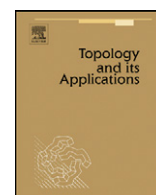




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Building suitable sets for locally compact groups by means of continuous selections

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ABSTRACT

If a discrete subset S of a topological group G with the identity 1 generates a dense subgroup of G and $S \cup \{1\}$ is closed in G , then S is called a *suitable set* for G . We apply Michael's selection theorem to offer a direct, self-contained, purely topological proof of the result of Hofmann and Morris [K.-H. Hofmann, S.A. Morris, Weight and c, J. Pure Appl. Algebra 68 (1–2) (1990) 181–194] on the existence of suitable sets in locally compact groups. Our approach uses only elementary facts from (topological) group theory.

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All topological groups considered in this paper are assumed to be Hausdorff, and all topological spaces are assumed to be Tychonoff.

1. Motivating background

Let G be a group. We use 1_G to denote the identity element of G . If X is a subset of G , then $\langle X \rangle$ will denote the smallest subgroup of G containing X , and we say that X (algebraically) *generates* $\langle X \rangle$.

Definition 1. ([2,13,8]) A subset X of a topological group G is called a *suitable set* for G provided that:

- (i) X is discrete,
- (ii) $X \cup \{1_G\}$ is closed in G ,
- (iii) $\langle X \rangle$ is dense in G .

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Suitable sets were considered first in the early sixties by Tate in the framework of Galois cohomology (see [2]). Tate proved² that every profinite group has a suitable set. This result has later been proved also by Mel'nikov [13]. Later on, Hofmann and Morris discovered the following fundamental theorem.

Theorem 2. ([8, Theorem 1.12]) *Every locally compact group has a suitable set.*

Let us briefly outline main points of the proof from [8]. The authors first prove the existence of suitable sets in compact connected Abelian groups. This is accomplished by using the full strength of the theory of free compact Abelian groups [6]. The theorem for compact connected groups then follows from the Abelian compact connected case and the result of Kuranishi [11] that every compact connected simple group has a dense subgroup generated by two elements. Since compact totally disconnected groups have suitable sets by the results of Tate and Mel'nikov (cited above), the authors of [8] then combine connected and totally disconnected cases together to get the conclusion for all compact groups by deploying a theorem of Lee [12]: Every compact group G contains a closed totally disconnected subgroup K such that $G = c(G) \cdot K$, where $c(G)$ is the connected component of G . Having proved the result in compact case, Hofmann and Morris then proceed to deduce the general case from the compact case using some structure theorems for locally compact groups.

The main purpose of this article is to offer a direct, self-contained, purely topological proof of Theorem 2 based on Michael's selection theorem. Our proof is in the spirit of [18,17], and uses only elementary facts from (topological) group theory.

Theorem 2 allowed Hofmann and Morris [8] to introduce the *generating rank*

$$s(G) = \min\{|X|: X \text{ is a suitable set for } G\}$$

of a locally compact group G . (For profinite groups, $s(G)$ has been already defined by Mel'nikov [13].) As witnessed by the fact that the whole Chapter 12 of the monograph [9] by Hofmann and Morris is devoted to the study of this cardinal function (and its relation to the weight), $s(G)$ is undoubtedly one of the most important cardinal invariants of a (locally) compact group G .

Let G be a topological group. Following [1] define the *topologically generating weight* $tgw(G)$ of G by

$$tgw(G) = \min\{w(F): F \text{ is closed in } G \text{ and } \langle F \rangle \text{ is dense in } G\},$$

where $w(X) = \min\{|\mathcal{B}|: \mathcal{B} \text{ is a base of } X\} + \omega$ is the *weight* of a space X . The two principle results of [1] are summarized in the following

Theorem 3. *Let G be a compact group. Then:*

- (i) $tgw(G) = s(G)$ whenever $s(G)$ is infinite, and
- (ii) $tgw(G) = w(G/c(G)) \cdot \sqrt[\omega]{w(c(G))}$, where $c(G)$ is the connected component of G and $\sqrt[\omega]{\tau}$ is defined to be the smallest infinite cardinal κ such that $\kappa^\omega \geq \tau$.

The proof of this theorem in [1] is essentially topological and completely self-contained *with the only exception* of Theorem 2 which is still necessary. Our present manuscript completes the job started in [1] by providing a self-contained, purely topological proof of Theorem 2. It is worth mentioning that in [1, Section 9] Theorem 3 has been used to deduce (as straightforward corollaries) a series of major results from Chapter 12 of the monograph [9] by Hofmann and Morris.

2. Necessary facts

In this section we collect (mostly) well-known facts that will be used in the proof.

Recall that a map $f: X \rightarrow Y$ is

- (i) *open* provided that $f(U)$ is open in Y for every open subset U of X ,
- (ii) *closed* provided that $f(F)$ is closed in Y for every closed subset F of X ,
- (iii) *perfect* if f is a closed map and $f^{-1}(y)$ is compact for every $y \in Y$.

Fact 4. ([3, Proposition 3.7.5]) *Assume that $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are continuous surjections and the map $g \circ f: X \rightarrow Z$ is perfect. Then g is also perfect.*

For every $i \in I$ let $f_i: X \rightarrow Y_i$ be a map. The *diagonal product* $\Delta\{f_i: i \in I\}$ of the family $\{f_i: i \in I\}$ is a map $f: X \rightarrow \prod\{Y_i: i \in I\}$ which assigns to every $x \in X$ the point $\{f_i(x)\}_{i \in I}$ of the Cartesian product $\prod\{Y_i: i \in I\}$. (More precisely, f assigns to each $x \in X$ the point $f(x) \in \prod\{Y_i: i \in I\}$ defined by $f(x)(i) = f_i(x)$ for all $i \in I$.)

² This proof is extremely condensed. Detailed proofs can be found in [16] and [7].

Fact 5. ([3, Theorem 3.7.10]) For every $i \in I$ let $f_i : X \rightarrow Y_i$ be a continuous perfect map. Then the diagonal product $\Delta\{f_i : i \in I\}$ is also a continuous perfect map.

Fact 6. ([5, Chapter II, Theorem 5.18]) If N is a compact normal subgroup of a topological group G , then the quotient map from G onto its quotient group G/N is perfect.

Fact 7. Let $\pi : G \rightarrow H$ be a continuous group homomorphism from a topological group G onto a topological group H . If π is a quotient map, then π is also an open map. In particular, if π is a perfect map, then π is an open map.

Proof. The first statement follows from [5, Chapter II, Theorem 5.17]. To prove the second statement note that a perfect map is a closed map, and every closed map is a quotient map [3, Corollary 2.4.8]. \square

Fact 8. ([5, Chapter II, Theorem 5.11]) A locally compact subgroup G of a topological group H is closed in H .

Recall that a topological group G is *compactly generated* provided that there exists a compact subset K of G such that $G = \langle K \rangle$.

Fact 9. ([10]) If U is an open subset of a compactly generated, locally compact group G , then there exists a compact normal subgroup $N \subseteq U$ of G such that G/N has a countable base.

We note that in [5, Chapter II, Theorem 8.7] one finds a purely topological, elementary proof of Fact 9 that does not use the structure theory of locally compact groups.

Definition 10. If D is an infinite set, then $S(D) = D \cup \{*\}$ will denote the one-point compactification of the discrete set of size $|D|$. (Here $* \notin D$.) That is, all points of D are isolated in $S(D)$, and the family $\{S(D) \setminus F : F \text{ is a finite subset of } D\}$ consists of open neighbourhoods of a single non-isolated point $*$.

Note that $S(D)$ can be characterized as a compact Hausdorff space of size $|D|$ having precisely one non-isolated point. The relevance of this space to our topic can be seen from the following folklore fact.

Fact 11. If X is an infinite suitable set for a compact group G , then the subspace $X \cup \{1_G\}$ of G is compact and homeomorphic to the space $S(X)$.

Proof. Indeed, $X \cup \{1_G\}$ is closed in G by item (ii) of Definition 1. Since G is compact, so is $X \cup \{1_G\}$. Since X is an infinite discrete subset of G by item (i) of Definition 1, the point 1_G cannot be isolated in $X \cup \{1_G\}$ (otherwise $X \cup \{1_G\}$ would become an infinite discrete compact space). Hence, $X \cup \{1_G\}$ is a compact space with a single non-isolated point 1_G , and thus $X \cup \{1_G\}$ is homeomorphic to $S(X)$. \square

Fact 12. Assume that X is a compact space with a single non-isolated point x and $f : X \rightarrow Y$ is a continuous surjection of X onto an infinite space Y . Then Y is a compact space with a single non-isolated point $f(x)$.

Proof. We are going to show first that $Y \setminus V$ is finite for every open subset V of Y containing $f(x)$. Indeed, since $f : X \rightarrow Y$ is continuous, $U = f^{-1}(V)$ is an open subset of X containing x . Since every point of X different from x is isolated, $X \setminus U$ consists of isolated points of X . Since X is compact, we conclude that the set $X \setminus U$ is finite. Therefore, the set $Y \setminus V$ must be finite as well. Since Y is an infinite set, V must be infinite. Thus, $f(x)$ is a non-isolated point of Y .

Let us show next that Y is compact. Let \mathcal{V} be an open cover of Y . There exists $V \in \mathcal{V}$ such that $f(x) \in V$. For every $y \in Y \setminus V$ choose $V_y \in \mathcal{V}$ with $y \in V_y$. Now $\{V_y : y \in Y \setminus V\} \cup \{V\}$ is a finite subcover of \mathcal{V} .

Finally, let $y \in Y \setminus \{f(x)\}$. Since Y is Hausdorff, there exist open subsets W and V of Y such that $y \in W$, $f(x) \in V$ and $W \cap V = \emptyset$. Then $W \subseteq Y \setminus V$, and hence W is finite. Since every singleton is a closed subset of Y , it now follows that y is an isolated point of Y . \square

Our next lemma, which is in a certain sense the “converse” of Fact 11, is the key to building suitable sets in (compact-like) topological groups.

Lemma 13. Suppose that G is a topological group, X is an infinite set and $f : S(X) \rightarrow G$ is a continuous map such that $f(*) = 1_G$ and $\langle f(S(X)) \rangle$ is dense in G . Then $S = f(S(X)) \setminus \{1_G\}$ is a suitable set for G such that $S \cup \{1_G\}$ is compact.

Proof. Suppose first that $f(S(X))$ is a finite set. Then S is discrete, $S \cup \{1_G\}$ is compact and closed (being finite), and $\langle S \rangle = \langle S \cup \{1_G\} \rangle = \langle f(S(X)) \rangle$ is dense in G . Therefore, S is a suitable set for G .

Assume now that $f(S(X))$ is infinite. As an infinite continuous image of the compact space $S(X)$ with a single non-isolated point $*$, the space $f(S(X))$ is also a compact space with a single non-isolated point $f(*) = 1_G$ (Fact 12). Therefore, $S = f(S(X)) \setminus \{1_G\}$ is a discrete set and $S \cup \{1_G\}$ is compact (and thus closed in G). Moreover, $\langle S \rangle = \langle f(S(X) \setminus \{1_G\}) \rangle = \langle f(S(X)) \rangle$. Since the latter set is dense in G , we conclude that S is a suitable set for G . \square

Note that $S(\mathbb{N})$ is (homeomorphic to) a non-trivial convergence sequence together with its limit. The next fact is a key ingredient in our proof, so to make our manuscript self-contained we include its proof adapted from [4].

Fact 14. ([4]) *Let G be a compactly generated metric group. Then there exists a continuous map $f : S(\mathbb{N}) \rightarrow G$ such that $f(*) = 1_G$ and $\langle f(S(\mathbb{N})) \rangle$ is dense in G .*

Proof. Fix a local base $\{V_n : n \in \mathbb{N}\}$ at 1_G such that $V_0 = G$ and $V_{n+1} \subseteq V_n$ for all $n \in \mathbb{N}$. Let $G = \langle K \rangle$, where K is a compact subset of G . One can easily see that G is separable, so let $D = \{d_n : n \in \mathbb{N}\}$ be a countable dense subset of G .

Fix $n \in \mathbb{N}$. Since $\{xV_{n+1} : x \in G\}$ is an open cover of G and K is a compact subset of G , $K \subseteq \bigcup\{xV_{n+1} : x \in F_n\}$ for some finite set F_n . Now we have

$$G = \langle K \rangle \subseteq \left\langle \bigcup\{xV_{n+1} : x \in F_n\} \right\rangle \subseteq \langle F_n \cup V_{n+1} \rangle. \tag{1}$$

By induction on n we will define a sequence $\{E_n : n \in \mathbb{N}\}$ of finite subsets of G with the following properties:

- (i_n) $E_n \subseteq V_n$,
- (ii_n) $G \subseteq \langle E_0 \cup E_1 \cup \dots \cup E_n \cup V_{n+1} \rangle$, and
- (iii_n) $d_n \in \langle E_0 \cup E_1 \cup \dots \cup E_n \rangle$.

To begin with, note that the set $E_0 = F_0 \cup \{d_0\}$ satisfies all three conditions (i₀)–(iii₀). Suppose that we have already defined finite sets E_0, E_1, \dots, E_{n-1} such that conditions (i₀), \dots , (i_{n-1}), (ii₀), \dots , (ii_{n-1}) and (iii₀), \dots , (iii_{n-1}) are satisfied. Condition (ii_{n-1}) implies that

$$F_n \cup \{d_n\} \subseteq \langle E_0 \cup E_1 \cup \dots \cup E_{n-1} \cup V_n \rangle,$$

and since F_n is finite, we can find a finite set $E_n \subseteq V_n$ such that

$$F_n \cup \{d_n\} \subseteq \langle E_0 \cup E_1 \cup \dots \cup E_{n-1} \cup E_n \rangle. \tag{2}$$

Conditions (i_n) and (iii_n) are clear, and (ii_n) follows from (1) and (2).

From (i_n) for $n \in \mathbb{N}$ it follows that the set $S = \bigcup\{E_n : n \in \mathbb{N}\}$ forms a sequence converging to 1_G . Since (iii_n) holds for every $n \in \mathbb{N}$, we get $D \subseteq \langle S \rangle$, and so $\langle S \rangle$ is dense in G . Now take any bijection $f : \mathbb{N} \rightarrow S$ and define also $f(*) = 1_G$. \square

Recall that a *set-valued map* is a map $F : Y \rightarrow Z$ which assigns to every point $y \in Y$ a non-empty closed subset $F(y)$ of Z . This set-valued map is *lower semicontinuous* if $V = \{y \in Y : F(y) \cap U \neq \emptyset\}$ is open in Y for every open subset U of Z . A (single-valued) map $f : Y \rightarrow Z$ is called a *selection of F* provided that $f(y) \in F(y)$ for all $y \in Y$.

We finish this section with the following special case of Michael's selection theorem [14, Theorem 2] (see also [15]).

Fact 15. *A lower semicontinuous set-valued map $F : Y \rightarrow Z$ from a zero-dimensional (para)compact space Y to a complete metric space Z has a continuous selection $f : Y \rightarrow Z$.*

3. Lifting lemmas based on Michael's selection theorem

Lemma 16. *Suppose that K_0, K_1 are topological groups, N is a subgroup of the product $K_0 \times K_1$, and for each $i = 0, 1$ let $q_i = p_i \upharpoonright_N : N \rightarrow K_i$ be the restriction to N of the projection $p_i : K_0 \times K_1 \rightarrow K_i$ onto the i th coordinate. Assume also that:*

- (1) $K_i = p_i(N)$ for each $i = 0, 1$,
- (2) q_0 is an open map,
- (3) q_1 is a closed map,
- (4) K_1 is a complete metric space,
- (5) Y is a (para)compact zero-dimensional space and $h : Y \rightarrow K_0$ is a continuous map.

Then there exists a continuous map $g : Y \rightarrow N$ such that $h = q_0 \circ g$.

Proof. For $y \in Y$ define $F(y) = \{z \in K_1 : (h(y), z) \in N\}$. Note that $N \cap (\{h(y)\} \times K_1)$ is a closed subset of N , and so the set $F(y) = q_1(N \cap (\{h(y)\} \times K_1))$ must be closed in $q_1(N) = p_1(N) = K_1$ by (1) and (3). Since $h(y) \in K_0 = p_0(N)$ by (1) and (5), it follows that $F(y) \neq \emptyset$. Therefore $F : Y \rightarrow K_1$ is a set-valued map.

We claim that F is lower semicontinuous. Indeed, let U be an open subset of K_1 . Since $N \cap (K_0 \times U)$ is an open subset of N , $q_0(N \cap (K_0 \times U))$ is an open subset of $q_0(N) = p_0(N) = K_0$ by (1) and (2). Since $h : Y \rightarrow K_0$ is a continuous map by (5), $V = h^{-1}(q_0(N \cap (K_0 \times U)))$ is an open subset of Y . Now note that $V = \{y \in Y : F(y) \cap U \neq \emptyset\}$ by definitions of F and V .

In view of (4), the assumptions of Fact 15 are satisfied if one takes K_1 as Z . Let $f : Y \rightarrow K_1$ be a (single-valued) continuous selection of F which exists by the conclusion of Fact 15.

Define $g : Y \rightarrow K_0 \times K_1$ by $g(y) = (h(y), f(y))$ for $y \in Y$. Since both h and f are continuous, so is g . If $y \in Y$, then $g(y) = (h(y), f(y)) \in \{h(y)\} \times F(y)$ because f is a selection of F , which yields $g(y) \in N$ by the definition of $F(y)$. Therefore, $g(Y) \subseteq N$. The equality $h = q_0 \circ g$ is obvious from our definition of g . \square

In the sequel we will only need a particular case when the previous lemma is applicable:

Lemma 17. Suppose that G is a locally compact group, K_0 is a topological group, K_1 is a metric group, $\chi_i : G \rightarrow K_i$ is a continuous group homomorphism for $i = 0, 1$, $\chi = \chi_0 \Delta \chi_1 : G \rightarrow K_0 \times K_1$ is the diagonal product of maps χ_0 and χ_1 , and $N = \chi(G)$. Assume also that:

- (a) $K_i = \chi_i(G)$ for each $i = 0, 1$,
- (b) each χ_i is a perfect map,
- (c) Y is a (para)compact zero-dimensional space and $h : Y \rightarrow K_0$ is a continuous map.

Then there exists a continuous map $g : Y \rightarrow N$ such that $h = q_0 \circ g$, where $q_0 = p_0 \upharpoonright_N : N \rightarrow K_0$ is the restriction to N of the projection $p_0 : K_0 \times K_1 \rightarrow K_0$.

Proof. It suffices to check that N , Y and h satisfy all the assumptions of Lemma 16. (1) follows from (a). Let $i = 0, 1$. Since both $\chi : G \rightarrow N$ and $q_i : N \rightarrow K_i$ are surjections, $\chi_i = q_i \circ \chi$ and χ_i is a perfect map by item (b), q_i is a perfect map (Fact 4), and so also an open map (Fact 7). This yields both (2) and (3). Being an open continuous image of a locally compact space G , K_1 is locally compact. Since a locally compact metric space admits a complete metric, we get (4). Finally, (5) coincides with (c). Now the conclusion of our lemma follows from the conclusion of Lemma 16. \square

4. Proof of Theorem 2

If G and H are groups and $f : G \rightarrow H$ is a group homomorphism, then $\ker f = \{x \in G : f(x) = 1_H\}$ denotes the kernel of f . Obviously, $\ker f$ is a normal subgroup of G .

We are now ready to prove a specific version of Theorem 2. Our proof is based on representing a compactly generated, locally compact group as a limit of some inverse spectra (aka a projective limit in the terminology of algebraists) of locally compact separable metric groups. In order to make an exposition easier to comprehend for readers not familiar with inverse (aka projective) limits, we have chosen the presentation using diagonal products of maps, thereby allowing for a much simpler visualization of such a limit.

Theorem 18. Let G be a topological group generated by its open subset with compact closure. Then G has a suitable set S such that $S \cup \{1_G\}$ is compact.

Proof. Fix a local base $\{U_\alpha : \alpha < \tau\}$ at 1_G . If $\tau \leq \omega$, then G is a compactly generated metric group, and hence G has the desired suitable set by Fact 14 and Lemma 13.

From now on we will assume that $\tau \geq \omega_1$. Let X be a set with $|X| = \tau$. For every ordinal $\alpha < \tau$, apply Fact 9 to choose a compact normal subgroup N_α of G such that $N_\alpha \subseteq U_\alpha$ and $H_\alpha = G/N_\alpha$ has a countable base, and let $\psi_\alpha : G \rightarrow H_\alpha$ be the quotient map. For every ordinal α satisfying $1 \leq \alpha \leq \tau$ define $\varphi_\alpha = \Delta\{\psi_\beta : \beta < \alpha\} : G \rightarrow \prod\{H_\beta : \beta < \alpha\}$ and $G_\alpha = \varphi_\alpha(G)$. For $1 \leq \beta \leq \alpha \leq \tau$ let $\varpi_\beta^\alpha : \prod\{H_\gamma : \gamma < \alpha\} \rightarrow \prod\{H_\gamma : \gamma < \beta\}$ be the natural projection, and define $\pi_\beta^\alpha = \varpi_\beta^\alpha \upharpoonright_{G_\alpha} : G_\alpha \rightarrow G_\beta$ to be the restriction of ϖ_β^α to $G_\alpha \subseteq \prod\{H_\gamma : \gamma < \alpha\}$. Note that π_β^α is a surjection. By our construction,

$$\varphi_\alpha \circ \pi_\beta^\alpha = \varphi_\beta \quad \text{and} \quad \pi_\gamma^\alpha = \pi_\gamma^\beta \circ \pi_\beta^\alpha \quad \text{whenever} \quad 1 \leq \gamma \leq \beta \leq \alpha \leq \tau. \quad (3)$$

Claim 19. φ_α is a perfect map for every α with $1 \leq \alpha \leq \tau$.

Proof. Each ψ_β is a perfect map by Fact 6, so the map $\varphi_\alpha = \Delta\{\psi_\beta : \beta < \alpha\}$ is also perfect by Fact 5. \square

By transfinite recursion on α , for every ordinal α satisfying $1 \leq \alpha \leq \tau$ we will define a continuous map $f_\alpha : S(X) \rightarrow G_\alpha$ satisfying the following properties:

- (i_α) $f_β = π_β^α ∘ f_α$ whenever $1 ≤ β < α$,
- (ii_α) $f_α(*) = 1_{G_α}$,
- (iii_α) $|\{x ∈ X : f_α(x) ≠ 1_{G_α}\}| ≤ ω · |α|$,
- (iv_α) $\langle f_α(S(X)) \rangle$ is dense in $G_α$.

To motivate these conditions, we mention that (ii_α) and (iv_α) guarantee that $f_α(S(X)) \setminus \{1_{G_α}\}$ is a suitable set for $G_α$ (Lemma 13). The other two conditions (i_α) and (iii_α) are technical and needed only for carrying out the recursion construction.

We start our recursion with $α = 1$. First of all note that $φ_1 = ψ_0$ and $G_1 = H_0$. Being a continuous homomorphic image of a compactly generated group G , G_1 itself is compactly generated. Let N be a countable subset of X . Since $S(N)$ and $S(\mathbb{N})$ are homeomorphic, applying Fact 14 we can find a continuous map $f : S(N) → G_1$ such that $f(*) = 1_{G_1}$ and $\langle f(S(N)) \rangle$ is dense in G_1 . We extend this map to the continuous map $f_1 : S(X) → G_1$ by defining $f_1(x) = 1_{G_1}$ for every $x ∈ X \setminus N$ and $f_1(y) = f(y)$ for $y ∈ S(N)$. Now note that f_1 satisfies properties (i₁)–(iv₁).

Suppose now that $α$ is an ordinal with $1 < α ≤ τ$. Assume also that a continuous map $f_β : S(X) → G_β$ satisfying properties (i_β)–(iv_β) has been already defined for every ordinal $β$ such that $1 ≤ β < α$. We are going to define a continuous map $f_α : S(X) → G_α$ satisfying properties (i_α)–(iv_α). As usual, we consider two cases.

Case 1. $α = β + 1$ is a successor ordinal. Clearly, a subspace

$$Y_β = \{x ∈ X : f_β(x) ≠ 1_{G_β}\} ∪ \{*\} \tag{4}$$

of $S(X)$ is closed in $S(X)$. Hence, $Y_β$ is a compact space with at most one non-isolated point. In particular, $Y_β$ is zero-dimensional.

We claim that $K_0 = G_β$, $K_1 = H_β$, $χ_0 = φ_β$, $χ_1 = ψ_β$, $N = G_α$, $Y = Y_β$ and $h = f_β \upharpoonright_{Y_β}$ satisfy the assumptions of Lemma 17. Indeed, $χ = χ_0 ∆ χ_1 = φ_β ∆ ψ_β = φ_α$, and so $N = G_α = φ_α(G) = χ(G)$. (a) holds trivially. The map $χ_0 = φ_β$ is perfect by Claim 19, while $χ_1 = ψ_β$ is a perfect map by Fact 6. This proves (b). Since $f_β$ is a continuous map, so is $h = f_β \upharpoonright_{Y_β}$. This establishes (c).

Let $g : Y_β → G_α$ be a continuous map satisfying $f_β \upharpoonright_{Y_β} = π_β^α ∘ g$ which exists according to the conclusion of Lemma 17. Define $g' : Y_β → G_α$ by $g'(y) = g(y) · g(*)^{-1}$ for $y ∈ Y_β$. Clearly, g' is a continuous map and $g'(*) = 1_{G_α}$. If $y ∈ Y_β$, then $π_β^α ∘ g(*) = f_β \upharpoonright_{Y_β} (*) = f_β(*) = 1_{G_β}$ by (ii_β), and so

$$π_β^α(g'(y)) = π_β^α(g(y) · g(*)^{-1}) = π_β^α(g(y)) · π_β^α(g(*))^{-1} = f_β \upharpoonright_{Y_β}(y) · (1_{G_β})^{-1} = f_β \upharpoonright_{Y_β}(y)$$

because $π_β^α$ is a group homomorphism. This gives

$$π_β^α ∘ g' = f_β \upharpoonright_{Y_β}. \tag{5}$$

Since $β ≥ 1$, from (3) we have $\ker π_β^α = φ_α(\ker φ_β) ⊆ φ_α(\ker ψ_0) ⊆ φ_α(N_0)$. Since N_0 is compact, so is $φ_α(N_0)$. Being a closed subspace of $φ_α(N_0)$, $\ker π_β^α$ must be compact. Since $\ker π_β^α ⊆ \{1_{G_β}\} × H_β$ and $H_β$ has a countable base, $\ker π_β^α$ is a compact metric group.

Note that $|Y_β| ≤ ω · |β| < τ$ by (iii_β), and since $τ ≥ ω_1$, we can choose a countable set $Z_β ⊆ X$ with $Y_β ∩ Z_β = ∅$. Since $Z_β ∪ \{*\}$ is naturally homeomorphic to $S(\mathbb{N})$, Fact 14 allows us to find a continuous map $θ : Z_β ∪ \{*\} → \ker π_β^α ⊆ G_α$ such that $θ(*) = 1_{G_α}$ and $\langle θ(Z_β) \rangle$ is dense in $\ker π_β^α$.

Now define the map $f_α : S(X) → G_α$ by

$$f_α(x) = \begin{cases} g'(x) & \text{if } x ∈ Y_β, \\ θ(x) & \text{if } x ∈ Z_β, \\ 1_{G_α} & \text{if } x ∈ S(X) \setminus (Y_β ∪ Z_β). \end{cases}$$

Since both g' and $θ$ are continuous maps, one can easily check that the map $f_α$ is continuous as well.

Claim 20. $f_β = π_β^α ∘ f_α$.

Proof. If $y ∈ Y_β$, then $π_β^α(f_α(y)) = π_β^α(g'(y)) = f_β \upharpoonright_{Y_β}(y) = f_β(y)$ by (5).

Suppose now that $x ∈ S(X) \setminus Y_β$. We claim that $π_β^α(x) = 1_{G_β}$. Indeed, if $x ∈ Z_β$, then $π_β^α(f_α(x)) = π_β^α(θ(x)) = 1_{G_β}$ because $θ(x) ∈ θ(Z_β) ⊆ \ker π_β^α$. If $x ∈ S(X) \setminus (Y_β ∪ Z_β)$, then $f_α(x) = 1_{G_α}$, and so $π_β^α(f_α(x)) = π_β^α(1_{G_α}) = 1_{G_β}$. Finally, (4) and (ii_β) yields $f_β(x) = 1_{G_β} = π_β^α(f_α(x))$ for $x ∈ S(X) \setminus Y_β$. □

Let us check now conditions (i_α)–(iv_α).

(i_α) Suppose that $1 ≤ γ < α = β + 1$. If $γ = β$, then Claim 20 applies. Suppose now that $1 ≤ γ < β$. Then $π_γ^α ∘ f_α = π_γ^β ∘ π_β^α ∘ f_α = π_γ^β ∘ f_β = f_γ$ by (3), Claim 20 and (i_β).

(ii_α) $f_α(*) = g'(*) = 1_{G_α}$.

(iii_α) From the definition of $f_α$ one has $\{x \in X: f_α(x) \neq 1_{G_α}\} \subseteq Y_β \cup Z_β$, and so $|\{x \in X: f_α(x) \neq 1_{G_α}\}| \leq |Y_β| \cdot |Z_β| \leq \omega \cdot |\beta| \cdot \omega \leq \omega \cdot |\alpha|$ by (iii_β).

(iv_α) Let F be the closure of $\langle f_α(S(X)) \rangle$ in $G_α$. We need to show that $F = G_α$. Observe that $\langle f_α(Z_β) \rangle \subseteq \langle f_α(S(X)) \rangle \subseteq F$. Since $\langle f_α(Z_β) \rangle$ is dense in $\ker \pi_β^α$, it now follows that $\ker \pi_β^α \subseteq F$. Since both $\varphi_α$ and $\pi_β^α$ are surjections and $\pi_β^α \circ \varphi_α = \varphi_β$ is a perfect map by (3) and Claim 19, Fact 4 allows us to conclude that $\pi_β^α$ is a perfect (and hence also closed) map. Therefore, $\pi_β^α(F)$ is a closed subset of $G_β$. From (i_α) one gets $\pi_β^α(f_α(S(X))) = f_β(S(X))$, and since $\pi_β^α$ is a group homomorphism, one also has $\langle f_β(S(X)) \rangle = \pi_β^α(\langle f_α(S(X)) \rangle) \subseteq \pi_β^α(F)$. According to (iv_β), the set $\langle f_β(S(X)) \rangle$ is dense in $G_β$, and since $\pi_β^α(F)$ is closed in $G_β$, this yields $\pi_β^α(F) = G_β$. Since F is a subgroup of $G_α$ satisfying both $\ker \pi_β^α \subseteq F$ and $\pi_β^α(F) = G_β = \pi_β^α(G_α)$, one obtains $F = G_α$.

Case 2. α is a limit ordinal. Define

$$L_α = \left\{ h \in \prod \{H_β: β < α\}: h \upharpoonright_β \in G_β \text{ whenever } 1 \leq β < α \right\}. \tag{6}$$

Claim 21. Suppose that $H \subseteq L_α$ and $\{h \upharpoonright_β: h \in H\}$ is dense in $G_β$ whenever $1 \leq β < α$. Then H is dense in $L_α$.

Proof. Let U be an open subset of the product $\prod \{H_β: β < α\}$ such that $U \cap L_α \neq \emptyset$. Pick arbitrarily $g \in U \cap L_α$. There exist $n \in \mathbb{N}$, pairwise distinct ordinals $\gamma_0, \gamma_1, \dots, \gamma_n < α$ and an open subset V_i of H_{γ_i} for every $i \leq n$ such that $g(\gamma_i) \in V_i$ for all $i \leq n$ and

$$\left\{ h \in \prod \{H_β: β < α\}: h(\gamma_i) \in V_i \text{ for all } i \leq n \right\} \subseteq U. \tag{7}$$

Since α is a limit ordinal, $\beta = \max\{\gamma_i: i \leq n\} + 1 < \alpha$. Note that

$$W = \left\{ h \in \prod \{H_\gamma: \gamma < \beta\}: h(\gamma_i) \in V_i \text{ for all } i \leq n \right\} \tag{8}$$

is an open subset of $\prod \{H_\gamma: \gamma < \beta\}$ and $g \upharpoonright_\beta \in W$. Since $g \in L_α$, one has $g \upharpoonright_\beta \in G_β$ by (6). It follows that $g \upharpoonright_\beta \in W \cap G_β \neq \emptyset$. By the assumption of our claim, there exists some $h \in H$ such that $h \upharpoonright_\beta \in W$. Now from (7), (8) and the choice of β we get $h \in U$. Thus $h \in H \cap U \neq \emptyset$. \square

Claim 22. $G_α \subseteq L_α$ and $G_α$ is dense in $L_α$.

Proof. Let $h \in G_α$. Then $h = \varphi_α(g)$ for some $g \in G$. For every ordinal β satisfying $1 \leq \beta < \alpha$ one has $h \upharpoonright_\beta = \varphi_β(g) \in G_β$, which yields $h \in L_α$ by (6). Thus, $G_α \subseteq L_α$.

Assume that β is an ordinal satisfying $1 \leq \beta < \alpha$. Let $h' \in G_β$. Then $h' = \varphi_β(g)$ for some $g \in G$. Now $h = \varphi_α(g) \in G_α$ and $h \upharpoonright_\beta = \varphi_β(g) = h'$. This yields $G_β \subseteq \{h \upharpoonright_\beta: h \in G_α\}$. The converse inclusion $\{h \upharpoonright_\beta: h \in G_α\} \subseteq G_β$ is trivial. This shows that $\{h \upharpoonright_\beta: h \in G_α\} = G_β$.

Therefore, $G_α$ (taken as H) satisfies the assumptions of Claim 21, so $G_α$ must be dense in $L_α$ by the conclusion of this claim. \square

Claim 23. $G_α = L_α$.

Proof. The map $\varphi_α$ is open by Claim 19 and Fact 7. As an open continuous image of a locally compact group G , the group $G_α = \varphi_α(G)$ is also locally compact. Since $L_α$ is a topological group containing $G_α$ (Claim 22), $G_α$ must be closed in $L_α$ (Fact 8). Since $G_α$ is also dense in $L_α$ (Claim 22), the conclusion of our claim follows. \square

We are now ready to define $f_α : S(X) \rightarrow G_α$. Let $x \in S(X)$ be arbitrary. Since (i_β) holds for every ordinal β satisfying $1 \leq \beta < \alpha$, there exists a unique $h_x \in L_α$ such that $h_x \upharpoonright_\beta = f_β(x)$ for all β with $1 \leq \beta < \alpha$. Now $h_x \in G_α$ by Claim 23, and so we can define $f_α(x)$ to be this unique h_x .

Let us check now conditions (i_α)–(iv_α). Condition (i_α) clearly holds. Since each $f_β$ is a continuous map, so is $f_α$. (ii_β) for $1 \leq \beta < \alpha$ trivially implies (ii_α). Similarly, (iii_β) for $1 \leq \beta < \alpha$ yields (iii_α). To check (iv_α) it suffices to show, in view of Claim 23, that $H = \langle f_α(S(X)) \rangle \subseteq G_α = L_α$ satisfies the assumptions of Claim 21. Indeed, assume $1 \leq \beta < \alpha$. Since $\pi_β^α$ is a group homomorphism, from (i_α) one has

$$\{h \upharpoonright_\beta: h \in H\} = \{\pi_β^α(h): h \in H\} = \pi_β^α(\langle f_α(S(X)) \rangle) = \langle \pi_β^α(f_α(S(X))) \rangle = \langle f_β(S(X)) \rangle,$$

and the latter set is dense in $G_β$ by (iv_β).

The recursive construction has been complete.

According to (ii_τ), we have $f_τ(*) = 1_{G_τ}$. According to (iv_τ), $\langle f_τ(S(X)) \rangle$ is dense in $G_τ$. From Lemma 13, we conclude that $S = f_τ(S(X)) \setminus \{1_{G_τ}\}$ is a suitable set for $G_τ$ such that $S \cup \{1_{G_τ}\}$ is compact.

Now observe that $\ker \varphi_\tau \subseteq \bigcap \{N_\alpha : \alpha < \tau\} \subseteq \bigcap \{U_\alpha : \alpha < \tau\} = \{1_G\}$, and hence $\varphi_\tau : G \rightarrow G_\tau$ is an algebraic isomorphism. Furthermore, φ_τ is a perfect map by Claim 19. Finally, note that a one-to-one continuous perfect map is a homeomorphism. Thus, G and G_τ are isomorphic as topological groups. \square

Proof of Theorem 2. Let H be a locally compact group. Take an open neighbourhood U of the identity 1_H that has a compact closure \bar{U} in H . Then $G = \langle U \rangle$ is an open (and thus closed [5, Chapter II, Theorem 5.5]) subgroup of H . In particular, $\bar{U} \subseteq \bar{G} = G$, and so G is generated by its open subset U with compact closure (in G). According to Theorem 18, G has a suitable set S . Choose $X \subseteq H \setminus G$ such that $\{xG : x \in X\}$ forms a (faithfully indexed) partition of $H \setminus G$. One can easily check now that $S \cup X$ is a suitable set for H . \square

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