\alpha, \beta \in S$ with $M = M(\alpha)$ and $N = N(\beta)$. Let $Y \in p_r(X)$. If the Y column of $[N, M](D_r)$ contains an idempotent, then the Y column of $[N, M](D_r)$ contains $r^{\varphi(\beta)-r}$* idempotents.

Proof. Letting $Y = \{a_i : i = 1, 2, ..., r\}$, assume that the Y column of $[N, M](D_r)$ contains an idempotent η. Then $M(\eta) = Y$ by Definition 2-(iii). Let $N = N(\beta) = \{B_i : i = 1, 2, ..., k\} \in \pi_k(X)$. Then there exists a subset I of the set $J = \{1, 2, ..., k\}$ with r elements such that $B_i (i \in I)$ contains just one element a_i of Y. Without loss of generality we may assume that $I = \{1, 2, ..., r\}$. Then we can see that $|B_j \cap Y| = 0$ for all j in $\{r + 1, r + 2, ..., k\}$. Taking a set union of two blocks B_i $(i \in I)$ and B_j $(j = r + 1, r + 2, ..., k)$, we can construct r^{k-r} distinct partitions $\{N_i \in \pi_r(X) : i = 1, 2, ..., r^{k-r}\}$ such that Y is a cross section of N_i for all i.

By Lemma 1, every H-class $H = (N_i, Y)$ determined by N_i and Y contains an idempotent, and hence the lemma follows.

DEFINITION 3. (i) Let $A = \{N_i \in \pi_r(X) : i = 1, 2, \ldots, k\}$.

$$\mathrm{Cross}\Big(\Big(\bigcup_{i=1}^{k} N_i\Big)M\Big)$$

denotes the collection of subsets Y of M such that Y is a cross section of a member N_i of A. $\mathrm{Cross}((\bigcap_{i=1}^{k} N_i)M)$ denotes the collection of all subsets Y of M such that Y is a cross section of all N_i of A.

(ii) Let $N \in \pi_r(X)$. We define $\pi_{r-k}(N) = \{N_i \in \pi_{r-k}(X) : N \subset N_i\}$ which will be called the k-th shadow of N. Let $M \in p(X)$. $\mathrm{Cross}(NM) = \{Y \subset M : Y \# N\}$. $\mathrm{Cross}(NM(k))$ denotes the collection of all $Y \in p_{r-k}(X)$ such that $Y \subset M$ and Y is a cross section of a member of $\pi_{r-k}(N)$.

LEMMA 3. *Let* $[N, M] = [\{V_1, V_2, \ldots, V_r\}, M]$ *be a partition-range with the section number* $(m_1, m_2, \ldots, m_r)$. *Assume that* $m_i \neq 0$ $(i = 1, 2, \ldots, r)$. $|\mathrm{Cross}(NM(k))| = f_{r-k}(m_i)$, *the* $(r-k)$*-th elementary symmetric function in* $m_1, m_2, \ldots,$ *and* m_r.

Proof. If $\{A_1, A_2, \ldots, A_{(r-k)}\}$ is a partition of the set $\{1, 2, \ldots, r\}$, then

$$\Big\{\bigcup_{i\in A_1} V_i, \bigcup_{i\in A_2} V_i, \ldots, \bigcup_{i\in A_{(r-k)}} V_i\Big\} = N'$$

s a partition of X of rank $(r-k)$ with $N \subset N'$; hence

$$|\mathrm{Cross}(N' M)| = \prod_{j=1}^{(r-k)} \Big(\sum_{i\in A_j} m_i\Big).$$

This last expression may be written as a sum of products of $(r-k)$ of the numbers $m_1, \ldots, m_r$ where no such product occurs more than once. Each such product corresponds to an unordered sequence of $(r-k)$ distinct elements $i(1), i(2), \ldots, i(r-k)$ of $\{1, 2, \ldots, r\}$ with $i(j) \in A_j$ for $1 \leqslant j \leqslant r-k$, and may be written as $m_{i(1)} m_{i(2)} \cdots m_{i(r-k)}$; this is just the number of cross sections $Y \subseteq M$ of N' such that $Y = \{u_1, u_2, \ldots, u_{(r-k)}\}$ with $u_j \in V_{i(j)}$ for $1 \leqslant j \leqslant r-k$. In other words,

$$\mathrm{Cross}(NM(k)) = \{\{u_{i(1)}, \ldots, u_{i(r-k)}\} \mid i(1), \ldots, i(r-k)$$

is an unordered sequence of $(r-k)$ distinct elements of $\{1, \ldots, r\}$ and $u_{i(j)} \in V_{i(j)}$ for $1 \leqslant j \leqslant r-k\}$, and hence

$$|\mathrm{Cross}(NM(k))| = f_{r-k}(m_1, m_2, \ldots, m_r),$$

the $(r-k)$-th elementary symmetric function in $m_1, m_2, \ldots, m_r$.

4. Theorem 1

THEOREM 1. *Let* $N \in \pi_r(X)$ *and* $M \in p(X)$ *with the section number* $(m_1, m_2, \ldots, m_r)$. *Assume that* $m_i \neq 0$ *for all* $i = 1, 2, \ldots, r$. *Then* $[N, M](D_{r-k})$ *contains* t *idempotent elements, where* $t = (r-k)^k f_{r-k}(m_i)$ *and* $f_{r-k}(m_i)$ *is the* $(r-k)$*-th elementary symmetric function in* $m_1, m_2, \ldots, m_r$.

The proof of Theorem 1 follows from Lemmas 1, 2, and 3.

Remark. In Theorem 1, we can remove the condition $m_i \neq 0$.

COROLLARY 1. *Every maximal principal right ideal of* S_n *contains*

$$\sum_{i=1}^{n-1} i^{n-i-1}\left(\binom{n-2}{i-1} + \binom{n-1}{i}\right)$$

idempotents, where $\binom{m}{k}$ *denotes a binominal coefficient.*

Proof. For a proper maximal principal right ideal R of S_n, there is a unique N_{ij} in $\pi_{n-1}(X)$ such that $R = [N_{ij}, X](S_n) = \{\alpha \in S_n : M(\alpha) \subset X$ and $N_{ij} \subset N(\alpha)\}$. Then the section number of $[N_{ij}, X]$ is $(2, 1, \ldots, 1) = (m_1, m_2, \ldots, m_{n-1})$. By Theorem 1, $\bigcup_{k=0}^{n-2} [N_{ij}, X](D_{n-1-k})$ contains $\sum_{k=0}^{n-2} (n-1-k)^k f_{n-1-k}(m_i)$ idempotents, and the corollary follows.

COROLLARY 2. *If* $H = (N, M)$ *is an H-class of rank* r *containing an idempotent, then* $\bigcup_{k=0}^{r-1} [N, M](D_{r-k})$ *contains* $\sum_{k=1}^{r} \binom{r}{k} k^{r-k}$ *idempotents.*

The proof is not difficult.

5. Idempotent Generated Semigroups

Howie [4] has defined an idempotent generated (IG) semigroup.

DEFINITION 4. A semigroup generated by idempotents will be called an IG semigroup.

Let $T_r = \{\alpha \in S_n : \varphi(\alpha) \leqslant r\}$. We state two lemmas without proofs:

LEMMA 4. $D_r D_r \supset D_{r-1}$ *for* $r \leqslant n-1$.

LEMMA 5. *Every element of* D_r $(r \leqslant n-1)$ *can be expressed as a product of idempotent elements in* D_r.

By Lemmas 4 and 5, we have the following theorem:

THEOREM 2. *T_r ($r \leqslant n - 1$) is an* IG *semigroup generated by the idempotents of D_r.*

6. MUTANTS IN THE SYMMETRIC SEMIGROUP S_n

We begin with a definition.

DEFINITION 5. A subset K of a semigroup T is said to be a mutant if $KK \subset T\backslash K$.

It is clear, by definition, that any mutant K of a semigroup T can not contain an idempotent. This indicates that mutants and the idempotents of a semigroup T have some kind of mutually exclusive relation. To give an explicit form of a maximal mutant of T_{n-1} ($4 \leqslant n$), we introduce the following notation:

Notation. Define $M_i = X\backslash u_i$ ($i = 1, 2,..., n$). $N_{ij} \in \pi_{n-1}(X)$ denotes a partition of rank $n - 1$ having one block consisting of two elements u_i and u_j ($i \neq j$).

(i) (i, j) denotes a sequence from the set $\{1, 2,..., n\}$ with $j > i$. Let (i, j) and (s, t) be two distinct sequences from the set $\{1, 2,..., n\}$. We write $(s, t) > (i, j)$ if either $t > j$ or $j = t$ and $s > i$. Letting $(m_1, m_2) > (n_1, n_2)$, define $[n_1n_2, m_1m_2] = \{N_{ij} \in \pi_{n-1}(X) : (n_1, n_2) \leqslant (i, j) \leqslant (m_1, m_2)\}$.

(ii) Let $t_2 > t_1$. Define $[t_1, t_2] = \{M_i \in p_{n-1}(X) : i = t_1, t_1 + 1,..., t_2\}$.

(iii) $[n_1n_2, m_1m_2][t] = \{[N_{ij}, M_t] \in \pi_{n-1}xp_{n-1}(X) : N_{ij} \in [n_1n_2, m_1m_2]\}$. $[n_1n_2][t_1, t_2] = \{[N_{n_1n_2}, M_t] : t = t_1, t_1 + 1,..., t_2\}$. $[n_1n_2, m_1m_2][t_1, t_2] = \{[N_{ij}, M_t] : N_{ij} \in [n_1n_2, m_1m_2]$ and $M_t \in [t_1, t_2]\}$.

(iv) $K_3 = \{[N_{12}, M_3], [N_{13}, M_2], [N_{23}, M_1]\}$. $K_4 = [12, 23][4] \cup [14, 24][3] \cup [34][1, 2]$.

(v) $K_5 = [12, 23][4, 5] \cup [14][2, 3] \cup [15][2, 3] \cup [24, 34][1] \cup [25, 45][1]$. $K_6 = [12, 34][5, 6] \cup [15, 25][3, 4] \cup [35, 16][2] \cup [26, 56][1]$. $K_7 = \cup A_i$, where $A_1 = [12, 45][6, 7]$, $A_2 = [16, 36][4, 5]$, $A_3 = [46, 17][2, 3]$, and $A_4 = [27, 67][1]$. $K_8 = \cup B_i$, where $B_1 = [12, 56][7, 8]$, $B_2 = [17, 47][5, 6]$, $B_3 = [57, 28][3, 4]$, and $B_4 = [38, 78][1, 2]$.

(vi) $A(n) = [1\ n-3, n-3\ n-2][n-1, n] \cup [1\ n-1, n-4\ n-1][n-3, n-2] \cup [n-3\ n-1, n-1\ n][n-5, n-4] \cup [n-5\ n, n-1\ n][n-7, n-6]$.

(vii) Let n be a positive integer of the form $n = 4m + 4 + i$ for $1 \leqslant i \leqslant 4$. We define $K_n = A(n) \cup A(n-4) \cup \cdots \cup A(n-(m-1)4) \cup K_{4+i}$. If $A \in \pi_rxp_r(X)$, we define $A(D_r) = \{\alpha \in D_r : [N(\alpha), M(\alpha)] \in A\}$.

Now we can state:

THEOREM 3. *Let* $3 \leqslant n$. *Let* S_n *be the symmetric semigroup on n letters* $u_1, u_2, \ldots,$ *and* u_n.

(i) $K_n(D_{n-1})$ *is a mutant in* S_n.

(ii) $K_n(D_{n-1})$ *is a maximal mutant of* $T_{n-1} = S_n \backslash D_n$, *that is, if* $\alpha \in (T_{n-1} \backslash K_n(D_{n-1}))$, *then* $\{\alpha\} \cup K_n(D_{n-1})$ *cannot be a mutant in* T_{n-1}.

We omit the proof of Theorem 3.

Remark. To prove Theorem 3, we may need the following, which taken from a generalized Clifford and Miller's theorem for S_n [7]:

THEOREM. *If* H_1 *and* H_2 *are two H-classes of* S_n, *then*

$$H_1 H_2 = \bigcup_{\substack{x \in H_1 \\ y \in H_2}} H_{xy}.$$

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