Idempotents in Symmetric Semigroups

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We count the number of idempotent elements in a certain section of the symmetric semigroup $S_n$ on $n$ letters. As a corollary of our result we have that every maximal principal right ideal of $S_n$ contains

$$
\sum_{i=1}^{n-1} i^{n-i-1} \binom{n-2}{i-1} \binom{n-1}{i}
$$

idempotent elements. Let $T_r$ $(1 \leq r \leq n-1)$ be the set of all elements of $S_n$ of rank less than or equal to $r$, and let $D_r$ denote the set of all elements of $S_n$ of rank $r$. Then $T_r$ is a semigroup generated by the idempotent elements of $D_r$. We shall obtain a maximal mutant of $T_{n-1} = S_n/D_n$.

1. INTRODUCTION

Harris and Schoenfeld [2] and Tainiter [10] have given different proofs of known formulas regarding the number of idempotents in subsets of the symmetric semigroup $S_n$ on $n$ letters. In Theorem 1, we shall get a counting formula for the number of idempotents in a certain section of the symmetric semigroup $S_n$. Any (known) counting formula for the number of idempotents in a subset of $S_n$ can be obtained as a corollary of our result (Theorem 1). In Section 2, we shall explain the meaning of the term "section" which appeared in the abstract and the above. In Sections 3 and 4, we shall count the number of idempotents in a certain section of $S_n$ and prove Theorem 1.

Howie [4] has proved that $T_{n-1}$ is a semigroup generated by the idempotents of $T_{n-1}$. In Section 5, we shall state Theorem 2 and then observe that Howie's result [4, Theorem 1] is a particular case of Theorem 2. Kim [6] has proved that, if $T$ is a topologic semigroup and $a \in T$ is not an idempotent, then there exists a maximal open mutant of $T$ containing $a$. (This does not give any information about the actual form of a maximal
mutant of a semigroup.) In Section 6, we shall give (in Theorem 3) an explicit form of a maximal mutant of the semigroup \( T_{n-1} \) \((n > 3)\).

2. NOTATION

Let \( S = S_n = S_X \) be the symmetric (or full transformation) semigroup on \( n \) letters \( \{u_i : i = 1, 2, ..., n\} = X \). The basic results on \( S \) can be found in [1, pp. 51–57]. If \( X = \{u_1, u_2, ..., u_n\} \) and \( \alpha \in S \), then we may use the (classical) notation

\[
\alpha = \begin{pmatrix}
  u_1 & u_2 & \cdots & u_n \\
v_1 & v_2 & \cdots & v_n
\end{pmatrix}
\]

to mean that \( \alpha \) is the mapping of \( X \) defined by \( u_i \alpha = v_i \) \((i = 1, 2, ..., n)\).

Let \( f_\alpha \) be the equivalence relation on \( X \) defined by \( uf_\alpha v(u, v \in X) \) iff \( u\alpha = v\alpha \). Then with each element \( \alpha \) in \( S \) we associate two sets: (1) the range \( N(\alpha) = X_{o\alpha} \) of \( \alpha \), and (2) the partition \( N(\alpha) = X/f_\alpha \), the equivalence classes of \( X \) mod \( f_\alpha \). \( \varphi(\alpha) \) denotes the rank of \( \alpha \), that is, \( \varphi(\alpha) = |M(\alpha)| \).

If \( M(\alpha) = \{v_i : i = 1, 2, ..., r\} \) and if we define

\[
v_i\alpha^{-1} = V_i = \{u \in X : u\alpha = v_i\}
\]

then we may write \( N(\alpha) = \{V_1, V_2, ..., V_r\} = \{V_i\} \), the partition on \( X \) corresponding to \( \alpha \); we may use the notation

\[
\alpha = \begin{pmatrix}
  V_1 & V_2 & \cdots & V_r \\
v_1 & v_2 & \cdots & v_r
\end{pmatrix} = (V_1, V_2, ..., V_r ; v_1, v_2, ..., v_r) = (V_i, v_i : i = 1, 2, ..., r).
\]

\( r \) is called the rank of the partition \( N(\alpha) \).

**DEFINITION 1.** \( \pi(X) \) denotes the collection of all partitions on \( X \) and \( \pi_r(X) = \{N \in \pi(X) : \text{the rank of } N \text{ is } r\} \).

(i) Let \( N \in \pi_r(X) \) and \( N = \{U_1, U_2, ..., U_r\} \). \( U_i \) is called a block of \( N \) (see Rota [9]).

(ii) Let \( N_1 = \{U_1, U_2, ..., U_r\} \) and \( N_2 = \{V_1, V_2, ..., V_s\} \) be elements of \( \pi(X) \). If, for every block \( U_i \) of \( N_1 \), there is a block \( V_j \) in \( N_2 \) such that \( U_i \) is a subset of \( V_j \), then we write \( N_1 \subset N_2 \).

(iii) \( p(X) \) denotes the collection of all non-empty subsets of \( X \), and \( p_r(X) \) denotes the collection of all subsets of \( X \) with \( r \) elements.

(iv) Let \( N \in \pi(X) \) and \( M \in p(X) \). Then a pair \([N, M]\) is called a partition-range. \( \pi_rxp_r(X) = \{[N, M] : N \in \pi_r(X) \text{ and } M \in p_r(X)\} \). \( D_r \) denotes the set of all elements \( \alpha \in S_n \) of rank \( r \).
Let \([N, M] \in \pi_r \times p_s(X)\). \([N, M](D_r) = \{\beta \in D_r : M(\beta) \subseteq M\} \text{ and } N(\beta) \supseteq N\}\) is called the \(NM\) section of \(D_r\). (Now we can describe the meaning of the term "section" which appeared in the abstract.)

We shall count the number of idempotents in \([N, M](D_r)\), where \(N \in \pi(X)\) and \(M \in p(X)\).

3. LEMMAS

Let \(\alpha \in S\). By Lemmas 2.5, 2.6, and 2.7 in [1], if \(L_\alpha, R_\alpha, H_\alpha, D_\alpha\) denote, respectively, the \(L, R, H, D\)-class containing \(\alpha\), then we can write \(N(\alpha) = N(H_\alpha) = N(R_\alpha)\) and \(M(\alpha) = M(H_\alpha) = M(L_\alpha)\). We shall use the notation \(NM(H_\alpha) = [N(\alpha), M(\alpha)]\) and \(H_\alpha = (V_1, V_2, ..., V_r; v_1, v_2, ..., v_r)\), where \(N(\alpha) = \{V_i : i = 1, 2, ..., r\}\) and \(M(\alpha) = \{v_i : i = 1, 2, ..., r\}\). We rewrite Theorem 2.10-(i) of [1] in the following:

**Lemma 1.** Let \(H\) be an \(H\)-class of rank \(r\) with \(H = (V_1, V_2, ..., V_r; v_1, v_2, ..., v_r)\). Then \(H\) contains an idempotent iff \(|V_i \cap M(H)| = 1\) for all blocks \(V_i\) \((i = 1, 2, ..., r)\).

**Definition 2.** (i) Let \(H\) be as in Lemma 1. \(N(H)\) is said to be a cross section of \(N(H)\) if \(|V_i \cap M(H)| = 1\) for every \(i = 1, 2, ..., r\) (see [1, p. 54]). \(N \# M\) means \(M\) is a cross section of \(N\).

(ii) Let \(N \in \pi_r(X)\) and \(N = \{V_i\}\). Let \(M \in p_r(X)\). Then an unordered arrangement \((m_1, m_2, ..., m_r)\) is called the section number of \([N, M]\), where \(m_i = |V_i \cap M| (i = 1, 2, ..., r)\).

(iii) Let \(Y \in p_r(X)\). By the \(Y\) column of \([N, M](D_r)\) is meant the set of all elements \(\beta\) in \([N, M](D_r)\) such that \(M(\beta) = Y\).

**Lemma 2.** Let \(\alpha, \beta \in S\) with \(M = M(\alpha)\) and \(N = N(\beta)\). Let \(Y \in p_r(X)\). If the \(Y\) column of \([N, M](D_r)\) contains an idempotent \(\eta\), then the \(Y\) column of \([N, M](D_r)\) contains \(r^{\phi(\beta)-r}\) idempotents.

**Proof.** Letting \(Y = \{a_i : i = 1, 2, ..., r\}\), assume that the \(Y\) column of \([N, M](D_r)\) contains an idempotent \(\eta\). Then \(M(\eta) = Y\) by Definition 2-(iii). Let \(N = N(\beta) = \{B_i : i = 1, 2, ..., k\} \in p_k(X)\). Then there exists a subset \(I\) of the set \(J = \{1, 2, ..., k\}\) with \(r\) elements such that \(B_i (i \in I)\) contains just one element \(a_i\) of \(Y\). Without loss of generality we may assume that \(I = \{1, 2, ..., r\}\). Then we can see that \(|B_j \cap Y| = 0\) for all \(j\) in \(\{r + 1, r + 2, ..., k\}\). Taking a set union of two blocks \(B_i (i \in I)\) and \(B_j (j = r + 1, r + 2, ..., k)\), we can construct \(r^{k-r}\) distinct partitions \(\{N_i \in \pi_r(X) : i = 1, 2, ..., r^{k-r}\}\) such that \(Y\) is a cross section of \(N_i\) for all \(i\).
By Lemma 1, every $H$-class $H = (N_i, Y)$ determined by $N_i$ and $Y$ contains an idempotent, and hence the lemma follows.

**Definition 3.** (i) Let $A = \{N_i \in \pi_r(X) : i = 1, 2, \ldots, k\}$.

$$\text{Cross}\left(\bigcup_{i=1}^{k} N_i \right)$$

denotes the collection of subsets $Y$ of $M$ such that $Y$ is a cross section of a member $N_i$ of $A$. $\text{Cross}\left(\bigcap_{i=1}^{k} N_i \right)M$ denotes the collection of all subsets $Y$ of $M$ such that $Y$ is a cross section of all $N_i$ of $A$.

(ii) Let $N \in \pi_r(X)$. We define $\pi_{r-k}(N) = \{N_i \in \pi_{r-k}(X) : N \subset N_i\}$ which will be called the $k$-th shadow of $N$. Let $M \in p(X)$. $\text{Cross}(NM) = \{Y \subset M : Y \neq N\}$. $\text{Cross}(NM(k))$ denotes the collection of all $Y \subset p_{r-k}(X)$ such that $Y \subset M$ and $Y$ is a cross section of a member of $\pi_{r-k}(N)$.

**Lemma 3.** Let $[N, M] = [\{V_1, V_2, \ldots, V_r\}, M]$ be a partition-range with the section number $(m_1, m_2, \ldots, m_r)$. Assume that $m_i \neq 0$ $(i = 1, 2, \ldots, r)$. $|\text{Cross}(NM(k))| = f_{r-k}(m_i)$, the $(r - k)$-th elementary symmetric function in $m_1, m_2, \ldots, m_r$.

**Proof.** If $\{A_1, A_2, \ldots, A_{(r-k)}\}$ is a partition of the set $\{1, 2, \ldots, r\}$, then

$$\bigcup_{i \in A_1} V_i, \bigcup_{i \in A_2} V_i, \ldots, \bigcup_{i \in A_{(r-k)}} V_i = N'$$

is a partition of $X$ of rank $(r - k)$ with $N \subset N'$; hence

$$|\text{Cross}(N'M)| = \left(\sum_{i \in A_j} m_i\right)^{(r-k)}.$$ 

This last expression may be written as a sum of products of $(r - k)$ of the numbers $m_1, \ldots, m_r$ where no such product occurs more than once. Each such product corresponds to an unordered sequence of $(r - k)$ distinct elements $i(1), i(2), \ldots, i(r - k)$ of $\{1, 2, \ldots, r\}$ with $i(j) \in A_j$ for $1 \leq j \leq r - k$, and may be written as $m_{i(1)}m_{i(2)} \cdots m_{i(r-k)}$; this is just the number of cross sections $Y \subset M$ of $N'$ such that $Y = \{u_1, u_2, \ldots, u_{(r-k)}\}$ with $u_j \in V_{i(j)}$ for $1 \leq j \leq r - k$. In other words,

$$\text{Cross}(NM(k)) = \{\{u_{i(1)}, \ldots, u_{i(r-k)}\} : i(1), \ldots, i(r - k)$$

is an unordered sequence of $(r - k)$ distinct elements of $\{1, \ldots, r\}$ and $u_{i(j)} \in V_{i(j)}$ for $1 \leq j \leq r - k$, and hence

$$|\text{Cross}(NM(k))| = f_{r-k}(m_1, m_2, \ldots, m_r),$$

the $(r - k)$-th elementary symmetric function in $m_1, m_2, \ldots, m_r$. 

THEOREM 1. Let \( N \in \pi_r(X) \) and \( M \in p(X) \) with the section number \( (m_1, m_2, \ldots, m_r) \). Assume that \( m_i \neq 0 \) for all \( i = 1, 2, \ldots, r \). Then \( [N, M](D_{r-k}) \) contains \( t \) idempotent elements, where \( t = (r - k)^k f_{r-k}(m_i) \) and \( f_{r-k}(m_i) \) is the \((r - k)\)-th elementary symmetric function in \( m_1, m_2, \ldots, m_r \).

The proof of Theorem 1 follows from Lemmas 1, 2, and 3.

**Remark.** In Theorem 1, we can remove the condition \( m_i \neq 0 \).

**COROLLARY 1.** Every maximal principal right ideal of \( S_n \) contains 
\[
\sum_{i=1}^{n-1} i^{n-i-1} \left( \binom{n-2}{i-1} + \binom{n-1}{i} \right)
\]
idempotents, where \( \binom{m}{k} \) denotes a binomial coefficient.

**Proof.** For a proper maximal principal right ideal \( R \) of \( S_n \), there is a unique \( N_{ij} \) in \( \pi_{n-1}(X) \) such that \( R = [N_{ij}, X](S_n) = \{ \alpha \in S_n : M(\alpha) \subseteq X \) and \( N_{ij} \subseteq N(\alpha) \}. \) Then the section number of \( [N_{ij}, X] \) is \((2, 1, \ldots, 1) = (m_1, m_2, \ldots, m_{n-1}) \). By Theorem 1, \( \bigcup_{k=0}^{n-2} [N_{ij}, X](D_{n-1-k}) \) contains \( \sum_{k=0}^{n-2} (n - 1 - k)^k f_{n-1-k}(m_i) \) idempotents, and the corollary follows.

**COROLLARY 2.** If \( H = (N, M) \) is an \( H \)-class of rank \( r \) containing an idempotent, then \( \bigcup_{k=0}^{r-1} [N, M](D_{r-k}) \) contains \( \sum_{k=1}^{r} \binom{r}{k} k^{r-k} \) idempotents.

The proof is not difficult.

### 5. IDEMPOTENT GENERATED SEMIGROUPS

Howie [4] has defined an idempotent generated (IG) semigroup.

**Definition 4.** A semigroup generated by idempotents will be called an IG semigroup.

Let \( T_r = \{ \alpha \in S_n : q_r(\alpha) \leq r \} \). We state two lemmas without proofs:

**Lemma 4.** \( D_r \cup D_{r-1} \) for \( r \leq n - 1 \).

**Lemma 5.** Every element of \( D_r \) (\( r \leq n - 1 \)) can be expressed as a product of idempotent elements in \( D_r \).

By Lemmas 4 and 5, we have the following theorem:
Theorem 2. \( T_r \ (r \leq n - 1) \) is an IG semigroup generated by the idempotents of \( D_r \).

6. MUTANTS IN THE SYMMETRIC SEMIGROUP \( S_n \)

We begin with a definition.

**Definition 5.** A subset \( K \) of a semigroup \( T \) is said to be a mutant if \( KK \subseteq T \setminus K \).

It is clear, by definition, that any mutant \( K \) of a semigroup \( T \) can not contain an idempotent. This indicates that mutants and the idempotents of a semigroup \( T \) have some kind of mutually exclusive relation. To give an explicit form of a maximal mutant of \( T_{n-1} \ (4 \leq n) \), we introduce the following notation:

**Notation.** Define \( M_i = X \setminus U_i \ (i = 1, 2, \ldots, n) \). \( N_{ij} \in \pi_{n-1}(X) \) denotes a partition of rank \( n - 1 \) having one block consisting of two elements \( u_i \) and \( u_j \ (i \neq j) \).

(i) \((i, j)\) denotes a sequence from the set \( \{1, 2, \ldots, n\} \) with \( j > i \). Let \((i, j)\) and \((s, t)\) be two distinct sequences from the set \( \{1, 2, \ldots, n\} \). We write \((s, t) > (i, j)\) if either \( t > j \) or \( t = j \) and \( s > i \). Letting \((m_1, m_2) > (n_1, n_2)\), define \( [n_1 n_2, m_1 m_2] = \{N_{ij} \in \pi_{n-1}(X) : (n_1, n_2) \leq (i, j) \leq (m_1, m_2)\} \).

(ii) Let \( t_2 > t_1 \). Define \( [t_1, t_2] = \{M_i \in p_{n-1}(X) : i = t_1, t_1 + 1, \ldots, t_2\} \).

(iii) \([n_1 n_2, m_1 m_2][t] = \{[N_{ij}, M_i] \in \pi_{n-1}xp_{n-1}(X) : N_{ij} \in [n_1 n_2, m_1 m_2]\} \).

(iv) \( K_3 = \{[N_{12}, M_3], [N_{13}, M_2], [N_{23}, M_1]\} \). \( K_3 = [12, 23][4] \cup [14, 24][3] \cup [34][1, 2] \).

(v) \( K_5 = [12, 23][4, 5] \cup [14][2, 3] \cup [15][2, 3] \cup [24, 34][1] \cup [25, 45][1] \). \( K_8 = [12, 34][5, 6] \cup [15, 25][3, 4] \cup [35, 16][2] \cup [26, 56][1] \).

(vi) \( K_4 = \cup A_i \), where \( A_1 = [12, 45][6, 7], A_2 = [16, 36][4, 5], A_3 = [46, 17][2, 3], \) and \( A_4 = [27, 67][1] \). \( K_8 = \cup B_i \), where \( B_1 = [12, 56][7, 8], B_2 = [17, 47][5, 6], B_3 = [57, 28][3, 4], \) and \( B_4 = [38, 78][1, 2] \).

(vii) Let \( n \) be a positive integer of the form \( n = 4m + 4 + i \) for \( 1 \leq i \leq 4 \). We define \( K_n = A(n) \cup A(n - 4) \cup \cdots \cup A(n - (m - 1)4) \cup K_{4+i} \). If \( A \in \pi_{n}xp_{n}(X) \), we define \( A(D_r) = \{\alpha \in D_r : [N(\alpha), M(\alpha)] \in A\} \).
Now we can state:

**Theorem 3.** Let $3 \leq n$. Let $S_n$ be the symmetric semigroup on $n$ letters $u_1, u_2, \ldots, u_n$.

(i) $K_n(D_{n-1})$ is a mutant in $S_n$.

(ii) $K_n(D_{n-1})$ is a maximal mutant of $T_{n-1} = S_n \setminus D_n$, that is, if $\alpha \in (T_{n-1} \setminus K_n(D_{n-1}))$, then $\{\alpha\} \cup K_n(D_{n-1})$ cannot be a mutant in $T_{n-1}$.

We omit the proof of Theorem 3.

**Remark.** To prove Theorem 3, we may need the following, which taken from a generalized Clifford and Miller's theorem for $S_n$ [7]:

**Theorem.** If $H_1$ and $H_2$ are two $H$-classes of $S_n$, then

$$H_1H_2 = \bigcup_{x \in H_1, y \in H_2} H_{xy}.$$

**References**