Axial loading of elliptical-section bonded rubber blocks

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Abstract

Closed-form expressions for the small axial deflection and stress distribution of axially loaded rubber blocks of elliptical cross-section, whose ends are bonded to rigid plates, are derived using a superposition approach. The governing equations and conditions are satisfied exactly, based upon the classical theory of elasticity. Easily calculable expressions are derived for the corresponding apparent Young’s modulus and the modified apparent Young’s modulus in forms analogous to those previously given for blocks of circular cross-section.

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1. Introduction

Rubber mountings which are bonded to rigid metallic end plates are used in a large variety of modern engineering components. The necessary design specifications are dependent upon the availability of readily calculable predictions of their stiffness and stress distribution under specified applied loads.

An analysis is presented here for a rubber block of right-elliptical cross-section with one end remaining in a fixed position while the other end is subjected to an axial load of constant magnitude. The analytical techniques used are generalizations of those developed by Horton et al. (2002, 2003) in studying blocks of circular and annular cross-sections.

Gent and Lindley (1959) and Gent (1994) developed widely used approximate relations for the apparent Young’s modulus, $E_a$, for bonded blocks of incompressible rubber subjected to compression. It was reasonably proposed by Gent and Lindley (1959) that the bulk compression of the block could be incorporated for blocks of high shape factor by introducing a modified apparent Young’s modulus, $E'_a$, given by

$$
\frac{1}{E'_a} = \frac{1}{E_a} + \frac{1}{K},
$$

(1)

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where $K$ is the bulk modulus of the rubber. They presented approximate expressions to $E_a$ for blocks of circular and long rectangular cross-section, which were derived not only with the reasonable assumption that plane cross-sections of the block normal to the axis of the block remain plane but also with the free lateral surfaces assumed to have parabolic deformed shapes. On the other hand, the analyses of Horton et al. (2002, 2003) were completed without the necessity of making this second assumption, and in fact they confirmed the invalidity of assuming a parabolic lateral profile especially for blocks whose shape factor is small. Subsequently, Gent and Meinecke (1970) suggested that the apparent Young’s modulus, $E_{a}^{GM}$, for a block of elliptical cross-section, with semi-major and minor axes $a$ and $b$, and of axial height $h$, could be determined from

$$E_{a}^{GM} = E \left[ \frac{4}{3} \left( \frac{ab + h^2}{a^2 + b^2 + 2h^2} \right) + \frac{a^2b^2}{(a^2 + b^2)h^2} \right]$$

with $E$ being the Young’s modulus of the rubber of the block.

The geometry of the axially loaded block and the fundamental technique of superposing two particular loading situations are formulated in Section 2. Then, in Sections 3 and 4, these loading cases are analyzed separately with explicit closed-form representations for the displacement of the loaded end being derived in each case, again with no assumption that the free lateral surfaces have parabolically deformed shapes. The corresponding apparent Young’s modulus and modified modulus are presented in Section 5 and compared with those previously suggested. Finally, the components of the displacement and stress created in the rubber are studied in detail in Sections 6 and 7.

2. Formulation

Consider a right-cylindrical rubber block of uniform elliptical cross-section with semi-major and semi-minor axes $a$ and $b$, respectively. A rectangular Cartesian coordinate system $(x, y, z)$ is defined relative to an origin $O$, with $Oz$ along the central axis of the block and its plane ends at $z = 0$ and $z = h$ and with the major and minor axes of the end at $z = 0$ along the $x$ and $y$ axes, respectively. The cylindrical polar coordinates $(r, \theta, z)$ of a point $P$ within the block are then related to its rectangular Cartesian coordinates by the equations:

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z.$$  

The rubber is assumed to be isotropic and homogeneous, with the displacement gradients remaining sufficiently small throughout the subsequent deformations for it to be permissible to apply the classical linear theory of elasticity. Rigid end plates are bonded to the rubber at $z = 0$ and $z = h$, which prevents any distortion of its end surfaces.

The end of the block at $z = h$ is subjected to a load of constant magnitude $F$ along the $z$-axis with the other end at $z = 0$ supposed held in a fixed position. This extends or compresses the block by a distance $d$. If the loading is tensile, the force-free lateral surface will be drawn inwards, as simplistically depicted in Fig. 1, but will bulge outwards if the load is compressive. The displacements created within the rubber are evaluated here by superposing the displacements occurring in two particular individual loading situations, as illustrated in Fig. 1.

In Case A, any distortion of the lateral surface is prevented by the application to it of a tensile stress of magnitude $\sigma_L$ while the block is subject to the axial tensile load. The end face at $z = h$ will be displaced a distance $d_A$, because of the extension created by the small bulk distortion. On the other hand, in Case B a compressive stress equal and opposite to that in Case A is applied to the lateral surface of the same block. This will depress the lateral surface and, with the rubber being regarded as incompressible since its Poisson’s ratio is very close to 1/2, will extend the block axially by a distance $d_B$.

The effects of the lateral surface loadings cancel each other if Cases A and B are superposed, with the total displaced distance, $d$, of the block end at $z = h$ then being given by $d_A + d_B$.

3. Case A: Undistorted lateral surface with axial end load

Suppose that a tensile stress of magnitude $\sigma_L$ is applied normal to the lateral surface of the block, with the end at $z = 0$ fixed, which restrains it to remain undistorted and parallel to the $z$-axis. An axial tensile load $F$ is applied on the plane end face at $z = h$. 
For an elliptical block of initial volume $V$, the resulting bulk dilation, $\delta V$, is then given by

$$ \frac{\delta V}{V} = \frac{F}{\pi a b K}, $$

with $K$ being the bulk modulus of the rubber, and the block material is everywhere in a state of hydrostatic tensile stress of magnitude, $\sigma$, given by

$$ \sigma = \sigma_L = \frac{F}{\pi a b}. $$

Since the cross-sectional area is restrained to remain constant, the resulting deflection, $d_A$, of the end at $z = h$ is

$$ d_A = \frac{F h}{\pi a b K}. $$

4. Case B: Loaded lateral surface

Suppose now that only a normal compressive stress of magnitude $\sigma_L$, given by Eq. (5), which is equal and opposite to that in Case A, is applied around the lateral surface of the block, with the peripheral shear stress equal to zero.

The displacement components at $P$ relative to the cylindrical polar coordinates are denoted by $u_r$, $u_\theta$, and $u_z$, and the corresponding strain and stress components by $\varepsilon_{ij}$ and $\sigma_{ij}$, respectively, with $i, j = r, \theta$ or $z$, in the usual notation.

The commonly adopted reasonable assumption that plane cross-sections normal to the $z$-axis remain plane is made here, so that

$$ \frac{\partial u_z}{\partial r} = \frac{\partial u_z}{\partial \theta} = 0, $$

and the incompressibility condition implies that, for small strains,

$$ \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz} = 0. $$

Then the strain-displacement gradient relations and the constitutive equations can be written as:
\[ \varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}, \]

\[ \varepsilon_{r\theta} = \frac{1}{2} \left( \frac{\partial u_\theta}{\partial r} - \frac{u_r}{r} \right), \]

\[ \varepsilon_{rz} = \varepsilon_{r\theta} = \frac{1}{2} \frac{\partial u_\theta}{\partial z}, \quad \varepsilon_{0z} = \varepsilon_{z0} = \frac{1}{2} \frac{\partial u_\theta}{\partial z}, \]

and

\[ \varepsilon_{rr} = \frac{1}{3\mu} \left[ \sigma_{rr} - \frac{1}{2} (\sigma_{\theta\theta} + \sigma_{zz}) \right], \quad \varepsilon_{\theta\theta} = \frac{1}{3\mu} \left[ \sigma_{\theta\theta} - \frac{1}{2} (\sigma_{rr} + \sigma_{zz}) \right], \]

\[ \varepsilon_{zz} = \frac{1}{3\mu} \left[ \sigma_{zz} - \frac{1}{2} (\sigma_{rr} + \sigma_{\theta\theta}) \right], \quad \sigma_{\theta\theta} = \sigma_{\theta r} = 2\mu\varepsilon_{\theta r}, \]

\[ \sigma_{rz} = \sigma_{zr} = 2\mu\varepsilon_{rz}, \quad \sigma_{0z} = \sigma_{z0} = 2\mu\varepsilon_{z0}, \]

where \( \mu \) is the shear modulus, and the equilibrium equations which must be fulfilled in the \( r \) and \( \theta \) directions are:

\[ \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \]

\[ \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{\theta r}}{r} = 0. \]  \( \tag{11} \)

In general, it would be reasonable to endeavour to construct trigonometric series representations for the corresponding displacement components \( u_r \) and \( u_\theta \) at the general point \( P \) in the rubber. However, in retrospect and for the sake of brevity, it is found to be sufficient for satisfying the required boundary conditions here to seek expressions for the displacement components in the forms:

\[ u_r = U_0(r,z) + U(r,z) \cos 2\theta, \quad u_\theta = V(r,z) \sin 2\theta, \quad u_z = W(z), \]  \( \tag{12} \)

with the functions \( U_0, U, \) and \( V \) depending upon \( r \) and \( z \), and the function \( W \) depending only upon \( z \).

By substituting these into the incompressibility condition (8), using Eqs. (9)–(9)3, it follows that:

\[ \frac{\partial U_0}{\partial r} + \frac{U_0}{r} + \frac{dW}{dz} + \left( \frac{\partial U}{\partial r} + \frac{U}{r} + \frac{2V}{r} \right) \cos 2\theta = 0. \]  \( \tag{13} \)

Since Eq. (13) must hold for all values of \( \theta \), it yields

\[ \frac{\partial U_0}{\partial r} + \frac{U_0}{r} + \frac{dW}{dz} = 0 \]  \( \tag{14} \)

together with

\[ \frac{\partial U}{\partial r} + \frac{U}{r} + \frac{2V}{r} = 0. \]  \( \tag{15} \)

It is necessary that \( u_r = 0 \) when \( r = 0 \), due to the symmetry of the loading, and thus by direct integration with respect to \( r \) of Eq. (14) and substitution into Eq. (12)1

\[ u_r = -\frac{r}{2} \frac{dW}{dz} + U \cos 2\theta \]  \( \tag{16} \)

with

\[ U(0,z) = 0. \]  \( \tag{17} \)

On the other hand, combination of the condition (15) with the form (12)2 gives

\[ u_\theta = -\frac{r}{2} \left( \frac{\partial U}{\partial r} + \frac{U}{r} \right) \sin 2\theta. \]  \( \tag{18} \)
Moreover, manipulation of the first two constitutive equations $(10)_1$ and $(10)_2$ produces the more convenient expressions:

$$\sigma_{00} = \sigma_{rr} - 2\mu(2\varepsilon_{rr} + \varepsilon_{zz}),$$
$$\sigma_{zz} = \sigma_{rr} - 2\mu(\varepsilon_{rr} + \varepsilon_{zz}).$$

Then, by combining the representations $(16)$ and $(18)$ into Eqs. $(19)$ and the constitutive equations $(10)_4$–$(10)_6$, using Eqs. $(9)$, the equilibrium equations $(11)$ can be written entirely in terms of $\sigma_{rr}, U, W$ and their derivatives as:

$$\frac{\partial \sigma_{rr}}{\partial r} - \mu \left[ \left( \frac{\partial^2 U}{\partial r^2} - \frac{3}{r^2} \frac{\partial U}{\partial r} + \frac{3U}{r^2} - \frac{\partial^2 U}{\partial z^2} \right) \cos 2\theta + \frac{r}{2} \frac{d^3 W}{dz^3} \right] = 0,$$  

$$\frac{\partial \sigma_{rr}}{\partial \theta} - \mu \left( \frac{\partial^2 U}{\partial \theta^2} + \frac{4\partial^2 U}{r^2 \partial r^2} - \frac{11}{r^2} \frac{\partial U}{\partial r} + \frac{3U}{r^3} + \frac{1}{r^2} \frac{\partial^2 U}{\partial z^2} + \frac{3\partial^2 U}{\partial r \partial z} \right) \sin 2\theta = 0.$$  

Elimination of $\sigma_r$ between Eqs. $(20)$ and $(21)$ leads to a governing differential equation for $U$ as

$$\frac{\partial^4 U}{\partial r^4} + \frac{6}{r} \frac{\partial^3 U}{\partial r^3} - \frac{3}{r^2} \frac{\partial^2 U}{\partial r^2} - \frac{9}{r^3} \frac{\partial U}{\partial r} - \frac{3}{r^4} \frac{\partial^2 U}{\partial r^2} + \frac{6}{r^2} \frac{\partial^2 U}{\partial r \partial z} + \frac{\partial^2 U}{\partial z^2} - \frac{3U}{r^2} - \frac{3U}{r^2} = 0.$$  

Now, this fourth-order equation can be integrated with respect to $r$ twice, to show that if an explicit expression for $U$ is required it will in general be necessary to solve the second-order equation

$$\frac{\partial^3 U}{\partial r^3} + \frac{3}{r} \frac{\partial^2 U}{\partial r^2} - \frac{3}{r^2} \frac{\partial U}{\partial r} - \frac{9}{r^3} \frac{\partial^2 U}{\partial r^2} + \frac{9}{r^4} \frac{\partial^2 U}{\partial r \partial z} + \frac{6}{r^2} \frac{\partial^2 U}{\partial r \partial z} + \frac{\partial^2 U}{\partial z^2} = \frac{f_1(z)}{r^3} + \frac{f_2(z)}{r^2},$$  

where $f_1$ and $f_2$ are arbitrary functions of $z$.

However, the first objective here is to derive a formula for the calculation of the axial stiffness of the block, and it transpires that in fact the axial stress component, $\sigma_{zz}$, which yields the differential equation from which the axial displacement component, $u_z$, can be found does not depend upon $U$. This is demonstrated by first finding an expression for the radial stress component, $\sigma_{rr}$.

Substitution of $\partial^2 U/\partial z^2$ as obtained from Eq. $(23)$ into Eqs. $(20)$ and $(21)$ yields the coupled equations:

$$\frac{\partial \sigma_{rr}}{\partial r} - \mu \left[ \left( \frac{2}{r^2} \frac{\partial U}{\partial r} - \frac{f_1(z)}{r^3} - \frac{f_2(z)}{r^2} \right) \cos 2\theta + \frac{r^2}{4} \frac{d^3 W}{dz^3} \right] = 0,$$  

$$\frac{\partial \sigma_{rr}}{\partial \theta} + \mu \left( \frac{4}{r} \frac{\partial U}{\partial r} + \frac{f_1(z)}{r^2} - \frac{f_2(z)}{r^3} \right) \sin 2\theta = 0.$$  

These can be directly integrated with respect to $r$ and $\theta$, respectively, to give two alternative forms for $\sigma_{rr}$, but for them to be equivalent and to ensure that $\sigma_{rr}$ remains finite when $r = 0$ it is necessary that $f_1(z) = 0$ and then

$$\sigma_{rr} = \mu \left[ \left( \frac{2}{r^2} \frac{\partial U}{\partial r} - \frac{f_2(z)}{2} \right) \cos 2\theta + \frac{r^2}{4} \frac{d^3 W}{dz^3} + g(z) \right],$$  

with $g$ being an arbitrary function of $z$. Thus, by substitution into Eqs. $(19)$, and recalling Eqs. $(9)_1, (9)_3$ and $(16)$, it follows that

$$\sigma_{00} = \mu \left[ \left( \frac{2}{r^2} \frac{\partial U}{\partial r} - \frac{f_2(z)}{2} \right) \cos 2\theta + \frac{r^2}{4} \frac{d^3 W}{dz^3} + g(z) \right]$$  

and

$$\sigma_{zz} = \mu \left[ \frac{r^2}{4} \frac{d^3 W}{dz^3} + \frac{3}{r^2} \frac{dW}{dz} - \frac{f_2(z)}{2} \cos 2\theta + g(z) \right].$$  

The expressions $(26)$–$(28)$ for $\sigma_{rr}, \sigma_{00},$ and $\sigma_{zz}$ have been derived to satisfy the equilibrium equations $(11)$, and the values of the functions $f_2(z)$ and $g(z)$ which are needed to fulfil the imposed boundary conditions on the lateral surface of the block can now be determined.
Consider the elemental right-angled triangle $BTR$ shown in Fig. 2 which is constructed in a $z = \text{constant}$ plane with $B$ as a point having cylindrical polar coordinates $(r_B, \theta, z)$ on the elliptical lateral surface of the block, $BT$ as a line tangential to the surface and $R$ as a point on the line $CB$. The line $AB$ is in the direction of the normal to the surface and makes an angle $\phi$ with the direction of the $x$-axis. The lateral surface is assumed to be subjected to a compressive stress of magnitude $\sigma_L$. It follows therefore, by resolving the forces acting on the faces $TB$, $RB$ and $TR$ in the directions normal and tangential to the surface and taking the limit as $TB \to 0$, that at the point $B$

$$- \sigma_L = (\sigma_{rr})_B \cos^2(\phi - \theta) + (\sigma_{\theta\theta})_B \sin^2(\phi - \theta) + (\sigma_{\phi\phi})_B \sin(\phi - \theta),$$

$$0 = -\frac{1}{2}[(\sigma_{rr})_B - (\sigma_{\theta\theta})_B] \sin 2(\phi - \theta) + (\sigma_{\phi\phi})_B \cos 2(\phi - \theta),$$

where throughout $(\cdot \cdot \cdot)_B$ denotes the value of $(\cdot \cdot \cdot)$ at the point $B$. Elimination of $(\sigma_{\phi\phi})_B$ from the coupled Eqs. (29) yields the boundary condition

$$(\sigma_{rr})_B = [(\sigma_{\theta\theta})_B + \sigma_L] \tan^2(\phi - \theta) - \sigma_L, \quad \text{for } 0 < z < h \text{ and } 0 < \theta < 2\pi.$$ (30)

Now, by the properties of an ellipse, with $k = a/b$ and $t = \tan \theta$,

$$r_B^2 = \frac{a^2(1 + t^2)}{1 + k^2 t^2}, \quad \tan(\phi - \theta) = \frac{(k^2 - 1)t}{1 + k^2 t^2},$$ (31)

which, upon using the relationship $\cos \theta = (1 - t^2)/(1 + t^2)$ and evaluating the stress components given by Eqs. (26) and (27) at $B$, enable the condition (30) to be written in the form

$$\left[ \frac{a^2}{4} \left( \frac{1 + t^2}{1 + k^2 t^2} \right) \frac{d^3 W}{dz^3} + \frac{\sigma_L}{\mu} + g(z) - \frac{a^2}{2} \left( \frac{1 - t^2}{1 + k^2 t^2} \right) f_2(z) \right] [k^4 t^4 - (k^4 - 4k^2 + 1)t^2 + 1]$$

$$+ 2 \left( \frac{1 - t^2}{1 + t^2} \right) [k^4 t^4 + (k^4 + 1)t^2 + 1] \left( \frac{\partial U}{\partial r} \right)_B = 0,$$ (32)

for $0 < z < h$ and all $t$. This is satisfied with

$$\left( \frac{\partial U}{\partial r} \right)_B = 0, \quad \text{for } 0 < z < h;$$ (33)

by

Fig. 2. Geometry of an elliptical cross-section through $z = 0$. 

\[ f_z(z) = \frac{1}{2} \left( \frac{k^2 - 1}{k^2 + 1} \right) d^3W/dz^3 \]  

(34)

and

\[ g(z) = -\frac{\sigma_L}{\mu} - \frac{a^2}{2(k^2 + 1)} d^3W/dz^3 \]  

(35)

and the axial stress component given by Eq. (28) then becomes

\[ \sigma_z = -\sigma_L - \mu \left[ \frac{a^2 - r^2(k^2 \sin^2 \theta + \cos^2 \theta)}{2(k^2 + 1)} d^3W/dz^3 - 3 \frac{dW}{dz} \right]. \]  

(36)

However, in this loading case, there is no axial force applied and so

\[ \int_0^2 \int_0^{\rho_0} \sigma_{zz} r dr d\theta = 0. \]  

(37)

Evaluating this integral, using Eq. (36), leads to the governing differential equation for \( W \) as

\[ \frac{d^3W}{dz^3} - \frac{12(k^2 + 1)}{a^2} \frac{dW}{dz} = \frac{-4(k^2 + 1)\sigma_L}{\mu a^2}. \]  

(38)

The rubber is bonded to rigid end plates, so that \( u_r = u_\theta = 0 \) for all \( r \) and \( \theta \) at \( z = 0 \) and \( z = h \), and hence, from Eqs. (16) and (18),

\[ U = \frac{\partial U}{\partial r} = \frac{dW}{dz} = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = h \quad \text{for all} \quad r \quad \text{and} \quad \theta. \]  

(39)

Further, since the end at \( z = 0 \) is fixed

\[ W = 0 \quad \text{at} \quad z = 0. \]  

(40)

The solution of Eq. (38) satisfying the conditions (39) and (40) can be written as

\[ u_z \equiv W(z) = \frac{\sigma_L}{3\mu} \left\{ z - \frac{2}{a} \sinh \frac{h}{a} \frac{\cosh \left[ \frac{a}{2}(h - z) \right]}{\cosh \frac{a}{2}} \right\}. \]  

(41)

where

\[ a^2 = \frac{12(k^2 + 1)}{\sigma_L}. \]  

(42)

With the Young’s modulus, \( E \), of the rubber given by \( E = 3\mu \) and noting the relation (5), the distance, \( d_B \), through which the end of the block at \( z = h \) is displaced is thus given by

\[ d_B = \frac{Fh}{\pi a b E_a} \left( 1 - \frac{2}{zh} \tan \frac{zh}{2} \right). \]  

(43)

It is instructive to note at this stage that the expressions (42) and (43) do indeed reduce to the corresponding results by Horton et al. (2002, Eqs. (48) and (49)) for blocks of circular cross-sections when \( k = 1 \).

5. Apparent Young’s modulus

When subjected to the axial load \( F \) only, the axial deflection, \( d = d_A + d_B \) of the end \( z = h \) of the block is found by superposition of the distances (6) and (43) obtained in Cases A and B. This can be expressed in terms of the modified apparent Young’s modulus, or “measured apparent Young’s modulus”, \( E'_a \), that was considered by Gent and Lindley (1959) as

\[ d = \frac{Fh}{\pi a b E'_a}, \]  

(44)
where therefore
\[ \frac{1}{E_a} = \frac{1}{E} \left( 1 - \frac{2}{zh} \tanh \frac{zh}{2} \right) + \frac{1}{K}. \] (45)

It follows that the apparent Young’s modulus, \( E_a \), of an elliptical block of incompressible rubber can be written as
\[ E_a = \frac{E}{1 - \frac{1}{zh} \tanh \frac{zh}{2}}. \] (46)

In comparison, Gent and Meinecke (1970) proposed the expression (2) as a “reasonable empirical form” for the apparent Young’s modulus. Clearly, for long blocks with \( h \gg a \), both \( E_a \rightarrow E \) and \( E_a^{GM} \rightarrow E \) as \( h \rightarrow \infty \). On the other hand, for short blocks with \( h \ll b \) the form (2) can be approximated by
\[ E_a^{GM \text{ approx}} = E \left[ \frac{4}{3} + \frac{2ab}{3(a^2 + b^2)} + \frac{a^2b^2}{(a^2 + b^2)h^2} \right], \] (47)

and, when the hyperbolic tangent is represented by the convenient rational function approximation
\[ \tanh x \approx \frac{x(15 + x^2)}{3(5 + 2x^2)}, \] (48)

the result (46) is very closely approximated by
\[ E_a^{\text{approx}} = E \left[ 1.2 + \frac{a^2b^2}{(a^2 + b^2)h^2} \right]. \] (49)

Moreover, in particular, from Eqs. (49) and (47), if \( a \gg b \)
\[ E_a^{\text{approx}} \rightarrow E \left( 1.2 + \frac{b^2}{h^2} \right), \] (50)
\[ E_a^{GM \text{ approx}} \rightarrow E \left( \frac{4}{3} + \frac{b^2}{h^2} \right) \] (51)
as \( h \rightarrow 0 \).

Further, it is interesting to note that analogously if \( a = b \)
\[ E_a^{\text{approx}} \rightarrow E \left( 1.2 + \frac{a^2}{2h^2} \right), \] (52)
\[ E_a^{GM \text{ approx}} \rightarrow E \left( 1 + \frac{a^2}{2h^2} \right) \] (53)
as \( h \rightarrow 0 \), which reproduce, respectively, the representations for a circular block previously derived by Horton et al. (2002, Eq. (56)) and Gent and Lindley (1959, Eq. (2)).

6. Solution for \( U(r, z) \)

It is now convenient to consider further the function \( U \) that, as was shown earlier, must satisfy the governing equation (23). Recalling that the boundary conditions were subsequently found to be fulfilled only with \( f_1(z) = 0 \) and \( f_2(z) \) given by Eq. (34), this becomes
\[ \frac{\partial^2 U}{\partial r^2} + \frac{3}{r} \frac{\partial U}{\partial r} - \frac{3U}{r^2} + \frac{\partial^2 U}{\partial z^2} = r \left( \frac{k^2 - 1}{k^2 + 1} \right) \frac{d^3 W}{dz^3}, \] (54)

with \( W \) given by Eq. (41). It can be shown that the solution of Eq. (54) which satisfies the boundary conditions (17), (39)\(_1\) and (39)\(_2\) can be written as
for arbitrary constants \( A_n \) with \( n = 1, 2, \ldots, \infty \), where \( I_2(x) \) is the modified Bessel function of order 2.

The final boundary condition upon \( U \), which enables the constants \( A_n \) to be determined, is given by Eq. (33). This is fulfilled if, for \( 0 < z < h \),

\[
\sum_{n=1}^{\infty} \frac{n\pi}{hr_B} A_n J_1 \left( \frac{n\pi r_B}{h} \right) \left[ 1 - \frac{3h}{n\pi r_B} I_1 \left( \frac{n\pi r_B}{h} \right) \right] \sin \left( \frac{n\pi z}{h} \right) = -\frac{F}{6\pi \mu ab} \left( \frac{k^2 - 1}{k^2 + 1} \right) \left[ 1 - \frac{\cosh \left[ \alpha (z - \frac{h}{2}) \right]}{\cosh \frac{h}{2}} \right]
\]

with \( J_1(x) \) being the modified Bessel function of order 1. The solutions for \( A_n \) are attained by straightforwardly expanding the right-hand side of Eq. (56) as a half-range Fourier sine series, which then ultimately yields the representation

\[
U = \frac{Fr}{6\pi \mu ab} \left( \frac{k^2 - 1}{k^2 + 1} \right) \left[ 1 - \frac{\cosh \left[ \alpha (z - \frac{h}{2}) \right]}{\cosh \frac{h}{2}} \right] - \frac{4hr_B}{\pi r^2} \sum_{p=1,3,\ldots}^{\infty} \frac{1}{p^2} \left[ 1 + \left( \frac{p^2}{2} \right)^2 \right] I_2 \left( \frac{n\pi r_B}{h} \right) \left[ 1 - \frac{3h}{n\pi r_B} I_1 \left( \frac{n\pi r_B}{h} \right) \right] \sin \left( \frac{n\pi z}{h} \right)
\]

Clearly, \( U = 0 \) when \( k = 1 \) and hence the radial displacement \( u_r \), given by Eqs. (16) and (41) does then reduce to that given previously by Horton et al. (2002, Eqs. (40) and (48)) for a circular block.

7. Stresses

If desired the normal stress components at any general point within the block that are created by the applied load \( F \) alone can be determined by superposition of those in Cases A and B using Eqs. (5), (26)–(28), (34), (35), (41), and (57).

However, it is instructive to note that the maximum values of these stress components occur at the bonded ends, \( z = 0 \) and \( z = h \) of the block, where, by Eqs. (39)2 and (39)3, \( \partial U/\partial r = 0 \) and \( dW/dz = 0 \). It thus follows from Eqs. (5), (26)–(28) that \( \sigma = \sigma_{r0} = \sigma_{00} = \sigma_{zz} \) at \( z = 0 \) and \( z = h \) is given by

\[
\sigma = \sigma_L + \mu \left[ \frac{r^2}{4} \frac{d^3 W}{dz^3} - f_2(z) \frac{r^2}{2} \cos 2\theta + g(z) \right].
\]

Evaluating this, using Eqs. (5), (34), (35), and (41), yields

\[
\sigma = \frac{2F}{\pi ab} \left( 1 - \frac{r^2}{r_B^2} \right),
\]

which is noted as independent of the height of the block. The maximum value, \( \sigma_{\text{max}} \), of these stress components is seen to occur on the axis, \( r = 0 \), of the block with \( \sigma_{\text{max}} = 2F/\pi ab \) there, whilst \( \sigma = 0 \) around the boundary, where \( r = r_B \).

The other significant stress component is that of the shear stress, \( \sigma_{rz} \). This has its maximum value at the boundary, \( r = r_B \), of the two bonded ends. There, from Eqs. (10), (9)5, and (16),

\[
\sigma_{rz} = \mu \left( -r \frac{d^2 W}{dz^2} + \frac{\partial U}{\partial z} \cos 2\theta \right) \quad \text{with} \quad z = h, \quad r = r_B.
\]

Evaluating this, using Eqs. (5), (41), and (57), yields

\[
\sigma_{rz} = \frac{Fr_B}{6\pi ab} \tanh \left( \frac{zh}{2} \right) \left[ 1 - \frac{k^2 - 1}{k^2 + 1} \left( 1 - \frac{4h^2\alpha}{\pi^3 r_B^2 \tanh \left( \frac{zh}{2} \right)} \sum_{p=1,3,\ldots}^{\infty} \frac{1}{p^2 + \left( \frac{2h}{2} \right)^2} I_1 \left( \frac{p\pi r_B}{h} \right) \left[ 1 - \frac{3h}{p\pi r_B} I_1 \left( \frac{2p\pi r_B}{h} \right) \right] \right] \cos 2\theta \right],
\]
at \( z = 0 \) or \( z = h \) and \( r = r_B \). However, it is well known (Abramowitz and Stegun, 1965, Eqs. (9.6.7) and (9.7.1)) that
\[
\frac{I_2(Z)}{I_1(Z)} \rightarrow 0 \quad \text{as } Z \rightarrow 0, \quad \frac{I_2(Z)}{I_1(Z)} \rightarrow 1 \quad \text{as } Z \rightarrow \infty.
\]
(62)

Hence, since \( h/r_B \) is small if \( h/b \) is small, and conversely \( h/r_B \) is large if \( h/a \) is large, it is evident that
\[
\sigma_{rz} = \frac{F r_B}{6\pi a b} \cdot \frac{z h}{2} \left( 1 - \frac{k^2 - 1}{k^2 + 1} \cos 2\theta \right) = \frac{F a}{3\pi b r_B (k^2 + 1)} \cdot \frac{z h}{2} \tan \frac{z h}{2} \quad \text{as } h/b \rightarrow 0 \text{ or } h/a \rightarrow \infty,
\]
(63)

which clearly attains its maximum value, \((\sigma_{rz})_{\max}\), when \( r_B = b \) with thus
\[
(\sigma_{rz})_{\max} \rightarrow \frac{F a}{3\pi b^3 (k^2 + 1)} \cdot \frac{z h}{2}, \quad \text{as } h/b \rightarrow 0 \text{ or } h/a \rightarrow \infty.
\]
(64)

Finally, it is noteworthy that if \( h/r_B \) is sufficiently small to allow the approximation \( \tanh(z h/2) \approx z h/2 \) then, using the relation (42), the expression (64) yields the approximation \((\sigma_{rz})_{\max} \approx 2F h/\pi a b^2\). This is in agreement with the finding of Gent and Meinecke (1970) that the maximum shear stress occurs on the bonded ends of the block at the ends of the minor axis.

References


