J. Differential Equations 252 (2012) 3695–3751

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Regularity of the global attractor and finite-dimensional behavior for the second grade fluid equations

Marius Paicu [a], Geneviève Raugel [b,*], Andrey Rekalo [c]

[a] Univ. Bordeaux, IMB, UMR 5251, F-33400 Talence, France

[b] CNRS, Laboratoire de Mathématiques d'Orsay, Orsay Cedex, F-91405; Univ Paris-Sud, Orsay Cedex, F-91405, France

[c] School of Mathematical and Computer Sciences, Heriot-Watt University, Riccarton, Edinburgh EH14 4AS, United Kingdom

ARTICLE INFO

Article history:
Received 23 June 2008
Revised 21 October 2011
Available online 9 January 2012

MSC:
primary 35Q30, 76D05, 46E35
secondary 35B65, 35K55

Keywords:
Second grade fluid equations
Global attractor
Regularity
Determining modes

ABSTRACT

This paper is devoted to the large time behavior and especially to the regularity of the global attractor of the second grade fluid equations in the two-dimensional torus. We first recall that, for any size of the material coefficient $\alpha > 0$, these equations are globally well posed and admit a compact global attractor $\mathcal{A}_\alpha$ in $(H^3(\mathbb{T}^2))^2$. We prove that, for any $\alpha > 0$, there exists $\beta(\alpha) > 0$, such that $\mathcal{A}_\alpha$ belongs to $(H^{3+\beta(\alpha)}(\mathbb{T}^2))^2$ if the forcing term is in $(H^{1+\beta(\alpha)}(\mathbb{T}^2))^2$. We also show that this attractor is contained in any Sobolev space $(H^{3+m}(\mathbb{T}^2))^2$ provided that α is small enough and the forcing term is regular enough. These arguments lead also to a new proof of the existence of the compact global attractor $\mathcal{A}_\alpha$. Furthermore we prove that on $\mathcal{A}_\alpha$, the second grade fluid system can be reduced to a finite-dimensional system of ordinary differential equations with an infinite delay. Moreover, the existence of a finite number of determining modes for the equations of the second grade fluid is established.

1. Introduction

For dissipative evolutionary partial differential equations, which enjoy asymptotic smoothness (also called asymptotic compactness) properties, the set $\mathcal{J}$ of all globally defined and bounded solutions for $t \in \mathbb{R}$, plays a special role. In general, the elements in this set $\mathcal{J}$ should enjoy certain regularity properties in space and the trajectories in $\mathcal{J}$ should be as smooth in the time variable as the non-linearity of the equation. We point out that, in the autonomous case, under additional dissipation hypotheses, this set $\mathcal{J}$ coincides with the compact global attractor of the equation.

* Corresponding author.
E-mail address: Genevieve.Raugel@math.u-psud.fr (G. Raugel).

0022-0396/$ – see front matter
doi:10.1016/j.jde.2011.10.015

These space or time regularity properties of $\mathcal{J}$ are obviously true for equations, which are smoothing in finite time, such as ordinary differential equations or semi-linear parabolic equations [29]. For example, the Navier–Stokes equations have this property of smoothing in finite time. When the system is not smoothing in finite time, regularity (in space or time) of the elements $\mathcal{J}$ can be very difficult to prove or could be even false. Note that regularity results are primordial in the theory of perturbations of invariant sets and in particular of periodic orbits, as shown in [25] (see also [26]).

Numerous authors have shown regularity properties for $\mathcal{J}$ in the case of dynamical systems which are not smoothing in finite time. For retarded functional differential equations in $\mathbb{R}^n$ with finite delay or neutral functional differential equations, such results were obtained already thirty years ago by Hale [20], Lopes [31], Nussbaum [34]. For dissipative evolutionary equations, which admit a compact global attractor, regularity results have been proved by several authors, using different methods (for the earliest results, see, for example, [17] for the damped wave equation, [18,19,35] for the weakly damped Schrödinger equations, [32] for the weakly damped, forced Korteweg–de Vries equation, and [36] for a review). In [17], in a same argument, Ghidaglia and Temam have shown space and time regularity in C^k-type spaces for the global attractor of the damped wave equation (from their proof, one could not deduce analyticity neither in time, nor in the spatial variables). In [18], Goubet showed the existence of the compact global attractor and its regularity in H^k-spaces for the one-dimensional weakly Schrödinger wave equation by using a Galerkin method. Applying the same Galerkin method, Oliver and Titi have shown that this compact global attractor belongs actually to a Gevrey regularity class.

In [24], Hale and Raugel have introduced a new type of Galerkin method, which, besides proving again the above mentioned regularity results, allowed to show also analyticity in time of the orbits on the global attractor and to reduce the system on the global attractor to a finite system of ordinary differential equations with infinite delay. They gave applications to semilinear or quasilinear equations. In this paper, in addition of showing spatial regularity of the elements of the global attractor for the nonlinear system of the grade two fluid equations, we extend the Galerkin method of Hale and Raugel to this system and reduce the second grade fluid equations on the global attractor to a finite system of ordinary differential equations with infinite delay. For an abstract formulation of this extension of the Galerkin method of Hale and Raugel, we also refer the reader to [23].

Before presenting the second grade fluids equations, we recall that one of the first abstract regularity results applying to partial differential equations was proved by Hale and Scheurle [27] in 1985. Consider the equation

$$\dot{u} = Au + f(u), \qquad u(0) = u_0 \in X, \tag{1.1}$$

on a Banach space X, where A is the generator of a (linear) C^0 semi-group and $f(\cdot)$ is a smooth map on X. It is well known that, under these assumptions, for any $u_0 \in X$, there exists a unique local mild solution $u(t) \in C^0([0,T); X)$ of (1.1). Assume that all these solutions exist on $[0,+\infty)$. Then one can define the dynamical system $S(t)$ on X, given by $S(t)u_0 = u(t)$ where $u(t)$ is the unique mild solution of (1.1). Assume that $S(t)$ has a compact invariant set $\mathcal{J}$ in X, that is, $S(t)\mathcal{J} = \mathcal{J}$, for any $t \in \mathbb{R}$. Then there exists a positive number η such that, if $\|Df(v)\|_{L(X,X)} \leqslant \eta$ for any v in a small neighborhood of $\mathcal{J}$, the mapping $t \in \mathbb{R} \to S(t)u \in X$ for any $u \in \mathcal{J}$ is as smooth as f. The smoothness in the time variable also implies smoothness in the spatial variable if (1.1) comes from a partial differential equation. For example, if the restriction of $S(t)$ to $\mathcal{J}$ is of class C^1, then $\mathcal{J}$ is bounded in the domain $D(A)$, which usually is a smoother space than X.

In this paper, we study an example of an asymptotically smooth system (arising in non-Newtonian fluid mechanics), which is not smoothing in finite time and, however, admits a compact global attractor. The main difficulty here comes from the fact that this system is not semilinear, but really nonlinear. Our goal is to show that this attractor is more regular than the phase space in which we are working and to exhibit finite-dimensional properties of the global attractor. The system of second grade fluids writes

$$\partial_t(u - \alpha\Delta u) - \nu\Delta u + \operatorname{rot}(u - \alpha\Delta u) \times u + \nabla p = f, \quad t > 0, \ x \in \mathbb{T}^2,$$

$$\operatorname{div} u = 0, \quad t > 0,\ x \in \mathbb{T}^2,$$

$$u(0,x) = u_0(x), \quad x \in \mathbb{T}^2, \tag{1.2}$$

where $\mathbb{T}^2$ is the two-dimensional torus $(0, 2\pi) \times (0, 2\pi)$ and where $\operatorname{rot} u \equiv \operatorname{curl} u = (0, 0, \partial_1 u_2 - \partial_2 u_1)$. In this paper, we identify each 2-component vector-field $u = (u_1, u_2)$ with the 3-component vector field $u = (u_1, u_2, 0)$ and each scalar m with the 3-component vector field $w = (0, 0, m)$ (see Appendix A for more details). In the above equations, u, f, and p denote the velocity vector field, the forcing term, and the pressure respectively.

Fluids of second grade are a particular class of non-Newtonian Rivlin–Ericksen fluids of differential type [37] and the above precise form has been justified by Dunn and Fosdick [11]. The local existence in time and uniqueness of strong solutions of Eqs. (1.2) in two-dimensional or three-dimensional bounded domains with no slip boundary conditions have been shown by Cioranescu and Ouazar [9]. Moreover, in the two-dimensional case, they obtained global existence of solutions (that is, existence on the time interval $[0, +\infty)$). Moise, Rosa and Wang have shown later that these equations admit a compact global attractor $\mathcal{A}_\alpha$ [33]. So the question of the regularity and finite-dimensional behavior of $\mathcal{A}_\alpha$ naturally arises.

Note that Eqs. (1.2) differ from the so-called α-Navier–Stokes system (see e.g. [12] or [13] and the references therein). Indeed, the α-Navier–Stokes model [13] contains the very regularizing term $-\nu\Delta(u - \alpha\Delta u)$ instead of $-\nu\Delta u$, and thus is a semilinear problem, which is much easier to solve. This is not the case for the second grade fluids equations where the dissipation is weaker. The α-models are used, in particular, as an alternative to the usual Navier–Stokes equations for numerical modelling of turbulence phenomena in pipes and channels. We emphasize that the physics underlying the second grade fluid equations and the α-models are quite different. There are numerous papers devoted to the asymptotic behavior of α-types models, including Camassa–Holm equations, α-Navier–Stokes equations, α-Bardina equations ([6,12] or [13]).

Let us now be more specific about the second grade fluid equations. We introduce the space V^m, $m \in \mathbb{N}$, which is the closure of the space

$$\left\{ u \in \left[C^\infty(\mathbb{T}^2)\right]^2 \;\middle|\; u \text{ is periodic, } \operatorname{div} u = 0,\ \int_{\mathbb{T}^2} u\,dx = 0 \right\},$$

in $H^m(\mathbb{T}^2)^2$. The space V^0 will simply be denoted by H. By classical interpolation theory, we also define the spaces V^θ, for $\theta \in \mathbb{R}^+$. We denote by $H^m_{per} \equiv H^m_{per}(\mathbb{T}^2)^2$ the space of vector fields $u \in H^m(\mathbb{T}^2)^2$, which are periodic and whose mean value vanishes.

To simplify the discussion, we will assume in a large part of what follows that the forcing term does not depend on the time variable. For any $\alpha > 0$, for any forcing term f in H^1_{per}, and for any initial data u_0 in V^3, the system (1.2) has a unique solution $u \in C^0([0, +\infty), V^3)$ (see [9,33] and Section 2 below). Actually, this solution $u(t)$ is in $C^0(\mathbb{R}, V^3)$. Thus, unlike the Navier–Stokes equations, the system (1.2) cannot be smoothing in finite time. However, as shown in [33], the system (1.2) is asymptotically smooth (also called asymptotically compact). The system (1.2) defines a continuous nonlinear semigroup (or dynamical system) $S_\alpha(t)$: $u_0 \in V^3 \to S_\alpha(t)u_0 = u(t)$, where $u(t)$ is the unique solution of (1.2). Actually, this semigroup is a continuous nonlinear group. According to [33], $S_\alpha(t)$ admits a compact global attractor $\mathcal{A}_\alpha$ in V^3. We recall that $\mathcal{A}_\alpha$ is a compact global attractor of $S_\alpha(t)$ in V^3 if $\mathcal{A}_\alpha$ is compact in V^3, invariant (i.e. $S_\alpha(t)\mathcal{A}_\alpha = \mathcal{A}_\alpha$, for any $t \geqslant 0$), and attracts all bounded sets of V^3, that is, for any $\varepsilon > 0$, for any bounded set B in V^3, there exists a time $T = T(\varepsilon, B)$ such that

$$S_\alpha(t)B \subset \mathcal{N}_{V^3}(\mathcal{A}_\alpha; \varepsilon), \quad \text{for any } t \geqslant T,$$

where $\mathcal{N}_{V^3}(\mathcal{A}_\alpha; \varepsilon)$ denotes the ε-neighborhood of $\mathcal{A}_\alpha$ in V^3.

We remark that, if f is time-dependent and belongs to $C_b^0(\mathbb{R}, H^1_{per})$, then similar results are true. Indeed, the existence results still hold and one can generalize the notion of global attractor and replace it by the notion of pullback attractor, for example. One can still study the set of all complete bounded trajectories, that is, trajectories which are bounded for all $t \in \mathbb{R}$. All the results that we present in this paper are still true in that case. We leave this easy generalization to the reader.

In Section 3, we prove the following regularity result.

Theorem 1.1. 1) *Let $f \in H^2_{per}$. If $a_1 \equiv 2\nu - 2\alpha \sup_{z\in\mathcal{A}_\alpha} \|\nabla z\|_{L^\infty} > 0$, the global attractor $\mathcal{A}_\alpha$ is bounded in V^4. Moreover, there exists a positive constant M_4 (independent of α), which can depend on a_1, such that, for any $u \in \mathcal{A}_\alpha$,*

$$\|u\|^2_{V^3} + \inf(1,\alpha)\|u\|^2_{V^4} \leqslant M_4 . \tag{1.3}$$

2) *For any $\alpha > 0$, there exists a positive number $\beta > 0$, $\beta \leqslant 1$, depending only on α and the norm $\|f\|_{H^1}$, such that, if f belongs to $H^{1+\beta}_{per}$, then $\mathcal{A}_\alpha$ is bounded in $V^{3+\beta}$. More precisely, there exists a positive constant M^*, depending on $\|f\|_{H^{1+\beta}}$ such that*

$$\|u\|^2_{V^{2+\beta}} + \inf(1,\alpha)\|u\|^2_{V^{3+\beta}} \leqslant M^*.$$

3) *For any $m \geqslant 2$, there exists a positive number d_m (which is a non-decreasing function of m), such that, for any $\alpha > 0$, if $f \in H^{m+1}_{per}$ and $a_m = 2\nu - 2d_m\alpha(\sup_{z\in\mathcal{A}_\alpha} \|\nabla z\|_{L^\infty}) > 0$, then the global attractor $\mathcal{A}_\alpha$ is bounded in V^{m+3}. Moreover, for any $\alpha_0 > 0$, there exists a positive constant $M_{m+3}(\alpha_0) = M_{m+3}(\lambda_1, \nu, f, m, a_m, \alpha_0)$, depending only on λ_1, ν, f, m, a_m and α_0, such that, if $0 < \alpha \leqslant \alpha_0$ and $a_m = 2\nu - 2d_m\alpha(\sup_{z\in\mathcal{A}_\alpha} \|\nabla z\|_{L^\infty}) > 0$, then, for any $u \in \mathcal{A}_\alpha$,*

$$\|u\|^2_{V^{m+2}} + \alpha\|u\|^2_{V^{m+3}} \leqslant M_{m+3}(\alpha_0). \tag{1.4}$$

If $a_{m-1} > 0$ and $a_m \leqslant 0$, then there exists $\theta_0 > 0$ such that the global attractor $\mathcal{A}_\alpha$ is bounded in $V^{m+2+\theta_0}$ and, for any $0 < \alpha \leqslant \alpha_0$, for any $u \in \mathcal{A}_\alpha$,

$$\|u\|^2_{V^{m+1+\theta_0}} + \alpha\|u\|^2_{V^{m+2+\theta_0}} \leqslant M_{m+2+\theta_0}(\alpha_0), \tag{1.5}$$

where $M_{m+2+\theta_0}(\alpha_0) = M(\lambda_1, \nu, f, m, \theta_0, a_{m-1}, \alpha_0)$ does not depend on α.

Remark. Let P be the Leray projection of $H^0_{per} \equiv (L^2_{per}(\mathbb{T}^2))^2$ onto H, that is, the orthogonal projection of $(L^2_{per}(\mathbb{T}^2))^2$ onto the subset of divergence free vectors fields. In Section 2.2, we are going to give various upper bounds of $\alpha \sup_{z\in\mathcal{A}_\alpha} \|\nabla z\|_{L^\infty}$ depending on α and $\|Pf\|_{H^1}$. These estimates show that $\nu - d_m\alpha \sup_{z\in\mathcal{A}_\alpha} \|\nabla z\|_{L^\infty}$ is strictly positive if for example α or $\|Pf\|_{H^1}$ are small enough (see the estimates (2.16) and (2.33) in Section 2.2).

Theorem 1.1 will be proved by decomposing system (1.2) into two affine non-autonomous systems. More precisely, let $u(t)$ be a trajectory of $S_\alpha(t)$ contained in the global attractor A_α. We write $u(t)$ as $u(t) = v_n(t) + w_n(t)$, where $v_n(t)$ and $w_n(t)$ are the solutions of the following two non-autonomous affine equations

$$\begin{gathered}\partial_t(v_n - \alpha\Delta v_n) - \nu\Delta v_n + \operatorname{rot}(v_n - \alpha\Delta v_n) \times u + \nabla p_n = f, \quad t > s_n,\ x \in \mathbb{T}^2,\\ \operatorname{div} v_n = 0, \quad t > s_n,\ x \in \mathbb{T}^2,\\ v_n(s_n, x) = 0, \quad x \in \mathbb{T}^2,\end{gathered} \tag{1.6}$$

and

$$\partial_t(w_n - \alpha\Delta w_n) - \nu\Delta w_n + \operatorname{rot}(w_n - \alpha\Delta w_n) \times u + \nabla\widetilde{p}_n = 0, \quad t > s_n,\ x \in \mathbb{T}^2,$$
$$\operatorname{div} w_n = 0, \quad t > s_n,\ x \in \mathbb{T}^2,$$
$$w_n(s_n, x) = u(s_n, x), \quad x \in \mathbb{T}^2, \tag{1.7}$$

where $s_n \in \mathbb{R}$ is an given "initial" time, which will go to $-\infty$. Since system (1.6) has zero initial data, we shall be able to prove that the solution of (1.6) is bounded in V^{m+3}, provided that f is bounded in H^{m+1}_{per} and α is sufficiently small. We will show that system (1.7) has a global solution $w_n(t)$ on the time interval $[s_n, +\infty)$, which is exponentially decaying to 0 in V^3 when $t - s_n$ goes to infinity. Notice that related decompositions of dynamical systems into two auxiliary systems, the first system admitting smoother solutions and the second one having exponentially decreasing solutions, have been used in earlier papers. For example, in order to show the existence of a compact global attractor for the damped wave equation with critical exponent, Arrieta, Carvalho and Hale have decomposed their equation into two nonlinear equations [1] (see also [2] for similar decompositions). For more details on the comparison between linear and nonlinear decompositions, we refer the reader to [23]. In their proofs of regularity of the global attractor, Goubet [18], Oliver and Titi [35], Moise and Rosa [32] and Hale and Raugel [24] have also split the systems under consideration into two equations, but in addition, they have used spectral projections. Here the difference lies in the fact that we do not need spectral projections and that we use "linear systems" instead of nonlinear ones. Another crucial remark is that, when estimating the solutions of (1.6) and (1.7), due to some cancellations, the "bad terms" $\operatorname{rot}(v_n - \alpha\Delta v_n) \times u$ and $\operatorname{rot}(w_n - \alpha\Delta w_n) \times u$ disappear. We emphasize that these cancellations will be used very often in this paper and that they are proved in Lemma A.1 of Appendix A.

At the end of the first part of this paper, that is, in Section 4, using the above regularity results, we study the convergence of the solutions of Eqs. (1.2) to those of the Navier–Stokes equations and give a V^s-estimate of the difference of their solutions, for $0 \leqslant s < 3$. From these estimates and the properties of the global attractor of the two-dimensional Navier–Stokes equations, we deduce the upper-semicontinuity of the attractors $\mathcal{A}_\alpha$ at $\alpha = 0$ in V^s, when f belongs to the space H^2_{per}, that is, we show that

$$\lim_{\alpha\to 0} \sup_{u_\alpha \in \mathcal{A}_\alpha} \inf_{u \in \mathcal{A}_0} \|u_\alpha - u\|_{V^s} = 0, \quad 0 \leqslant s < 3,$$

where $\mathcal{A}_0$ is the compact global attractor of the Navier–Stokes system with forcing term f.

In the second part of this paper, assuming that $a_1^* \equiv 2\nu - 4\alpha(\sup_{z\in\mathcal{A}_\alpha} \|\nabla z\|_{L^\infty}) > 0$, we concentrate on "finite-dimensional properties" of the global attractor $\mathcal{A}_\alpha$. Our main goal is to reduce the second grade fluid equations (1.2) on the global attractor to a finite system of ordinary differential equations with infinite delay. In particular, we want to show that every complete trajectory $u(t)$, $t \in \mathbb{R}$, which is contained in $\mathcal{A}_\alpha$, is uniquely determined by the low modes part of it. Here we use the construction and the Galerkin method introduced by Hale and Raugel [24] in the general frame of semilinear equations. As we have already indicated, in the case of the second grade fluid equations, we have to face the problem that these nonlinear equations are not semilinear. Before stating our main results in this direction, we will describe the strategy of the proof in the semilinear case considered in [24].

In [24], the following general equation has been considered:

$$u_t = Bu + h(u), \qquad u(0) = u_0 \in Y, \tag{1.8}$$

where B is the generator of a C^0 semi-group on a Banach space Y and h is a nonlinear C^1-map from Y into Y. Assume that Eq. (1.8) admits a compact global attractor $\mathcal{A}$ in Y. Let P_n be the projector onto an appropriate subspace of Y of dimension n (usually P_n is a spectral projection), such that P_n converges strongly to the identity as n goes to infinity. For sake of simplicity, we assume that $BP_n = P_nB$. We set $Q_n = Id - P_n$. If $u(t)$, $t \in \mathbb{R}$, is a complete trajectory contained in the global attractor $\mathcal{A}$, $v(t) = P_n u(t)$ and $q(t) = Q_n u(t)$ satisfy the equations

$$\begin{cases} v_t = Bv + P_n h(v+q), \\ q(t) = \displaystyle\int_{-\infty}^{t} e^{(t-s)B} Q_n h\big(v(s)+q(s)\big)\,ds, \end{cases} \tag{1.9}$$

which means that $q(t)$ is a fixed point of the map

$$w \in C^0_{bu}(\mathbb{R}, Q_n Y) \to \int_{-\infty}^{t} e^{(t-s)B} Q_n h\big(v(s)+w(s)\big)\,ds \in C^0_{bu}(\mathbb{R}, Q_n Y), \tag{1.10}$$

where $C^0_{bu}(\mathbb{R}, Q_n Y)$ denotes the space of all bounded, uniformly continuous functions from $\mathbb{R}$ into $Q_n Y$. In [24], the authors have proved that, under some additional hypotheses, for any $v(t)$ in the subset $C^0_{bu}(\mathbb{R}, \mathcal{N})$ of $C^0_{bu}(\mathbb{R}, P_n Y)$, where $\mathcal{N}$ is a neighborhood of $P_n\mathcal{A}_\alpha$ in $P_n Y$, the map defined by (1.10) is a strict contraction and thus admits a unique fixed point q_v in a small neighborhood of 0 in $C^0_{bu}(\mathbb{R}, Q_n Y)$, provided n is large enough. We remark that, by construction, $q_v(t)$ depends on $v(s)$, $s \leqslant t$. This implies that, on $\mathcal{A}$, Eq. (1.8) reduces to the following finite-dimensional system of retarded functional differential equations with infinite delay

$$v_t = Bv + P_n h(v + q_v).$$

The above construction can somehow be considered as a generalization of the construction of an inertial manifold and an inertial form (in the case of parabolic equations, see the generalization of inertial manifold of Debussche and Temam [10]).

Here we follow a similar strategy. However, our proofs are more complex because of the presence of the nonlinear term $-\operatorname{rot}(\alpha\Delta u(t)) \times u(t)$. We point out that our construction of the fixed point differs somehow from the previous one. Indeed, due to the special properties of the quadratic nonlinearity in the system (1.2), q_v will be the fixed point of an appropriate affine time-dependent map instead of a nonlinear map as in (1.10).

Among other properties, in Section 5, we shall prove Theorem 1.2 below. We recall that P denotes the classical orthogonal projection of $(L^2_{per}(\mathbb{T}^2))^2$ onto the subspace H of L^2-divergence-free vector fields. In what follows, we introduce the orthogonal projection P_n in H onto the space spanned by the eigenfunctions corresponding to the first n eigenvalues of the Stokes operator $A = -P\Delta$. Finally, we introduce the projection $Q_n = I - P_n$. Hereafter, $B_{Q_n V^3}(0,r)$ denotes the ball of center 0 and radius $r > 0$ in $Q_n V^3$, where $Q_n V^3$ is equipped with the norm $(\|\cdot\|^2_{V^2} + \alpha\|\cdot\|^2_{V^3})^{\frac{1}{2}}$. Let B be a bounded subset of a metric space X; we denote $C^0_b(\mathbb{R}, B)$ (respectively $C^0_{bu}(\mathbb{R}, B)$) the space of bounded and continuous (respectively uniformly continuous) functions from $\mathbb{R}$ into B.

Theorem 1.2. *Let f be given in H^{1+d}_{per}, $0 < d \leqslant 1$.*

We assume that $a_1^* \equiv 2\nu - 4\alpha(\sup_{z\in\mathcal{A}_\alpha} \|\nabla z\|_{L^\infty}) > 0$. Then, there exist an integer N_1 and a small positive constant r such that, for $n \geqslant N_1$, each solution $u(t)$ of (1.2), which belongs to the attractor $\mathcal{A}_\alpha$ for $t \in \mathbb{R}$, can be represented as

$$u = v_n + q^n(v_n), \quad v_n \in P_n\mathcal{A}_\alpha,$$

where q^n maps the subset $C^0_b(\mathbb{R}, \mathcal{N})$ of $C^0_b(\mathbb{R}, P_n V^{3+d})$ into $C^0_b(\mathbb{R}, B_{Q_n V^3}(0,r))$ and where $\mathcal{N}$ is an appropriate neighborhood of $P_n\mathcal{A}_\alpha$ in $P_n V^{3+d}$. Furthermore, $q^n(v_n)(t)$ depends only upon $v_n(s)$, $s \leqslant t$ and Eqs. (1.2) on $\mathcal{A}_\alpha$ reduce to the following system of n retarded functional differential equations

$$\partial_t(v_n - \alpha\Delta v_n) - \nu\Delta v_n + P_n P\Big(\operatorname{rot}\big(v_n + q^n(v_n) - \alpha\Delta\big(v_n + q_n(v_n)\big)\big) \times \big(v_n + q(v_n)\big)\Big) = P_n P f.$$

In the case where d vanishes, the following slightly weaker property holds (see Section 5 for more details).

Remark 1.1. Assume now that f is given in H^{1+d}_{per} with $d = 0$ (that is, $f \in H^1_{per}$) and that $a^*_1 \equiv 2\nu - 4\alpha(\sup_{z\in\mathcal{A}_\alpha} \|\nabla z\|_{L^\infty}) > 0$. Then, there exist an integer N_1 and a small positive constant r such that, for $n \geqslant N_1$, the properties described in Theorem 1.2 hold, provided that $0 < \alpha \leqslant \alpha_n$, where α_n is a small positive number, which may depend on n.

Theorem 1.2 will be made more precise in a series of lemmas in Section 5.

Like in [24], we deduce from Theorem 1.2 the so-called "finite number of determining modes property" for the system (1.2), when α is small enough. The property of "finite number of determining modes" was introduced and proved for the two-dimensional Navier–Stokes equations by Foias and Prodi in 1967 [14]. This property tells that the asymptotic behavior in time of the second grade fluid system depends only on a finite number of parameters (called the determining modes).

From Theorem 1.2 and from the proof of Theorem 2.7 in [24], we at once deduce the following result

Theorem 1.3. *Let f be given in H^{1+d}_{per}, $d > 0$.*

We assume that $2\nu - 4\alpha(\sup_{z\in\mathcal{A}_\alpha} \|\nabla z\|_{L^\infty}) > 0$. Then system (1.2) has the property of finite number of determining modes, that is, there exists a positive integer N_0 such that, for any u_0, u_1 in V^3, the property

$$\left\| P_{N_0} S_\alpha(t) u_0 - P_{N_0} S_\alpha(t) u_1 \right\|_{V^3} \to_{t\to+\infty} 0$$

implies that

$$\left\| S_\alpha(t) u_0 - S_\alpha(t) u_1 \right\|_{V^3} \to_{t\to+\infty} 0.$$

We point out that we also could directly prove Theorem 1.3, without applying Theorem 1.2, by performing appropriate a priori estimates. But, showing Theorem 1.3 as a consequence of Theorem 1.2 and of the proof of [24, Theorem 2.7] is much shorter.

The paper is organized as follows. In Section 2, we recall the global existence and uniqueness of solutions of (1.2) in V^3 and the existence of a compact global attractor $\mathcal{A}_\alpha$ in V^3. We give various estimates of the solutions of (1.2), which improve the previously known estimates. We also prove different estimates for appropriate non-autonomous affine equations associated to (1.2). We use these estimates to show the property of propagation of regularity for the second grade fluid equations, that is, we prove that, if the initial data are more regular, then the solution of (1.2) is uniformly bounded in a smoother space. Section 3 is devoted to the regularity properties of the compact global attractor. In particular, we show that there exists $\beta > 0$ such that $\mathcal{A}_\alpha \subset V^{3+\beta}$ if the forcing term f belongs to $H^{1+\beta}_{per}$. Furthermore, we prove that $\mathcal{A}_\alpha \subset V^{m+3}$ provided that the forcing term f belongs to H^{m+1}_{per}, $m \geqslant 1$, and that the material coefficient α is small enough. These arguments also lead to another proof of existence of the compact global attractor (see Remark 3.2). Section 4 deals with the upper semicontinuity of the attractors $\mathcal{A}_\alpha$, when α goes to zero. In Section 5, we show that the high modes component of any solution on the attractor is uniquely determined by the low modes component and prove that the dynamics on the attractor can be completely described by a finite-dimensional system of retarded ODE's with infinite time delay.

2. Global existence of solutions and compact global attractor

2.1. Global existence of solutions and group property

The first result of existence and uniqueness of global solutions of Eqs. (1.2) is due to Cioranescu and Ouazar [9]. Before stating it, we recall a few notations and the definition of strong solutions of (1.2).

We recall that V^m, $m \in \mathbb{N}$ denotes the closure of the space

$$\left\{u \in \left[C^\infty\left(\mathbb{T}^2\right)\right]^2 \;\middle|\; u \text{ is periodic, } \operatorname{div} u = 0, \int_{\mathbb{T}^2} u \, dx = 0\right\},$$

in $H^m(\mathbb{T}^2)^2$ and that we simply write $H = V^0$. We equip the space V^m with the classical H^m-norm, denoted $\|\cdot\|_{V^m} \equiv \|\cdot\|_{H^m}$. We denote by $(\cdot,\cdot)$ the usual $L^2(\mathbb{T}^2)^2$-scalar product. We also introduce the space

$$W = \left\{u \in V^1 \;\middle|\; \operatorname{rot}(u - \alpha \Delta u) \in L^2\left(\mathbb{T}^2\right)\right\},$$

equipped with the scalar product

$$(u, w)_W = \int_{\mathbb{T}^2} \big(u \cdot w + \operatorname{rot}(u - \alpha \Delta u) \cdot \operatorname{rot}(w - \alpha \Delta w)\big) \, dx.$$

As already remarked in [8], if u belongs to W, then u is in V^3. Moreover, there exists a positive constant C_0 independent of α such that, for any $u \in W$,

$$\|\nabla u\|_{L^2}^2 + 2\alpha \|\Delta u\|_{L^2}^2 + \alpha^2 \|\nabla \Delta u\|_{L^2}^2 \leqslant C_0 \big\|\operatorname{rot}(u - \alpha \Delta u)\big\|_{L^2}^2. \tag{2.1}$$

In what follows, we will use this inequality, without further notice.

Definition 2.1. For given $f \in L^\infty((0,T); H^1_{per})$ and $u_0 \in V^3$, we say that the vector field $u = u(t,x)$ is a strong solution of Eq. (1.2) on the interval $[0,T]$, $T > 0$, if $u \in C^0([0,T]; V^3)$, $\partial_t u \in L^\infty((0,T), V^2)$, $u(0) = u_0$, and the following equation holds, for any $w \in H$,

$$\big(\partial_t\big(u(t) - \alpha \Delta u(t)\big) - \nu \Delta u(t) + \operatorname{rot}\big(u(t) - \alpha \Delta u(t)\big) \times u(t), w\big) = \big(f(t), w\big). \tag{2.2}$$

In 1982, Cioranescu and Ouazar [9] showed the global existence and uniqueness of a solution $u(t)$ of (1.2) in $L^\infty((0,+T), V^3)$ (with $\partial_t u \in L^\infty((0,T), W'))$, when u_0 belongs to V^3 and f is in $L^\infty((0,T), V^1)$ (in the three-dimensional case, they showed local existence and uniqueness of the solution). When f and u_0 are more regular, then these solutions are classical, as shown by Galdi and Sequeira in [15] (see also [16] for example). In [8], Cioranescu and Girault proved that, in the three-dimensional case, if the data are small enough, then the solution is also global.

In summary, using the existing results and their proofs, one can easily deduce the following existence and uniqueness results.

Theorem 2.1. *Let $\alpha > 0$ and $T > 0$.*

1. *For any $f \in L^\infty((0,T); H^1_{per})$ (respectively any $f \in L^\infty((0,+\infty); H^1_{per})$) and any $u_0 \in V^3$, there exists a unique (strong) solution u in $C^0([0,T], V^3) \cap W^{1,\infty}((0,T), V^2)$ (resp. $u \in C^0_b([0,\infty), V^3) \cap W^{1,\infty}((0,\infty), V^2)$) of system (1.2). Moreover, for any $t \in [0,T]$, the map $u_0 \in V^3 \mapsto u(t) \in V^3$ is continuous.*
2. *Likewise, if $f \in L^\infty([-T,0]; H^1_{per})$, then, for every $u_0 \in V^3$, there exists a unique strong solution $u(t) \in C^0([-T,0]; V^3) \cap W^{1,\infty}((-T,0), V^2)$ of Eqs. (1.2) for $t \in [-T,0]$, with initial data $u(0) = u_0$. Moreover, for any $t \in [-T,0]$, the map $u_0 \in V^3 \mapsto u(t) \in V^3$ is continuous.*

As indicated, the global existence and uniqueness of the solution $u(t) \in L^\infty((0,T), V^3)$ of (1.2) are proved in [9], for $T > 0$. The existence and uniqueness results for negative times are straightforward and are shown in [33], by reversing the time in system (1.2). The fact that the solution $u(t)$ belongs to $C^0([0,T), V^3)$ has been proved in [33, Section 4.1]. Note that the continuity property with respect to t of $u(t)$ in V^3 follows from the continuity of the norm $\|u(t)\|_{V^3}$ and from the fact that $u(t)$ is weakly continuous from $[0,+\infty)$ into V^3. The continuity of the V^3-norm is a consequence of the following "energy equality", valid for any $t \in [0,T]$,

$$\big\|\big(\mathrm{rot}(u-\alpha\Delta u)\big)(t)\big\|_{L^2}^2 = e^{-\frac{2\nu}{\alpha}t}\big\|\big(\mathrm{rot}(u-\alpha\Delta u)\big)(0)\big\|_{L^2}^2 + \int_0^t e^{-\frac{2\nu}{\alpha}(t-s)}\bigg(2\,\mathrm{rot}\, f(s) + \frac{2\nu}{\alpha}\,\mathrm{rot}\, u(s), \mathrm{rot}(u-\alpha\Delta u)(s)\bigg)\, ds. \quad (2.3)$$

In Theorem 2.3, we will give another proof of the continuity of $u(t)$ with respect to the time variable t. The continuity of the map $u_0 \in V^3 \mapsto u(t) \in V^3$, where $u(t)$ is the solution of system (1.2) is also shown in [33, Section 4.1] and uses the energy equality (2.3) as well.

Notice that the second statement of Theorem 2.1 rules out the possibility of a smoothing effect in finite time for Eqs. (1.2), when $\alpha > 0$. This is an important difference with the Navier–Stokes equations.

In the largest part of what follows, we assume that the forcing term f is independent of t. In this case, Theorem 2.1 implies in particular that the map $S_\alpha(t) : u_0 \mapsto u(t)$, where $u(t)$ is the strong solution of (1.2) with initial data u_0, defines a continuous flow (or C^0-nonlinear group) on V^3. Applying the method of functionals introduced by J. Ball in 1992 (see [3,4]), Moise, Rosa, and Wang [33] have shown that the dynamical system $S_\alpha(t)$ is asymptotically smooth in the sense of Hale [22] or asymptotically compact in the sense of Ladyzenskaya. This property together with the fact that $S_\alpha(t)$ admits a bounded absorbing set in V^3 implies that $S_\alpha(t)$ admits a compact global attractor in V^3. For more details on the notions of asymptotic smoothness, asymptotic compactness, absorbing sets, etc., see [2,21,22] or [36] for example.

Theorem 2.2. *Assume that f belongs to H^1_{per} and does not depend on the time variable. Then, for any $\alpha > 0$, $S_\alpha(t)$ has a compact global attractor $\mathcal{A}_\alpha$ in V^3.*

In Section 3 below, we will prove that Eqs. (1.2) have an asymptotic smoothing effect, namely, that the global attractors $\mathcal{A}_\alpha$ are more regular if the forcing term is more regular. We will show that these global attractors are as smooth as one wishes, provided that the coefficient α is sufficiently small and f is sufficiently regular. In Section 3, we will also give another simple proof of the asymptotic compactness of $S_\alpha(t)$ if f belongs to H^{1+d}_{per}, $d > 0$ (see Remark 3.2).

2.2. Various a priori estimates

In this section, we show a few a priori estimates, which are more or less contained in earlier papers (see [9,33], as well as [15] and [8] in the three-dimensional case). We will also show that the norm $\|\cdot\|_{V^2} + (\inf(1,\sqrt{\alpha}))\|\cdot\|_{V^3}$ is uniformly bounded on the global attractor $\mathcal{A}_\alpha$ by a positive constant C, which is independent of α.

Let λ_n, $n \geqslant 1$, be the eigenvalues (in increasing order) of the Stokes operator $A = -P\Delta$ corresponding to the eigenfunctions in V^3. Since the considered eigenfunctions have mean value zero, λ_1 is strictly positive. In particular, we have, for any $u \in V^1$,

$$\|u\|_{L^2}^2 + \alpha\|\nabla u\|_{L^2}^2 \leqslant \big(\lambda_1^{-1} + \alpha\big)\|\nabla u\|_{L^2}^2. \quad (2.4)$$

In what follows, we will establish several formal a priori estimates. All these a priori estimates can be rigorously justified by the use of a classical Galerkin method. Thus, without loss of generality, we may assume here that u is regular enough. Taking the inner product in $L^2(\mathbb{T}^2)^2$ of Eq. (1.2) with u and applying (2.4), we get the following inequalities, for any $t \geqslant 0$,

$$\frac{1}{2}\partial_t\big(\|u\|_{L^2}^2+\alpha\|\nabla u\|_{L^2}^2\big)+\frac{\nu}{2}\|\nabla u\|_{L^2}^2 \leqslant \frac{1}{2\lambda_1\nu}\|f\|_{L^2}^2, \tag{2.5}$$

and,

$$\partial_t\big(\|u\|_{L^2}^2+\alpha\|\nabla u\|_{L^2}^2\big)+\frac{\nu}{(\lambda_1^{-1}+\alpha)}\big(\|u\|_{L^2}^2+\alpha\|\nabla u\|_{L^2}^2\big) \leqslant \frac{1}{\nu\lambda_1}\|f\|_{L^2}^2.$$

Integrating the previous estimate and using the Gronwall inequality, we obtain, for $t \geqslant 0$,

$$\begin{aligned}\big\|u(t)\big\|_{L^2}^2+\alpha\big\|\nabla u(t)\big\|_{L^2}^2 &\leqslant \exp\left(-\frac{\nu t}{(\lambda_1^{-1}+\alpha)}\right)\big(\big\|u(0)\big\|_{L^2}^2+\alpha\big\|\nabla u(0)\big\|_{L^2}^2\big)\\ &\quad+\frac{(\lambda_1^{-1}+\alpha)}{\nu^2\lambda_1}\|f\|_{L_t^\infty(L^2)}^2,\end{aligned} \tag{2.6}$$

where, for any Banach space X, $L_t^\infty(X)$ denotes the set $L^\infty((0,t),X)$, while $L^\infty(X)$ denotes the set $L^\infty((0,+\infty),X)$. In particular, if f does not depend on the time variable, any element $u \in \mathcal{A}_\alpha$ satisfies the following estimate

$$\|u\|_{L^2}^2+\alpha\|\nabla u\|_{L^2}^2 \leqslant \frac{2(\lambda_1^{-1}+\alpha)}{\nu^2\lambda_1}\|f\|_{L^2}^2. \tag{2.7}$$

We remark that the estimate (2.7) together with the Poincaré inequality imply that the norm $\|u\|_{L^2}$ is bounded by a constant independent of $\alpha > 0$.

Integrating the inequality (2.5) from t to $t+\tau$, $\tau > 0$, and taking into account the estimate (2.6), we also get, for $t \geqslant 0$,

$$\begin{aligned}&\big\|u(t+\tau)\big\|_{L^2}^2+\alpha\big\|\nabla u(t+\tau)\big\|_{L^2}^2+\nu\int_t^{t+\tau}\big\|\nabla u(s)\big\|_{L^2}^2\,ds\\ &\quad\leqslant \exp\left(-\frac{\nu t}{(\lambda_1^{-1}+\alpha)}\right)\big(\big\|u(0)\big\|_{L^2}^2+\alpha\big\|\nabla u(0)\big\|_{L^2}^2\big)+\left(\frac{\lambda_1^{-1}+\alpha}{\nu^2\lambda_1}+\frac{\tau}{\nu\lambda_1}\right)\|f\|_{L^\infty(L^2)}^2.\end{aligned} \tag{2.8}$$

We next want to obtain a priori estimates of the H^2 and H^3 norms of the solutions of Eqs. (1.2). Assume that $u(t)$ is a smooth enough solution of (1.2). Taking the vorticity of the first equation in (1.2), we obtain the equation

$$\partial_t \operatorname{rot}(u-\alpha\Delta u)-\nu\Delta\operatorname{rot} u+\operatorname{rot}\big(\operatorname{rot}(u-\alpha\Delta u)\times u\big)=\operatorname{rot} f. \tag{2.9}$$

Taking the L^2-inner product of (2.9) with $\operatorname{rot}(u-\alpha\Delta u)$ and using the identities (A.2) and (A.5) of Appendix A lead to the following equality

$$\frac{1}{2}\partial_t\big\|\operatorname{rot}(u-\alpha\Delta u)\big\|_{L^2}^2+\nu\big(\|\nabla\operatorname{rot} u\|_{L^2}^2+\alpha\|\Delta\operatorname{rot} u\|_{L^2}^2\big)=\big(\operatorname{rot} f,\operatorname{rot}(u-\alpha\Delta u)\big). \tag{2.10}$$

Remarking that,

$$\|\mathrm{rot}\, u\|_{L^2}^2 + 2\alpha\|\nabla\, \mathrm{rot}\, u\|_{L^2}^2 + \alpha^2\|\Delta\, \mathrm{rot}\, u\|_{L^2}^2 \leqslant \big(\lambda_1^{-1} + 2\alpha\big)\big(\|\nabla\, \mathrm{rot}\, u\|_{L^2}^2 + \alpha\|\Delta\, \mathrm{rot}\, u\|_{L^2}^2\big), \quad (2.11)$$

we deduce from the previous equality (2.10) that, for $t \geqslant 0$,

$$\partial_t \big\|\mathrm{rot}(u - \alpha\Delta u)\big\|_{L^2}^2 + \frac{\nu}{2(\lambda_1^{-1} + 2\alpha)}\big\|\mathrm{rot}(u - \alpha\Delta u)\big\|_{L^2}^2 + \frac{\nu}{2}\big(\|\nabla\, \mathrm{rot}\, u\|_{L^2}^2 + \alpha\|\Delta\, \mathrm{rot}\, u\|_{L^2}^2\big)$$
$$\leqslant \frac{(\lambda_1^{-1} + 2\alpha)}{\nu}\|\mathrm{rot}\, f\|_{L^2}^2. \quad (2.12)$$

Integrating this inequality and using Gronwall Lemma, we obtain, for $t \geqslant 0$ and for any $0 < \beta_0 \leqslant \frac{\nu}{2(\lambda_1^{-1}+2\alpha)}$,

$$\big\|\mathrm{rot}\, u(t)\big\|_{L^2}^2 + 2\alpha\big\|\nabla\, \mathrm{rot}\, u(t)\big\|_{L^2}^2 + \alpha^2\big\|\Delta\, \mathrm{rot}\, u(t)\big\|_{L^2}^2$$
$$+ \frac{\nu}{2}\int_0^t \exp\big(\beta_0(s-t)\big)\big(\big\|\nabla\, \mathrm{rot}\, u(s)\big\|_{L^2}^2 + \alpha\big\|\Delta\, \mathrm{rot}\, u(s)\big\|_{L^2}^2\big)\, ds$$
$$\leqslant \exp(-\beta_0 t)\big\|\mathrm{rot}\big(u(0) - \alpha\Delta u(0)\big)\big\|_{L^2}^2 + \frac{\lambda_1^{-1} + 2\alpha}{\beta_0\nu}\|\mathrm{rot}\, f\|_{L_t^\infty(L^2)}^2. \quad (2.13)$$

The continuous Sobolev embedding $V^3 \subset (W^{1,\infty}(\mathbb{T}^2))^2$ implies that, for $t \geqslant 0$,

$$\alpha\big\|\nabla u(t)\big\|_{L^\infty} \leqslant C_S\bigg(\exp\bigg(-\frac{\beta_0}{2}t\bigg)\big\|\mathrm{rot}\big(u(0) - \alpha\Delta u(0)\big)\big\|_{L^2}$$
$$+ \bigg(\frac{(\lambda_1^{-1} + 2\alpha)}{\beta_0\nu}\bigg)^{1/2}\|\mathrm{rot}\, f\|_{L_t^\infty(L^2)}\bigg). \quad (2.14)$$

In particular, if f is time-independent, any element $u \in \mathcal{A}_\alpha$ satisfies the following bound,

$$\big\|\mathrm{rot}(u - \alpha\Delta u)\big\|_{L^2}^2 = \|\mathrm{rot}\, u\|_{L^2}^2 + 2\alpha\|\nabla\, \mathrm{rot}\, u\|_{L^2}^2 + \alpha^2\|\Delta\, \mathrm{rot}\, u\|_{L^2}^2$$
$$\leqslant \frac{2(\lambda_1^{-1} + 2\alpha)^2}{\nu^2}\|\mathrm{rot}\, f\|_{L^2}^2. \quad (2.15)$$

We at once deduce from (2.15) and the Poincaré inequality that, for any $u \in \mathcal{A}_\alpha$, the norm $\|\mathrm{rot}\, u\|_{L^2}$ is bounded by a constant depending only on $\|\mathrm{rot}\, f\|_{L^2}$, and that, for $\alpha > 0$, we have

$$\|\Delta\, \mathrm{rot}\, u\|_{L^2} \leqslant \frac{\sqrt{2}}{\nu}\bigg(\frac{\lambda_1^{-1}}{\alpha} + 2\bigg)\|\mathrm{rot}\, f\|_{L^2},$$
$$\alpha\|\nabla u\|_{L^\infty} \leqslant C_S\frac{\sqrt{2}(\lambda_1^{-1} + 2\alpha)}{\nu}\|\mathrm{rot}\, f\|_{L^2}. \quad (2.16)$$

Integrating the estimate (2.12) between t and $t+1$, we deduce from the inequality (2.13) that

$$\begin{aligned}
&\left\|\operatorname{rot}(u-\alpha\Delta u)(t+1)\right\|_{L^2}^2+\frac{\nu}{2}\int_t^{t+1}\left(\left\|\nabla\operatorname{rot}u(s)\right\|_{L^2}^2+\alpha\left\|\Delta\operatorname{rot}u(s)\right\|_{L^2}^2\right)ds\\
&\quad\leqslant\exp\left(-\frac{\nu t}{2(\lambda_1^{-1}+2\alpha)}\right)\left\|\operatorname{rot}(u-\alpha\Delta u)(0)\right\|_{L^2}^2\\
&\qquad+\frac{(\lambda_1^{-1}+2\alpha)}{\nu}\left(1+\frac{2(\lambda_1^{-1}+2\alpha)}{\nu}\right)\|\operatorname{rot}f\|_{L^\infty_{t+1}(L^2)}^2. \qquad (2.17)
\end{aligned}$$

If f depends on the time variable and is square-integrable in time, the estimate (2.13) is simply replaced by the following one. Integrating the estimate (2.12) between 0 and t, we obtain, for $t\geqslant 0$,

$$\begin{aligned}
&\left\|\operatorname{rot}(u-\alpha\Delta u)(t)\right\|_{L^2}^2+\frac{\nu}{2}\int_0^{t}\left(\left\|\nabla\operatorname{rot}u(s)\right\|_{L^2}^2+\alpha\left\|\Delta\operatorname{rot}u(s)\right\|_{L^2}^2\right)ds\\
&\quad\leqslant\left\|\operatorname{rot}\bigl(u(0)-\alpha\Delta u(0)\bigr)\right\|_{L^2}^2+\frac{\lambda_1^{-1}+2\alpha}{\nu}\|\operatorname{rot}f\|_{L^2_t(L^2)}^2. \qquad (2.18)
\end{aligned}$$

When $\alpha\lambda_1\leqslant 1$ for example, the estimates (2.13) and (2.15) can be improved. Indeed, taking the L^2-inner product of (2.9) with $-\Delta\operatorname{rot}u$ and noticing that, by the properties (A.2) and (A.5) of Appendix A,

$$\bigl(\operatorname{rot}(\operatorname{rot}\Delta u\times u),\operatorname{rot}\Delta u\bigr)=0, \qquad (2.19)$$

we get,

$$\begin{aligned}
&\frac{1}{2}\partial_t\left(\|\nabla\operatorname{rot}u\|_{L^2}^2+\alpha\|\Delta\operatorname{rot}u\|_{L^2}^2\right)+\nu\|\Delta\operatorname{rot}u\|_{L^2}^2-\bigl(\operatorname{rot}(\operatorname{rot}u\times u),\Delta\operatorname{rot}u\bigr)\\
&\quad=-(\operatorname{rot}f,\Delta\operatorname{rot}u). \qquad (2.20)
\end{aligned}$$

The properties (A.2) and (A.3) in Appendix A imply that

$$\bigl(\operatorname{rot}(\operatorname{rot}u\times u),\Delta\operatorname{rot}u\bigr)=\sum_{i,j=1}^{2}\int_{\mathbb{T}^2}\partial_j u_i\Delta u_i\Delta u_j\,dx.$$

Using the Gagliardo–Nirenberg and Young inequalities, we thus obtain,

$$\begin{aligned}
\left|\bigl(\operatorname{rot}(\operatorname{rot}u\times u),\Delta\operatorname{rot}u\bigr)\right|&\leqslant C\|\nabla u\|_{L^2}\|\nabla\operatorname{rot}u\|_{L^4}^2\leqslant C\|\nabla u\|_{L^2}\|\nabla\operatorname{rot}u\|_{H^{1/2}}^2\\
&\leqslant C\|\nabla u\|_{L^2}\|\nabla\operatorname{rot}u\|_{L^2}\|\Delta\operatorname{rot}u\|_{L^2}. \qquad (2.21)
\end{aligned}$$

The properties (2.20) and (2.21) together with the condition $\alpha\lambda_1\leqslant 1$ imply that

$$\begin{aligned}
&\partial_t\left(\|\nabla\operatorname{rot}u\|_{L^2}^2+\alpha\|\Delta\operatorname{rot}u\|_{L^2}^2\right)+\frac{\nu\lambda_1}{4}\left(\|\nabla\operatorname{rot}u\|_{L^2}^2+\alpha\|\Delta\operatorname{rot}u\|_{L^2}^2\right)+\frac{\nu}{2}\|\Delta\operatorname{rot}u\|_{L^2}^2\\
&\quad\leqslant\frac{2}{\nu}\|\operatorname{rot}f\|_{L^\infty_t(L^2)}^2+\frac{C}{\nu}\|\nabla u\|_{L^2}^2\|\nabla\operatorname{rot}u\|_{L^2}^2. \qquad (2.22)
\end{aligned}$$

Let

$$\beta_1=\frac{\nu\lambda_1}{4(1+2\lambda_1\alpha)}. \qquad (2.23)$$

Integrating the inequality (2.22) and applying Gronwall inequality, we get, for $t \geqslant 0$,

$$\begin{aligned}
&\big\|\nabla \operatorname{rot} u(t)\big\|_{L^2}^2 + \alpha\big\|\Delta \operatorname{rot} u(t)\big\|_{L^2}^2 + \frac{\nu}{2}\int_0^t \exp\big(\beta_1(s-t)\big)\big\|\Delta \operatorname{rot} u(s)\big\|_{L^2}^2\,ds \\
&\quad\leqslant \exp(-\beta_1 t)\big(\big\|\nabla \operatorname{rot} u(0)\big\|_{L^2}^2 + \alpha\big\|\Delta \operatorname{rot} u(0)\big\|_{L^2}^2\big) + \frac{2}{\beta_1\nu}\|\operatorname{rot} f\|_{L_t^\infty(L^2)}^2 \\
&\qquad + \frac{C}{\nu}\|\nabla u\|_{L_t^\infty(L^2)}^2\int_0^t \exp\big(\beta_1(s-t)\big)\big\|\nabla \operatorname{rot} u(s)\big\|_{L^2}^2\,ds. \qquad (2.24)
\end{aligned}$$

Taking into account the estimate (2.13), we deduce from the previous inequality that, for $t \geqslant 0$,

$$\begin{aligned}
&\big\|\nabla \operatorname{rot} u(t)\big\|_{L^2}^2 + \alpha\big\|\Delta \operatorname{rot} u(t)\big\|_{L^2}^2 + \frac{\nu}{2}\int_0^t \exp\big(\beta_1(s-t)\big)\big\|\Delta \operatorname{rot} u(s)\big\|_{L^2}^2\,ds \\
&\quad\leqslant \exp(-\beta_1 t)\big(\big\|\nabla \operatorname{rot} u(0)\big\|_{L^2}^2 + \alpha\big\|\Delta \operatorname{rot} u(0)\big\|_{L^2}^2\big) + \frac{2}{\beta_1\nu}\|\operatorname{rot} f\|_{L_t^\infty(L^2)}^2 \\
&\qquad + \frac{C}{\nu^2}\big\|\operatorname{rot}\big(u(0) - \alpha\Delta u(0)\big)\big\|_{L^2}^4 + \frac{C(\lambda_1^{-1} + 2\alpha)^2}{\beta_1^2\nu^4}\|\operatorname{rot} f\|_{L_t^\infty(L^2)}^4. \qquad (2.25)
\end{aligned}$$

To prove that $\|\nabla \operatorname{rot} u(t)\|_{L^2}^2 + \alpha\|\Delta \operatorname{rot} u(t)\|_{L^2}^2$ decays exponentially fast to some constant when $t \to +\infty$, we can apply the uniform Gronwall inequality. We recall that if, for $t \geqslant 0$,

$$\frac{dy}{dt}(t) \leqslant g(t)y(t) + h(t),$$

where $y(t)$, $g(t)$ and $h(t)$ are non-negative locally integrable functions, then, for $t \geqslant 0$ and $r > 0$,

$$y(t+r) \leqslant \left(\frac{1}{r}\int_t^{t+r} y(s)\,ds + \int_t^{t+r} h(s)\,ds\right)\exp\int_t^{t+r} g(s)\,ds. \qquad (2.26)$$

Integrating the inequality (2.22) from t to $t+1$, applying the uniform Gronwall inequality and using the estimates (2.13) and (2.17) yield, for $t \geqslant 0$,

$$\begin{aligned}
\big\|\nabla \operatorname{rot} u(t+1)\big\|_{L^2}^2 + \alpha\big\|\Delta \operatorname{rot} u(t+1)\big\|_{L^2}^2 &\leqslant \int_t^{t+1}\big(\big\|\nabla \operatorname{rot} u(s)\big\|_{L^2}^2 + \alpha\big\|\Delta \operatorname{rot} u(s)\big\|_{L^2}^2\big)\,ds \\
&\quad + \frac{C}{\nu}\int_t^{t+1}\big\|\nabla u(s)\big\|_{L^2}^2\big\|\nabla \operatorname{rot} u(s)\big\|_{L^2}^2\,ds \\
&\quad + \frac{2}{\nu}\|\operatorname{rot} f\|_{L^\infty(L^2)}^2,
\end{aligned}$$

and thus,

$$\big\|\nabla \operatorname{rot} u(t+1)\big\|_{L^2}^2+\alpha\big\|\Delta \operatorname{rot} u(t+1)\big\|_{L^2}^2 \leqslant \frac{2}{\nu}\left(1+\frac{C}{\nu}E(t)\right)\left[E(t)+\frac{1+2\alpha\lambda_1}{\nu\lambda_1}\|\operatorname{rot} f\|_{L^\infty(L^2)}^2\right] + \frac{2}{\nu}\|\operatorname{rot} f\|_{L^\infty(L^2)}^2, \tag{2.27}$$

where

$$E(t)\equiv \exp\left(-\frac{\nu\lambda_1 t}{2(1+2\alpha\lambda_1)}\right)\big\|\operatorname{rot}(u-\alpha\Delta u)(0)\big\|_{L^2}^2+\frac{2(1+2\alpha\lambda_1)^2}{\nu^2\lambda_1^2}\|\operatorname{rot} f\|_{L^\infty(L^2)}^2. \tag{2.28}$$

The Brézis–Gallouët inequality (see [5]) implies that, for $t\geqslant 0$,

$$\begin{aligned}\big\|\nabla u(t)\big\|_{L^\infty}^2 &\leqslant C_{BG}\big\|\nabla \operatorname{rot} u(t)\big\|_{L^2}^2\left(1+\ln\left(1+\frac{\|\Delta \operatorname{rot} u(t)\|_{L^2}^2}{\|\nabla \operatorname{rot} u(t)\|_{L^2}^2}\right)\right)\\ &\leqslant C_{BG}\big\|\nabla \operatorname{rot} u(t)\big\|_{L^2}^2\left(1+\ln\left(1+\frac{\|\nabla \operatorname{rot} u(t)\|_{L^2}^2+\alpha\|\Delta \operatorname{rot} u(t)\|_{L^2}^2}{\alpha\|\nabla \operatorname{rot} u(t)\|_{L^2}^2}\right)\right).\end{aligned} \tag{2.29}$$

Since the function $x\ln(1+\frac{y}{\alpha x})$ is an increasing function of x (where $0<x\leqslant y$), we deduce from the inequalities (2.29) and (2.25) that, for $t\geqslant 0$,

$$\begin{aligned}\big\|\nabla u(t)\big\|_{L^\infty}^2 \leqslant C_{BG}\left(1+\ln\left(1+\frac{1}{\alpha}\right)\right)\Big(&\exp(-\beta_1 t)\big(\big\|\nabla \operatorname{rot} u(0)\big\|^2+\alpha\big\|\Delta \operatorname{rot} u(0)\big\|^2\big)\\ &+\frac{2}{\beta_1\nu}\|\operatorname{rot} f\|_{L_t^\infty(L^2)}^2+\frac{C}{\nu^2}\big\|\operatorname{rot}\big(u(0)-\alpha\Delta u(0)\big)\big\|_{L^2}^4\\ &+\frac{C(\lambda_1^{-1}+2\alpha)^2}{\beta_1^2\nu^4}\|\operatorname{rot} f\|_{L_t^\infty(L^2)}^4\Big).\end{aligned} \tag{2.30}$$

We also infer from the inequalities (2.29) and (2.27) that, for $t\geqslant 0$,

$$\begin{aligned}\big\|\nabla u(t+1)\big\|_{L^\infty}^2 \leqslant C_{BG}\left(1+\ln\left(1+\frac{1}{\alpha}\right)\right)\Big(&\frac{2}{\nu}\left(1+\frac{C}{\nu}E(t)\right)\\ &\times\left[E(t)+\frac{1+2\alpha\lambda_1}{\nu\lambda_1}\|\operatorname{rot} f\|_{L^\infty(L^2)}^2\right]+\frac{2}{\nu}\|\operatorname{rot} f\|_{L^\infty(L^2)}^2\Big).\end{aligned} \tag{2.31}$$

When $u(s)$ is a bounded solution of (1.2) on the whole line $\mathbb{R}$, the above estimates become much simpler. If f is time-independent, in the case $\alpha\lambda_1\leqslant 1$, the estimates (2.25) and (2.30) together with (2.15) imply that every element $u\in\mathcal{A}_\alpha$ satisfies

$$\|\nabla \operatorname{rot} u\|_{L^2}^2+\alpha\|\Delta \operatorname{rot} u\|_{L^2}^2\leqslant C_0\left(\frac{1}{\nu^2\lambda_1}\|\operatorname{rot} f\|_{L^2}^2+\frac{1}{\nu^6\lambda_1^4}\|\operatorname{rot} f\|_{L^2}^4\right), \tag{2.32}$$

and also,

$$\|\nabla u\|_{L^\infty}^2\leqslant C_0C_{BG}\left(1+\ln\left(1+\frac{1}{\alpha}\right)\right)\left(\frac{1}{\nu^2\lambda_1}\|\operatorname{rot} f\|_{L^2}^2+\frac{1}{\nu^6\lambda_1^4}\|\operatorname{rot} f\|_{L^2}^4\right), \tag{2.33}$$

where C_0 is a positive constant independent of α and f.

The Poincaré inequality and the estimates (2.16) and (2.32) show in particular that, if u belongs to the global attractor $\mathcal{A}_\alpha$, the quantity $\|\nabla \operatorname{rot} u\|_{L^2}$ and thus $\|u\|_{V^2}$ can be bounded by a constant, which is independent of α and depends only on $\|\operatorname{rot} f\|_{L^2}$.

Remark 2.1. The estimates (2.13), (2.14) and (2.18) still hold if $u(t)$ is a smooth enough solution of the following non-autonomous affine equation

$$\begin{aligned} &\partial_t(u-\alpha\Delta u)-\nu\Delta u+\operatorname{rot}(u-\alpha\Delta u)\times u^*+\nabla p=f, \quad t>0,\ x\in\mathbb{T}^2,\\ &\operatorname{div} u=0, \quad t>0,\ x\in\mathbb{T}^2,\\ &u(0,x)=u_0(x), \quad x\in\mathbb{T}^2, \end{aligned} \tag{2.34}$$

where u_0 is an element of V^3 and u^* belongs to $C^0([0,+\infty),V^2)$.

In the case $\alpha\lambda_1 \leqslant 1$, the estimate (2.24) is still valid for the solution u of (2.34) provided the term $\|\nabla u\|_{L^\infty_t(L^2)}$ is replaced by $\|\nabla u^*\|_{L^\infty_t(L^2)}$ in (2.24). And the inequality (2.25) is replaced by

$$\begin{aligned} &\big\|\nabla \operatorname{rot} u(t)\big\|_{L^2}^2+\alpha\big\|\Delta \operatorname{rot} u(t)\big\|_{L^2}^2+\frac{\nu}{2}\int_0^t \exp\big(\beta_1(s-t)\big)\big\|\Delta \operatorname{rot} u(s)\big\|_{L^2}^2\,ds\\ &\quad\leqslant \exp(-\beta_1 t)\big(\big\|\nabla \operatorname{rot} u(0)\big\|_{L^2}^2+\alpha\big\|\Delta \operatorname{rot} u(0)\big\|_{L^2}^2\big)+\frac{2}{\beta_1\nu}\|\operatorname{rot} f\|_{L^\infty_t(L^2)}^2\\ &\qquad+\frac{C}{\nu^2}\big\|\nabla u^*\big\|_{L^\infty_t(L^2)}^2\bigg(\exp(-\beta_1 t)\big\|\operatorname{rot}\big(u(0)-\alpha\Delta u(0)\big)\big\|_{L^2}^2+\frac{\lambda_1^{-1}+2\alpha}{\nu\beta_1}\|\operatorname{rot} f\|_{L^\infty_t(L^2)}^2\bigg). \end{aligned} \tag{2.35}$$

Likewise, the inequality (2.27) is replaced by

$$\begin{aligned} &\big\|\nabla \operatorname{rot} u(t+1)\big\|_{L^2}^2+\alpha\big\|\Delta \operatorname{rot} u(t+1)\big\|_{L^2}^2\\ &\quad\leqslant \frac{2}{\nu}\bigg(1+\frac{C}{\nu}\big\|\nabla u^*\big\|_{L^\infty_{t+1}(L^2)}^2\bigg)\bigg[E(t)+\frac{1+2\alpha\lambda_1}{\nu\lambda_1}\|\operatorname{rot} f\|_{L^\infty(L^2)}^2\bigg]+\frac{2}{\nu}\|\operatorname{rot} f\|_{L^\infty(L^2)}^2. \end{aligned} \tag{2.36}$$

2.3. Case of smoother initial data: V^4-regularity

In the next two sections, we assume that the initial data u_0 belong to V^{m+3}, $m>0$, and that f is in H^{m+1}_{per}. We prove that the solution $u(t)$ of (1.2) is then in V^{m+3} and we give uniform estimates in t under additional conditions. To this end, we consider the following auxiliary affine problem,

$$\begin{aligned} &\partial_t\big(w^*-\alpha\Delta w^*\big)-\nu\Delta w^*+\operatorname{rot}\big(w^*-\alpha\Delta w^*\big)\times u^*+\nabla p^*=f, \quad t>0,\ x\in\mathbb{T}^2,\\ &\operatorname{div} w^*=0, \quad t>0,\ x\in\mathbb{T}^2,\\ &w^*(0,x)=u_0(x), \quad x\in\mathbb{T}^2, \end{aligned} \tag{2.37}$$

where $f\in H^{m+1}_{per}$ and $u^*\in L^\infty((0,+\infty),V^{m+2})\cap C^0([0,+\infty),V^2)$. Once we have proved regularity results and appropriate estimates for the solution w^* of (2.37), we set $u^*=u$ in the above equations in order to deduce the regularity properties and corresponding estimates for the solution u of (1.2).

We begin with the case $m=1$. Arguing as in the case of the nonlinear equation (1.2), one easily shows that, for any $u_0\in V^3$, there exists a unique global solution (in the sense of Definition 2.1) $w^*(t)\in C^0([0,+\infty),V^3)$ of (2.37), which also satisfies the inequality (2.13). In what follows, we are going to prove that actually $w^*(t)$ belongs to $C^0([0,+\infty),V^4)$. Notice that the propagation of the

V^4-regularity has already been proved in [8] for the local solutions of the system (1.2) in the three-dimensional case.

Theorem 2.3. *Assume that f and $u^*(t)$ belong respectively to the spaces $L^\infty((0,+\infty), H^2_{per})$ and $L^\infty((0,+\infty), V^3) \cap C^0([0,+\infty), V^2)$. Then, for any $\alpha > 0$ and for any $u_0 \in V^4$, there exists a unique solution $w^* \in C^0([0,+\infty), V^4)$ of* (2.37)*.*

Furthermore, if $a_1 = 2\nu - 2\alpha\|\nabla u^*\|_{L^\infty(L^\infty)} > 0$, w^* belongs to the space $C^0_b([0,+\infty), V^4)$, that is, w^* is uniformly bounded in V^4, for $t \geqslant 0$ and the estimates (2.51) *and* (2.53) *below hold. In the case where $w^*(t)$ and $u^*(t)$ belong the global attractor and $\alpha\lambda_1 < 1$, the estimate* (2.53) *is replaced by the better estimate* (2.61)*.*

If $2\nu - 2\alpha\|\nabla u^*\|_{L^\infty(L^\infty)} \leqslant 0$, then w^* belongs to the space $C^0_b([0,+\infty), V^{3+\theta})$, for any θ satisfying the condition (2.70) *below and the estimates* (2.72) *and* (2.74) *below hold.*

Finally, if moreover f and ∇u^* belong to $L^2((0,+\infty), H^2_{per})$ and $L^2((0,+\infty), L^\infty)$ respectively, then, for any $\alpha > 0$, w^* belongs to $C^0([0,+\infty), V^4)$ and the inequality (2.63) *below holds. In particular, if f is in $L^2((0,+\infty), H^2_{per})$, for any $u_0 \in V^4$, for any $\alpha > 0$, the solutions $u(t)$ of* (1.2) *belong to $C^0_b([0,+\infty), V^4)$ and the estimate* (2.65) *below holds.*

Proof. In this proof, we shall mainly discuss the uniform boundedness of $w^*(t)$ in V^4. We will prove the property that $t \mapsto w^*(t) \in V^4$ is a continuous map at the end of this proof.

Arguing as in [8, Section 5], one can prove the V^4-regularity of the solution w^* of (2.37). For this reason, we shall not give the details of the proof, but emphasize the a priori estimates that we need for the purpose of this paper. In order to show that, if f and u_0 are more regular, the solution $w^*(t)$ is also smoother, we begin by performing a Galerkin scheme. We denote by $w^*_n(t) = P_n w^*(t)$ the solution of the equation

$$\begin{aligned} &\partial_t\big(w^*_n - \alpha\Delta w^*_n\big) - \nu\Delta w^*_n + P_n P\big(\mathrm{rot}\big(w^*_n - \alpha\Delta w^*_n\big) \times u^*\big) = P_n P f, \quad t > 0,\ x \in \mathbb{T}^2, \\ &\mathrm{div}\, w^*_n = 0, \quad t > 0,\ x \in \mathbb{T}^2, \\ &w^*_n(0,x) = P_n u_0(x), \quad x \in \mathbb{T}^2, \end{aligned} \tag{2.38}$$

where P_n is the orthogonal projection (in H) onto the span of the eigenfunctions of the Stokes operator $-P\Delta$, corresponding to the first n eigenvalues $0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_n$.

Considering the vorticity of this equation, we obtain the equality

$$\partial_t\,\mathrm{rot}\big(w^*_n - \alpha\Delta w^*_n\big) - \nu\Delta\,\mathrm{rot}\,P_n P w^*_n + \mathrm{rot}\,P_n P\big\{\mathrm{rot}\big(w^*_n - \alpha\Delta w^*_n\big) \times u^*\big\} = \mathrm{rot}\,P_n P f. \tag{2.39}$$

Taking the L^2-inner product of this equation with $-\mathrm{rot}(\Delta w^*_n - \alpha\Delta^2 w^*_n)$, using the equality (A.2), and remarking that, by (A.5), the term $(\mathrm{rot}(\mathrm{rot}\,\Delta w^*_n \times u^*), \mathrm{rot}\,\Delta w^*_n)$ vanishes, we get the following equality,

$$\begin{aligned} &\frac{1}{2}\partial_t\big\|\nabla\big(\mathrm{rot}\,w^*_n - \alpha\,\mathrm{rot}\,\Delta w^*_n\big)\big\|^2_{L^2} + \nu\big(\big\|\Delta\,\mathrm{rot}\,w^*_n\big\|^2_{L^2} + \alpha\big\|\Delta^2 w^*_n\big\|^2_{L^2}\big) + \big(\Delta w^*_n \times u^*, \mathrm{rot}\,\Delta w^*_n\big) \\ &\quad - \alpha\big(\Delta w^*_n \times u^*, \mathrm{rot}\,\Delta^2 w^*_n\big) + \alpha^2\big(\Delta^2 w^*_n \times u^*, \mathrm{rot}\,\Delta^2 w^*_n\big) \\ &= -\big(\mathrm{rot}\,P_n P f, \mathrm{rot}\big(\Delta w^*_n - \alpha\Delta^2 w^*_n\big)\big). \end{aligned} \tag{2.40}$$

We begin with the estimation of the term $\alpha^2(\Delta^2 w^*_n \times u^*, \mathrm{rot}\,\Delta^2 w^*_n)$. We first recall the equality (A.3) of Appendix A,

$$(v \times u, \mathrm{rot}\,v) = -\sum_{i,j=1}^{2}\int_{\mathbb{T}^2}\partial_i u_j v_i v_j\,dx, \tag{2.41}$$

valid for any smooth enough vectors v and w. Using this equality with $u = u^*$ and $v = \Delta^2 w_n^*$, we obtain the following bound

$$\alpha^2 \big|\big(\Delta^2 w_n^* \times u^*, \operatorname{rot} \Delta^2 w_n^*\big)\big| \leqslant \alpha^2 \big\|\nabla u^*\big\|_{L^\infty} \big\|\Delta^2 w_n^*\big\|_{L^2}^2. \tag{2.42}$$

Likewise, using the equality (2.41) with $u = u^*$ and $v = \Delta w_n^*$, we can write,

$$\big|\big(\Delta w_n^* \times u^*, \operatorname{rot} \Delta w_n^*\big)\big| \leqslant \big\|\nabla u^*\big\|_{L^\infty} \big\|\Delta w_n^*\big\|_{L^2}^2. \tag{2.43}$$

Performing an integration by parts, we get the following estimate for the term $\alpha|(\Delta w_n^* \times u^*, \operatorname{rot} \Delta^2 w_n^*)|$,

$$\alpha \big|\big(\Delta w_n^* \times u^*, \operatorname{rot} \Delta^2 w_n^*\big)\big| \leqslant \alpha\big(\big\|u^*\big\|_{L^\infty} \big\|\nabla \Delta w_n^*\big\|_{L^2} + \big\|\nabla u^*\big\|_{L^\infty} \big\|\Delta w_n^*\big\|_{L^2}\big) \big\|\Delta^2 w_n^*\big\|_{L^2}. \tag{2.44}$$

Finally, we estimate the term containing the forcing term as follows,

$$\big|\big(\operatorname{rot} P_n P f, \operatorname{rot}\big(\Delta w_n^* - \alpha \Delta^2 w_n^*\big)\big)\big| \leqslant \|\operatorname{rot} P f\|_{L^2} \big\|\operatorname{rot} \Delta w_n^*\big\|_{L^2} + \alpha \|\Delta P f\|_{L^2} \big\|\Delta^2 w_n^*\big\|_{L^2}. \tag{2.45}$$

The equality (2.40) and the estimates (2.42) to (2.45) imply that, for $t \geqslant 0$,

$$\begin{aligned}
&\partial_t \big\|\nabla\big(\operatorname{rot} w_n^* - \alpha \operatorname{rot} \Delta w_n^*\big)\big\|_{L^2}^2 + \nu \big\|\Delta \operatorname{rot} w_n^*\big\|_{L^2}^2 + (2-\beta_2)\nu\alpha \big\|\Delta^2 w_n^*\big\|_{L^2}^2 \\
&\quad \leqslant \frac{3\alpha}{\beta_2 \nu}\big(\big\|u^*\big\|_{L^\infty}^2 \big\|\nabla \Delta w_n^*\big\|_{L^2}^2 + \big\|\nabla u^*\big\|_{L^\infty}^2 \big\|\Delta w_n^*\big\|_{L^2}^2 + \|\Delta P f\|_{L^2}^2\big) + \frac{2}{\nu}\|\operatorname{rot} P f\|_{L^2}^2 \\
&\qquad + 2\alpha^2 \big\|\nabla u^*\big\|_{L^\infty} \big\|\Delta^2 w_n^*\big\|_{L^2}^2 + 2\min\left(1, \frac{1}{\lambda_1 \nu}\big\|\nabla u^*\big\|_{L^\infty}\right) \big\|\nabla u^*\big\|_{L^\infty} \big\|\Delta w_n^*\big\|_{L^2}^2,
\end{aligned} \tag{2.46}$$

where $0 < \beta_2 \leqslant 2$.

We notice that, taking the L^2-inner product of Eq. (2.39) with $\operatorname{rot} \Delta^2 w_n^*$ and arguing exactly as above, we also obtain the following inequality, for $t \geqslant 0$,

$$\begin{aligned}
&\partial_t \big(\big\|\operatorname{rot} \Delta w_n^*\big\|_{L^2}^2 + \alpha \big\|\Delta^2 w_n^*\big\|_{L^2}^2\big) + (2-\beta_2)\nu \big\|\Delta^2 w_n^*\big\|_{L^2}^2 \\
&\quad \leqslant \frac{3}{\beta_2 \nu}\big(\big\|u^*\big\|_{L^\infty}^2 \big\|\nabla \Delta w_n^*\big\|_{L^2}^2 + \big\|\nabla u^*\big\|_{L^\infty}^2 \big\|\Delta w_n^*\big\|_{L^2}^2 + \|\Delta P f\|_{L^2}^2\big) \\
&\qquad + 2\alpha \big\|\nabla u^*\big\|_{L^\infty} \big\|\Delta^2 w_n^*\big\|_{L^2}^2,
\end{aligned} \tag{2.47}$$

where $0 < \beta_2 \leqslant 2$.

Now, in the inequalities (2.46) and (2.47), we have to distinguish two cases according to the sign of the quantity $a_1 \equiv 2\nu - 2\alpha \|\nabla u^*\|_{L^\infty(L^\infty)}$.

First case: $a_1 \equiv 2\nu - 2\alpha \|\nabla u^*\|_{L^\infty(L^\infty)} > 0$**.**
In this case, we choose $\beta_2 = a_1/2\nu$ in the inequality (2.46) and we obtain the following bound, for $t \geqslant 0$,

$$\begin{aligned}
&\partial_t \big\|\nabla\big(\operatorname{rot} w_n^* - \alpha \operatorname{rot} \Delta w_n^*\big)\big\|_{L^2}^2 + \nu \big\|\operatorname{rot} \Delta w_n^*\big\|_{L^2}^2 + \frac{a_1 \alpha}{2} \big\|\Delta^2 w_n^*\big\|_{L^2}^2 \\
&\quad \leqslant \frac{6\alpha}{a_1}\big(\big\|u^*\big\|_{L^\infty}^2 \big\|\nabla \Delta w_n^*\big\|_{L^2}^2 + \big\|\nabla u^*\big\|_{L^\infty}^2 \big\|\Delta w_n^*\big\|_{L^2}^2 + \|\Delta P f\|_{L^2}^2\big) \\
&\qquad + 2\min\left(1, \frac{1}{\lambda_1 \nu}\big\|\nabla u^*\big\|_{L^\infty}\right) \big\|\nabla u^*\big\|_{L^\infty} \big\|\Delta w_n^*\big\|_{L^2}^2 + \frac{2}{\nu}\|\operatorname{rot} P f\|_{L^2}^2.
\end{aligned} \tag{2.48}$$

Remarking that

$$\left\|\nabla\left(\operatorname{rot} w_n^* - \alpha \operatorname{rot}\Delta w_n^*\right)\right\|_{L^2}^2 \leqslant \left(\lambda_1^{-1} + 2\alpha\right)\left(\left\|\operatorname{rot}\Delta w_n^*\right\|_{L^2}^2 + \alpha\left\|\Delta^2 w_n^*\right\|_{L^2}^2\right), \tag{2.49}$$

and integrating in time the inequality (2.48), we get, for $t \geqslant 0$,

$$\begin{aligned}
&\left\|\nabla\left(\operatorname{rot} w_n^*(t) - \alpha \operatorname{rot}\Delta w_n^*(t)\right)\right\|_{L^2}^2 + \int_0^t \exp\left(\beta_3(s-t)\right)\left(\frac{\nu}{2}\left\|\operatorname{rot}\Delta w_n^*(s)\right\|_{L^2}^2 + \frac{a_1\alpha}{4}\left\|\Delta^2 w_n^*(s)\right\|_{L^2}^2\right) ds \\
&\quad \leqslant \exp(-\beta_3 t)\left\|\nabla\left(\operatorname{rot} w_n^*(0) - \alpha \operatorname{rot}\Delta w_n^*(0)\right)\right\|_{L^2}^2 + \frac{6\alpha}{a_1\beta_3}\|\Delta Pf\|_{L_t^\infty(L^2)}^2 + \frac{2}{\nu\beta_3}\|\operatorname{rot} Pf\|_{L_t^\infty(L^2)}^2 \\
&\qquad + \frac{6\alpha}{a_1}\int_0^t \exp\left(\beta_3(s-t)\right)\left(\left\|u^*(s)\right\|_{L^\infty}^2\left\|\nabla\Delta w_n^*(s)\right\|_{L^2}^2 + \left\|\nabla u^*(s)\right\|_{L^\infty}^2\left\|\Delta w_n^*(s)\right\|_{L^2}^2\right) ds \\
&\qquad + 2\int_0^t \exp\left(\beta_3(s-t)\right)\min\left(1, \frac{1}{\lambda_1\nu}\left\|\nabla u^*(s)\right\|_{L^\infty}\right)\left\|\nabla u^*(s)\right\|_{L^\infty}\left\|\Delta w_n^*(s)\right\|_{L^2}^2 ds,
\end{aligned} \tag{2.50}$$

where $0 < \beta_3 \leqslant \frac{a_1}{4(\lambda_1^{-1}+2\alpha)}$. In the case where u^* is a general smooth divergence-free vector field, we proceed as follows. Taking into account the estimate (2.13) (see also Remark 2.1) and the fact that $2\nu - 2\alpha\|\nabla u^*\|_{L^\infty(L^\infty)} > 0$, we derive from the inequality (2.50) that, for $t \geqslant 0$,

$$\begin{aligned}
&\left\|\nabla\left(\operatorname{rot} w_n^*(t) - \alpha \operatorname{rot}\Delta w_n^*(t)\right)\right\|_{L^2}^2 + \int_0^t \exp\left(\beta_3(s-t)\right)\left(\frac{\nu}{2}\left\|\operatorname{rot}\Delta w_n^*(s)\right\|_{L^2}^2 + \frac{a_1\alpha}{4}\left\|\Delta^2 w_n^*(s)\right\|_{L^2}^2\right) ds \\
&\quad \leqslant \exp(-\beta_3 t)\left\|\nabla\left(\operatorname{rot} w_n^*(0) - \alpha \operatorname{rot}\Delta w_n^*(0)\right)\right\|_{L^2}^2 \\
&\qquad + \frac{24\alpha(\lambda_1^{-1}+2\alpha)}{a_1^2}\|\Delta Pf\|_{L_t^\infty(L^2)}^2 + \frac{8(\lambda_1^{-1}+2\alpha)}{a_1\nu}\|\operatorname{rot} Pf\|_{L_t^\infty(L^2)}^2 \\
&\qquad + \frac{2\nu - a_1}{\alpha\nu}\left[\frac{3(2\nu - a_1)}{a_1\alpha}\left(\lambda_1^{-1}+\alpha\right) + 2\min\left(1, \frac{2\nu - a_1}{2\alpha\nu\lambda_1}\right)\right] \\
&\qquad \times \left(\exp(-\beta_3 t)\left\|\operatorname{rot}\left(w_n^*(0) - \alpha\Delta w_n^*(0)\right)\right\|_{L^2}^2 + \frac{4(\lambda_1^{-1}+2\alpha)^2}{a_1\nu}\|\operatorname{rot} f\|_{L_t^\infty(L^2)}^2\right),
\end{aligned} \tag{2.51}$$

where $\beta_3 = a_1(\lambda_1^{-1}+2\alpha)^{-1}/4$ and C is a positive constant, independent of α, u_0, f and u^*.

The inequality (2.51) shows that $w_n^*(t)$ is uniformly bounded in V^4, for $t \geqslant 0$. Performing the Galerkin procedure in the classical way, we deduce that the solution $w^*(t)$ of (2.37) belongs to $L^\infty((0,+\infty), V^4)$ and satisfies the bound (2.51) (we leave to the reader all the classical arguments concerning the Galerkin procedure). We notice that the inequality (2.51) is a good estimate, uniform in α, when α is bounded away from zero.

Notice that, if we choose $\beta_2 = a_1/2\nu$ in the inequality (2.47) and argue as above, we obtain the following bound, for $t \geqslant 0$,

$$\left\|\operatorname{rot}\Delta w_n^*(t)\right\|_{L^2}^2 + \alpha\left\|\Delta^2 w_n^*(t)\right\|_{L^2}^2 + \frac{a_1}{4}\int_0^t \exp\left(\widetilde{\beta}_3(s-t)\right)\left\|\Delta^2 w_n^*\right\|_{L^2}^2 ds$$

$$\leqslant \exp(-\widetilde{\beta}_3 t)\big(\big\|\operatorname{rot}\Delta w_n^*(0)\big\|_{L^2}^2+\alpha\big\|\Delta^2 w_n^*(0)\big\|_{L^2}^2\big)+\frac{6}{a_1\widetilde{\beta}_3}\|\Delta Pf\|_{L_t^\infty(L^2)}^2$$
$$+\frac{6}{a_1}\int_0^t \exp\big(\widetilde{\beta}_3(s-t)\big)\big(\|u^*\|_{L^\infty}^2\|\nabla\Delta w_n^*\|_{L^2}^2+\|\nabla u^*\|_{L^\infty}^2\|\Delta w_n^*\|_{L^2}^2\big)\,ds, \tag{2.52}$$

where $\widetilde{\beta}_3 \leqslant a_1/4$. Like above, we derive from the inequality (2.52) that, for $t \geqslant 0$,

$$\big\|\operatorname{rot}\Delta w_n^*(t)\big\|_{L^2}^2+\alpha\big\|\Delta^2 w_n^*(t)\big\|_{L^2}^2+\frac{a_1}{4}\int_0^t \exp\big(\beta_4(s-t)\big)\big\|\Delta^2 w_n^*\big\|_{L^2}^2\,ds$$
$$\leqslant \exp(-\beta_4 t)\big(\big\|\operatorname{rot}\Delta w_n^*(0)\big\|_{L^2}^2+\alpha\big\|\Delta^2 w_n^*(0)\big\|_{L^2}^2\big)+\frac{6}{a_1\beta_4}\|\Delta Pf\|_{L_t^\infty(L^2)}^2$$
$$+\frac{C(2\nu-a_1)^2}{\nu a_1\alpha^3\lambda_1}\left(\exp(-\beta_4 t)\big\|\operatorname{rot}\big(w_n^*(0)-\alpha\Delta w_n^*(0)\big)\big\|_{L^2}^2+\frac{(\lambda_1^{-1}+2\alpha)}{\beta_4\nu}\|\operatorname{rot} f\|_{L_t^\infty(L^2)}^2\right), \tag{2.53}$$

where $0<\beta_4 \leqslant \inf(\frac{\nu}{2(\lambda_1^{-1}+2\alpha)}, \frac{a_1}{4})$ and C is a positive constant, independent of α, u_0, f and u^*.

Actually, we are mainly interested in the special case where $u^*(t)=S_\alpha(t)u_0^*$ is a solution of the nonlinear system (1.2). In this case, according to (2.14), the condition

$$\nu - C_S\left(\big\|\operatorname{rot}\big(u_0^*-\alpha\Delta u_0^*\big)\big\|_{L^2}+\frac{\sqrt{2}(\lambda_1^{-1}+2\alpha)}{\nu}\|\operatorname{rot} f\|_{L^\infty(L^2)}\right)>0, \tag{2.54}$$

implies that $a_1>0$. In the case where f does not depend on the time variable and u_0^* belongs to the global attractor $\mathcal{A}_\alpha$, due to (2.16), the condition

$$\nu - C_S\frac{\sqrt{2}(\lambda_1^{-1}+2\alpha)}{\nu}\|\operatorname{rot} f\|_{L^2}>0, \tag{2.55}$$

is sufficient to imply that $a_1>0$. When $\alpha\lambda_1 \leqslant 1$ and $u^*(t)=S_\alpha(t)u_0^*$ is a solution of the nonlinear system (1.2), one deduces from (2.30) that the condition $a_1>0$ is satisfied if

$$\nu-\alpha C_{BG}^{1/2}\left(1+\ln\left(1+\frac{1}{\alpha}\right)\right)^{1/2}\bigg[\big\|\nabla\operatorname{rot} u_0^*\big\|^2+\alpha\big\|\Delta\operatorname{rot} u_0^*\big\|^2+\frac{C_0}{\nu^2}\big\|\operatorname{rot}\big(u_0^*-\alpha\Delta u_0^*\big)\big\|_{L^2}^4$$
$$+C_0\left(\frac{1}{\nu^2\lambda_1}\|\operatorname{rot} f\|_{L^\infty(L^2)}^2+\frac{1}{\nu^6\lambda_1^4}\|\operatorname{rot} f\|_{L^\infty(L^2)}^4\right)\bigg]^{1/2}>0. \tag{2.56}$$

Moreover, in the case where f does not depend on the time variable and u_0^* belongs to the global attractor $\mathcal{A}_\alpha$, due to (2.33), the condition $a_1>0$ holds if

$$\nu-\alpha C_0^{1/2}C_{BG}^{1/2}\left(1+\ln\left(1+\frac{1}{\alpha}\right)\right)^{1/2}\left(\frac{1}{\nu^2\lambda_1}\|\operatorname{rot} f\|_{L^2}^2+\frac{1}{\nu^6\lambda_1^4}\|\operatorname{rot} f\|_{L^2}^4\right)^{1/2}>0. \tag{2.57}$$

In the above estimates, the positive constant C_0 depends only on the constants appearing in the Poincaré and Sobolev inequalities.

As we already noticed, the estimate of $\|\Delta^2 w_n^*\|_{L^2}$ in the inequalities (2.51) or (2.53) is not uniform in α, when α goes to zero. We next want to improve the estimate of $\|\mathrm{rot}\,\Delta w_n^*(t)\|_{L^2}^2 + \alpha\|\Delta^2 w_n^*(t)\|_{L^2}^2$ (and thus of $\|\mathrm{rot}\,\Delta w^*(t)\|_{L^2}^2 + \alpha\|\Delta^2 w^*(t)\|_{L^2}^2$) in the case where $\alpha\lambda_1 \leqslant 1$, when $u^*(t) = S_\alpha(t)u_0^*$ is a solution of (1.2). First we deduce from (2.52) that, for $t \geqslant 0$,

$$\begin{aligned}
&\big\|\mathrm{rot}\,\Delta w_n^*(t)\big\|_{L^2}^2 + \alpha\big\|\Delta^2 w_n^*(t)\big\|_{L^2}^2 + \frac{a_1}{4}\int_0^t \exp\big(\beta_5(s-t)\big)\big\|\Delta^2 w_n^*\big\|_{L^2}^2\,ds \\
&\quad\leqslant \exp(-\beta_5 t)\big(\big\|\mathrm{rot}\,\Delta w^*(0)\big\|_{L^2}^2 + \alpha\big\|\Delta^2 w^*(0)\big\|_{L^2}^2\big) + \frac{6}{a_1\beta_5}\|\Delta Pf\|_{L_t^\infty(L^2)}^2 \\
&\qquad + \frac{C_S}{a_1}\Bigg[\sup_{0\leqslant s\leqslant t}\big(\big\|\Delta u^*(s)\big\|_{L^2}^2\big)\int_0^t \exp\beta_5(s-t)\big\|\nabla\Delta w_n^*(s)\big\|_{L^2}^2\,ds \\
&\qquad + \sup_{0\leqslant s\leqslant t}\big(\big\|\Delta w_n^*(s)\big\|_{L^2}^2\big)\int_0^t \exp\beta_5(s-t)\big\|\nabla\Delta u^*(s)\big\|_{L^2}^2\,ds\Bigg],
\end{aligned}\tag{2.58}$$

where $0 < \beta_5 \leqslant \inf(\beta_1, a_1/4)$, with $\beta_1 = \frac{\nu\lambda_1}{4(1+2\lambda_1\alpha)}$, and C_S is a positive constant coming from the classical Sobolev embeddings. Taking into account the estimates (2.13), (2.25), and (2.35), we deduce from the inequality (2.58) that, for $t \geqslant 0$,

$$\begin{aligned}
&\big\|\mathrm{rot}\,\Delta w_n^*(t)\big\|_{L^2}^2 + \alpha\big\|\Delta^2 w_n^*(t)\big\|_{L^2}^2 + \frac{a_1}{4}\int_0^t \exp\big(\beta_5(s-t)\big)\big\|\Delta^2 w_n^*\big\|_{L^2}^2\,ds \\
&\quad\leqslant \exp(-\beta_5 t)\big(\big\|\mathrm{rot}\,\Delta w^*(0)\big\|_{L^2}^2 + \alpha\big\|\Delta^2 w^*(0)\big\|_{L^2}^2\big) + \frac{6}{a_1\beta_5}\|\Delta Pf\|_{L_t^\infty(L^2)}^2 \\
&\qquad + \frac{C_S}{\nu a_1}L_1\Bigg[L_2 + L_3\Bigg(\big\|\mathrm{rot}\big(u^*(0) - \alpha\Delta u^*(0)\big)\big\|_{L^2}^2 + \frac{2(\lambda_1^{-1}+2\alpha)^2}{\nu^2}\|\mathrm{rot}\,f\|_{L_t^\infty(L^2)}^2\Bigg)\Bigg],
\end{aligned}\tag{2.59}$$

where

$$\begin{aligned}
L_1 &= \big\|\nabla\,\mathrm{rot}\,u^*(0)\big\|_{L^2}^2 + \alpha\big\|\Delta\,\mathrm{rot}\,u^*(0)\big\|_{L^2}^2 + \frac{2}{\beta_5\nu}\|\mathrm{rot}\,f\|_{L_t^\infty(L^2)}^2 \\
&\quad + \frac{C}{\nu^2}\big\|\mathrm{rot}\big(u^*(0) - \alpha\Delta u^*(0)\big)\big\|_{L^2}^4 + \frac{C(\lambda_1^{-1}+2\alpha)^2}{\beta_5^2\nu^4}\|\mathrm{rot}\,f\|_{L_t^\infty(L^2)}^4, \\
L_2 &= \big\|\nabla\,\mathrm{rot}\,w^*(0)\big\|_{L^2}^2 + \alpha\big\|\Delta\,\mathrm{rot}\,w^*(0)\big\|_{L^2}^2 + \frac{2}{\beta_5\nu}\|\mathrm{rot}\,f\|_{L_t^\infty(L^2)}^2, \\
L_3 &= \frac{C}{\nu^2}\Bigg(\big\|\mathrm{rot}\big(w^*(0) - \alpha\Delta w^*(0)\big)\big\|_{L^2}^2 + \frac{\lambda_1^{-1}+2\alpha}{\nu\beta_5}\|\mathrm{rot}\,f\|_{L_t^\infty(L^2)}\Bigg).
\end{aligned}\tag{2.60}$$

Performing a classical Galerkin method, we obtain the same estimates for the limit w^*. If f does not depend on t and that $u^*(t) = S_\alpha(t)u_0$ and $w^*(t) = S_\alpha(t)w_0$ belong to the global attractor, the estimate (2.59) can be simplified. Indeed, one then deduces from the estimates (2.58), (2.15), (2.32), and (2.25), that, for $t \geqslant 0$,

$$
\begin{aligned}
&\big\|\operatorname{rot}\Delta w^*(t)\big\|_{L^2}^2+\alpha\big\|\Delta^2 w^*(t)\big\|_{L^2}^2+\frac{a_1}{4}\int_0^t \exp\big(\beta_5(s-t)\big)\big\|\Delta^2 w^*\big\|_{L^2}^2\,ds\\
&\quad\leqslant \exp(-\beta_5 t)\big(\big\|\operatorname{rot}\Delta w^*(0)\big\|_{L^2}^2+\alpha\big\|\Delta^2 w^*(0)\big\|_{L^2}^2\big)+\frac{6}{a_1\beta_5}\|\Delta Pf\|_{L^2}^2\\
&\qquad+\frac{C_1 C_S}{\nu a_1}L_4\bigg[L_4+\frac{1}{\beta_5\nu}\|\operatorname{rot} f\|_{L^2}^2+\frac{1}{\beta_5^2\nu^4\lambda_1^2}\|\operatorname{rot} f\|_{L^2}^4\bigg],
\end{aligned}
\tag{2.61}
$$

where C_1 is a positive constant independent of α, f, t and $w^*(0)$ and where

$$
L_4=\frac{1}{\nu^2\lambda_1}\|\operatorname{rot} f\|_{L^2}^2+\frac{1}{\nu^6\lambda_1^4}\|\operatorname{rot} f\|_{L^2}^4.
$$

We next consider the second case, that is, the case where $a_1\leqslant 0$.

Second case: $a_1\equiv 2\nu-2\alpha\|\nabla u^*\|_{L^\infty(L^\infty)}\leqslant 0$**.**
In this case, setting $\beta_2=1/2$ in the inequality (2.46), using the Young inequality, taking into account the inequality (2.49) and integrating from 0 to t, yields, for $t\geqslant 0$,

$$
\begin{aligned}
&\big\|\nabla\big(\operatorname{rot} w_n^*-\alpha\operatorname{rot}\Delta w_n^*\big)(t)\big\|_{L^2}^2+\frac{\nu}{\lambda_1^{-1}+2\alpha}\int_0^t\big\|\nabla\big(\operatorname{rot} w_n^*-\alpha\operatorname{rot}\Delta w_n^*\big)(s)\big\|_{L^2}^2\,ds\\
&\quad\leqslant\big\|\nabla\big(\operatorname{rot} w_n^*-\alpha\operatorname{rot}\Delta w_n^*\big)(0)\big\|_{L^2}^2+\frac{6\alpha}{\nu}\|\Delta Pf\|_{L_t^2(L^2)}^2+\frac{2}{\nu}\|\operatorname{rot} Pf\|_{L_t^2(L^2)}^2\\
&\qquad+\int_0^t\big\|\nabla\big(\operatorname{rot} w_n^*-\alpha\operatorname{rot}\Delta w_n^*\big)(s)\big\|_{L^2}^2\bigg(\frac{6}{\nu}\big\|u^*(s)\big\|_{L^\infty}^2+\frac{6}{\nu}\inf\big(\lambda_1^{-1},\alpha\big)\big\|\nabla u^*(s)\big\|_{L^\infty}^2\\
&\qquad+2\bigg[\inf\bigg(1,\frac{\alpha}{\nu}\big\|\nabla u^*(s)\big\|_{L^\infty}\bigg)+\inf\bigg(1,\frac{1}{\lambda_1\nu}\big\|\nabla u^*(s)\big\|_{L^\infty}\bigg)\bigg]\big\|\nabla u^*(s)\big\|_{L^\infty}\bigg)\,ds.
\end{aligned}
\tag{2.62}
$$

Applying the Gronwall inequality to the estimate (2.62), we get, for $t\geqslant 0$,

$$
\begin{aligned}
&\big\|\nabla\big(\operatorname{rot} w_n^*-\alpha\operatorname{rot}\Delta w_n^*\big)(t)\big\|_{L^2}^2\\
&\quad\leqslant\bigg(\big\|\nabla\big(\operatorname{rot} w_n^*-\alpha\operatorname{rot}\Delta w_n^*\big)(0)\big\|_{L^2}^2+\frac{6\alpha}{\nu}\|\Delta Pf\|_{L_t^2(L^2)}^2+\frac{2}{\nu}\|\operatorname{rot} Pf\|_{L_t^2(L^2)}^2\bigg)\\
&\qquad\times\exp\bigg(-\frac{\nu t}{\lambda_1^{-1}+2\alpha}+\int_0^t\bigg(\frac{6}{\nu}\big\|u^*(s)\big\|_{L^\infty}^2+\frac{6}{\nu}\inf\big(\lambda_1^{-1},\alpha\big)\big\|\nabla u^*(s)\big\|_{L^\infty}^2\\
&\qquad+2\bigg[\inf\bigg(1,\frac{\alpha}{\nu}\big\|\nabla u^*(s)\big\|_{L^\infty}\bigg)+\inf\bigg(1,\frac{1}{\lambda_1\nu}\big\|\nabla u^*(s)\big\|_{L^\infty}\bigg)\bigg]\big\|\nabla u^*(s)\big\|_{L^\infty}\bigg)\,ds\bigg).
\end{aligned}
\tag{2.63}
$$

For any fixed $T>0$, the inequality (2.63) shows that $w_n^*(t)$ is uniformly bounded in V^4, for $0\leqslant t\leqslant T$. Performing the Galerkin procedure in the classical way, we deduce that the solution $w^*(t)$ of (2.37) belongs to $L^\infty((0,T),V^4)$, for any $0<T<+\infty$, and satisfies the bound (2.63). If f and ∇u^* belong to $L^2(H^2_{per})$ and $L^2(L^2)$ respectively, we conclude that $w^*(t)$ belongs to $L^\infty((0,+\infty),V^4)$. If this hypothesis is not satisfied, the estimates (2.63) do not allow to conclude that $w^*(t)$ belongs to $L^\infty((0,+\infty), V^4)$, since the right-hand side of (2.63) is exponentially growing.

If $\alpha > \alpha_0 > 0$, the inequality (2.63) implies an estimate of $\|w^*\|_{L^\infty((0,T),V^4)}$, which is uniform in α. When $\alpha\lambda_1 \leqslant 1$, the estimate of $\|w_n^*\|_{L^\infty((0,T),V^4)}$ (and of $\|w^*\|_{L^\infty((0,T),V^4)}$) can be improved by going back to the inequality (2.47). Indeed, arguing as above, setting $\beta_2 = 1/2$ in the inequality (2.47), integrating from 0 to t and, using the Gronwall lemma, we obtain, for $t \geqslant 0$,

$$\begin{aligned}
&\big\|\operatorname{rot}\Delta w_n^*(t)\big\|_{L^2}^2 + \alpha\big\|\Delta^2 w_n^*(t)\big\|_{L^2}^2 \\
&\quad\leqslant \Big(\big\|\operatorname{rot}\Delta w^*(0)\big\|_{L^2}^2 + \alpha\big\|\Delta^2 w^*(0)\big\|_{L^2}^2 + \frac{6\alpha}{\nu}\|\Delta P f\|_{L_t^2(L^2)}^2\Big) \\
&\qquad\times \exp\Bigg(-\nu t + \int_0^t \bigg(\frac{6}{\nu}\big\|u^*(s)\big\|_{L^\infty}^2 + \frac{6\lambda_1^{-1}}{\nu}\big\|\nabla u^*(s)\big\|_{L^\infty}^2 \\
&\qquad + 2\inf\Big(1, \frac{\alpha}{\nu}\big\|\nabla u^*(s)\big\|_{L^\infty}\Big)\big\|\nabla u^*(s)\big\|_{L^\infty}\bigg)\,ds\Bigg).
\end{aligned} \tag{2.64}$$

We remark that, in the special case where f belongs to the space $L^2(H^2_{per})$, we immediately deduce from the estimates (2.18) and (2.63) that, for any $\alpha > 0$, any $u_0 \in V^4$, the solution of the nonlinear equations (1.2) is bounded in $C_b^0((0,+\infty), V^4)$. Indeed, these estimates imply that, for $t \geqslant 0$,

$$\begin{aligned}
&\big\|\nabla(\operatorname{rot} u - \alpha\operatorname{rot}\Delta u)(t)\big\|_{L^2}^2 \\
&\quad\leqslant \bigg(\big\|\nabla(\operatorname{rot} u_0 - \alpha\operatorname{rot} u_0)\big\|_{L^2}^2 + \frac{6\alpha}{\nu}\|\Delta P f\|_{L_t^2(L^2)}^2 + \frac{2}{\nu}\|\operatorname{rot} P f\|_{L_t^2(L^2)}^2\bigg) \\
&\qquad\times \exp\frac{C}{\nu^2}\bigg(\big\|\operatorname{rot}(u_0 - \alpha\Delta u_0)\big\|_{L^2}^2 + \frac{\lambda_1^{-1} + 2\alpha}{\nu}\|\operatorname{rot} P f\|_{L^2(L^2)}^2\bigg),
\end{aligned} \tag{2.65}$$

where C is a positive constant independent of α, f and u_0. In the case where $\alpha > 0$ is small, this estimate can be improved by using the inequality (2.64) instead of (2.63). We leave the details of this improvement to the reader.

We now go back to the general case where f does not belong to the space $L^2(H^2_{per})$. According to Remark 2.1 and the estimates (2.13), the V^3-norm of $w_n^*(t)$ decays exponentially fast to some constant. On the contrary, according to the estimate (2.63), the V^4-norm of $w_n^*(t)$ could grow exponentially fast. These properties imply that, in some interpolated space, we still have exponential decay.

To show it, we proceed as follows. We set f equal to zero in Eq. (2.38), that is, we consider the following equation in the finite-dimensional space $P_n V^3$, for $t \geqslant s$,

$$\begin{aligned}
&\partial_t w_n - \nu(I+\alpha A)^{-1}P_n\Delta w_n + (I+\alpha A)^{-1}P_n P\big(\operatorname{rot}(w_n - \alpha\Delta w_n)\times u^*\big) = 0, \\
&w_n(s,x) = w_{s,n}(x) \in P_n V^3,
\end{aligned} \tag{2.66}$$

where P is the Leray orthogonal projection in $L^2(\mathbb{T}^2)^2$ onto H and $A = -P\Delta$ is the classical Stokes operator. We denote $\Sigma_{\alpha,n}(t,s)w_{s,n}$ the solution of Eq. (2.66). According to the estimates (2.13) and (2.63), we have the following inequalities:

$$\big\|\Sigma_{\alpha,n}(t,s)w_n(s) - \alpha\Delta\Sigma_{\alpha,n}(t,s)w_n(s)\big\|_{V^1} \leqslant C_P\exp\big(-\gamma_0(t-s)\big)\big\|w_n(s) - \alpha\Delta w_n(s)\big\|_{V^1} \tag{2.67}$$

and also

$$\begin{aligned}
&\left\|\Sigma_{\alpha,n}(t,s)w_n(s)-\alpha\Delta\Sigma_{\alpha,n}(t,s)w_n(s)\right\|_{V^2}\\
&\quad\leqslant C_P\exp\left(\left[-\gamma_0+\left(2+\frac{6C_P}{\nu}\left\|\nabla u^*\right\|_{L^\infty(L^\infty)}\right)\left\|\nabla u^*\right\|_{L^\infty(L^\infty)}\right](t-s)\right)\\
&\qquad\times\left\|w_n(s)-\alpha\Delta w_n(s)\right\|_{V^2},
\end{aligned}\tag{2.68}$$

where C_P is a positive constant depending only on the constant λ_1^{-1} and $\gamma_0=\frac{\nu}{4(\lambda_1^{-1}+2\alpha)}$.

Interpolating between the Hilbert spaces V^1 and V^2, we conclude that, for any $0\leqslant\theta\leqslant 1$, we obtain,

$$\begin{aligned}
&\left\|\Sigma_{\alpha,n}(t,s)w_n(s)-\alpha\Delta\Sigma_{\alpha,n}(t,s)w_n(s)\right\|_{V^{1+\theta}}\\
&\quad\leqslant C_P\exp\left(\left[-\gamma_0+\theta\left(2+\frac{6C_P}{\nu}\left\|\nabla u^*\right\|_{L^\infty(L^\infty)}\right)\left\|\nabla u^*\right\|_{L^\infty(L^\infty)}\right](t-s)\right)\\
&\qquad\times\left\|w_n(s)-\alpha\Delta w_n(s)\right\|_{V^{1+\theta}}.
\end{aligned}\tag{2.69}$$

Thus the norm of $\|\Sigma_{\alpha,n}(t,s)w_n(s)-\alpha\Delta\Sigma_{\alpha,n}(t,s)w_n(s)\|_{V^{1+\theta}}$ decays exponentially fast, if

$$-\gamma_\theta\equiv-\frac{\nu}{4(\lambda_1^{-1}+2\alpha)}+\theta\left(2+\frac{6C_P}{\nu}\left\|\nabla u^*\right\|_{L^\infty(L^\infty)}\right)\left\|\nabla u^*\right\|_{L^\infty(L^\infty)}<0.\tag{2.70}$$

We now go back to the solution $w^*(t)$ of Eq. (2.38) and choose θ_0 so that (2.70) holds for $\theta=\theta_0$. We notice that, since w_n^* is a solution of the finite-dimensional system (2.38) of ordinary differential equations, w_n^* is given by the Duhamel formula (or variation of constants formula), that is, w_n^* can be written as

$$w_n^*(t)=\Sigma_{\alpha,n}(t,0)P_nu_0+\int_0^t\Sigma_{\alpha,n}(t,s)(I+\alpha A)^{-1}P_nPf(s)\,ds,\tag{2.71}$$

which implies, by (2.69) that, for $t\geqslant 0$,

$$\left\|w_n^*(t)-\alpha\Delta w_n^*(t)\right\|_{V^{1+\theta_0}}\leqslant\exp(-\gamma_{\theta_0}t)\|u_0-\alpha\Delta u_0\|_{V^{1+\theta_0}}+\gamma_{\theta_0}^{-1}\|Pf\|_{L^\infty(V^{1+\theta_0})}.\tag{2.72}$$

Next performing the Galerkin procedure in the classical way, we deduce that the solution $w^*(t)$ of (2.37) also satisfies the inequality (2.72).

Remark 2.2. The dependence with respect to α in estimate (2.72) is very good when α is bounded away from 0. At first glance, the dependence with respect to α is less good when α tends to 0. But, actually, when α is very close to 0, we will never consider this estimate, since then $a_1\geqslant 0$ and we use the estimates (2.59) and (2.61).

Notice also that from estimate (2.72), we deduce that, for $t\geqslant 0$,

$$\begin{aligned}
\left\|w^*(t)\right\|_{V^{2+\theta_0}}+\sqrt{\alpha}\left\|w^*(t)\right\|_{V^{3+\theta_0}}\leqslant{}&\frac{C}{\sqrt{\alpha}}\exp(-\gamma_{\theta_0}t)\|u_0-\alpha\Delta u_0\|_{V^{1+\theta_0}}\\
&+\frac{C}{\sqrt{\alpha}}\gamma_{\theta_0}^{-1}\|Pf\|_{L^\infty(V^{1+\theta_0})}.
\end{aligned}\tag{2.73}$$

We derive from the inequality (2.73) and the condition $a_1 \leqslant 0$ (that is, $\alpha^{-1} \leqslant \nu^{-1}\|\nabla u^*\|_{L^\infty(L^\infty)}$), that, for $t \geqslant 0$,

$$\|w^*(t)\|_{V^{2+\theta_0}} + \sqrt{\alpha}\|w^*(t)\|_{V^{3+\theta_0}} \leqslant C\nu^{-1/2}\|\nabla u^*\|^{1/2}_{L^\infty(L^\infty)}\big(\exp(-\gamma_{\theta_0}t)\|u_0 - \alpha\Delta u_0\|_{V^{1+\theta_0}} + \gamma_{\theta_0}^{-1}\big)\|Pf\|_{L^\infty(V^{1+\theta_0})}. \tag{2.74}$$

It remains to prove that $w^*(t)$ is a continuous function from $[0,+\infty)$ into V^4. To this end, for any integer m, we decompose $w^*(t)$ into a sum of two functions $w^*(t) = v_m(t) + z_m(t)$, where $v_m(t)$ and $z_m(t)$ are the solutions of the following systems of equations

$$\begin{aligned}\partial_t(v_m - \alpha\Delta v_m) - \nu\Delta v_m + P\big(\mathrm{rot}(v_m - \alpha\Delta v_m) \times u^*\big) &= P_m Pf,\\ v_m(0,x) &= P_m u_0(x),\end{aligned} \tag{2.75}$$

and

$$\begin{aligned}\partial_t(z_m - \alpha\Delta z_m) - \nu\Delta z_m + P\big(\mathrm{rot}(z_m - \alpha\Delta z_m) \times u^*\big) &= (I - P_m)Pf,\\ z_m(0,x) &= (I - P_m)u_0(x).\end{aligned} \tag{2.76}$$

Let $t_0 > 0$ and $\eta > 0$ be fixed. We can choose an integer m_η such that, for $m \geqslant m_\eta$, $(I - P_m)u_0$ and $(I - P_m)Pf$ are small enough and that, according to the estimate (2.63), we have, for $0 \leqslant t \leqslant 2t_0$, $m \geqslant m_\eta$,

$$2\sup_{0\leqslant s\leqslant t}\|z_m(t)\|_{V^4} \leqslant \eta/2. \tag{2.77}$$

Arguing as in the proof of Theorem 2.1 (by using a Galerkin method, see [33]), one shows that v_m belongs to $C^0([0,+\infty), V^2)$ at least. Furthermore, since $P_m u_0$ and $P_m Pf$ belong to V^5 and $L^\infty((0,+\infty), H^3_{per})$ respectively, we can show by arguing as above (see also the proof of Theorem 2.4 below) that the solution v_m of system (2.75) belongs to $L^\infty((0,2t_0), V^5)$. By interpolation, we deduce that v_m belongs to $C^0([0,2t_0), V^4)$. In particular, v_{m_η} belongs to $C^0([0,2t_0), V^4)$ and there exists a positive real number δ_η such that, if $|t - t_0| \leqslant \delta_\eta$, then

$$\|v_{m_\eta}(t) - v_{m_\eta}(t_0)\|_{V^4} \leqslant \eta/2. \tag{2.78}$$

The estimates (2.77) and (2.78) imply that $\|w^*(t) - w^*(t_0)\|_{V^4} \leqslant \eta$. The continuity in V^4 is shown and thus Theorem 2.3 is proved. □

Remark 2.3. We point out that, in Theorem 2.1, we can prove, in the same way, that the solution $u(t)$ of (1.2) is continuous with values in V^3. Indeed, in the proof of Theorem 2.1, the Galerkin method easily implies that $u(t)$ belongs to $C^0([0,+\infty), V^2)$ (see [33]). Thus, the same arguments as above allow to show that $u(t)$ actually belongs to $C^0([0,+\infty), V^3)$. We leave the details to the reader.

2.4. Case of smoother initial data: V^{m+3}-regularity, $m \geqslant 2$

We next assume that f belongs to the space $L^\infty((0,+\infty), H^{m+1}_{per})$ and that u^* belongs to $L^\infty((0,+\infty), V^{m+2}) \cap C^0([0,+\infty), V^2)$, for $m \geqslant 2$. If $a_m \equiv 2\nu - 2d_m\alpha\|\nabla u^*\|_{L^\infty(L^\infty)} > 0$, where $d_m > 0$ is defined in the proof of Theorem 2.4 below (see (2.84)), we will show that, for any $u_0 \in V^{m+3}$, the solution w^* of Eq. (2.37) is uniformly bounded, with respect to $t \geqslant 0$, in the space V^{m+3}.

In the case where $a_m < 0$, we can still prove that w^* is uniformly bounded, with respect to t, in the space V^{m+3} if moreover f and u^* belong to the space $L^2((0,+\infty), H^{m+1}_{per})$ and $L^2((0,+\infty), V^{m+2})$. More precisely, we show the following theorem.

Theorem 2.4. *Let $\alpha > 0$ and $m \geqslant 2$. Assume that f and $u^*(t)$ belong to the spaces $L^\infty((0,+\infty), H^{m+1}_{per})$ and $L^\infty((0,+\infty), V^{m+2}) \cap C^0([0,+\infty), V^2)$ respectively. Then, for any $\alpha > 0$, for any $u_0 \in V^{m+3}$, the unique solution w^* of* (2.37) *belongs to $C^0([0,+\infty), V^{m+3})$.*

Moreover, if $a_m = 2\nu - 2d_m\alpha\|\nabla u^*\|_{L^\infty(L^\infty)} > 0$, where $d_m > 0$ is a non-decreasing function of m, then w^* belongs to the space $C^0_b([0,+\infty), V^{m+3})$ and satisfies, for $t \geqslant 0$

$$\begin{aligned}
&\|w^*(t)\|^2_{V^{m+2}} + \alpha\|w^*(t)\|^2_{V^{m+3}} + \frac{a_m}{4}\int_0^t \exp\bigl(-\beta_m(s-t)\bigr)\|w^*(s)\|^2_{V^{m+3}}\,ds \\
&\quad\leqslant \|u_0\|^2_{V^{m+2}} + \alpha\|u_0\|^2_{V^{m+3}} + K_m\|f\|^2_{L^\infty_t(H^{m+1})} \\
&\qquad + \sum_{i=1}^{m-3}\bigl(\|u_0\|^2_{V^{m+2-i}} + \alpha\|u_0\|^2_{V^{m+3-i}} + K_{m-i}\|f\|^2_{L^\infty_t(H^{m+1-i})}\bigr) \\
&\qquad \times\left(\prod_{j=1}^{i} K_{m+1-j}\right)\left(\prod_{j=1}^{i}\bigl(\|u^*\|^2_{L^\infty_t(V^{m+2-j})} + \alpha\|u^*\|^2_{L^\infty_t(V^{m+3-j})}\bigr)\right) \\
&\qquad + \Biggl(\|u_0\|^2_{V^4} + \alpha\|u_0\|^2_{V^5} + K_2\|f\|^2_{L^\infty_t(H^3)} \\
&\qquad + K_2\int_0^t \exp\bigl(-\beta_2(s-t)\bigr)\bigl(\alpha^2\|w^*(s)\|^2_{V^4}\|u^*(s)\|^2_{V^4} + \|w^*(s)\|^2_{V^4}\|u^*(s)\|_{V^3}\bigr)\,ds\Biggr) \\
&\qquad \times\left(\prod_{j=1}^{m-3} K_{m+1-j}\right)\left(\prod_{j=1}^{m-3}\bigl(\|u^*\|^2_{L^\infty_t(V^{m+2-j})} + \alpha\|u^*\|^2_{L^\infty_t(V^{m+3-j})}\bigr)\right),
\end{aligned} \tag{2.79}$$

where K_i are positive constants depending only on i and a_i, and where $0 < \beta_i \leqslant a_i/4$.

If $a_{m-1} > 0$ and $a_m \leqslant 0$, there still exists $\theta > 0$ such that w^* is uniformly bounded in $V^{m+1+\theta}$.

In particular, if $a_m > 0$ and $u^*(t) = S_\alpha(t)u_0^*$, the following estimate holds

$$\begin{aligned}
&\|w^*(t)\|^2_{V^{m+2}} + \alpha\|w^*(t)\|^2_{V^{m+3}} + \frac{a_m}{4}\int_0^t \exp\bigl(-\beta_m(s-t)\bigr)\|w^*(s)\|^2_{V^{m+3}}\,ds \\
&\quad\leqslant \|u_0\|^2_{V^{m+2}} + \alpha\|u_0\|^2_{V^{m+3}} + K_m\|f\|^2_{L^\infty_t(H^{m+1})} \\
&\qquad + Q_m(x_2,\dots,x_{m+1},y_2,\dots,y_{m+1},z_1,\dots,z_m),
\end{aligned} \tag{2.80}$$

where $x_j = \|u_0\|^2_{V^j} + \alpha\|u_0\|^2_{V^{j+1}}$, $y_j = \|u_0^*\|^2_{V^j} + \alpha\|u_0^*\|^2_{V^{j+1}}$, $z_j = \|f\|_{L^\infty_t(H^j)}$, and where $Q_m(x_2,\dots,x_{m+1}, y_2,\dots,y_{m+1},z_1,\dots,z_m)$ is a polynomial of $x_2,\dots,x_{m+1}$, $y_2,\dots,y_{m+1}$, and $z_1,\dots,z_m$, whose coefficients depend only on m and a_i, $i=1,\dots,m$.

Proof. Assume that $m \geqslant 2$. We will prove the propagation of the regularity and the estimate (2.79) by recursion on m. The propagation of the $V^{\ell+3}$ regularity and the estimate (2.79) have been proved in Theorem 2.3 for $\ell = 1$. Assume now that the propagation of the $V^{\ell+3}$-regularity and the estimate

(2.79) have been proved for $\ell \leqslant m-1$, with $m \geqslant 2$, and let us prove these properties for $\ell = m$. As in the proof of Theorem 2.3, we need to use a Galerkin method and first obtain estimates for w_n^* in V^{m+3}. To simplify the notation, we denote w_n^* by w^* and we assume that m is an odd integer. The proof is similar in the case where m is even.

Taking the inner product in $L^2(\mathbb{T}^2)$ of Eq. (2.39) with $\operatorname{rot}\Delta^{m+1}w^*$, we obtain the equality

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\|\operatorname{rot}\Delta^{\frac{m+1}{2}}w^*\|_{L^2}^2+\alpha\|\Delta^{\frac{m+3}{2}}w^*\|_{L^2}^2\big)+\nu\|\Delta^{\frac{m+3}{2}}w^*\|_{L^2}^2-\big(\Delta w^*\times u^*,\operatorname{rot}\Delta^{m+1}w^*\big)\\&\quad+\alpha\big(\big(\Delta^2w^*\times u^*\big),\operatorname{rot}\Delta^{m+1}w^*\big)=\big(\operatorname{rot}P_nPf,\operatorname{rot}\Delta^{m+1}w^*\big).\end{aligned}\tag{2.81}$$

On the one hand, performing several integrations by parts, using the Leibnitz formula, the equality (A.5) of Lemma A.1, and the classical Sobolev embeddings (or the Moser inequalities), we get,

$$\begin{aligned}\big(\Delta w^*\times u^*,\operatorname{rot}\Delta^{m+1}w^*\big)&=\big(\Delta^{\frac{m+1}{2}}\big(\Delta w^*\times u^*\big),\operatorname{rot}\Delta^{\frac{m+1}{2}}w^*\big)\\&=\big(\big(\Delta\Delta^{\frac{m+1}{2}}w^*\big)\times u^*\big),\operatorname{rot}\Delta^{\frac{m+1}{2}}w^*\big)+B=B,\end{aligned}\tag{2.82}$$

where

$$\begin{aligned}B&=\big(\Delta^{\frac{m+1}{2}}\big(\Delta w^*\times u^*\big)-\big(\big(\Delta\Delta^{\frac{m+1}{2}}w^*\big)\times u^*\big),\operatorname{rot}\Delta^{\frac{m+1}{2}}w^*\big)\\&\leqslant C_1(m)\big(\|\Delta w^*\|_{L^\infty}\|u^*\|_{H^{m+1}}+\|w^*\|_{H^{m+2}}\|\nabla u^*\|_{L^\infty}\big)\|\nabla\Delta^{\frac{m+1}{2}}w^*\|_{L^2}\\&\leqslant C_2(m)\big(\|w^*\|_{H^{7/2}}\|u^*\|_{H^{m+1}}+\|w^*\|_{H^{m+2}}\|u^*\|_{H^{5/2}}\big)\|\nabla\Delta^{\frac{m+1}{2}}w^*\|_{L^2},\end{aligned}\tag{2.83}$$

and where $C_1(m)$ and $C_2(m)$ are positive constants depending only on m.

On the other hand, again integrating by parts, using the Leibnitz formula, and the equality (A.5) of Lemma A.1 as well as classical Sobolev and interpolation inequalities, we deduce that, for $t \geqslant 0$,

$$\begin{aligned}&\alpha\big(\big(\Delta^2w^*\times u^*\big),\operatorname{rot}\Delta^{m+1}w^*\big)\\&\quad=-\sum_{i=1}^{2}\alpha\big(\Delta^{\frac{m+1}{2}-1}\partial_i\big(\Delta^2w^*\times u^*\big),\operatorname{rot}\Delta^{\frac{m+1}{2}}\partial_iw^*\big)\\&\quad=-\alpha\sum_{i=1}^{2}\big(\big(\Delta\partial_i\Delta^{\frac{m+1}{2}}w^*\big)\times u^*,\operatorname{rot}\Delta^{\frac{m+1}{2}}\partial_iw^*\big)-\alpha B^*\\&\qquad-\alpha d_m\sum_{j=1}^{2}\sum_{i=1}^{2}\big(\big(\Delta\Delta^{\frac{m+1}{2}}w^*\big)\times\partial_ju^*,\operatorname{rot}\Delta^{\frac{m+1}{2}}\partial_iw^*\big)\\&\quad=-\left(\alpha B^*+\alpha d_m\sum_{j=1}^{2}\sum_{i=1}^{2}\big(\big(\Delta\Delta^{\frac{m+1}{2}}w^*\big)\times\partial_ju^*,\operatorname{rot}\Delta^{\frac{m+1}{2}}\partial_iw^*\big)\right)\end{aligned}\tag{2.84}$$

where

$$\begin{aligned}\alpha B^*&\leqslant\alpha C_3(m)\sum_{i=1}^{2}\sum_{|a+b|\leqslant m,\,|b|\geqslant 2}\big|\big(D^a\Delta^2w^*\times D^bu^*,\operatorname{rot}\Delta^{\frac{m+1}{2}}\partial_iw^*\big)\big|\\&\leqslant\alpha C_4(m)\big(\|w^*\|_{V^4}\|u^*\|_{V^{m+2}}+\|w^*\|_{V^{m+2}}\|u^*\|_{V^4}\big)\|\Delta^{\frac{m+3}{2}}w^*\|_{L^2},\end{aligned}\tag{2.85}$$

and where d_m, $C_3(m)$, and $C_4(m)$ are positive constants depending only on m.

The estimates (2.83), (2.85), and the equalities (2.81), (2.82) and (2.84) imply that, for $t \geqslant 0$,

$$\begin{aligned}
&\frac{d}{dt}\big(\big\|\Delta^{\frac{m+1}{2}}\operatorname{rot} w^*\big\|_{L^2}^2+\alpha\big\|\Delta^{\frac{m+3}{2}} w^*\big\|_{L^2}^2\big)+\nu\big(2-\beta^*\big)\big\|\Delta^{\frac{m+3}{2}} w^*\big\|_{L^2}^2\\
&\quad\leqslant \frac{3\alpha^2 C_4(m)^2}{\beta^*\nu}\big(\big\|w^*\big\|_{V^4}^2\big\|u^*\big\|_{V^{m+2}}^2+\big\|w^*\big\|_{V^{m+2}}^2\big\|u^*\big\|_{V^4}^2\big)\\
&\qquad+\frac{3}{\beta^*\nu}\|Pf\|_{H^{m+1}}^2+2\alpha d_m\big\|\nabla u^*\big\|_{L^\infty}\big\|\Delta^{\frac{m+3}{2}} w^*\big\|_{L^2}^2\\
&\qquad+2C_2(m)\big(\big\|w^*\big\|_{V^4}\big\|u^*\big\|_{V^{m+1}}+\big\|w^*\big\|_{V^{m+2}}\big\|u^*\big\|_{V^3}\big)\big\|w^*\big\|_{V^{m+2}},
\end{aligned}\tag{2.86}$$

where $0<\beta^*<2$. If $a_m=2\nu-2d_m\alpha\|\nabla u^*\|_{L_t^\infty(L^\infty)}>0$, we choose $\beta^*=a_m/2\nu$ and, from (2.86), we deduce the following estimate, for $t\geqslant 0$,

$$\begin{aligned}
&\frac{d}{dt}\big(\big\|\Delta^{\frac{m+1}{2}}\operatorname{rot} w^*\big\|_{L^2}^2+\alpha\big\|\Delta^{\frac{m+3}{2}} w^*\big\|_{L^2}^2\big)+\frac{a_m}{2}\big\|\Delta^{\frac{m+3}{2}} w^*\big\|_{L^2}^2\\
&\quad\leqslant \frac{6\alpha^2 C_4(m)^2}{a_m}\big(\big\|w^*\big\|_{V^4}^2\big\|u^*\big\|_{V^{m+2}}^2+\big\|w^*\big\|_{V^{m+2}}^2\big\|u^*\big\|_{V^4}^2\big)+\frac{6}{a_m}\|Pf\|_{H^{m+1}}^2\\
&\qquad+2C_2(m)\big(\big\|w^*\big\|_{V^4}\big\|u^*\big\|_{V^{m+1}}+\big\|w^*\big\|_{V^{m+2}}\big\|u^*\big\|_{V^3}\big)\big\|w^*\big\|_{V^{m+2}}.
\end{aligned}\tag{2.87}$$

Integrating in time the inequality (2.87), we obtain, for $t\geqslant 0$,

$$\begin{aligned}
&\big\|\Delta^{\frac{m+1}{2}}\operatorname{rot} w^*(t)\big\|_{L^2}^2+\alpha\big\|\Delta^{\frac{m+3}{2}} w^*(t)\big\|_{L^2}^2+\frac{a_m}{4}\int_0^t\exp\big(\beta(s-t)\big)\big\|\Delta^{\frac{m+3}{2}} w^*(s)\big\|_{L^2}^2\,ds\\
&\quad\leqslant \exp(-\beta t)\big(\big\|\Delta^{\frac{m+1}{2}}\operatorname{rot} u_0\big\|_{L^2}^2+\alpha\big\|\Delta^{\frac{m+3}{2}} u_0\big\|_{L^2}^2\big)+\frac{6}{a_m\beta}\|Pf\|_{L_t^\infty(H^{m+1})}^2\\
&\qquad+\frac{\alpha^2 C_m^2}{a_m}\int_0^t\exp\big(\beta(s-t)\big)\big(\big\|w^*(s)\big\|_{V^4}^2\big\|u^*(s)\big\|_{V^{m+2}}^2+\big\|w^*(s)\big\|_{V^{m+2}}^2\big\|u^*(s)\big\|_{V^4}^2\big)\,ds\\
&\qquad+C_m\int_0^t\exp\big(\beta(s-t)\big)\big(\big\|w^*(s)\big\|_{V^4}\big\|u^*(s)\big\|_{V^{m+1}}\\
&\qquad+\big\|w^*(s)\big\|_{V^{m+2}}\big\|u^*(s)\big\|_{V^3}\big)\big\|w^*(s)\big\|_{V^{m+2}}\,ds,
\end{aligned}\tag{2.88}$$

where $0<\beta\leqslant a_m/4$ and C_m is a positive constant depending only on m. The estimate (2.79) is now a direct consequence of (2.88) and of the estimates (2.79), with m replaced by ℓ, $2\leqslant\ell\leqslant m-1$.

If $a_m\leqslant 0$, one proceeds like in the proof of Theorem 2.3, that is, we set $\beta^*=1/2$ in the estimate (2.86). Integrating then the resulting estimate, using the Gronwall lemma as well as the recursion hypothesis that w^* belongs to $L^\infty_{loc}((0,+\infty),V^{m+2})$, we prove that w^* is actually in the space $L^\infty_{loc}((0,+\infty),V^{m+3})$. The obvious details are left to the reader. If $a_m\leqslant 0$ and $a_{m-1}>0$, then proceeding by interpolation like in the proof of Theorem 2.3, one shows that there exists $\theta>0$ such that $w^*\in L^\infty((0,+\infty),V^{m+2+\theta})$. The details are also left to the reader.

The continuity of $w^*: t\in[0,+\infty)\mapsto w^*(t)\in V^{m+3}$ is proved by using the same arguments as in the proof of Theorem 2.3. □

Inequality (2.80) is also proved by recursion. The estimate (2.80) is a direct consequence of (2.88), (2.53) and (2.61) when $m = 2$. Assume now that the inequality (2.80) is true for $\ell \leqslant m-1$. Then the estimate (2.80) for $\ell = m$ is a direct consequence of the estimate (2.79) and the inequalities (2.80) for $\ell \leqslant m-1$.

3. Regularity of the global attractor in V^{3+m}, $m > 0$

In this section, we shall prove Theorem 1.1 about the regularity of the global attractor $\mathcal{A}_\alpha$ if f belongs to H^{m+1}_{per}, $m > 0$. As explained in the introduction, we shall prove it by decomposing the system (1.2) into two affine non-autonomous systems. Let $u(t)$ be an orbit of $S_\alpha(t)$ contained in the global attractor A_α. We decompose $u(t)$ as $u(t) = v_n(t) + w_n(t)$, where $v_n(t)$ and $w_n(t)$ are the solutions of the following non-autonomous affine equations

$$\begin{aligned} &\partial_t(v_n - \alpha\Delta v_n) - \nu\Delta v_n + \operatorname{rot}(v_n - \alpha\Delta v_n)\times u + \nabla p_n = f, \quad t > s_n,\ x \in \mathbb{T}^2,\\ &\operatorname{div} v_n = 0, \quad t > s_n,\ x\in\mathbb{T}^2,\\ &v_n(s_n, x) = 0, \quad x \in \mathbb{T}^2, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} &\partial_t(w_n - \alpha\Delta w_n) - \nu\Delta w_n + \operatorname{rot}(w_n - \alpha\Delta w_n)\times u + \nabla \widetilde{p}_n = 0, \quad t > s_n,\ x \in \mathbb{T}^2,\\ &\operatorname{div} w_n = 0, \quad t > s_n,\ x\in\mathbb{T}^2,\\ &w_n(s_n, x) = u(s_n, x), \quad x \in \mathbb{T}^2, \end{aligned} \tag{3.2}$$

where $s_n \in \mathbb{R}$ is a given initial time, which will go to $-\infty$.

For sake of clarity, we shall distinguish the case where the forcing term f belongs to $H^{1+\theta}_{per}$, $0 < \theta \leqslant 1$ from the case where f belongs to H^{m+1}_{per}, $m > 1$.

Theorem 3.1. *The following regularity properties of the global attractor $\mathcal{A}_\alpha$ hold.*

1) *Assume that $f \in H^2_{per}$ and that $a_1 = 2\nu - 2\alpha(\sup_{z\in\mathcal{A}_\alpha}\|\nabla z\|_{L^\infty}) > 0$, then the global attractor $\mathcal{A}_\alpha$ is bounded in V^4. Moreover, for any $u \in \mathcal{A}_\alpha$,*

$$\|u\|^2_{V^3} + \inf(1,\alpha)\|u\|^2_{V^4} \leqslant M_4, \tag{3.3}$$

where $M_4 = M_4(\lambda_1, \nu, f, a_1)$ does not depend on α.

2) *For any $\alpha > 0$, there exists a positive number θ_0, $0 < \theta_0 \leqslant 1$, depending only on α and the norm $\|f\|_{H^1}$, such that, if f belongs to $H^{1+\theta_0}_{per}$, then $\mathcal{A}_\alpha$ is bounded in $V^{3+\theta_0}$. Moreover, for any $u \in \mathcal{A}_\alpha$,*

$$\|u\|^2_{V^{2+\theta_0}} + \inf(1,\alpha)\|u\|^2_{V^{3+\theta_0}} \leqslant M_{3+\theta_0}, \tag{3.4}$$

where $M_{3+\theta_0} = M_{3+\theta_0}(\lambda_1, \nu, f)$ does not depend on α.

Proof. Let $u(t)$ be a trajectory on the global attractor. Due to the uniqueness of the solution of Eqs. (1.2), we at once notice that $u = v_n + w_n$. We also remark that v_n (resp. w_n) is the solution of the system (2.37) with $u^*(t) = u(t) = S_\alpha(t)u(0)$, $v_n(s_n, x) = 0$ (resp. $w_n(s_n, x) = u(s_n, x)$) and forcing term f (resp. 0). Then, by Remark 2.1, v_n and w_n satisfy the inequality (2.13). In particular, the following estimate holds for w_n

$$\begin{aligned}\big\|\operatorname{rot}\big(w_n(t)-\alpha\Delta w_n(t)\big)\big\|_{L^2}^2 &\leqslant \exp\biggl(-\frac{\nu}{2(\lambda_1^{-1}+2\alpha)}(t-s_n)\biggr)\big\|\operatorname{rot}\big(u(s_n)-\alpha\Delta u(s_n)\big)\big\|_{L^2}^2\\ &\leqslant \exp\biggl(-\frac{\nu}{2(\lambda_1^{-1}+2\alpha)}(t-s_n)\biggr)K\big(1+\alpha^2\big),\end{aligned}\tag{3.5}$$

where, according to the bounds (2.16) or (2.32) satisfied by the elements on the global attractor, $K = K(\|f\|_{H^1})$ is a positive constant depending only on $\|f\|_{H^1}$.

Assume first that $2\nu - 2\alpha \sup_{z\in\mathcal{A}_\alpha}\|\nabla z\|_{L^\infty} > 0$. Then, by Theorem 2.3 and its proof, $v_n(t)$ belongs to $C_b^0([s_n,+\infty), V^4)$ and the inequality (2.51) implies that, for $t \geqslant s_n$,

$$\begin{aligned}&\big\|\nabla\big(\operatorname{rot} v_n(t)-\alpha\operatorname{rot}\Delta v_n(t)\big)\big\|_{L^2}^2\\ &\quad\leqslant \frac{C(1+\alpha)}{a_1}\biggl[\frac{\alpha}{a_1}\|\Delta f\|_{L^2}^2+\biggl(1+\frac{(1+\alpha)^2}{a_1\alpha^2}+\frac{(1+\alpha)}{\alpha}\biggr)\|\operatorname{rot} f\|_{L^2}^2\biggr],\end{aligned}\tag{3.6}$$

where $C \equiv C(\lambda_1,\nu)$ is a positive constant independent of α, f, a_1 and v_n, but depending on λ_1 and ν.

If $\alpha\lambda_1 < 1$, taking into account the estimate (2.59), we can improve the previous estimate and replace it by the following inequality,

$$\big\|\operatorname{rot}\Delta v_n(t)\big\|_{L^2}^2+\alpha\big\|\Delta^2 v_n(t)\big\|_{L^2}^2 \leqslant \frac{C}{a_1\beta}\biggl[\|\Delta f\|_{L^2}^2+\|\operatorname{rot} f\|_{L^2}^4\big(1+\|\operatorname{rot} f\|_{L^2}^2\big)^2\biggl(1+\frac{1}{\beta}\biggr)^2\biggr],\tag{3.7}$$

where $C \equiv C(\lambda_1,\nu)$ is a positive constant independent of α, f, a_1 and v_n, but depending on λ_1 and ν, and where $\beta = \inf(a_1, 1)$.

The properties (3.5) and (3.6) (or (3.7)) imply that $v_n(t)$ converges to $u(t)$ in V^3 as n goes to infinity and that $v_n(t)$ is uniformly bounded in V^4 with respect to n. Therefore there exists a subsequence $v_{n_k}(t)$ which converges weakly to $u(t)$ in V^4. Hence, $u(t)$ belongs to V^4 and satisfies the estimate (3.6) (or (3.7)), where v_n is replaced by $u(t)$. Statement 1) of Theorem 3.1 is proved.

To prove statement 2), one proceeds in the same way. But, since

$$a_1 \equiv 2\nu - 2\alpha\Bigl(\sup_{z\in\mathcal{A}_\alpha}\|\nabla z\|_{L^\infty}\Bigr) \leqslant 0,$$

one cannot longer prove that v_n is bounded in V^4. However, by Theorem 2.3 and its proof (see condition (2.70) and estimate (2.74)), for any $\theta_0 \in (0,1)$ such that

$$-\gamma_{\theta_0} \equiv -\frac{\nu}{4(\lambda_1^{-1}+2\alpha)}+\theta_0\sup_{z\in\mathcal{A}_\alpha}\biggl[\biggl(2+\frac{6C_P}{\nu}\|\nabla z\|_{L^\infty}\biggr)\|\nabla z\|_{L^\infty}\biggr] < 0\tag{3.8}$$

holds, $v_n(t)$ belongs to $C_b^0([s_n,+\infty), V^{3+\theta_0})$ and the inequality (2.72) implies that, for $t \geqslant s_n$,

$$\big\|v_n(t)-\alpha\Delta v_n(t)\big\|_{V^{1+\theta_0}} \leqslant \gamma_{\theta_0}^{-1}\|Pf\|_{V^{1+\theta_0}}.\tag{3.9}$$

In the case where $\alpha > 0$ is very close to zero (in particular $\alpha\lambda_1 < 1$ and $\alpha < 1$), one argues as follows. The inequality (2.73) of Remark 2.2 implies that

$$\big\|v_n(t)\big\|_{V^{2+\theta_0}}+\sqrt{\alpha}\big\|v_n(t)\big\|_{V^{3+\theta_0}} \leqslant \frac{C}{\sqrt{\alpha}}\gamma_{\theta_0}^{-1}\|Pf\|_{V^{1+\theta_0}}.\tag{3.10}$$

Now one distinguishes two cases. If

$$\nu - \alpha\left(1 + \ln\left(1 + \frac{1}{\alpha}\right)\right)^{1/2} K_0\big(\|\mathrm{rot}\, f\|_{L^2}\big) > 0,$$

where

$$K_0\big(\|\mathrm{rot}\, f\|_{L^2}\big) \equiv C_0^{1/2} C_{BG}^{1/2}\left(\frac{1}{\nu^2\lambda_1}\|\mathrm{rot}\, f\|_{L^2}^2 + \frac{1}{\nu^6\lambda_1^4}\|\mathrm{rot}\, f\|_{L^2}^4\right)^{1/2},$$

then, by the estimate (2.33), $a_1 > 0$. In this case, we already proved that $\mathcal{A}_\alpha$ is bounded in V^4 and that the estimate (3.3) holds. If

$$\nu - \alpha\left(1 + \ln\left(1 + \frac{1}{\alpha}\right)\right)^{1/2} K_0\big(\|\mathrm{rot}\, f\|_{L^2}\big) \leqslant 0,$$

then,

$$\sqrt{\alpha} \geqslant c_0\alpha\left(1 + \ln\left(1 + \frac{1}{\alpha}\right)\right)^{1/2} \geqslant \frac{c_0\nu}{K_0(\|\mathrm{rot}\, f\|_{L^2})},$$

where c_0 is a positive constant, and therefore

$$\big\|v_n(t)\big\|_{V^{2+\theta_0}} + \sqrt{\alpha}\big\|v_n(t)\big\|_{V^{3+\theta_0}} \leqslant \frac{CK_0\big(\|\mathrm{rot}\, f\|_{L^2}\big)}{c_0\nu}\gamma_{\theta_0}^{-1}\|Pf\|_{V^{1+\theta_0}}. \tag{3.11}$$

Finally, one concludes the proof like in the case 1). □

Theorem 3.1 and the estimates (2.16) and (2.33) at once imply the following corollary.

Corollary 3.2. *Let $\alpha > 0$. Assume that $f \in H^2_{per}$ and that, either*

$$\nu - C_S\frac{\sqrt{2}(\lambda_1^{-1} + 2\alpha)}{\nu}\|\mathrm{rot}\, f\|_{L^2} > 0$$

or, when $\alpha\lambda_1 < 1$, that

$$\nu - \alpha C_0^{1/2} C_{BG}^{1/2}\left(1 + \ln\left(1 + \frac{1}{\alpha}\right)\right)^{1/2}\left(\frac{1}{\nu^2\lambda_1}\|\mathrm{rot}\, f\|_{L^2}^2 + \frac{1}{\nu^6\lambda_1^4}\|\mathrm{rot}\, f\|_{L^2}^4\right)^{1/2} > 0,$$

holds, then the global attractor $\mathcal{A}_\alpha$ is bounded in V^4 and the estimate (3.3) *holds.*

Remark 3.1. If one is only interested in the V^4-regularity of a given trajectory $u(t) \subset \mathcal{A}_\alpha$, one can replace the condition $\nu - \alpha(\sup_{z\in\mathcal{A}_\alpha})\|\nabla z\|_{L^\infty} > 0$ by the weaker hypothesis

$$\nu - \alpha\big\|\nabla u(t)\big\|_{L^\infty(L^\infty)} > 0.$$

Under this condition, one shows as above that the trajectory $u(t)$ is uniformly bounded in V^4.

One easily generalizes the previous regularity result to the case where f belongs to H^{m+1}_{per}, $m \geqslant 2$.

Theorem 3.3. *For any $m \geqslant 2$, there exists a positive number d_m (which is a non-decreasing function of m), such that, for any $\alpha > 0$, the following regularity properties hold. If $f \in H^{m+1}_{per}$ and $a_m = 2\nu - 2d_m\alpha(\sup_{z\in\mathcal{A}_\alpha} \|\nabla z\|_{L^\infty}) > 0$, then the global attractor $\mathcal{A}_\alpha$ is bounded in V^{m+3}. Moreover, for any $\alpha_0 > 0$, there exists a positive constant $M_{m+3}(\alpha_0) = M_{m+3}(\lambda_1, \nu, f, m, a_m, \alpha_0)$, depending only on λ_1, ν, f, m, a_m and α_0, such that, if $0 < \alpha \leqslant \alpha_0$ and $a_m = 2\nu - 2d_m\alpha(\sup_{z\in\mathcal{A}_\alpha} \|\nabla z\|_{L^\infty}) > 0$, then, for any $u \in \mathcal{A}_\alpha$,*

$$\|u\|^2_{V^{m+2}} + \alpha\|u\|^2_{V^{m+3}} \leqslant M_{m+3}(\alpha_0). \tag{3.12}$$

If $a_{m-1} > 0$ and $a_m \leqslant 0$, then there exists $\theta_0 > 0$ such that the global attractor $\mathcal{A}_\alpha$ is bounded in $V^{m+2+\theta_0}$ and, for any $0 < \alpha \leqslant \alpha_0$, for any $u \in \mathcal{A}_\alpha$,

$$\|u\|^2_{V^{m+1+\theta_0}} + \alpha\|u\|^2_{V^{m+2+\theta_0}} \leqslant M_{m+2+\theta_0}(\alpha_0), \tag{3.13}$$

where $M_{m+2+\theta_0}(\alpha_0) = M(\lambda_1, \nu, f, m, \theta_0, a_{m-1}, \alpha_0)$ does not depend on α.

Proof. The proof follows the same lines as the proof Theorem 3.1; but, instead of only using Theorem 2.3 in order to estimate the solution v_n of (3.1), one applies Theorem 2.4 together with Theorem 2.3. □

Remark 3.2. We conclude this section by pointing out that, if f belongs to H^{1+d}_{per}, $d > 0$, then we can prove the asymptotic compactness by using a decomposition similar to the one introduced in the proof of Theorem 3.1. Assume first that f is in H^2_{per}; for any $u_0 \in V^3$, we write $S(t)u_0 = v(t) + w(t) \equiv L_0(t)u_0 + K_0(t)u_0$, where $v(t)$ and $w(t)$ are the solutions of the following systems

$$\begin{gathered}
\partial_t(v - \alpha\Delta v) - \nu\Delta v + \operatorname{rot}(v - \alpha\Delta v) \times u + \nabla p = f, \quad t > 0\, x \in \mathbb{T}^2, \\
\operatorname{div} v = 0, \quad t > 0,\ x \in \mathbb{T}^2, \\
v(0, x) = 0, \quad x \in \mathbb{T}^2,
\end{gathered} \tag{3.14}$$

and

$$\begin{gathered}
\partial_t(w - \alpha\Delta w) - \nu\Delta w + \operatorname{rot}(w - \Delta w) \times u + \nabla\widetilde{p} = 0, \quad t > 0,\ x \in \mathbb{T}^2, \\
\operatorname{div} w = 0, \quad t > 0,\ x \in \mathbb{T}^2, \\
w(0, x) = u_0(x), \quad x \in \mathbb{T}^2.
\end{gathered} \tag{3.15}$$

Like in the proof of Theorem 3.1, applying Remark 2.1, we show that w satisfies the following exponential decay, for $t \geqslant 0$,

$$\big\|\operatorname{rot}\big(w(t) - \alpha\Delta w(t)\big)\big\|^2_{L^2} \leqslant \exp\left(-\frac{\nu t}{2(\lambda_1^{-1} + 2\alpha)}\right)\big\|\operatorname{rot}(u_0 - \alpha\Delta u_0)\big\|^2_{L^2}. \tag{3.16}$$

On the other hand, we deduce from the estimates (2.63) and (2.13) that, for $t \geqslant 0$,

$$\big\|\nabla\operatorname{rot}\big(v(t) - \alpha\Delta v(t)\big)\big\|^2_{L^2} \leqslant C_1\|Pf\|^2_{H^2}\exp\big(C_1 t\big(\big\|\operatorname{rot}(u_0 - \alpha\Delta u_0)\big\|^2_{L^2} + \|\operatorname{rot} f\|^2_{L^2}\big)\big), \tag{3.17}$$

where $C_1 = C_1(\lambda_1, \nu, \alpha)$ is a positive constant depending only of λ_1, ν and α. This shows that $S(t)u_0$ can be written as the sum of two maps $L_0(t)u_0$ and $K_0(t)u_0$, where $L_0(t)$ is asymptotically contracting and $K_0(t)$ is a compact map. Thus $S(t)$ is asymptotically smooth (see for example [22]). Since $S(t)$

admits a bounded absorbing set in V^3 (by the property (2.13)), we can conclude that $S(t)$ admits a compact global attractor $\mathcal{A}_\alpha$ in V^3. If f belongs only to H^{1+d}_{per}, $0 < d < 1$, using an interpolation argument between V^3 and V^4 as in the proof of Theorem 2.3, we can replace the estimate (3.17) by the following inequality

$$\left\| v(t) - \alpha\Delta v(t)\right\|^2_{V^{1+d}} \leqslant C_2\|Pf\|^2_{H^{1+d}} \exp\bigl(C_2 t\bigl(\left\|\operatorname{rot}(u_0 - \alpha\Delta u_0)\right\|^2_{L^2} + \|\operatorname{rot} f\|^2_{L^2}\bigr)\bigr), \tag{3.18}$$

where $C_2 = C_2(\lambda_1, \nu, \alpha)$ is a positive constant depending only of λ_1, ν and α. This again implies that $K_0(t)$ is a compact map.

4. Convergence to the Navier–Stokes equations

If we set $\alpha = 0$ in Eqs. (1.2), we recover the Navier–Stokes equations. The Navier–Stokes equations are not only a "formal" limit of Eqs. (1.2), as it has already been remarked by Iftimie, who proved a weak convergence result in [30], in any space dimension. In this section, we want to compare the strong solutions of the second grade fluid equations (1.2) with those of the corresponding Navier–Stokes equations when α goes to zero. We also want to give upper-semicontinuity results for the corresponding global attractors. We assume here that the forcing term f belongs to $C^\theta(H^1_{per})$, where $0 < \theta \leqslant 1$. Let us recall that the Navier–Stokes equations

$$\begin{aligned} \partial_t u - \nu\Delta u + \operatorname{rot} u \times u + \nabla p = f, \quad t > 0, \quad x \in \mathbb{T}^2,\\ \operatorname{div} u = 0, \quad t > 0,\ x \in \mathbb{T}^2,\\ u(0,x) = u_0(x), \quad x \in \mathbb{T}^2, \end{aligned} \tag{4.1}$$

have a unique global solution $S_0(t)u_0 \equiv u(t) \in C^0([0,+\infty), V^3) \cap C^1([0,+\infty), V^1)$ if u_0 belongs to V^3. Moreover, if f does not depend on the time variable, $S_0(t)$ admits a compact global attractor $\mathcal{A}_0$ in V^3. If f belongs to $C^\theta(H^2_{per})$, $0 < \theta \leqslant 1$, then $S_0(t)u_0$ is in $C^0([0,+\infty), V^3) \cap C^0((0,+\infty), V^4) \cap C^1((0,+\infty), V^2)$. If, in addition, u_0 belongs to V^4, then $S_0(t)u_0$ is in $C^0([0,+\infty), V^4)$, and in the autonomous case, the global attractor $\mathcal{A}_0$ in V^3 is actually also the compact global attractor in V^4. If $u_\alpha(t) = S_\alpha(t)u_0$ is the solution of the grade two fluid equations, then $z = u_\alpha - u$ satisfies the following equations

$$\begin{aligned} &\partial_t(z - \alpha\Delta z) - \nu\Delta z + \operatorname{rot} z \times u_\alpha + \operatorname{rot} u \times z + \nabla(p_\alpha - p)\\ &\quad = \alpha\partial_t\Delta u + \alpha \operatorname{rot}\Delta u_\alpha \times u_\alpha, \quad t > 0,\ x \in \mathbb{T}^2,\\ &\quad \operatorname{div} z = 0, \quad t > 0,\ x \in \mathbb{T}^2,\\ &\quad z(0,x) = 0, \quad x \in \mathbb{T}^2. \end{aligned} \tag{4.2}$$

Taking the inner product in $L^2(\mathbb{T}^2)$ of (4.2) with z, applying the Young inequality and using the classical Sobolev inequalities, we obtain the following estimate

$$\begin{aligned} &\frac{1}{2}\partial_t\bigl(\|z\|^2_{L^2} + \alpha\|\nabla z\|^2_{L^2}\bigr) + \nu\|\nabla z\|^2_{L^2}\\ &\quad\leqslant \frac{2\alpha^2}{\nu}\|\partial_t\nabla u\|^2_{L^2} + \frac{\nu}{4}\|\nabla z\|^2_{L^2} + \frac{2}{\nu}\|u_\alpha\|^2_{L^4}\|z\|^2_{L^4}\\ &\qquad + \alpha\|\operatorname{rot}\Delta u_\alpha\|_{L^2}\|u_\alpha\|_{L^4}\|z\|_{L^4}\\ &\quad\leqslant \frac{2\alpha^2}{\nu}\|\partial_t\nabla u\|^2_{L^2} + \frac{\nu}{4}\|\nabla z\|^2_{L^2} + \frac{2C_S^4}{\nu}\|u_\alpha\|_{L^2}\|\nabla u_\alpha\|_{L^2}\|z\|_{L^2}\|\nabla z\|_{L^2}\\ &\qquad + \alpha C_S^2\|\operatorname{rot}\Delta u_\alpha\|_{L^2}\|u_\alpha\|^{1/2}_{L^2}\|\nabla u_\alpha\|^{1/2}_{L^2}\|z\|^{1/2}_{L^2}\|\nabla z\|^{1/2}_{L^2} \end{aligned}$$

$$\leqslant \frac{2\alpha^2}{\nu}\|\partial_t \nabla u\|_{L^2}^2 + \frac{\nu}{2}\|\nabla z\|_{L^2}^2 + \frac{8C_S^8}{\nu^2}\|u_\alpha\|_{L^2}^2\|\nabla u_\alpha\|_{L^2}^2\|z\|_{L^2}^2 + \frac{\alpha^2 C_S^4}{2}\|\operatorname{rot}\Delta u_\alpha\|_{L^2}^2\|u_\alpha\|_{L^2}\|\nabla u_\alpha\|_{L^2} + \frac{1}{2\nu}\|z\|_{L^2}^2.$$

From the above estimate and the Poincaré inequality (2.4), we deduce that, for $t \geqslant 0$,

$$\begin{aligned} &\partial_t\big(\big\|z(t)\big\|_{L^2}^2 + \alpha\big\|\nabla z(t)\big\|_{L^2}^2\big) + \frac{\nu}{\alpha+\lambda_1^{-1}}\big(\big\|z(t)\big\|_{L^2}^2 + \alpha\big\|\nabla z(t)\big\|_{L^2}^2\big) \\ &\quad\leqslant \frac{4\alpha^2}{\nu}\big\|\partial_t\nabla u(t)\big\|_{L^2}^2 + \left(\frac{1}{\nu} + \frac{16C_S^8}{\nu^2}\big\|u_\alpha(t)\big\|_{L^2}^2\big\|\nabla u_\alpha(t)\big\|_{L^2}^2\right)\big\|z(t)\big\|_{L^2}^2 \\ &\qquad + \alpha^2 C_S^4\big\|\operatorname{rot}\Delta u_\alpha(t)\big\|_{L^2}^2\big\|u_\alpha(t)\big\|_{L^2}\big\|\nabla u_\alpha(t)\big\|_{L^2}. \end{aligned} \tag{4.3}$$

Integrating the inequality (4.3) from 0 to t and using the Gronwall inequality, we obtain, for $t \geqslant 0$,

$$\begin{aligned} \big\|z(t)\big\|_{L^2}^2 + \alpha\big\|\nabla z(t)\big\|_{L^2}^2 \leqslant{}& \left(\frac{4\alpha^2}{\nu}\int_0^t \big\|\partial_t\nabla u(s)\big\|_{L^2}^2\,ds \right. \\ &\left. + \alpha^2 C_S^4\int_0^t \big\|\operatorname{rot}\Delta u_\alpha(s)\big\|_{L^2}^2\big\|u_\alpha(s)\big\|_{L^2}\big\|\nabla u_\alpha(s)\big\|_{L^2}\,ds\right) \\ &\times \exp\left(\int_0^t \left(\frac{1}{\nu} + \frac{16C_S^8}{\nu^2}\big\|u_\alpha(s)\big\|_{L^2}^2\big\|\nabla u_\alpha(s)\big\|_{L^2}^2\right)ds\right). \end{aligned} \tag{4.4}$$

When $\alpha\lambda_1 \leqslant 1$, the inequality (4.4) together with the estimates (2.8), (2.18), (2.22) and a classical estimate for the solution of the Navier–Stokes equations imply that, for $t \geqslant 0$,

$$\begin{aligned} \big\|z(t)\big\|_{L^2}^2 + \alpha\big\|\nabla z(t)\big\|_{L^2}^2 \leqslant{}& \alpha^2 K_0\big(\|\operatorname{rot} f\|_{L_t^\infty(L^2)}, \|\nabla\operatorname{rot} u_0\|_{L^2}^2 + \alpha\|\Delta\operatorname{rot} u_0\|_{L^2}^2\big) \\ &\times \exp\big(1 + K_1\big(\|\operatorname{rot} f\|_{L_t^\infty(L^2)}, \|u_0\|_{L^2}^2 + \alpha\|\nabla u_0\|_{L^2}^2\big)\big)t, \end{aligned} \tag{4.5}$$

where K_0 and respectively K_1 are positive constants depending only on $\|\operatorname{rot} f\|_{L_t^\infty(L^2)}$ and on $\|\nabla\operatorname{rot} u_0\|_{L^2}^2 + \alpha\|\Delta\operatorname{rot} u_0\|_{L^2}^2$ (respectively $\|u_0\|_{L^2}^2 + \alpha\|\nabla u_0\|_{L^2}^2$). The property (2.25) and analogous properties of the solutions of the Navier–Stokes equations imply that, for $t \geqslant 0$,

$$\big\|\nabla\operatorname{rot} z(t)\big\|_{L^2}^2 + \alpha\big\|\Delta\operatorname{rot} z(t)\big\|_{L^2}^2 \leqslant K_2\big(\|\operatorname{rot} f\|_{L_t^\infty(L^2)}, \|\nabla\operatorname{rot} u_0\|_{L^2}^2 + \alpha\|\Delta\operatorname{rot} u_0\|_{L^2}^2\big), \tag{4.6}$$

where K_2 is a positive constant depending only on $\|\operatorname{rot} f\|_{L_t^\infty(L^2)}$ and on $\|\nabla\operatorname{rot} u_0\|_{L^2}^2 + \alpha\|\Delta\operatorname{rot} u_0\|_{L^2}^2$. Thus, by interpolation, we obtain, for $0 \leqslant s \leqslant 2$ and $t \geqslant 0$,

$$\begin{aligned} \big\|z(t)\big\|_{V^s}^2 + \alpha\big\|z(t)\big\|_{V^{s+1}}^2 \leqslant{}& \alpha^{2(1-s/2)} K_2^{s/2}\big(\|\operatorname{rot} f\|_{L_t^\infty(L^2)}, \|\nabla\operatorname{rot} u_0\|_{L^2}^2 + \alpha\|\Delta\operatorname{rot} u_0\|_{L^2}^2\big) \\ &\times K_0^{1-s/2}\big(\|\operatorname{rot} f\|_{L_t^\infty(L^2)}, \|\nabla\operatorname{rot} u_0\|_{L^2}^2 + \alpha\|\Delta\operatorname{rot} u_0\|_{L^2}^2\big) \\ &\times \exp\left(1 - \frac{s}{2}\right)\big(1 + K_1\big(\|\operatorname{rot} f\|_{L_t^\infty(L^2)}, \|u_0\|_{L^2}^2 + \alpha\|\nabla u_0\|_{L^2}^2\big)\big)t. \end{aligned} \tag{4.7}$$

Let us now assume that f belongs to $C^{\theta}((0,+\infty), H^2_{per})$, $\theta > 0$, and that u_0 is an element in V^4. Then, the solution $S_0(t)u_0$ of the Navier–Stokes equations (4.1) belongs to the space $C^0_b([0,+\infty), V^4)$. Moreover, for $\alpha > 0$ small enough so that $2\nu - 2\alpha\|S_\alpha(t)u_0\|_{L^\infty(L^\infty)} > 0$, the solution $S_\alpha(t)u_0$ of Eqs. (1.2) is bounded in V^4, for $t \geqslant 0$, as described in (2.59). We thus obtain, for $t \geqslant 0$,

$$\|\operatorname{rot}\Delta z(t)\|^2_{L^2} + \alpha\|\Delta^2 \operatorname{rot} z(t)\|^2_{L^2} \leqslant K_3\big(\|f\|_{L^\infty_t(H^2)}, \|\operatorname{rot}\Delta u_0\|^2_{L^2} + \alpha\|\Delta^2 u_0\|^2_{L^2}\big), \tag{4.8}$$

where K_3 is a positive constant depending only on $\|f\|_{L^\infty_t(H^2)}$ and on $\|\operatorname{rot}\Delta u_0\|^2_{L^2} + \alpha\|\Delta^2 u_0\|^2_{L^2}$. Interpolating between the inequalities (4.8) and (4.5), we obtain that, for $0 \leqslant s \leqslant 3$ and $t \geqslant 0$,

$$\begin{aligned}\|z(t)\|^2_{V^s} + \alpha\|z(t)\|^2_{V^{s+1}} &\leqslant \alpha^{2(1-s/3)}K_3^{s/3}\big(\|f\|_{L^\infty_t(H^2)}, \|\operatorname{rot}\Delta u_0\|^2_{L^2} + \alpha\|\Delta^2 u_0\|^2_{L^2}\big)\\ &\quad\times K_0^{1-s/3}\big(\|\operatorname{rot} f\|_{L^\infty_t(L^2)}, \|\nabla \operatorname{rot} u_0\|^2_{L^2} + \alpha\|\Delta \operatorname{rot} u_0\|^2_{L^2}\big)\\ &\quad\times \exp\left(1-\frac{s}{3}\right)\big(1 + K_1\big(\|\operatorname{rot} f\|_{L^\infty_t(L^2)}, \|u_0\|^2_{L^2} + \alpha\|\nabla u_0\|^2_{L^2}\big)\big)t. \end{aligned}\tag{4.9}$$

We can improve the estimate (4.9) by arguing as follows. Taking the inner product in $L^2(\mathbb{T}^2)$ of (4.2) with $-\Delta z$, using the equality (A.5) of Appendix A, applying the Young inequality and using the classical Sobolev inequalities, we obtain the following estimate

$$\begin{aligned}\partial_t\big(\|\nabla z\|^2_{L^2} + \alpha\|\Delta z\|^2_{L^2}\big) + \nu\|\Delta z\|^2_{L^2} &\leqslant \frac{3}{\nu}\big(\alpha^2\|\partial_t\Delta u\|^2_{L^2} + \alpha^2\|\operatorname{rot}\Delta u_\alpha\|^2_{L^2}\|u_\alpha\|^2_{L^\infty}\\ &\quad + \|\operatorname{rot} u\|^2_{L^4}\|z\|^2_{L^4}\big)\\ &\leqslant \frac{3}{\nu}\big(\alpha^2\|\partial_t\Delta u\|^2_{L^2} + \alpha^2 C_S^2\|\operatorname{rot}\Delta u_\alpha\|^2_{L^2}\|\Delta u_\alpha\|^2_{L^2}\\ &\quad + C_S^4\|\Delta u\|^2_{L^2}\|\nabla z\|^2_{L^2}\big),\end{aligned}$$

where C_S is a positive constant coming from the Sobolev inequalities. From the above estimate and the Poincaré inequality (2.4), we infer that, for $t \geqslant 0$,

$$\begin{aligned}&\partial_t\big(\|\nabla z(t)\|^2_{L^2} + \alpha\|\Delta z(t)\|^2_{L^2}\big) + \frac{\nu\lambda_1}{1+\alpha\lambda_1}\big(\|\nabla z(t)\|^2_{L^2} + \alpha\|\Delta z(t)\|^2_{L^2}\big)\\ &\quad\leqslant \frac{3}{\nu}\big(\alpha^2\|\partial_t\Delta u(t)\|^2_{L^2} + \alpha^2 C_S^2\|\operatorname{rot}\Delta u_\alpha(t)\|^2_{L^2}\|\Delta u_\alpha(t)\|^2_{L^2} + C_S^4\|\Delta u(t)\|^2_{L^2}\|\nabla z(t)\|^2_{L^2}\big).\end{aligned}\tag{4.10}$$

Integrating the inequality (4.10) from 0 to t and using the Gronwall lemma, we get, for $t \geqslant 0$,

$$\begin{aligned}\|\nabla z(t)\|^2_{L^2} + \alpha\|\Delta z(t)\|^2_{L^2} &\leqslant \frac{3}{\nu}\left(\alpha^2\int_0^t\big(\|\partial_t\Delta u(\tau)\|^2_{L^2} + C_S^2\|\operatorname{rot}\Delta u_\alpha(\tau)\|^2_{L^2}\|\Delta u_\alpha(\tau)\|^2_{L^2}\big)\,d\tau\right)\\ &\quad\times\exp\left(C_S^4\int_0^t\|\Delta u(\tau)\|^2_{L^2}\,d\tau\right).\end{aligned}\tag{4.11}$$

When $\alpha\lambda_1 \leqslant 1$, the inequality (4.11) together with the estimates (2.18), (2.22) and the regularity properties of the solution $u(t)$ of the Navier–Stokes equations (4.1), imply that, for $t \geqslant 0$,

$$\big\|\nabla z(t)\big\|_{L^2}^2+\alpha\big\|\Delta z(t)\big\|_{L^2}^2\leqslant\alpha^2K_4\big(\|f\|_{L_t^\infty(H^2)},\|\nabla\operatorname{rot}u_0\|_{L^2}^2+\alpha\|\Delta\operatorname{rot}u_0\|_{L^2}^2\big)$$
$$\times\exp K_4\big(\|f\|_{L_t^\infty(H^2)},\|\nabla\operatorname{rot}u_0\|_{L^2}^2+\alpha\|\Delta\operatorname{rot}u_0\|_{L^2}^2\big)t,\quad(4.12)$$

where $K_4(\|f\|_{L_t^\infty(H^2)},\|\nabla\operatorname{rot}u_0\|_{L^2}^2+\alpha\|\Delta\operatorname{rot}u_0\|_{L^2}^2)$ is a positive constant depending only on $\|f\|_{L_t^\infty(H^2)}$ and $\|\nabla\operatorname{rot}u_0\|_{L^2}^2+\alpha\|\Delta\operatorname{rot}u_0\|_{L^2}^2$. By interpolation, we deduce from the estimates (4.8) and (4.12) that, for $t\geqslant0$, for $0\leqslant s\leqslant2$,

$$\begin{aligned}&\big\|z(t)\big\|_{V^{1+s}}^2+\alpha\big\|z(t)\big\|_{V^{2+s}}^2\\&\quad\leqslant\alpha^{2(1-s/2)}K_3^{s/2}\big(\|f\|_{L_t^\infty(H^2)},\|\operatorname{rot}\Delta u_0\|_{L^2}^2+\alpha\big\|\Delta^2u_0\big\|_{L^2}^2\big)\\&\qquad\times K_4^{1-s/2}\big(\|f\|_{L_t^\infty(H^2)},\|\nabla\operatorname{rot}u_0\|_{L^2}^2+\alpha\|\Delta\operatorname{rot}u_0\|_{L^2}^2\big)\\&\qquad\times\exp\big[(1-s/2)\big(K_4\big(\|f\|_{L_t^\infty(H^2)},\|\nabla\operatorname{rot}u_0\|_{L^2}^2+\alpha\|\Delta\operatorname{rot}u_0\|_{L^2}^2\big)t\big)\big].\end{aligned}\quad(4.13)$$

We have thus proved the following result.

Theorem 4.1. *Assume that $\alpha\lambda_1\leqslant1$.*

1) *Assume that f belongs to $C^\theta((0,+\infty),H_{per}^1)$, $\theta>0$, and let $u_0\in V^3$. Then, for every $0\leqslant s\leqslant2$, the following estimate holds*

$$\begin{aligned}&\big\|S_0(t)u_0-S_\alpha(t)u_0\big\|_{V^s}^2+\alpha\big\|S_0(t)u_0-S_\alpha(t)u_0\big\|_{V^{s+1}}^2\\&\quad\leqslant\alpha^{2(1-s/2)}\exp tK_5\big(\|\operatorname{rot}f\|_{L_t^\infty(L^2)},\|\nabla\operatorname{rot}u_0\|_{L^2}^2+\alpha\|\Delta\operatorname{rot}u_0\|_{L^2}^2\big),\end{aligned}\quad(4.14)$$

where $K_5(\|\operatorname{rot}f\|_{L_t^\infty(L^2)},\|\nabla\operatorname{rot}u_0\|_{L^2}^2+\alpha\|\Delta\operatorname{rot}u_0\|_{L^2}^2)$ is a positive constant depending only on $\|\operatorname{rot}f\|_{L_t^\infty(L^2)}$ and $\|\nabla\operatorname{rot}u_0\|_{L^2}^2+\alpha\|\Delta\operatorname{rot}u_0\|_{L^2}^2$.

2) *Assume moreover that f belongs to $C^\theta((0,+\infty),H_{per}^2)$, $\theta>0$, that $\alpha>0$ is small enough and that u_0 is in V^4. Then, for every $0\leqslant s\leqslant2$, the following estimate holds*

$$\begin{aligned}&\big\|S_0(t)u_0-S_\alpha(t)u_0\big\|_{V^{s+1}}^2+\alpha\big\|S_0(t)u_0-S_\alpha(t)u_0\big\|_{V^{s+2}}^2\\&\quad\leqslant\alpha^{2(1-s/2)}\exp tK_6\big(\|f\|_{L_t^\infty(H^2)},\|\Delta\operatorname{rot}u_0\|_{L^2}^2+\alpha\big\|\Delta^2u_0\big\|_{L^2}^2\big),\end{aligned}\quad(4.15)$$

where $K_6(\|f\|_{L_t^\infty(H^2)},\|\Delta\operatorname{rot}u_0\|_{L^2}^2+\alpha\|\Delta^2u_0\|_{L^2}^2)$ is a positive constant depending only on $\|f\|_{L_t^\infty(H^2)}$ and $\|\Delta\operatorname{rot}u_0\|_{L^2}^2+\alpha\|\Delta^2u_0\|_{L^2}^2$.

This theorem at once implies the upper-semicontinuity of the global attractors in V^s, for $0\leqslant s<2$ (respectively $0\leqslant s<3$).

Corollary 4.2. 1) *Assume that f belongs to H_{per}^1. Let $\mathcal{A}_0$ and $\mathcal{A}_\alpha$ be the compact global attractors of Eqs.* (4.1) *and* (1.2)*. Then, the attractors $\mathcal{A}_\alpha$ are upper-semicontinuous at $\alpha=0$ in V^s, for $0\leqslant s<2$, that is*

$$\lim_{\alpha\to0}\sup_{u_\alpha\in\mathcal{A}_\alpha}\inf_{u\in\mathcal{A}_0}\|u_\alpha-u\|_{V^s}=0.$$

2) *Assume moreover that f belongs to H_{per}^2, then the attractors $\mathcal{A}_\alpha$ are upper-semicontinuous at $\alpha=0$ in V^s, for $0\leqslant s<3$.*

Proof. Although the proof of this corollary is classical, we give it for the reader's convenience. Assume that $\alpha\lambda_1 \leqslant 1$. Then, the global attractor $\mathcal{A}_\alpha$ is bounded in V^2 by a positive constant C_2 independent of α. Let $\eta > 0$ be a small positive number. Since $\mathcal{A}_0$ is the compact global attractor of the Navier–Stokes equations in V^3 and in V^2, there exists a positive time T_η such that, for any $u_0 \in \mathcal{A}_\alpha$, $S_0(t)u_0$ belongs to the $\eta/2$-neighborhood of $\mathcal{A}_0$ in V^2, for $t \geqslant T_\eta$. By Theorem 4.1, since $\|\mathrm{rot}\, u_0\|_{L^2}^2 + \alpha\|\Delta\,\mathrm{rot}\, u_0\|_{L^2}^2$ is uniformly bounded by a constant C_3 independent of α, for any s, $0 \leqslant s < 2$, there exists $\alpha_s > 0$, such that, for $0 < \alpha \leqslant \alpha_s$, for any $u_0 \in \mathcal{A}_\alpha$, we have,

$$\big\|S_0(T_\eta)u_0 - S_\alpha(T_\eta)u_0\big\|_{V^s} \leqslant \frac{\eta}{2}.$$

The above properties imply that $S_\alpha(T_\eta)\mathcal{A}_\alpha$ is included in the η-neighborhood of $\mathcal{A}_0$ in V^s. Since $\mathcal{A}_\alpha$ is invariant under $S_\alpha(t)$, we deduce that A_α is included in the η-neighborhood of $\mathcal{A}_0$ in V^s.

The second statement of the corollary is proved exactly in the same way. But now, we use the fact that, by the results of Section 2, $\|\mathrm{rot}\,\Delta u_0\|_{L^2}^2 + \alpha\|\Delta^2 u_0\|_{L^2}^2$ is uniformly bounded by a constant C_4 independent of α, for any $u_0 \in \mathcal{A}_\alpha$. □

More generally, by using the regularity results of the global attractors $\mathcal{A}_\alpha$ of Section 3, we can prove the following upper-semicontinuity result, the proof of which is left to the reader.

Corollary 4.3. *Assume that f belongs to H^m_{per}, $m \geqslant 1$. Let $\mathcal{A}_0$ and $\mathcal{A}_\alpha$ be the compact global attractors of Eqs.* (4.1) *and* (1.2)*. Then, the attractors $\mathcal{A}_\alpha$ are upper-semicontinuous at $\alpha = 0$ in V^s, for $0 \leqslant s < m+1$, that is*

$$\lim_{\alpha\to 0} \sup_{u_\alpha\in\mathcal{A}_\alpha} \inf_{u\in\mathcal{A}_0} \|u_\alpha - u\|_{V^s} = 0.$$

5. Determining modes and asymptotic dynamics

The main goal of this section is the proof of Theorem 1.2 and its consequences. Theorem 1.2 will be proved in several steps and will be the consequence of several lemmas. Except in Lemma 5.1, we will need to impose the condition $2\nu - 4\alpha \sup_{z\in\mathcal{A}_\alpha} \|\nabla z\|_{L^\infty} > 0$.

Before entering into the details of the proof of Theorem 1.2, we quickly explain the lines of its proof and the construction of the map $q(v)$. As we have already explained in the introduction, here we follow the strategy developed by Hale and Raugel in [24]. We have to face an additional difficulty due to the low regularity of the nonlinear term. Let $n \geqslant 1$ be a fixed integer. We recall that P denotes the Leray orthogonal projection of $L^2(\mathbb{T}^2)^2$ onto H and that P_n is the orthonormal projection in H onto the space generated by the eigenfunctions corresponding to the first n eigenvalues of the Stokes operator A and that $Q_n = I - P_n$. As in [24], we remark that, if $u(t) \in C_b^0(\mathbb{R}, V^3)$ is a solution of (1.2), contained in the attractor $\mathcal{A}_\alpha$, then $v(t) = P_n u(t)$ and $q(t) = Q_n u(t)$ are solutions to the following systems, for all $t \in \mathbb{R}$,

$$\partial_t(v - \alpha\Delta v) - \nu\Delta v + P_n P\Big(\big\{\mathrm{rot}\big(q + v - \alpha\Delta(q+v)\big) \times (v+q)\big\}\Big) = P_n P f, \tag{5.1}$$

and

$$\partial_t(q - \alpha\Delta q) - \nu\Delta q + Q_n P\Big(\big\{\mathrm{rot}\big(q + v - \alpha\Delta(q+v)\big) \times (v+q)\big\}\Big) = Q_n P f. \tag{5.2}$$

We will see below that, if n is large enough, for every "bounded curve" (also called bounded trajectory) $v(t) \in C_b^0(\mathbb{R}, P_n\mathcal{A}_\alpha)$, there exists a unique solution $q_n(v)(t) \in C_b^0(\mathbb{R}, Q_n V^3)$ of Eq. (5.2) and that, if $v(t) = P_n u(t)$ where $u(t)$ is a solution of (1.2) contained in the attractor $\mathcal{A}_\alpha$, then $q_n(v)(t) = Q_n u(t)$ (to simplify the notation, we write $q_n(v)(t) \equiv q(v)(t)$ below). In Theorem 1.2, we want to show that

Eqs. (1.2) reduce to a finite system of ordinary differential equations with an infinite delay term involving $q(v)$. In order to give a sense to this reduced system, we have to define the mapping $q(v)$ not only for trajectories $v(t)$ contained in $P_n\mathcal{A}_\alpha$, but also for all "curves" contained in a neighborhood $\mathcal{V}_n$ of $P_n\mathcal{A}_\alpha$ in P_nV^{3+d}, $d>0$. The most natural idea is to define $q(v)$ as "the" solution of Eq. (5.2). We will show below that, for any element $v \in C_b^0(\mathbb{R},\mathcal{V}_n)$, and for n large enough and α small enough, the solution $q(v) \in C_b^0(\mathbb{R}, Q_nV^3)$ of (5.2) is obtained as the (unique) fixed point of the contraction mapping defined as follows. Let $w(t)$ be given in $C_b^0(\mathbb{R}, Q_nV^2) \cap L^\infty(\mathbb{R}, Q_nV^3)$, we consider the problem: to find $q \in C_b^0(\mathbb{R}, V^3)$ such that,

$$\partial_t(q-\alpha\Delta q) - \nu\Delta q + Q_nP\big(\big\{\mathrm{rot}\big(q+v-\alpha\Delta(q+v)\big)\times(v+w)\big\}\big) = Q_nPf, \quad t\in\mathbb{R}. \tag{5.3}$$

In a first step (see Lemma 5.1 below), we show that, if f belongs to H^1_{per}, and if v and w belong to $C_b^0(\mathbb{R}, P_nV^3)$ and $C_b^0(\mathbb{R}, Q_nV^2)\cap L^\infty(\mathbb{R}, Q_nV^3)$ respectively, Eq. (5.3) has a unique solution $q^n_{v,w}$ in the space $C_b^0(\mathbb{R}, Q_nV^3)$ (to simplify, we write $q_{v,w}$ instead of $q^n_{v,w}$ if there is no possible confusion).

In a second step, we assume that f belongs to H^{1+d}_{per}, $d>0$, and we show that, if v belongs to $C^0(\mathbb{R}, \mathcal{N}_{P_nV^{3+d}}(P_n\mathcal{A}_\alpha, r_0))$, where $\mathcal{N}_{P_nV^{3+d}}(P_n\mathcal{A}_\alpha, r_0)$ is the r_0-neighborhood of $P_n\mathcal{A}_\alpha$ in P_nV^{3+d}, equipped with the norm $(\|\cdot\|^2_{V^{2+d}} + \alpha\|\cdot\|^2_{V^{3+d}})^{\frac{1}{2}}$, and where $r_0>0$ is small enough, then, for n large enough, for $R=R_n>0$ large enough, and for α small enough, the map $w \in W_n(R_n) \to q^n_{v,w} \in W_n(R_n)$ is a strict contraction and thus has a unique fixed point $q_n(v) \equiv q(v) \in W_n(R_n)$, where $W_n(R_n)$ is defined in (5.16) below. Moreover, we prove that the map $v \to q(v)$ is Lipschitz-continuous. These properties allow us to reduce system (1.2) on the attractor $\mathcal{A}_\alpha$ to a system of ordinary differential equations containing an infinite delay term. This system is well-posed since it satisfies the hypotheses of the Cauchy–Lipschitz theorem.

5.1. Proof of Theorem 1.2

5.1.1. Step 1 of the proof of Theorem 1.2

The first step consists in proving that Eq. (5.3) has a unique solution in $C_b^0(\mathbb{R}, Q_nV^3)$. In this step we do not need any smallness assumption of α.

Lemma 5.1. *Let $f \in H^1_{per}$ be fixed. We assume that $v(t)$ and $w(t)$ belong to the spaces $C_b^0(\mathbb{R}, P_nV^3)$ and $C_b^0(\mathbb{R}, Q_nV^2)\cap L^\infty(\mathbb{R}, Q_nV^3)$ respectively. Then, for any integer n, there exists a unique solution $q^n_{v,w} \in C_b^0(\mathbb{R}, Q_nV^3)$ of Eq. (5.3). Moreover, $q^n_{v,w}(t)$ depends on the values $v(s)$ and $w(s)$, for $s \leqslant t$ only.*

Proof. The proof consists in three steps.
a) Existence of a solution: As usually, we show the existence of a solution q in $C^0(\mathbb{R}, Q_nV^3)$ by considering a Galerkin approximation of Eq. (5.3):

$$\partial_t(q_m-\alpha\Delta q_m) - \nu\Delta q_m + P_mQ_nP\big\{\mathrm{rot}(q_m-\alpha\Delta q_m)\times(v+w)\big\} = g_m, \quad t\in\mathbb{R}, \tag{5.4}$$

where

$$g_m = P_mQ_nPf - P_mQ_nP\big\{\mathrm{rot}(v-\alpha\Delta v)\times(v+w)\big\}.$$

Eq. (5.4) can be rewritten in the equivalent form

$$\partial_t q_m + A_m(t)q_m = (I+\alpha A)^{-1}g_m(t), \quad t\in\mathbb{R}, \tag{5.5}$$

where

$$A_m(t)q_m = -\nu(I+\alpha A)^{-1}P_mQ_n\Delta q_m + (I+\alpha A)^{-1}P_mQ_nP\big\{\mathrm{rot}(q_m-\alpha\Delta q_m)\times(v+w)\big\}.$$

Note that (5.5) is a system of linear non-autonomous ODE's in the finite-dimensional space $\overline{H} = P_m Q_n P H^3(\mathbb{T}^2) = P_m Q_n V^3$, that $A_m(t) \in C^0(\mathbb{R}, \mathcal{L}(\overline{H}))$ is a continuous matrix-valued function and that $g_m(t) \in C^0(\mathbb{R}, \overline{H})$. Consider the Cauchy problem for the matrix-valued function $U_m(t,s)$

$$\partial_t U_m + A_m(t) U_m = 0, \quad t \geqslant s, \qquad U_m(s,s) = I. \tag{5.6}$$

As is well known from the theory of linear ODE's, this problem admits a unique continuous solution $U_m(t,s) \in C(D, \mathcal{L}(\overline{H}))$ where $D = \{(t,s) \in \mathbb{R}^2\colon\ t \geqslant s\}$. We remark that, for any $t \in \mathbb{R}$, for any $U \in \mathcal{L}(\overline{H})$,

$$\big(\mathrm{rot}(I+\alpha A)A_m(t)U, \mathrm{rot}(U - \alpha\Delta U)\big) = \nu\big(\|\Delta U\|_{L^2}^2 + \alpha\|\nabla\Delta U\|_{L^2}^2\big).$$

Therefore, using the inequality (2.11), we deduce from the above equality that the solution $U_m(t,s)$ of (5.6), satisfies, for any $(t,s) \in D$,

$$\partial_t \big\|\mathrm{rot}(U_m - \alpha\Delta U_m)\big\|_{L^2}^2 + \frac{\nu}{\lambda_1^{-1} + 2\alpha}\big\|\mathrm{rot}(U_m - \alpha\Delta U_m)\big\|_{L^2}^2 \leqslant 0, \tag{5.7}$$

and therefore

$$\big\|\mathrm{rot}\big(U_m(t,s)q_0 - \alpha\Delta U_m(t,s)q_0\big)\big\|_{L^2}^2 \leqslant e^{-\gamma_1(t-s)}\big\|\mathrm{rot}(q_0 - \alpha\Delta q_0)\big\|_{L^2}^2, \quad (t,s) \in D, \tag{5.8}$$

where $\gamma_1 = \frac{\nu}{\lambda_1^{-1}+2\alpha}$. It is easy to see now that the function

$$q_m(t) = \int_{-\infty}^{t} U_m(t,s)(I+\alpha A)^{-1} g_m(s)\,ds$$

is well defined on $\mathbb{R}$, belongs to the space $C^1(\mathbb{R}, \overline{H})$ and satisfies (5.5). We remark that $q_m(t)$ depends on the values of $v(s)$ and $w(s)$, for $s \leqslant t$ only. Moreover, the estimate (5.8) implies that

$$\begin{aligned} &\big\|\mathrm{rot}(q_m - \alpha\Delta q_m)(t)\big\|_{L^2} \\ &\quad \leqslant \int_{-\infty}^{t} e^{-\frac{\gamma_1}{2}(t-s)}\big\|\mathrm{rot}\big(f - \big(\mathrm{rot}\big(v(s) - \alpha\Delta v(s)\big)\big) \times (v+w)(s)\big)\big\|_{L^2}\,ds \leqslant C, \end{aligned} \tag{5.9}$$

where C is a positive constant, which does not depend on m. We have thus proved that the sequence $q_m(t)$ is uniformly bounded (with respect to m) in the space $L^\infty(\mathbb{R}, V^3)$. Applying standard Galerkin approximation arguments, we infer that there exists a weak limit $q_{v,w}^n \equiv q \in L^\infty(\mathbb{R}, V^3)$ which satisfies (5.3). Arguing as in Remark 2.3, one shows that $q(t)$ belongs to the space $C^0(\mathbb{R}, V^3)$.

b) Remark: We note that $Q_n q = q$, where q is any solution in $L^\infty(\mathbb{R}; V^3)$ of Eq. (5.3). To prove this property, it suffices to show that $P_n q = 0$. But $P_n q$ is the solution of the following linear equation

$$\partial_t(P_n q - \alpha\Delta P_n q) - \nu\Delta P_n q = 0.$$

Taking the inner product in H of this equation with q in H, we get

$$\frac{1}{2}\partial_t\big(\|P_n q\|_{L^2}^2 + \alpha\|\mathrm{rot}\, P_n q\|_{L^2}^2\big) + \nu\|\mathrm{rot}\, P_n q\|_{L^2}^2 = 0, \quad t \in \mathbb{R}.$$

This implies that, for some positive constant $\gamma > 0$, we have, for $t \geqslant s$,

$$\|P_n q(t)\|_{L^2}^2 + \alpha \|\operatorname{rot} P_n q(t)\|_{L^2}^2 \leqslant e^{\gamma(s-t)}\big(\|P_n q(s)\|^2 + \alpha \|\operatorname{rot} P_n q(s)\|^2\big).$$

Since $q(s)$ belongs to the space $L^\infty(\mathbb{R}, V^3)$, $P_n q$ belongs to the same space and the above inequality implies that $P_n q(t) = 0$ for all $t \in \mathbb{R}$.

c) Uniqueness of the solution: Assume now that there exist two solutions $q_1(t)$, $q_2(t)$ of problem (5.3), which belong to the space $C_b(\mathbb{R}, V^2) \cap L^\infty(\mathbb{R}, V^3)$. Writing the equation for the difference $q^* = q_1 - q_2$, we obtain the equality

$$\partial_t\big(q^* - \alpha\Delta q^*\big) - \nu\Delta q^* + Q_n P\big(\operatorname{rot}\big(q^* - \alpha\Delta q^*\big) \times (v + w)\big) = 0, \quad \forall t \in \mathbb{R}. \tag{5.10}$$

Taking the vorticity of the equality (5.10), we get the equation,

$$\partial_t \operatorname{rot}\big(q^* - \alpha\Delta q^*\big) - \nu \operatorname{rot}\Delta q^* + \operatorname{rot} Q_n P\big(\operatorname{rot}\big(q^* - \alpha\Delta q^*\big) \times (v + w)\big) = 0, \quad \forall t \in \mathbb{R}. \tag{5.11}$$

Formally, taking the scalar product in H of this equation with $\operatorname{rot}(q^* - \alpha\Delta q^*)$ we obtain,

$$\frac{1}{2}\partial_t \big\|\operatorname{rot}\big(q^* - \alpha\Delta q^*\big)\big\|_{L^2}^2 + \nu\big(\|\Delta q^*\|^2 + \alpha\|\operatorname{rot}\Delta q^*\|^2\big) = 0, \quad \forall t \in \mathbb{R},$$

which implies, after integration in time, that, for $t \geqslant s$,

$$\big\|\operatorname{rot}\big(q^*(t) - \alpha\Delta q^*(t)\big)\big\|_{L^2}^2 \leqslant e^{-\gamma(t-s)}\big\|\operatorname{rot}\big(q^*(s) - \alpha\Delta q^*(s)\big)\big\|_{L^2}^2$$

where γ is a positive number. Since $q^*(s)$ belongs to $L^\infty(\mathbb{R}, V^3)$, by taking $s \to -\infty$, one deduces from this inequality that $q^*(t) = 0$, for all $t \in \mathbb{R}$. We emphasize that, arguing in the same way as above, one shows that the solution $q(t)$ of (5.3) depends on the values of $v(s)$ and $w(s)$, for $s \leqslant t$ only.

In order to justify the above formal computation, we proceed as follows. For any integer N, for any $g \in L^2(\mathbb{T}^2)$, we introduce the operator

$$J_N g = \sum_{n \equiv (n_1, n_2)} \varphi\left(\frac{|n|}{2^N}\right) \hat{g}_n e^{i(n_1 x_1 + n_2 x_2)},$$

where $\hat{g}_n$ is the Fourier coefficient of order n of g and φ is a classical truncation function. For example, we choose a symmetric function $\varphi \in C^\infty(\mathbb{R}, [0,1])$ such that the support of φ is contained in $[-4/3, 4/3]$ and such that $\varphi = 1$ on $[-3/4, 3/4]$. In the same way, we define the operator J_N for vectors in $L^2(\mathbb{T}^2)^2$. For any vector $u \in L^2(\mathbb{T}^2)^2$, we introduce the operator

$$J_N u = \sum_{n \equiv (n_1, n_2)} \varphi\left(\frac{|n|}{2^N}\right) \hat{u}_n e^{i(n_1 x_1 + n_2 x_2)}.$$

Applying the operator J_N to Eq. (5.10) and taking the vorticity of the resulting equation, we get the equality

$$\partial_t \operatorname{rot}\big(J_N q^* - \alpha\Delta J_N q^*\big) - \nu \operatorname{rot}\Delta J_N q^* + \operatorname{rot}\big(\operatorname{rot}\big(J_N q^* - \alpha\Delta J_N q^*\big) \times (v + w)\big) = h_N \tag{5.12}$$

where

$$h_N = \operatorname{rot}\big(\operatorname{rot}\big(J_N q^* - \alpha \Delta J_N q^*\big) \times (v+w)\big) - \operatorname{rot} J_N\big(\operatorname{rot}\big(q^* - \alpha \Delta q^*\big) \times (v+w)\big).$$

Then, we take the inner product in H of Eq. (5.12) with $\operatorname{rot}(J_N q^* - \alpha \Delta J_N q^*)$ and we perform the same computation as above. We obtain, for $t \geqslant s$,

$$\begin{aligned}\big\|\operatorname{rot}\big(J_N q^*(t) - \alpha \Delta J_N q^*(t)\big)\big\|_{L^2}^2 &\leqslant e^{-\gamma(t-s)}\big\|\operatorname{rot}\big(J_N q^*(s) - \alpha \Delta J_N q^*(s)\big)\big\|_{L^2}^2 \\ &\quad + \int_s^t e^{-\gamma(t-\tau)}\big(h_N(\tau), \operatorname{rot}\big(J_N q^* - \alpha \Delta J_N q^*\big)(\tau)\big)\, d\tau. \end{aligned} \tag{5.13}$$

We note that

$$\int_{\mathbb{T}^2} h_N \cdot \operatorname{rot}\big(J_N q^* - \alpha \Delta J_N q^*\big) = -\int_{\mathbb{T}^2} \big(J_N(b \times a) - b \times J_N a\big) \cdot \operatorname{rot} J_N a,$$

where $b = v + w$ and $a = \operatorname{rot}(q^* - \alpha \Delta q^*) \in L^2(\mathbb{T}^2)$. Arguing as in [7] for example, one proves the following commutator estimate

$$\big\|J_N(b \times a) - b \times J_N a\big\|_{L^2} \leqslant C 2^{-N}\|a\|_{L^2}\|\nabla b\|_{L^\infty},$$

which implies that,

$$\big|\big(h_N, \operatorname{rot}\big(J_N q^* - \alpha \Delta J_N q^*\big)\big)\big| \leqslant C 2^{-N}\|\operatorname{rot} J_N a\|_{L^2}\|\nabla b\|_{L^\infty}\|a\|_{L^2}.$$

Using Parseval theorem together with Lebesgue convergence theorem, we easily show that, for every $a \in L^2(\mathbb{T}^2)$, $2^{-N}\|\nabla J_N a\|_{L^2}$ converges to zero when N goes to infinity. Using this property and taking the limit as $N \to \infty$, we obtain the justification of the above computations. □

Remark 5.1. If one does not wish to introduce the above operator J_N and to use commutator inequalities, one can also prove the uniqueness (under some restrictions on the size of w and on α) as follows. These conditions are not really restrictive, since they will be satisfied in the next steps.

Let $\lambda_n > 0$, $n \geqslant 1$, be the sequence of eigenvalues of the Stokes operator A. Let $R_0 > 0$, $\rho_0 > 0$ be given positive constants. Let R_n, $n \geqslant 1$, be a sequence of positive numbers such that $R_n \lambda_{n+1}^{\delta - 1/2}$, $\delta > 0$, converges to zero when n goes to infinity. Assume that $v(t)$ and $w(t)$ belong to $C_b^0(\mathbb{R}, B_{P_n V^3}(0, R_0))$ and $L^\infty(\mathbb{R}, B_{Q_n V^3}(0, R_n)) \cap C_b^0(\mathbb{R}, V^2)$ respectively and that $\|\nabla v\|_{L^\infty(\mathbb{R}, L^\infty(\mathbb{T}^2)^2)} \leqslant \rho_0$, with $2\nu - 4\alpha\rho_0 > 0$.

Taking the inner product in H of Eq. (5.10) with q^*, we get, for any t,

$$\begin{aligned}&\partial_t\big(\big\|q^*(t)\big\|_{L^2}^2 + 2\alpha\big\|\nabla q^*(t)\big\|_{L^2}^2\big) + \big\|\nabla q^*(t)\big\|_{L^2} \\ &\quad \leqslant 2\big|\big(\operatorname{rot}\big(q^*(t) - \alpha \Delta q^*(t)\big) \times \big(v(t) + w(t)\big), q^*(t)\big)\big|. \end{aligned} \tag{5.14}$$

Using the equalities (A.4) and (A.6) of Appendix A and classical Sobolev imbedding theorems, we obtain the following estimate,

$$\begin{aligned}\left|\left(\operatorname{rot}\left(q^*-\alpha\Delta q^*\right)\times(v+w),q^*\right)\right| &\leqslant \left\|q^*\right\|_{L^2}^2\left\|\nabla(v+w)\right\|_{L^\infty}+\alpha\left\|q^*\right\|_{L^4}\left\|\nabla q^*\right\|_{L^2}(\|\Delta v+\Delta w\|_{L^4})\\ &\quad+2\alpha\left\|\nabla q^*\right\|_{L^2}^2\left(\|\nabla v\|_{L^\infty}+\|\nabla w\|_{L^\infty}\right)\\ &\leqslant C_S\lambda_{n+1}^{-1}\left\|\nabla q^*\right\|_{L^2}^2(R_0+R_n)\\ &\quad+\alpha C_S\lambda_{n+1}^{-1/4}\left\|\nabla q^*\right\|_{L^2}^2\left(R_0+\lambda_{n+1}^{-1/4}R_n\right)\\ &\quad+2\alpha C_{S,\delta}\lambda_{n+1}^{\delta-1/2}\left\|\nabla q^*\right\|_{L^2}^2R_n+2\alpha\left\|\nabla q^*\right\|_{L^2}^2\rho_0, \end{aligned}\tag{5.15}$$

where C_S, $C_{S,\delta}$ are positive constants coming from the Sobolev estimates. Since $\beta_0 \equiv \nu - 2\alpha\rho_0 > 0$, that λ_n goes to infinity and that $R_n\lambda_n^{\delta-1/2}$ goes to zero, when n tends to infinity, one deduces from the inequalities (5.14) and (5.15) that there exists n_0 large enough so that, for $n \geqslant n_0$,

$$\left\|q^*(t)\right\|^2+\alpha\left\|\nabla q^*(t)\right\|^2 \leqslant e^{-\beta_0(t-s)}\left(\left\|q^*(s)\right\|^2+\alpha\left\|\operatorname{rot} q^*(s)\right\|^2\right), \quad \forall t \geqslant s.$$

Since $q^*(s)$ belongs to $L^\infty(\mathbb{R}, V^3)$, this inequality implies that $q^*(t) = 0$, for any $t \in \mathbb{R}$.

5.1.2. Step 2 of the proof of Theorem 1.2

For any integer n, we introduce the space

$$\begin{aligned}\mathcal{W}_n &\equiv W_n(R_n) \equiv \mathcal{W}\\ &= \left\{w \in L^\infty\left(\mathbb{R}, Q_nV^3\right)\cap C^0\left(\mathbb{R}, V^2\right) \mid \|\Delta w\|_{L^\infty(\mathbb{R},H)}^2+\alpha\|\nabla\Delta w\|_{L^\infty(\mathbb{R},H)}^2 \leqslant R_n^2\right\}, \end{aligned}\tag{5.16}$$

where $R_n > 0$ is a large enough positive number, which may depend on n. We endow $\mathcal{W}$ with the topology induced by the norm

$$\left(\|w\|_{C_b^0(\mathbb{R},H)}^2+\alpha\|\operatorname{rot} w\|_{C_b^0(\mathbb{R},H)}^2\right)^{1/2},$$

which makes $\mathcal{W}$ a complete metric space.

In this step, we need to assume that $2\nu - 4\alpha\sup_{z\in\mathcal{A}_\alpha}\|\nabla z\|_{L^\infty} > 0$. Under this assumption, we know by Section 3, that, if f belongs to V^{1+d}, $0 < d \leqslant 1$, then the attractor is bounded in V^{3+d}. Here we prove that, for a fixed element $v(t)$ in $C^0(\mathbb{R}, \mathcal{N}_{P_nV^{3+d}}(P_n\mathcal{A}_\alpha, r_0))$, where $r_0 > 0$ is small enough, there exists n_0 large enough so that, for $n \geqslant n_0$, for $R_n > 0$ large enough, the map $w \in \mathcal{W} \mapsto q_{v,w} \equiv q^n_{v,w} \in \mathcal{W}$ is a strict contraction, provided α is small enough. We recall that $\mathcal{N}_{P_nV^{3+d}}(P_n\mathcal{A}_\alpha, r_0)$ is the r_0-neighborhood of $P_n\mathcal{A}_\alpha$ in P_nV^{3+d}, equipped with the norm $(\|\cdot\|^2_{V^{2+d}} + \alpha\|\cdot\|^2_{V^{3+d}})^{\frac{1}{2}}$

Lemma 5.2. *Assume that $f \in H^{1+d}_{per}$, $0 < d \leqslant 1$ and that $2\nu - 4\alpha\sup_{z\in\mathcal{A}_\alpha}\|\nabla z\|_{L^\infty} > 0$.*

There exist a real number $r_0 > 0$ small enough, an integer n_0 large enough and, for $n \geqslant n_0$, a large enough number R_n, such that, for $n \geqslant n_0$, for v in $C^0(\mathbb{R}, \mathcal{N}_{P_nV^{3+d}}(P_n\mathcal{A}_\alpha, r_0))$, the map $w \in \mathcal{W}_n \mapsto q^n_{v,w}$, where $q^n_{v,w}$ is the unique solution of (5.3)*, is a strict contraction on $\mathcal{W}_n$ and thus admits a unique fixed point $q^n(v)$ in $\mathcal{W}_n \cap C^0(\mathbb{R}, V^3)$. Moreover, $q^n(v)$ depends on the values $v(s)$, for $s \leqslant t$ only.*

Proof. a) Remark that the condition $a_1^* = 2\nu - 4\alpha\sup_{z\in\mathcal{A}_\alpha}\|\nabla z\|_{L^\infty} > 0$ implies that we can assume without loss of generality that $\alpha \leqslant \alpha_0$, for some adequate α_0.

Let $\delta_0 < 1$ and $\delta_1 < 1$ be two small fixed positive constants, which will be made more precise later. We assume that δ_0 is so small that

$$8\alpha_0^{1/2}\delta_0 < a_1^*. \tag{5.17}$$

If f belongs to H^{1+d}_{per}, $0 < d \leqslant 1$, then, by the results of Sections 2 and 3, there exists a positive number ρ_0 such that, for any $u \in \mathcal{A}_\alpha$, and thus, for any integer n

$$\|u\|^2_{V^{2+d}} + \alpha \|u\|^2_{V^{3+d}} \leqslant \rho_0^2,$$
$$\|P_n u\|^2_{V^{2+d}} + \alpha \|P_n u\|^2_{V^{3+d}} \leqslant \rho_0^2. \tag{5.18}$$

Since $\mathcal{A}_\alpha$ is compact in V^3, there exists an integer n_0 such that, for any integer $n \geqslant n_0$, for any element $u \in \mathcal{A}_\alpha$,

$$\alpha^{1/2} \|Q_n \nabla u\|_{L^\infty} \leqslant \frac{\delta_0}{4} \tag{5.19}$$

and, therefore,

$$2\nu - 4\alpha \|P_n \nabla u\|_{L^\infty} \geqslant a_1^* - \alpha^{1/2}\delta_0. \tag{5.20}$$

We will also choose n_0 large enough so that, for $n \geqslant n_0$,

$$\lambda_n^{-1/4} \leqslant \delta_1 < 1, \tag{5.21}$$

and

$$\|Q_n P f\|^2_{L^2} \leqslant \frac{\delta_0^2}{2}. \tag{5.22}$$

We choose $r_0 > 0$ small enough so that, for any $n \geqslant n_0$, for $v \in C^0(\mathbb{R}, \mathcal{N}_{P_n V^{3+d}}(P_n \mathcal{A}_\alpha, r_0))$,

$$2\nu - 4\alpha \|\nabla v\|_{L^\infty} \geqslant a_1^* - 2\alpha^{1/2}\delta_0 \geqslant \frac{3}{4} a_1^*. \tag{5.23}$$

b) We set

$$R_n^2 = \left(\lambda_1^{-1} + \alpha_0\right)\left(\frac{8}{\nu^2}\|\mathrm{rot}\, f\|^2_{L^2} + \frac{16 \sup(C_S^2, C_{S,d}^2)}{\nu^2}(\rho_0 + r_0)^4\left(1 + \alpha_0 \lambda_n^{1-d}\right)\right) + 2\rho_0^2, \tag{5.24}$$

where C_S is the positive constant in the Sobolev estimates below and where $C_{S,d}$ is the positive constant in the Sobolev estimate,

$$\|U\|_{L^\infty} \leqslant C_{S,d} \|U\|_{H^{1+d}}, \tag{5.25}$$

valid for any $U \in H^{1+d}$.

Let v be fixed in $C^0(\mathbb{R}, \mathcal{N}_{P_n V^{3+d}}(P_n \mathcal{A}_\alpha, r_0))$. Due to Lemma 5.1 and to the choice of R_n, there exists an integer n_1, such that, for $n \geqslant n_1$, for $w \in \mathcal{W}$, Eq. (5.3) has a unique solution $q^n_{v,w}$. We first prove that there exists an integer $n_0 \geqslant n_1$ such that, for $n \geqslant n_0$, the solution $q^n_{v,w}$ belongs to $\mathcal{W}$. To simplify the notation, we set $q \equiv q^n_{v,w}$. The following a priori estimates on q can be rigorously justified by a classical Galerkin method (we let the details to the reader). For this reason, we can assume without loss of generality that q is regular enough. Then, applying the curl operator to Eq. (5.3), taking the inner product in H of the resulting equation with $\mathrm{rot}\,\Delta q$, and using the equality (A.5), we obtain the following equality

$$\begin{aligned}
&\frac{1}{2}\partial_t\big(\|\nabla \operatorname{rot} q\|^2+\alpha\|\Delta \operatorname{rot} q\|^2\big)+\nu\|\Delta \operatorname{rot} q\|^2 \\
&\quad=(\operatorname{rot} f,\Delta \operatorname{rot} q)+\big(\Delta q\times(v+w),\Delta \operatorname{rot} q\big) \\
&\qquad-\big(\operatorname{rot}\big(\operatorname{rot}(v-\alpha\Delta v)\times(v+w)\big),\Delta \operatorname{rot} q\big).
\end{aligned} \tag{5.26}$$

Using the equality (A.3) of Lemma A.1 and applying the classical Sobolev estimates, we obtain, for $t\in\mathbb{R}$,

$$\begin{aligned}
\big|\big(\Delta q\times(v+w),\Delta \operatorname{rot} q\big)\big| &\leqslant \|\Delta q\|_{L^4}^2\big\|\nabla(v+w)\big\|_{L^2} \\
&\leqslant C_S^2\lambda_{n+1}^{-1/2}\big(\rho_0+r_0+\lambda_{n+1}^{-1/2}R_n\big)\|\Delta \operatorname{rot} q\|_{L^2}^2,
\end{aligned} \tag{5.27}$$

where C_S is a constant coming from the Sobolev inequalities.

Next we estimate the term $|(\operatorname{rot}(\operatorname{rot}(v-\alpha\Delta v)\times(v+w)),\Delta \operatorname{rot} q)|$. Using the equality (A.2), we can write, for $t\in\mathbb{R}$,

$$\begin{aligned}
\big|\big(\operatorname{rot}\big(\operatorname{rot}(v-\alpha\Delta v)\big)\times w,\Delta \operatorname{rot} q\big)\big| &\leqslant \big(\|v\|_{V^2}\|w\|_{L^\infty}+\alpha\big\|\Delta^2 v\big\|_{L^2}\|w\|_{L^\infty}\big)\|\operatorname{rot}\Delta q\|_{L^2} \\
&\leqslant C_{S,\delta}\big(\|v\|_{V^2}\lambda_{n+1}^{-1/2+\delta}+\alpha^{1/2}\|v\|_{V^3}\lambda_n^{1/2}\lambda_{n+1}^{-1+\delta}\big)R_n\|\operatorname{rot}\Delta q\|_{L^2} \\
&\leqslant C_{S,\delta}(\rho_0+r_0)R_n\lambda_{n+1}^{-1/2+\delta}\|\operatorname{rot}\Delta q\|_{L^2} \\
&\leqslant \frac{\nu}{8}\|\operatorname{rot}\Delta q\|_{L^2}^2+\frac{2C_{S,\delta}^2}{\nu}(\rho_0+r_0)^2R_n^2\lambda_{n+1}^{-1+2\delta},
\end{aligned} \tag{5.28}$$

where $C_{S,\delta}>0$ is the constant in the Sobolev estimate $\|w\|_{L^\infty}\leqslant C_{S,\delta}\|w\|_{H^{1+\delta}}$. Arguing in the same way, we obtain, for any $t\in\mathbb{R}$,

$$\begin{aligned}
\big|\big(\operatorname{rot}\big(\operatorname{rot}(v-\alpha\Delta v)\times v\big),\Delta \operatorname{rot} q\big)\big| &\leqslant \big(\|v\|_{V^2}+\alpha\big\|\Delta^2 v\big\|_{L^2}\big)\|v\|_{L^\infty}\|\operatorname{rot}\Delta q\|_{L^2} \\
&\leqslant C_{S,d}\big(\|v\|_{V^2}+\alpha\|v\|_{V^{3+d}}\lambda_n^{\frac{1-d}{2}}\big)\|v\|_{V^2}\|\operatorname{rot}\Delta q\|_{L^2} \\
&\leqslant C_{S,d}(\rho_0+r_0)^2\big(1+\alpha^{1/2}\lambda_n^{\frac{1-d}{2}}\big)\|\operatorname{rot}\Delta q\|_{L^2} \\
&\leqslant \frac{\nu}{8}\|\Delta \operatorname{rot} q\|_{L^2}^2+\frac{4C_{S,d}^2}{\nu}(\rho_0+r_0)^4\big(1+\alpha\lambda_n^{1-d}\big).
\end{aligned} \tag{5.29}$$

Finally, we have, for $t\in\mathbb{R}$,

$$\big|(\operatorname{rot} f,\Delta \operatorname{rot} q)\big|\leqslant\frac{\nu}{8}\|\Delta \operatorname{rot} q\|_{L^2}^2+\frac{2}{\nu}\|\operatorname{rot} f\|_{L^2}^2. \tag{5.30}$$

From the equality (5.26) and the estimates (5.27) to (5.30), we deduce that, for $t\in\mathbb{R}$,

$$\begin{aligned}
&\frac{1}{2}\partial_t\big(\|\nabla \operatorname{rot} q\|_{L^2}^2+\alpha\|\Delta \operatorname{rot} q\|_{L^2}^2\big)+\left(\frac{5}{8}\nu-C_S^2\lambda_{n+1}^{-1/2}\big(\rho_0+r_0+\lambda_{n+1}^{-1/2}R_n\big)\right)\|\Delta \operatorname{rot} q\|_{L^2}^2 \\
&\quad\leqslant\frac{2}{\nu}\|\operatorname{rot} f\|_{L^2}^2+\frac{4C_{S,d}^2}{\nu}(\rho_0+r_0)^4\big(1+\alpha\lambda_n^{1-d}\big)+\frac{2C_{S,\delta}^2}{\nu}\lambda_{n+1}^{-1+2\delta}R_n^2(\rho_0+r_0)^2.
\end{aligned} \tag{5.31}$$

We first remark that, due to the choice of R_n, there exists a positive constant R_0 such that, for any integer n,

$$\lambda_{n+1}^{d/2-1/2}R_n\leqslant R_0. \tag{5.32}$$

We now set $\delta = 1/4$ for example. We next choose the integer n_0 large enough (and δ_1 small enough) so that, for $n \geqslant n_0$,

$$\lambda_{n+1}^{-1/2} C_S^2(\rho_0 + r_0 + R_0) \leqslant \delta_1^2 C_S^2(\rho_0 + r_0 + R_0) \leqslant \frac{\nu}{8},$$
$$\frac{4C_{S,\delta}^2(\lambda_1^{-1} + \alpha)}{\nu^2} \lambda_{n+1}^{-1/2}(\rho_0 + r_0)^2 \leqslant \frac{4C_{S,\delta}^2(\lambda_1^{-1} + \alpha)}{\nu^2} \delta_1^2(\rho_0 + r_0)^2 \leqslant \frac{1}{2}. \tag{5.33}$$

Taking into account the conditions (5.33), the definition (5.24) of R_n and the inequality (2.4), we deduce from (5.31) that, for $t \in \mathbb{R}$,

$$\partial_t\big(\|\nabla \operatorname{rot} q\|_{L^2}^2 + \alpha\|\Delta \operatorname{rot} q\|_{L^2}^2\big) + \frac{\nu}{\lambda_1^{-1} + \alpha}\big(\|\nabla \operatorname{rot} q\|_{L^2}^2 + \alpha\|\Delta \operatorname{rot} q\|_{L^2}^2\big) \leqslant \frac{\nu}{\lambda_1^{-1} + \alpha} R_n^2. \tag{5.34}$$

Integrating the inequality (5.34) from $-\infty$ to t, we obtain finally that, for $t \in \mathbb{R}$,

$$\big\|\nabla \operatorname{rot} q(t)\big\|_{L^2}^2 + \alpha\big\|\Delta \operatorname{rot} q(t)\big\|_{L^2}^2 \leqslant R_n^2, \tag{5.35}$$

which proves that the mapping $w \mapsto q_{v,w}$ maps $\mathcal{W}_n$ into itself.

b) Next we prove that the map $w \in \mathcal{W}_n \mapsto q_{v,w} \in \mathcal{W}_n$ is a strict contraction. Let $w_1 \in \mathcal{W}_n$, $w_2 \in \mathcal{W}_n$ and let $q_1 = q_{v,w_1}$, $q_2 = q_{v,w_2}$ be the corresponding solutions of Eq. (5.3). The difference $q^* = q_1 - q_2$ satisfies the following equation, for any $t \in \mathbb{R}$,

$$\begin{aligned}
&\partial_t\big(q^* - \alpha\Delta q^*\big) - \nu\Delta q^* + Q_n P\big(\operatorname{rot}\big(q^* - \alpha\Delta q^*\big) \times (v + w_1)\big) \\
&\quad = -Q_n P\big(\operatorname{rot}(v - \alpha\Delta v) \times (w_1 - w_2)\big) - Q_n P\big(\operatorname{rot}(q_2 - \alpha\Delta q_2) \times (w_1 - w_2)\big).
\end{aligned} \tag{5.36}$$

Taking the inner product in H of the equality (5.36) with q^*, we obtain the following inequality, for any $t \in \mathbb{R}$,

$$\begin{aligned}
\frac{1}{2}\partial_t\big(\big\|q^*\big\|_{L^2}^2 + \alpha\big\|\nabla q^*\big\|_{L^2}^2\big) + \nu\big\|\nabla q^*\big\|_{L^2}^2 &\leqslant \big|\big(\operatorname{rot}\big(q^* - \alpha\Delta q^*\big) \times (v + w_1), q^*\big)\big| \\
&\quad + \big|\big(\operatorname{rot}(v - \alpha\Delta v) \times (w_1 - w_2), q^*\big)\big| \\
&\quad + \big|\big(\operatorname{rot}(q_2 - \alpha\Delta q_2) \times (w_1 - w_2), q^*\big)\big|.
\end{aligned} \tag{5.37}$$

In order to bound the three terms in the right-hand side of the estimate (5.37), we proceed as follows. Using the classical Sobolev inequalities, we easily show that, for $t \in \mathbb{R}$,

$$\begin{aligned}
&\big|\big(\operatorname{rot}(v - \alpha\Delta v) \times (w_1 - w_2), q^*\big)\big| \\
&\quad \leqslant \big(\|\operatorname{rot} v\|_{L^4}\|w_1 - w_2\|_{L^2} + \alpha\|\operatorname{rot}\Delta v\|_{L^2}\|w_1 - w_2\|_{L^4}\big)\big\|q^*\big\|_{L^4} \\
&\quad \leqslant \lambda_{n+1}^{-1/4}\big\|\nabla q^*\big\|_{L^2} C_S^2\big(\|w_1 - w_2\|_{L^2}\|v\|_{V^2} + \alpha\lambda_{n+1}^{-1/4}\big\|\nabla(w_1 - w_2)\big\|_{L^2}\|\operatorname{rot}\Delta v\|_{L^2}\big) \\
&\quad \leqslant \frac{\delta_1}{2}\big\|\nabla q^*\big\|_{L^2}^2 + \delta_1 C_S^4\big(\|w_1 - w_2\|_{L^2}^2 + \delta_1^2\alpha\big\|\nabla(w_1 - w_2)\big\|_{L^2}^2\big)(r_0 + \rho_0)^2.
\end{aligned} \tag{5.38}$$

Likewise, using classical Sobolev estimates and the inequality (5.32), we have, for $t \in \mathbb{R}$,

$$\begin{aligned}
&\big|\big(\mathrm{rot}(q_2-\alpha\Delta q_2)\times(w_1-w_2),q^*\big)\big|\\
&\quad\leqslant\big(\|\mathrm{rot}\,q_2\|_{L^4}\|w_1-w_2\|_{L^2}+\alpha\|\mathrm{rot}\,\Delta q_2\|_{L^2}\|w_1-w_2\|_{L^4}\big)\big\|q^*\big\|_{L^4}\\
&\quad\leqslant C_S^2\lambda_{n+1}^{-1/2}\big\|\nabla q^*\big\|_{L^2}\big(\|\Delta q_2\|_{L^2}\|w_1-w_2\|_{L^2}+\alpha\|\mathrm{rot}\,\Delta q_2\|_{L^2}\big\|\nabla(w_1-w_2)\big\|_{L^2}\big)\\
&\quad\leqslant C_S^2\lambda_{n+1}^{-1/2}\big\|\nabla q^*\big\|_{L^2}R_n\big(\|w_1-w_2\|_{L^2}+\alpha^{1/2}\big\|\nabla(w_1-w_2)\big\|_{L^2}\big)\\
&\quad\leqslant\lambda_{n+1}^{-d/2}\big\|\nabla q^*\big\|_{L^2}^2+\lambda_{n+1}^{-1+d/2}R_n^2C_S^4\big(\|w_1-w_2\|_{L^2}^2+\alpha\big\|\nabla(w_1-w_2)\big\|_{L^2}^2\big)\\
&\quad\leqslant\lambda_{n+1}^{-d/2}\big\|\nabla q^*\big\|_{L^2}^2+\lambda_{n+1}^{-d/2}R_0^2C_S^4\big(\|w_1-w_2\|_{L^2}^2+\alpha\big\|\nabla(w_1-w_2)\big\|_{L^2}^2\big).
\end{aligned}\tag{5.39}$$

It remains to bound the term $|(\mathrm{rot}(q^*-\alpha\Delta q^*)\times(v+w_1),q^*)|$. Using the equality (A.4) of Appendix A and the classical Sobolev embeddings, we can write

$$\begin{aligned}
\big|\big(\mathrm{rot}\,q^*\times(v+w_1),q^*\big)\big|&\leqslant\big\|q^*\big\|_{L^2}\big\|q^*\big\|_{L^4}\big\|\nabla(v+w_1)\big\|_{L^4}\\
&\leqslant\big\|\nabla q^*\big\|_{L^2}^2\lambda_{n+1}^{-3/4}C_S^2\big((r_0+\rho_0)+\lambda_{n+1}^{-1/4}R_n\big).
\end{aligned}\tag{5.40}$$

Next, using the equality (A.6) of Appendix A and the classical Sobolev embeddings, we estimate the term $\alpha|(\mathrm{rot}\,\Delta q^*\times(v+w_1),q^*)|$ as follows,

$$\begin{aligned}
&\alpha\big|\big(\mathrm{rot}\,\Delta q^*\times(v+w_1),q^*\big)\big|\\
&\quad\leqslant\alpha\big\|\nabla q^*\big\|_{L^2}\big\|q^*\big\|_{L^4}\big(\|\Delta v\|_{L^4}+\|\Delta w_1\|_{L^4}\big)+2\alpha\big\|\nabla q^*\big\|_{L^2}^2\big(\|\nabla v\|_{L^\infty}+\|\nabla w_1\|_{L^\infty}\big)\\
&\quad\leqslant\big\|\nabla q^*\big\|_{L^2}^2\big[\alpha^{1/2}\lambda_{n+1}^{-1/4}C_S^2\big(r_0+\rho_0+\lambda_{n+1}^{-1/4}R_n\big)+2\alpha\|\nabla v\|_{L^\infty}+2\alpha^{1/2}\lambda_{n+1}^{-1/2+\delta}R_nC_{S,\delta}\big].
\end{aligned}\tag{5.41}$$

We choose $\delta=d/4$. From the estimates (5.37) to (5.41) and the inequality (2.4), we deduce that, for $t\in\mathbb{R}$,

$$\begin{aligned}
&\partial_t\big(\big\|q^*\big\|^2+\alpha\big\|\nabla q^*\big\|^2\big)+\big(\lambda_1^{-1}+\alpha\big)^{-1}\big(2\nu-4\alpha\|\nabla v\|_{L^\infty}-L\big)\big(\big\|q^*\big\|^2+\alpha\big\|\nabla q^*\big\|^2\big)\\
&\quad\leqslant 2C_S^4\big(\lambda_{n+1}^{-1/4}(r_0+\rho_0)^2+\lambda_{n+1}^{-d/2}R_0^2\big)\big(\|w_1-w_2\|_{L^2}^2+\alpha\big\|\nabla(w_1-w_2)\big\|_{L^2}^2\big),
\end{aligned}\tag{5.42}$$

where

$$\begin{aligned}
L=\lambda_{n+1}^{-1/4}+2\lambda_{n+1}^{-d/2}+2\lambda_{n+1}^{-3/4}C_S^2\big((r_0+\rho_0)+\lambda_{n+1}^{-1/4}R_n\big)&\\
+2\big[\alpha^{1/2}C_S^2\big(\lambda_{n+1}^{-1/4}(r_0+\rho_0)+\lambda_{n+1}^{-1/2}R_n\big)+2\alpha^{1/2}\lambda_{n+1}^{-1/2+d/4}R_nC_{S,\delta}\big].&
\end{aligned}\tag{5.43}$$

Due to the condition (5.23), we have

$$2\nu-4\alpha\|\nabla v\|_{L^\infty}-L\geqslant a_1^*-2\alpha^{1/2}\delta_0-L.\tag{5.44}$$

We recall that $\lambda_{n+1}^{d/2-1/2}R_n$ is bounded from above by a positive constant R_0. We now choose the integer n_0 large enough so that the following two inequalities hold, for $n\geqslant n_0$,

$$\begin{aligned}
&\lambda_{n+1}^{-1/4}+2\lambda_{n+1}^{-d/2}+2\lambda_{n+1}^{-1/2}C_S^2\big(\lambda_{n+1}^{-1/4}(r_0+\rho_0)+\lambda_{n+1}^{-d/2}R_0\big)\\
&\quad+2\big[\alpha^{1/2}C_S^2\big(\lambda_{n+1}^{-1/4}(r_0+\rho_0)+\lambda_{n+1}^{-d/2}R_0\big)+2\alpha^{1/2}\lambda_{n+1}^{-d/4}R_0C_{S,\delta}\big]\leqslant\frac{a_1^*}{4},
\end{aligned}\tag{5.45}$$

and

$$2C_S^4\big(\lambda_{n+1}^{-1/4}(r_0+\rho_0)^2+\lambda_{n+1}^{-d/2}R_0^2\big)\leqslant\frac{a_1^*}{4(\lambda_1^{-1}+\alpha)}. \tag{5.46}$$

The estimate (5.42), together with the property (5.44), the conditions (5.17), (5.45) and (5.46), implies that, for $t\in\mathbb{R}$,

$$\begin{aligned}&\partial_t\big(\|q^*(t)\|^2+\alpha\|\nabla q^*(t)\|^2\big)+\frac{a_1^*}{2(\lambda_1^{-1}+\alpha)}\big(\|q^*(t)\|^2+\alpha\|\nabla q^*(t)\|^2\big)\\&\quad\leqslant\frac{a_1^*}{4(\lambda_1^{-1}+\alpha)}\big(\|(w_1-w_2)(t)\|_{L^2}^2+\alpha\|\big(\nabla(w_1-w_2)\big)(t)\|_{L^2}^2\big).\end{aligned} \tag{5.47}$$

Integrating this inequality from $-\infty$ to t, we finally obtain that, for $t\in\mathbb{R}$,

$$\|q^*(t)\|^2+\alpha\|\nabla q^*(t)\|^2\leqslant\frac{1}{2}\sup_{s\leqslant t}\big(\|(w_1-w_2)(s)\|_{L^2}^2+\alpha\|\nabla(w_1-w_2)(s)\|_{L^2}^2\big),$$

which means that the map $w\in\mathcal{W}_n\mapsto q^n_{v,w}\in\mathcal{W}_n$ is a strict contraction and thus admits a unique fixed point $q^n(v)\in\mathcal{W}_n$. Lemma 5.2 is proved. □

Remark 5.2. In general, if $0<d<1$, the fixed point $q(v)$ could be not uniformly bounded in V^3 with respect to n. If Pf belongs to V^2, then d is equal to 1 and $R_n=R$ is independent of n, that is, $q^n(v)$ is bounded in V^3, uniformly in n, for $n\geqslant n_0$.

Remark 5.3. In order to simplify the proofs, we have assumed in Lemma 5.2 that the forcing term f belongs to H^{1+d}_{per} with $d>0$. Looking carefully at the proof of this lemma and replacing the Sobolev inequality (5.25) by the Brézis–Gallouët inequality, one easily shows that, if f belongs only to H^1_{per}, the properties of Lemma 5.2 still hold provided that $0<\alpha<\alpha_n$, where α_n satisfies the following condition

$$\lambda_{n+1}^{-1/2}R_n\big(1+\alpha_n^{1/2}\ln(1+\lambda_{n+1})\big)\leqslant\delta_1, \tag{5.48}$$

where, in the definition of R_n, the constant $C_{S,d}$ has been replaced by the constant of the Brézis–Gallouët inequality.

5.1.3. Step 3 of the proof of Theorem 1.2

In this step we prove that the map $v\mapsto q^n(v)$ is a Lipschitz-continuous map.

Lemma 5.3. *We assume that the hypotheses of Lemma* 5.2 *hold. One can choose the integer* n_0 *large enough so that, for* $n\geqslant n_0$*, the mapping* $v\in C^0(\mathbb{R},\mathcal{N}_{P_nV^{3+d}}(P_n\mathcal{A}_\alpha,r_0))\mapsto q^n(v)\in\mathcal{W}_n$ *is Lipschitz-continuous. More precisely, for* v_1 *and* v_2 *in* $C^0(\mathbb{R},\mathcal{N}_{P_nV^{3+d}}(P_n\mathcal{A}_\alpha,r_0))$*, we have the estimates, for any* $t\in\mathbb{R}$*,*

$$\begin{aligned}&\sup_{s\leqslant t}\big(\|\big(q^n(v_1)-q^n(v_2)\big)(s)\|_{L^2}^2+\alpha\|\nabla\big(q^n(v_1)-q^n(v_2)\big)(s)\|_{L^2}^2\big)\\&\quad\leqslant\frac{C_L}{a_1^{*2}}\Big[\sup_{s\leqslant t}\big(\|(v_1-v_2)(s)\|_{L^2}^2+\alpha\|\nabla(v_1-v_2)(s)\|_{L^2}^2\big)+\alpha^2\sup_{s\leqslant t}\|\Delta(v_1-v_2)(s)\|_{L^2}^2\Big],\end{aligned} \tag{5.49}$$

and

$$\sup_{s\leqslant t}\big(\big\|\nabla\big(q^n(v_1)-q^n(v_2)\big)(s)\big\|_{L^2}^2+\alpha\big\|\Delta\big(q^n(v_1)-q^n(v_2)\big)(s)\big\|_{L^2}^2\big)$$
$$\leqslant\frac{C_L}{a_1^{*2}}\big(1+\alpha R_n^2\big)\Big[\sup_{s\leqslant t}\big(\big\|(v_1-v_2)(s)\big\|_{V^2}^2+\alpha\big\|(v_1-v_2)(s)\big\|_{V^3}^2\big)\Big], \tag{5.50}$$

where $C_L\equiv C_L(r_0,\rho_0)$ is a positive constant depending on r_0 and ρ_0 and where ρ_0 has been defined in (5.18).

Proof. 1) We begin by proving the inequality (5.49). To simplify the notations, we set

$$q_i(s)=q^n\big(v_i(s)\big),\quad i=1,2,\qquad v^*(s)=v_1(s)-v_2(s),\qquad q^*(s)=q_1(s)-q_2(s).$$

The difference q^* satisfies the equation

$$\partial_t\big(q^*-\alpha\Delta q^*\big)-\nu\Delta q^*+Q_nP\big(\mathrm{rot}\big(q^*-\alpha\Delta q^*\big)\times(q_1+v_1)\big)+Q_nP\big(\mathrm{rot}(q_2-\alpha\Delta q_2)\times\big(q^*+v^*\big)\big)$$
$$=-Q_nP\big(\mathrm{rot}\big(v^*-\alpha\Delta v^*\big)\times(q_1+v_1)\big)-Q_nP\big(\mathrm{rot}(v_2-\alpha\Delta v_2)\times\big(q^*+v^*\big)\big)-\nabla p^*. \tag{5.51}$$

Taking the inner product in H of Eq. (5.51) with q^*, we obtain the following inequality, for any $t\in\mathbb{R}$,

$$\partial_t\big(\|q^*\|^2+\alpha\|\nabla q^*\|^2\big)+2\nu\|\nabla q^*\|^2$$
$$\leqslant 2\Big[\big|\big(\mathrm{rot}\,q^*\times(q_1+v_1),q^*\big)\big|+\alpha\big|\big(\mathrm{rot}\,\Delta q^*\times(q_1+v_1),q^*\big)\big|$$
$$+\big|\big((\mathrm{rot}\,q_2)\times v^*,q^*\big)\big|+\alpha\big|\big(\mathrm{rot}\,\Delta q_2\times v^*,q^*\big)\big|$$
$$+\big|\big(\mathrm{rot}\big(v^*-\alpha\Delta v^*\big)\times(v_1+q_1),q^*\big)\big|+\big|\big(\mathrm{rot}(v_2-\alpha\Delta v_2)\times v^*,q^*\big)\big|\Big]. \tag{5.52}$$

Now we proceed like in the proof of Lemma 5.2 to estimate the various terms in the right-hand side of the inequality (5.52). Due to the condition (5.21) and the property (5.32), the first term $|(\mathrm{rot}\,q^*\times(q_1+v_1),\Delta q^*)|$ is estimated as follows, for any $t\in\mathbb{R}$,

$$\big|\big(\mathrm{rot}\,q^*\times(q_1+v_1),q^*\big)(t)\big|\leqslant\big\|q^*(t)\big\|_{L^4}\big\|\mathrm{rot}\,q^*(t)\big\|_{L^2}\big(\big\|v_1(t)\big\|_{L^4}+\big\|q_1(t)\big\|_{L^4}\big)$$
$$\leqslant C_S^2\big\|\nabla q^*(t)\big\|_{L^2}^2\lambda_{n+1}^{-1/4}\big(r_0+\rho_0+\lambda_{n+1}^{-3/4}R_n\big)$$
$$\leqslant C_S^2\big\|\nabla q^*(t)\big\|_{L^2}^2\delta_1\big(r_0+\rho_0+\lambda_{n+1}^{-1/4}R_0\big). \tag{5.53}$$

Using the equality (A.6) of Lemma A.1 and the classical Sobolev embeddings, and taking into account the property (5.32), we can write, for any $t\in\mathbb{R}$,

$$\alpha\big|\big(\mathrm{rot}\,\Delta q^*\times(q_1+v_1),q^*\big)(t)\big|$$
$$\leqslant\alpha\big\|\nabla q^*(t)\big\|_{L^2}\big(\big\|q^*(t)\big\|_{L^4}\big\|\Delta(v_1+q_1)(t)\big\|_{L^4}+\big\|\nabla q^*(t)\big\|_{L^2}\big\|\nabla(v_1+q_1)(t)\big\|_{L^\infty}\big)$$
$$\leqslant\big\|\nabla q^*(t)\big\|_{L^2}^2\big[\alpha^{1/2}C_S^2\lambda_{n+1}^{-1/4}\big(r_0+\rho_0+\lambda_{n+1}^{-1/4}R_n\big)+\alpha\big\|\nabla v_1(t)\big\|_{L^\infty}+\alpha^{1/2}C_{S,d/2}\lambda_{n+1}^{-\frac{1}{2}+\frac{d}{4}}R_n\big]$$
$$\leqslant\big\|\nabla q^*(t)\big\|_{L^2}^2\big[\alpha^{1/2}C_S^2\big(\lambda_{n+1}^{-1/4}(r_0+\rho_0)+\lambda_{n+1}^{-d/2}R_0\big)$$
$$+\alpha\big\|\nabla v_1(t)\big\|_{L^\infty}+\alpha^{1/2}C_{S,d/2}\lambda_{n+1}^{-\frac{d}{4}}R_0\big]. \tag{5.54}$$

Applying classical Sobolev embeddings and using the property (5.32) again, we obtain, for any $t\in\mathbb{R}$,

$$\begin{aligned}\big|\big((\operatorname{rot} q_2)\times v^*, q^*\big)(t)\big| &\leqslant \big\|q^*(t)\big\|_{L^4}\big\|\operatorname{rot} q_2(t)\big\|_{L^4}\big\|v^*(t)\big\|_{L^2}\\ &\leqslant C_S^2\lambda_{n+1}^{-1/2}R_n\big\|\nabla q^*(t)\big\|_{L^2}\big\|v^*(t)\big\|_{L^2}\\ &\leqslant C_S^2\lambda_{n+1}^{-d/2}R_0\big\|\nabla q^*(t)\big\|_{L^2}\big\|v^*(t)\big\|_{L^2}. \end{aligned}\tag{5.55}$$

Likewise, we get, for any $t\in\mathbb{R}$,

$$\begin{aligned}\alpha\big|\big(\operatorname{rot}\Delta q_2\times v^*, q^*\big)(t)\big| &\leqslant \alpha\big\|\operatorname{rot}\Delta q_2(t)\big\|_{L^2}\big\|q^*(t)\big\|_{L^{\frac{4}{2-d}}}\big\|v^*(t)\big\|_{L^{\frac{4}{d}}}\\ &\leqslant \widetilde{C}_{S,d/4}^2\lambda_{n+1}^{-1/2+d/4}\big\|\nabla q^*(t)\big\|_{L^2}R_n\alpha^{1/2}\big\|\nabla v^*(t)\big\|_{L^2}\\ &\leqslant \widetilde{C}_{S,d/4}^2\lambda_{n+1}^{-d/4}R_0\big\|\nabla q^*(t)\big\|_{L^2}\alpha^{1/2}\big\|\nabla v^*(t)\big\|_{L^2}, \end{aligned}\tag{5.56}$$

where $\widetilde{C}_{S,d/4}^2$ is a constant occurring in the Sobolev estimates depending on $d/4$. We next estimate the term $|(\operatorname{rot}(v^*-\alpha\Delta v^*)\times(v_1+q_1), q^*)|$. Integrating by parts, using classical Sobolev estimates and the property (5.32) yields, for any $t\in\mathbb{R}$,

$$\begin{aligned}&\big|\big(\operatorname{rot} v^*\times(v_1+q_1), q^*\big)(t)\big|\\ &\quad\leqslant \big\|v^*(t)\big\|_{L^2}\big(\big\|\nabla q^*(t)\big\|_{L^2}\big\|(v_1+q_1)(t)\big\|_{L^\infty}+\big\|q^*(t)\big\|_{L^4}\big\|\nabla(v_1+q_1)(t)\big\|_{L^4}\big)\\ &\quad\leqslant \big\|v^*(t)\big\|_{L^2}\big\|\nabla q^*(t)\big\|_{L^2}\big(\big(C_S+C_S^2\big)(r_0+\rho_0)+C_{S,d/2}\lambda_{n+1}^{-\frac{1}{2}+\frac{d}{4}}R_n+C_S^2\lambda_{n+1}^{-1/2}R_n\big)\\ &\quad\leqslant \big\|v^*(t)\big\|_{L^2}\big\|\nabla q^*(t)\big\|_{L^2}\big(\big(C_S+C_S^2\big)(r_0+\rho_0)+C_{S,d/2}\lambda_{n+1}^{-\frac{d}{4}}R_0+C_S^2\lambda_{n+1}^{-\frac{d}{2}}R_0\big). \end{aligned}\tag{5.57}$$

The estimate of the term $|\alpha(\operatorname{rot}\Delta v^*\times(v_1+q_1), q^*)(t)|$ is the only one, involving the norm of $\|\Delta v^*\|_{L^2}$ in the right-hand side. Integrating by parts, using classical Sobolev inequalities, the property (5.32) once more and the fact that $\lambda_n/\lambda_{n+1}\leqslant 1$, we obtain, for any $t\in\mathbb{R}$,

$$\begin{aligned}&\big|\alpha\big(\operatorname{rot}\Delta v^*\times(v_1+q_1), q^*\big)(t)\big|\\ &\quad\leqslant \alpha\big\|\Delta v^*(t)\big\|_{L^2}\big[\big\|\nabla q^*(t)\big\|_{L^2}\big\|(v_1+q_1)(t)\big\|_{L^\infty}+\big\|q^*(t)\big\|_{L^4}\big\|\nabla(v_1+q_1)(t)\big\|_{L^4}\big]\\ &\quad\leqslant \alpha^{1/2}\big\|\Delta v^*(t)\big\|_{L^2}\big\|\nabla q^*(t)\big\|_{L^2}\big[\alpha^{1/2}\big(C_S+C_S^2\big)(r_0+\rho_0)+C_{S,1+d/2}\lambda_{n+1}^{-1+d/4}R_n+C_S^2\lambda_{n+1}^{-1}R_n\big]\\ &\quad\leqslant \big\|\nabla q^*(t)\big\|_{L^2}\big[\alpha\big\|\Delta v^*(t)\big\|_{L^2}\big(C_S+C_S^2\big)(r_0+\rho_0)\\ &\qquad+C_0\alpha^{1/2}\big\|\nabla v^*(t)\big\|_{L^2}R_0\big(C_{S,d/2}\lambda_{n+1}^{-d/4}+C_S^2\lambda_{n+1}^{-d/2}\big)\big]. \end{aligned}\tag{5.58}$$

Finally, we consider the term $|(\operatorname{rot}(v_2-\alpha\Delta v_2)\times v^*, q^*)|$. Due to the classical Sobolev estimates, we can write, for $t\in\mathbb{R}$,

$$\begin{aligned}&\big|\big(\operatorname{rot}(v_2-\alpha\Delta v_2)\times v^*, q^*\big)(t)\big|\\ &\quad\leqslant \big\|\operatorname{rot} v_2(t)\big\|_{L^4}\big\|q^*(t)\big\|_{L^4}\big\|v^*(t)\big\|_{L^2}+\alpha\big\|\operatorname{rot}\Delta v_2(t)\big\|_{L^2}\big\|v^*(t)\big\|_{L^4}\big\|q^*(t)\big\|_{L^4}\\ &\quad\leqslant C_S^2(r_0+\rho_0)\lambda_{n+1}^{-1/4}\big\|\nabla q^*(t)\big\|_{L^2}\big(\big\|v^*(t)\big\|_{L^2}+\alpha^{1/2}\big\|\nabla v^*(t)\big\|_{L^2}\big). \end{aligned}\tag{5.59}$$

Using the Young inequality $2ab\leqslant\eta^{-1}a^2+\eta b^2$, with $\eta=\frac{a_1^*}{48}$, we deduce from the estimates (5.52) to (5.59) that, for $t\in\mathbb{R}$,

$$\partial_t\big(\|q^*(t)\|^2+\alpha\|\nabla q^*(t)\|^2\big)+K_1\|\nabla q^*(t)\|^2$$
$$\leqslant \frac{K_2}{a_1^*}\big(\|v^*(t)\|_{L^2}^2+\alpha\|\nabla v^*(t)\|_{L^2}^2\big)+\frac{C_1\alpha^2}{a_1^*}\|\Delta v^*(t)\|_{L^2}^2(r_0+\rho_0)^2, \tag{5.60}$$

where C_1 is a positive constant depending only on the Sobolev estimates, where

$$K_1=2\nu-2\alpha\sup_s\|\nabla v_1(s)\|_{L^\infty}-2C_S^2\lambda_{n+1}^{-1/4}\big(r_0+\rho_0+\lambda_{n+1}^{-1/4}R_0\big)-2\alpha^{1/2}C_{S,d/2}\lambda_{n+1}^{-d/4}R_0$$
$$-2\alpha^{1/2}C_S^2\big(\lambda_{n+1}^{-1/4}(r_0+\rho_0)+\lambda_{n+1}^{-d/2}R_0\big)-6\left(\frac{a_1^*}{48}\right)$$
$$\equiv 2\nu-2\alpha\sup_s\|\nabla v_1(s)\|_{L^\infty}-\frac{a_1^*}{8}-\lambda_{n+1}^{-d/4}K_1^*, \tag{5.61}$$

and

$$K_2=48\big[\big(C_S^4+\widetilde{C}_{S,d/4}^4\big)\lambda_{n+1}^{-d/2}R_0^2+4\big(C_S+C_S^2\big)^2\big(\lambda_{n+1}^{-1/2}+1\big)(r_0+\rho_0)^2$$
$$+R_0^2\big(2+C_0^2\big)\big(C_{S,d/2}\lambda_{n+1}^{-d/4}+C_S^2\lambda_{n+1}^{-d/2}\big)^2\big]. \tag{5.62}$$

We remark that, by the condition (5.23),

$$K_1\geqslant\frac{3a_1^*}{4}-\frac{a_1^*}{8}-\lambda_{n+1}^{-d/4}K_1^*\geqslant\frac{5a_1^*}{8}-\lambda_{n+1}^{-d/4}K_1^*. \tag{5.63}$$

Clearly, we can choose n_0 large enough so that $\lambda_{n+1}^{-d/4}K_1^*\leqslant a_1^*/8$, which implies that

$$K_1\geqslant\frac{a_1^*}{2}. \tag{5.64}$$

The estimates (5.60) and (5.64), together with the inequality (2.4), imply that, for any $t\in\mathbb{R}$,

$$\partial_t\big(\|q^*(t)\|^2+\alpha\|\nabla q^*(t)\|^2\big)+\frac{a_1^*}{2(\lambda_1^{-1}+\alpha)}\big(\|q^*(t)\|^2+\alpha\|\nabla q^*(t)\|^2\big)$$
$$\leqslant \frac{K_2}{a_1^*}\big(\|v^*(t)\|_{L^2}^2+\alpha\|\nabla v^*(t)\|_{L^2}^2\big)+\frac{C_1\alpha^2}{a_1^*}\|\Delta v^*(t)\|_{L^2}^2(r_0+\rho_0)^2, \tag{5.65}$$

which, after integration in time from $-\infty$ to t, yields, for any $t\in\mathbb{R}$,

$$\|q^*(t)\|_{L^2}^2+\alpha\|\nabla q^*(t)\|_{L^2}^2\leqslant 2\big(\lambda_1^{-1}+\alpha\big)\left[\frac{K_2}{a_1^{*2}}\sup_{s\leqslant t}\big(\|v^*(s)\|_{L^2}^2+\alpha\|\nabla v^*(s)\|_{L^2}^2\big)\right.$$
$$\left.+\frac{C_1\alpha^2}{a_1^{*2}}(r_0+\rho_0)^2\sup_{s\leqslant t}\|\Delta v^*(s)\|_{L^2}^2\right]. \tag{5.66}$$

The inquality (5.49) is thus proved.

2) In order to prove the inequality (5.50), we take the inner product in H of the equality (5.51) with $-\Delta q^*$ and obtain the following inequality, for any $t\in\mathbb{R}$,

$$\begin{aligned}
&\partial_t\big(\|\nabla q^*\|^2+\alpha\|\Delta q^*\|^2\big)+2\nu\|\Delta q^*\|^2\\
&\quad\leqslant 2\big[\big|\big(\operatorname{rot} q^*\times(q_1+v_1),\Delta q^*\big)\big|+\alpha\big|\big(\operatorname{rot}\Delta q^*\times(q_1+v_1),\Delta q^*\big)\big|\\
&\qquad+\big|\big(\operatorname{rot}(q_2+v_2)\times q^*,\Delta q^*\big)\big|+\alpha\big|\big(\operatorname{rot}\Delta(q_2+v_2)\times q^*,\Delta q^*\big)\big|\\
&\qquad+\big|\big((\operatorname{rot} q_2)\times v^*,\Delta q^*\big)\big|+\alpha\big|\big(\operatorname{rot}\Delta q_2\times v^*,\Delta q^*\big)\big|\\
&\qquad+\big|\big(\operatorname{rot}(v^*-\alpha\Delta v^*)\times(v_1+q_1),\Delta q^*\big)\big|+\big|\big(\operatorname{rot}(v_2-\alpha\Delta v_2)\times v^*,\Delta q^*\big)\big|\big].
\end{aligned}\tag{5.67}$$

Since the terms on the right-hand side of the inequality (5.67) are estimated by arguing in the same way as for estimating the terms in the right-hand side of the inequality (5.52), we do not give the details.

The proof of Lemma 5.3 is completed. □

Remark 5.4. Like in Lemma 5.2, in order to simplify the proofs, we have assumed that the forcing term f belongs to H^{1+d}_{per} with $d>0$. Looking carefully at the proof of Lemma 5.3 and replacing the Sobolev inequality (5.25) by the Brézis–Gallouët inequality, one easily shows that, if f belongs only to H^1_{per}, the mapping $v\in C^0(\mathbb{R};\mathcal{N}_{P_nV^{3+d}}(P_n\mathcal{A}_\alpha,r_0))\mapsto q^n(v)\in\mathcal{W}_n$ is still Lipschitz-continuous.

5.1.4. Step 4 of the proof of Theorem 1.2

Under the hypotheses of Lemma 5.2, we can choose the integer n_0 large enough (in Lemma 5.2) so that, for any $0<\alpha<\alpha_0$, if $u(\mathbb{R})\subset\mathcal{A}_\alpha$, then, for any $n\geqslant n_0$, the "trajectory" $w_n=Q_nu\equiv(I-P_n)u$ belongs to $\mathcal{W}_n$ and, thus, by uniqueness of the solution of the system (1.2), u is represented as

$$u(t)=P_nu(t)+Q_nu(t)=P_nu(t)+q^n\big(P_nu(t)\big),\tag{5.68}$$

where $q^n(\cdot)$ is the fixed point defined in Lemma 5.2 and where v_n is the solution of the finite-dimensional system (5.1); that is, $v_n\equiv P_nu$ satisfies the system

$$\begin{aligned}
\partial_t v_n&=P_n(I+\alpha A)^{-1}P_nP\big(\nu\Delta v_n-\operatorname{rot}\big(v_n+q(v_n)-\alpha\Delta\big(v_n+q^n(v_n)\big)\big)\times\big(v+q^n(v_n)\big)+f\big)\\
&\equiv F_n(v_n),
\end{aligned}\tag{5.69}$$

where A denotes the Stokes operator $-P\Delta$.

Since the map $v\mapsto q^n(v)\in\mathcal{W}_n$ is defined for any $v\in C^0(\mathbb{R},\mathcal{N}_{P_nV^{3+d}}(P_n\mathcal{A}_\alpha,r_0))$ and, by Lemma 5.3, is even a Lipschitzian mapping, the map $F_n(v)$ is well defined and, as we shall see below, is also Lipschitzian. Therefore, it is interesting to study the finite-dimensional system (5.68) for " continuous curves" v with values in $\mathcal{N}_{P_nV^{3+d}}(P_n\mathcal{A}_\alpha,r_1)$ where $0<r_1<r_0$. As in [27] and in [24], one approach consists in considering the following differential equation

$$\partial_s v=F_n(v),\qquad v(0)=v_0,\tag{5.70}$$

in the Banach space $C^0(\mathbb{R},P_nV^{3+d})$. Since F_n is a Lipschitzian map (see Lemma 5.4 below), by the classical theorem of existence of solutions, for any $v_0\in C^0(\mathbb{R},\mathcal{N}_{P_nV^{3+d}}(P_n\mathcal{A}_\alpha,r_0/2))$, there are a positive constant s^* and a unique solution $v^*:(v_0,s)\in C^0(\mathbb{R},\mathcal{N}_{P_nV^{3+d}}(P_n\mathcal{A}_\alpha,r_0/2))\times[0,s^*)\mapsto v^*(v_0)(s)\in C^0(\mathbb{R},\mathcal{N}_{P_nV^{3+d}}(P_n\mathcal{A}_\alpha,2r_0/3))$ of (5.70). Let now $u_0(t)=P_nu_0(t)+Q_nu_0(t)\subset\mathcal{A}_\alpha$ be a solution of (1.2) and let $z(s)(t)=P_nu_0(t+s)$. By assumption, $z(0)=P_nu_0(t)\in C^0(\mathbb{R},\mathcal{N}_{P_nV^{3+d}}(P_n\mathcal{A}_\alpha,r_0/2))$. One easily checks that $z(s)$ is a solution of (5.70) and, thus, by uniqueness of the solution of (5.70), $z(s)(t)=P_nu_0(t+s)=v^*(v_0)(s)(t)$ for any $s\in\mathbb{R}$. This point of view allowed, in [27] and in [24], to prove time-regularity results for the trajectories contained in compact global attractors, in the case of general dissipative dynamical systems.

Since the map $q^n(v)$ depends on the values $v(s)$, for $s\leqslant t$ only, we may also consider the system (5.69) as a system of differential equations with infinite delay in the following way.

Lemma 5.4. *Assume that the hypotheses of Lemma 5.2 are satisfied. Then, for any $n \geqslant n_0$, there exists a time $T_0(n) > 0$ such that, for any $v_0 \in C^0((-\infty,0];\mathcal{N}_{P_nV^{3+d}}(P_n\mathcal{A}_\alpha, r_0/2))$, the following finite-dimensional system of differential equations*

$$\begin{aligned}\partial_t v &= F_n(v), \quad 0 \leqslant t \leqslant T_0(n),\\ v(s) &= v_0(s), \quad s \leqslant 0,\end{aligned} \tag{5.71}$$

where

$$F_n(v) = P_n(I+\alpha A)^{-1}P_nP\big(\nu\Delta v_n - \mathrm{rot}\big(v_n + q(v_n) - \alpha\Delta\big(v_n + q^n(v_n)\big)\big) \times \big(v + q^n(v_n)\big) + f\big),$$

admits a unique solution $v \in C^0((-\infty, T_0(n)];\mathcal{N}_{P_nV^{3+d}}(P_n\mathcal{A}_\alpha, 2r_0/3))$. Moreover, the dependence of v with respect to the initial data v_0 is continuous.

Remarks 5.5. 1) In Lemma 5.4, we have taken $\sigma_0 = 0$ as initial time. Of course, due to Lemmata 5.2 and 5.3, the same well-posedness result holds if 0 is replaced by any real number σ_0.

2) As we have remarked at the beginning of this section, we can choose n_0 large enough so that, for any $0 < \alpha \leqslant \alpha_0$, if $u(\mathbb{R}) \subset \mathcal{A}_\alpha$, then, for $n \geqslant n_0$,

$$u(t) = P_nu(t) + q^n\big(P_nu(t)\big),$$

where $v(t) = P_nu(t)$ is the solution of the finite-dimensional system (5.71). This allows to say that, on the global attractor, Eqs. (1.2) reduce to the functional-differential equation (5.71) with infinite delay. For the properties of functional-differential equations with infinite delay, we refer the reader to the book [28] for example.

Proof of Lemma 5.4. Since this lemma is proved by using the strict contraction fixed point theorem in a very classical way, we will not give all the details of the proof and let them to the reader. First, we remark that the system (5.71) is equivalent to the following integral system

$$\begin{aligned}v(t) &= v_0(0) + \int_0^t F_n\big(v(\sigma)\big)\,d\sigma, \quad 0 \leqslant t \leqslant T_0,\\ v(s) &= v_0(s), \quad s \leqslant 0.\end{aligned} \tag{5.72}$$

We emphasize that, due to Lemma 5.2, for any T, the map $F_n(v)$ is well-defined if v belongs to $B_{C^0((-\infty,T];P_nV^{3+d})}(P_n\mathcal{A}_\alpha, r_0)$. We also remark that Lemma 5.2 at once implies that there exists a positive number $M_0(n)$ such that, for any T and for any $v \in C^0((-\infty,T];\mathcal{N}_{P_nV^{3+d}}(P_n\mathcal{A}_\alpha, r_0))$, we have the following bound

$$\sup_{\sigma \leqslant T}\big\|F_n\big(v(\sigma)\big)\big\|_{V^{3+d}} \leqslant M_0(n). \tag{5.73}$$

We next choose a positive time $T_0 \equiv T_0(n)$ satisfying the condition

$$T_0M_0(n) < \frac{r_0}{3}, \tag{5.74}$$

as well as the condition (5.76) below.

Let v_0 be fixed in $C^0((-\infty,0];\mathcal{N}_{P_nV^{3+d}}(P_n\mathcal{A}_\alpha, r_0/2))$. In view of solving the integral system (5.72), we introduce the convex subset

$$E_0 = \big\{v \in C^0\big((-\infty,T_0];\mathcal{N}_{P_nV^{3+d}}(P_n\mathcal{A}_\alpha, 2r_0/3)\big)\ \big|\ v(s) = v_0(s),\ \forall s \leqslant 0\big\},$$

of the Banach space E_{T_0} where E_T is the Banach space $C_b^0((-\infty,T];P_nV^{3+d})$, endowed with the norm

$$\|v\|_{E_T}=\sup_{s\leqslant T}\big\|v(s)\big\|_{L^2}.$$

One next introduces the mapping $\mathcal{F}_0$ from E_0 into E_{T_0}, defined for any $y\in E_0$ by

$$\mathcal{F}_0(y)(t)=v_0(t),\quad t\leqslant 0,$$
$$\mathcal{F}_0(y)(t)=v_0(0)+\int_0^t F_n\big(y(\sigma)\big)\,d\sigma,\quad 0\leqslant t\leqslant T_0.$$

Since T_0 satisfies the condition (5.74), $\mathcal{F}_0$ is a mapping from E_0 into E_0. If in addition $\mathcal{F}_0$ is a strict contraction from E_0 into E_0, the strict contraction fixed point theorem will imply that $\mathcal{F}_0$ has a unique fixed point in E_0 and thus we will have proved the existence of a solution of (5.72). In order to show that $\mathcal{F}_0$ is a strict contraction, we prove that, for any T, F_n is a Lipschitz-continuous mapping from $\mathcal{N}\equiv C^0((-\infty,T];\mathcal{N}_{P_nV^{3+d}}(P_n\mathcal{A}_\alpha,r_0))$ into E_T. More precisely, we will show that, for any elements v_1, v_2 in $\mathcal{N}$,

$$\big\|F_n(v_1)-F_n(v_2)\big\|_{E_T}\leqslant M_1(n)\|v_1-v_2\|_{E_T},\tag{5.75}$$

where $M_1(n)$ does not depend on T, but may depend on n.

If we choose T_0 satisfying the condition (5.74) and the inequality (5.76) below,

$$T_0M_1(n)\leqslant\frac{1}{2},\tag{5.76}$$

we will have shown that $\mathcal{F}_0$ is a strict contraction from E_0 into E_0.

It thus remains to prove the property (5.75). We set $v=v_1-v_2$, $q=q^n(v_1)-q^n(v_2)$, and $q_i=q^n(v_i)$, $i=1,2$. The map $F_n(v_1)-F_n(v_2)$ satisfies the equality

$$\begin{aligned}F_n(v_1)-F_n(v_2)=P_n(I+\alpha A)^{-1}P_nP\big(\nu\Delta v-\operatorname{rot}\big(q+v-\alpha\Delta(q+v)\big)\times(q_1+v_1)\\-\operatorname{rot}\big(q_2+v_2-\alpha\Delta(q_2+v_2)\big)\times(q+v)\big).\end{aligned}\tag{5.77}$$

Clearly, for any $v\in P_nH$, we have

$$\big\|(I+\alpha A)^{-1}P_nP\Delta v\big\|_{L^2}\leqslant C(\alpha)\|v\|_{L^2}.\tag{5.78}$$

Next, using Lemma 5.2, the inequality (5.50) of Lemma 5.3 as well as the inequality (A.9), we obtain that, for any $t\leqslant T$,

$$\begin{aligned}&\big\|(1+\alpha A)^{-1}P_nP\big[\operatorname{rot}\big(q(t)+v(t)-\alpha\Delta\big(q(t)+v(t)\big)\big)\times\big(q_1(t)+v_1(t)\big)\big]\big\|_{L^2}\\&\leqslant\frac{C}{\alpha^{1/2}}\big(\big\|\Delta q_1(t)\big\|_{L^2}+\big\|\Delta v_1(t)\big\|_{L^2}\big)\big\|q(t)+v(t)-\alpha\Delta\big(q(t)+v(t)\big)\big\|_{L^2}\\&\leqslant\frac{C}{\alpha^{1/2}}(R_n+r_0+\rho_0)\big(\big\|v(t)\big\|_{L^2}+\alpha\big\|\Delta v(t)\big\|_{L^2}+\big\|q(t)\big\|_{L^2}+\alpha\big\|\Delta q(t)\big\|_{L^2}\big)\\&\leqslant\frac{C}{\alpha^{1/2}}(R_n+r_0+\rho_0)\sup_{s\leqslant t}\big\|v(s)\big\|_{L^2}\bigg[(1+\alpha\lambda_n)+\frac{C_L^{1/2}}{a_1^*}\big(1+\alpha R_n^2\big)^{1/2}\big(\lambda_n^2+\alpha\lambda_n^3\big)^{1/2}\bigg].\end{aligned}\tag{5.79}$$

In the same way, we obtain, for all $t \leqslant T$,

$$\begin{aligned}
&\big\|(1+\alpha A)^{-1} P_n P\big[\operatorname{rot}\big(q_2(t)+v_2(t)-\alpha\Delta\big(q_2(t)+v_2(t)\big)\big)\times\big(q(t)+v(t)\big)\big]\big\|_{L^2} \\
&\quad\leqslant \frac{C}{\alpha^{1/2}}\big\|q_2(t)+v_2(t)-\alpha\Delta\big(q_2(t)+v_2(t)\big)\big\|_{L^2}\big(\big\|\Delta q(t)\big\|_{L^2}+\big\|\Delta v(t)\big\|_{L^2}\big) \\
&\quad\leqslant \frac{C}{\alpha^{1/2}}\left(\frac{R_n}{\lambda_{n+1}^{1/2}}+r_0+\rho_0\right)\sup_{s\leqslant t}\big\|v(s)\big\|_{L^2}\left[\lambda_n+\frac{C_L^{1/2}}{a_1^*}\big(1+\alpha R_n^2\big)^{1/2}\big(\lambda_n^2+\alpha\lambda_n^3\big)^{1/2}\right].
\end{aligned} \tag{5.80}$$

The equality (5.77) and the estimates (5.78), (5.79), and (5.80) imply the Lipschitz property (5.75).

The proof of the uniqueness and continuity with respect to the initial data are really elementary and classical and are left to the reader. Lemma 5.4 is proved. □

Acknowledgments

This work was done in part during the post-doctoral stay of Andrey Rekalo at the Université Paris-Sud in 2006. Andrey Rekalo is thankful to the University Paris-Sud for the financial support. The kind hospitality of the Department of Mathematics at Orsay is also gratefully acknowledged.

Appendix A. Proof of auxiliary equalities and estimates

We first recall some formulas of vectorial calculus. For any vectors $u = (u^1, u^2, u^3)^t$, $v = (v^1, v^2, v^3)^t$ in $\mathbb{R}^3$, the vector product $w = u \times v$ is defined as follows.

$$w = \big(u^2v^3 - u^3v^2, u^3v^1 - u^1v^3, u^1v^2 - u^2v^1\big)^t.$$

For any vector $u = (u^1, u^2, u^3)^t$ in $\mathbb{R}^3$, we define the curl of u as follows

$$\operatorname{curl} u = \operatorname{rot} u = \big(\partial_2 u^3 - \partial_3 u^2, \partial_3 u^1 - \partial_1 u^3, \partial_1 u^2 - \partial_2 u^1\big)^t.$$

We recall that, in this paper, all scalar and vector fields functions are defined on $\mathbb{T}^2$. We identify the 2-component vector field $u = (u^1, u^2)^t$ with the 3-component vector field $\bar{u} = (u^1, u^2, 0)^t$. The scalar w is identified with the 3-component vector $\widehat{w} = (0, 0, w)^t$.

Since the vectorial functions u considered here do not depend on the x_3-variable, $\partial_3 u^1 = \partial_3 u^2 = 0$. Consequently,

$$\operatorname{rot} u = \big(0, 0, \partial_1 u^2 - \partial_2 u^1\big)^t.$$

For any $m \in \mathbb{N}$ the iterated operator rot^{m+1} is defined as $\operatorname{rot}^{m+1} u = \operatorname{rot}(\operatorname{rot}^m u)$, $\operatorname{rot}^1 u = \operatorname{rot} u$.

In the case where $\operatorname{div} u = 0$, we notice that

$$\operatorname{rot}(\operatorname{rot} u) = -\Delta u. \tag{A.1}$$

Throughout this paper, we frequently use the following identity

$$\operatorname{rot}\{\operatorname{rot} u \times v\} = -\Delta u \times v, \tag{A.2}$$

which holds for any (regular enough) divergence-free vector fields u and v.

Throughout this paper we also often use the following identities, without mentioning it explicitly.

Lemma A.1. *Let $u=(u^1,u^2)^t$, $v=(v^1,v^2)^t$ be smooth periodic divergence-free vector fields defined on $\mathbb{T}^2$. With the above conventions, the following identities hold*:

$$(v\times u,\operatorname{rot}v)=-\sum_{i,j=1}^{2}\int_{\mathbb{T}^2}\partial_i u^j v^i v^j\,dx, \tag{A.3}$$

$$(\operatorname{rot}v\times u,v)=\int_{\mathbb{T}^2}\big(\partial_1u^1\big(v^1\big)^2+\big(\partial_1u^2+\partial_2u^1\big)v^1v^2+\partial_2u^2\big(v^2\big)^2\big)\,dx, \tag{A.4}$$

$$\big(\operatorname{rot}\{\operatorname{rot}v\times u\},\operatorname{rot}v\big)=-(\Delta v\times u,\operatorname{rot}v)=0, \tag{A.5}$$

$$(\operatorname{rot}\Delta v\times u,v)=\int_{\mathbb{T}^2}\operatorname{rot}v\big(\Delta u^1v^2-\Delta u^2v^1\big)\,dx+2\int_{\mathbb{T}^2}\operatorname{rot}v\big(\nabla u^1\cdot\nabla v^2-\nabla u^2\cdot\nabla v^1\big)\,dx, \tag{A.6}$$

$$\big(\Delta u\times u,\Delta^2\operatorname{rot}u\big)=2(\Delta\nabla u\times\nabla u,\Delta\operatorname{rot}u), \tag{A.7}$$

where

$$\Delta\nabla u\times\nabla u=-\nabla u^1\cdot\nabla\Delta u^2+\nabla u^2\cdot\nabla\Delta u^1.$$

Proof. 1) Integrating by parts and taking into account the property $\operatorname{div}v=0$, we get

$$\begin{aligned}
&(v\times u,\operatorname{rot}v)\\
&\quad=\int_{\mathbb{T}^2}\big(u^1v^2\partial_2v^1+u^2v^1\partial_1v^2-u^1v^2\partial_1v^2-u^2v^1\partial_2v^1\big)\,dx\\
&\quad=-\int_{\mathbb{T}^2}\bigg(u^1v^1\partial_2v^2+u^2v^2\partial_1v^1+v^1v^2\partial_1u^2+v^1v^2\partial_2u^1+\frac12u^1\partial_1\big(v^2\big)^2+\frac12u^2\partial_2\big(v^1\big)^2\bigg)\,dx\\
&\quad=-\int_{\mathbb{T}^2}\big(\partial_1u^2+\partial_2u^1\big)v^1v^2\,dx+\frac12\int_{\mathbb{T}^2}\big(u^1\partial_1\big(v^1\big)^2+u^2\partial_2\big(v^2\big)^2-u^1\partial_1\big(v^2\big)^2-u^2\partial_2\big(v^1\big)^2\big)\,dx.
\end{aligned} \tag{A.8}$$

The second integral in the right-hand side is equal to

$$\int_{\mathbb{T}^2}\big(\big\{-\partial_1u^1+\partial_2u^2\big\}\big(v^1\big)^2+\big\{\partial_1u^1-\partial_2u^2\big\}\big(v^2\big)^2\big)\,dx=-2\int_{\mathbb{T}^2}\big(\partial_1u^1\big(v^1\big)^2+\partial_2u^2\big(v^2\big)^2\big)\,dx.$$

Combining both equalities, we obtain (A.3).

2) A similar calculation gives (A.4).

3) A direct calculation yields

$$\Delta v\times u=u^2\Delta v^1-u^1\Delta v^2=-u^2\partial_2\operatorname{rot}v-u^1\partial_1\operatorname{rot}v.$$

Therefore,

$$(\Delta v\times u,\operatorname{rot}v)=\frac12\int_{\mathbb{T}^2}(\operatorname{rot}v)^2\operatorname{div}u\,dx=0.$$

4) Integrating by parts, we get

$$\begin{aligned}(\operatorname{rot}\Delta v\times u, v) &= \int_{\mathbb{T}^2}\Delta\operatorname{rot} v\big(u^1v^2-u^2v^1\big)\,dx = \int_{\mathbb{T}^2}\operatorname{rot} v\,\Delta\big(u^1v^2-u^2v^1\big)\,dx\\ &= \int_{\mathbb{T}^2}\operatorname{rot} v\big(\Delta u^1v^2-\Delta u^2v^1\big)\,dx + 2\int_{\mathbb{T}^2}\operatorname{rot} v\big(\nabla u^1\cdot\nabla v^2-\nabla u^2\cdot\nabla v^1\big)\,dx\\ &\quad+\int_{\mathbb{T}^2}\operatorname{rot} v(u\cdot\nabla)\operatorname{rot} v\,dx.\end{aligned}$$

The third integral in the right-hand side is equal to

$$-\frac{1}{2}\int_{\mathbb{T}^2}(\operatorname{rot} v)^2\operatorname{div} u\,dx = 0.$$

5) Due to (A.5) and to the identity $\Delta u\times\Delta u = 0$, we can write

$$\begin{aligned}\big(\Delta u\times u, \Delta^2\operatorname{rot} u\big) &= \big(\Delta^2 u\times u, \Delta\operatorname{rot} u\big) + 2(\Delta\nabla u\times\nabla u, \Delta\operatorname{rot} u) + (\Delta u\times\Delta u, \Delta\operatorname{rot} u)\\ &= 2(\Delta\nabla u\times\nabla u, \Delta\operatorname{rot} u).\end{aligned}$$

Thus Lemma A.1 is proved. □

Lemma A.2. *Let u, v be smooth periodic divergence-free vector fields and let*

$$w = (1+\alpha A)^{-1}P\{\operatorname{rot} u\times v\},$$

where $A = -P\Delta$ is the Stokes operator. Then there exists a positive constant C such that

$$\begin{aligned}\|w\|_{L^2}^2+\alpha\|\nabla w\|_{L^2}^2 &\leqslant C\|\nabla u\|_{L^2}^2\|\Delta v\|_{L^2}^2,\\ \|w\|_{L^2}^2+\alpha\|\nabla w\|_{L^2}^2 &\leqslant \frac{C}{\alpha}\|u\|_{L^2}^2\|\Delta v\|_{L^2}^2.\end{aligned}\tag{A.9}$$

Proof. We take the L^2-inner product of the equation

$$(1+\alpha A)w = P(\operatorname{rot} u\times v)$$

with w in $L^2(\mathbb{T}^2)^2$ to get

$$\|w\|_{L^2}^2+\alpha\|\nabla w\|_{L^2}^2 = (\operatorname{rot} u\times v, w).\tag{A.10}$$

On the one hand, the first inequality in (A.9) is a direct consequence of (A.10) and the classical Sobolev inequalities. On the other hand, performing an integration by parts yields

$$\begin{aligned}(\operatorname{rot} u\times v, w) &= \int_{\mathbb{T}^2}\big(w^1v^2\big(\partial_2u^1-\partial_1u^2\big)+w^2v^1\big(\partial_1u^2-\partial_2u^1\big)\big)\,dx\\ &= \int_{\mathbb{T}^2}\big(\big(-\partial_2w^1v^2+\partial_2w^2v^1-w^1\partial_2v^2+w^2\partial_2v^1\big)u^1\\ &\quad+\big(\partial_1w^1v^2-\partial_1w^2v^1+w^1\partial_1v^2-w^2\partial_1v^1\big)u^2\big)\,dx.\end{aligned}$$

Therefore, (A.10) implies that

$$\|w\|_{L^2}^2 + \alpha\|\nabla w\|_{L^2}^2 \leqslant \int_{\mathbb{T}^2} \big(|\nabla v| \cdot |w| + |v| \cdot |\nabla w|\big)|u|\,dx \leqslant \big(\|\nabla v\|_{L^4}\|w\|_{L^4} + \|v\|_{L^\infty}\|\nabla w\|_{L^2}\big)\|u\|_{L^2}$$

$$\leqslant c_0\|u\|_{L^2}\|\Delta v\|_{L^2}\|\nabla w\|_{L^2} \leqslant \frac{\alpha}{2}\|\nabla w\|_{L^2}^2 + \frac{c_0^2}{2\alpha}\|u\|_{L^2}^2\|\Delta v\|_{L^2}^2.$$

The lemma is proved. □

References

[1] J. Arrieta, A. Carvalho, J.K. Hale, A damped hyperbolic equation with critical exponent, Comm. Partial Differential Equations 17 (1992) 841–866.
[2] A.V. Babin, M.I. Vishik, Attractors of Evolutionary Equations, North-Holland, 1989.
[3] J.M. Ball, Attractors of damped wave equations, Conference at Oberwolfach, 1992.
[4] J.M. Ball, Global attractors for damped semilinear wave equations, Discrete Contin. Dyn. Syst. 10 (2004) 31–52.
[5] H. Brézis, T. Gallouët, Nonlinear Schrödinger evolution equation, Nonlinear Anal. 4 (1980) 677–681.
[6] Y. Cao, E.M. Lunasin, E.S. Titi, Global well-posedness of three-dimensional viscous and inviscid simplified Bardina turbulence models, Commun. Math. Sci. 4 (2006) 823–848.
[7] J.-Y. Chemin, Fluides parfaits incompressibles, Astérisque 230 (1995).
[8] D. Cioranescu, V. Girault, Weak and classical solutions of a family of second grade fluids, Internat. J. Non-Linear Mech. 32 (2) (1997) 317–335.
[9] D. Cioranescu, E.H. Ouazar, Existence and uniqueness for fluids of second grade, in: Collège de France Seminar, vol. VI, Paris, 1982/1983, Pitman, Boston, MA, 1984, pp. 178–197.
[10] A. Debussche, R. Temam, Some new generalizations of inertial manifolds, Discrete Contin. Dyn. Syst. 2 (1996) 543–558.
[11] J.E. Dunn, R.L. Fosdick, Thermodynamics, stability and boundedness of fluids of complexity 2 and fluids of second grade, Arch. Ration. Mech. Anal. 56 (1974) 191–252.
[12] C. Foias, D. Holm, E.S. Titi, The three-dimensional viscous Camassa–Holm equations and their relation to the Navier–Stokes equations and turbulence theory, J. Dynam. Differential Equations 14 (2002) 1–35.
[13] C. Foias, D. Holm, E.S. Titi, The Navier–Stokes-alpha model of fluid turbulence, Special Issue in Honor of V.E. Zakharov on the Occasion of His 60th Birthday, Phys. D 152 (2001) 505–519.
[14] C. Foias, G. Prodi, Sur le comportement global des solutions non stationnaires des équations de Navier–Stokes en dimension deux, Rend. Semin. Mat. Univ. Padova 39 (1967) 1–34.
[15] G.P. Galdi, A. Sequeira, Further existence results for classical solutions of the equations of second grade fluids, Arch. Ration. Mech. Anal. 128 (1994) 297–312.
[16] G.P. Galdi, M. Grobbelaarvandalsen, N. Sauer, Existence and uniqueness of classical-solutions of the equations of motion for 2nd-grade fluids, Arch. Ration. Mech. Anal. 124 (1993) 221–237.
[17] J.-M. Ghidaglia, R. Temam, Regularity of the solutions of second order evolution equations and their attractors, Ann. Sc. Norm. Super. Pisa Cl. Sci. IV 14 (1987) 485–511.
[18] O. Goubet, Regularity of the attractor for a weakly damped nonlinear Schrödinger equation, Appl. Anal. 60 (1996) 99–119.
[19] O. Goubet, Asymptotic smoothing effect for a weakly damped nonlinear Schrödinger equation in T^2, J. Differential Equations 165 (2000) 96–122.
[20] J.K. Hale, Smoothing properties of neutral equations, An. Acad. Brasil. Cienc. 45 (1973) 49–50.
[21] J.K. Hale, Asymptotic behavior and dynamics in infinite dimensions, in: Res. Notes Math., vol. 132, Pitman, Boston, 1985, pp. 1–41.
[22] J.K. Hale, Asymptotic Behavior of Dissipative Systems, Math. Surveys Monogr., vol. 25, Amer. Math. Soc., Providence, RI, 1988.
[23] J.K. Hale, R. Joly, G. Raugel, Infinite Dimensional Dissipative Systems, book, manuscript.
[24] J.K. Hale, G. Raugel, Regularity, determining modes and Galerkin method, J. Math. Pures Appl. 82 (2003) 1075–1136.
[25] J.K. Hale, G. Raugel, A modified Poincaré method for the persistence of periodic orbits and applications, J. Dynam. Differential Equations 22 (2010) 3–68.
[26] J.K. Hale, G. Raugel, Persistence of periodic orbits for perturbed dissipative dynamical systems, Fields Inst. Commun., in press.
[27] J.K. Hale, J. Scheurle, Smoothness of bounded solutions of nonlinear evolution equations, J. Differential Equations 56 (1985) 142–163.
[28] J.K. Hale, S. Verduyn-Lunel, Introduction to Functional Differential Equations, Appl. Math. Sci., vol. 99, Springer-Verlag, 1993.
[29] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Math., vol. 840, Springer-Verlag, 1981.
[30] D. Iftimie, Remarques sur la limite $\alpha \to 0$ pour les fluides de grade 2, C. R. Acad. Sci. Paris Sér. I Math. 334 (1) (2002) 83–86.
[31] O. Lopes, Asymptotic fixed point theorems and forced oscillations in neutral equations, PhD thesis, Brown University, Providence, RI, June 1973.

[32] I. Moise, R. Rosa, On the regularity of the global attractor of a weakly damped, forced Korteweg–de Vries equation, Adv. Differential Equations 2 (1997) 257–296.
[33] I. Moise, R. Rosa, X. Wang, Attractors for non-compact semigroups via energy equations, Nonlinearity 11 (1998) 1369–1393.
[34] R. Nussbaum, Periodic solutions of analytic functional differential equations are analytic, Michigan Math. J. 20 (1973) 249–255.
[35] M. Oliver, E. Titi, Analyticity of the attractor and the number of determining nodes for a weakly damped driven nonlinear Schrödinger equation, Indiana Univ. Math. J. 47 (1998) 49–73.
[36] G. Raugel, Global attractors in partial differential equations, in: B. Fiedler (Ed.), Handbook of Dynamical Systems, vol. 2, North-Holland, 2002, pp. 885–982.
[37] R.S. Rivlin, J.L. Ericksen, Stress-deformation relations for isotropic materials, J. Ration. Mech. Anal. 4 (4) (1955) 323–425.