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Note

# No-hole $k$-tuple $(r + 1)$-distant colorings of odd cycles

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**Abstract**

We give a necessary and sufficient condition for the existence of near-optimal $N_r^k$-colorings of cycles. Troxell (preprint) studied near-optimal $N_r^k$-colorings and proved most of the result presented here; our contribution is to complete the proof for odd cycles.

## 1. Introduction

*T-colorings* were originally introduced in connection with a radio frequency assignment problem. A $T$-coloring of a graph is a function $f: V \to \mathbb{N}$ from the vertices of a graph to the non-negative integers, such that for adjacent vertices $v$ and $w$, $|f(v) - f(w)| \notin T$. Frequently $T$ is chosen to be $\{0, \ldots, r\}$ for some $r$, in which case the coloring is also called an $(r + 1)$-distant coloring.

The $T$-coloring idea can be extended to set colorings of graphs, that is, to functions $f: V \to \binom{\mathbb{N}}{k}$, where $\binom{\mathbb{N}}{k}$ is the collection of all $k$-subsets of $\mathbb{N}$. A (*set*) *T-coloring* is a set coloring $f$ for which $|c_v - c_w| \notin T$ whenever $v$ and $w$ are adjacent, $c_v \in f(v)$ and $c_w \in f(w)$. Define $\mathrm{ds}(A, B)$ to be the minimum value (over all elements of $A$ and $B$) of $|a - b|$. If $T = \{0, \ldots, r\}$, then a (*set*) *T-coloring* is a set coloring for which $\mathrm{ds}(f(v), f(w)) > r$ whenever $v$ and $w$ are adjacent. We also refer to a set coloring $f: V \to \binom{\mathbb{N}}{k}$ as a $k$-tuple coloring.

Finally, a *no-hole*, *k-tuple*, $(r + 1)$-*distant coloring* is one for which $x \in f(v)$ for some $v$ if and only if $x \in \{0, \ldots, c\}$ for some $c$. The modification "no-hole" of course refers to the fact that every color in the set is used. Troxell [4] has studied no-hole $k$-tuple, $(r + 1)$-distant colorings of graphs, determining necessary and sufficient conditions for the existence of such colorings. She also introduced the concept of a "near-optimal" coloring in this context, and obtained some results for paths, cycles and interval graphs. Following Troxell, we say that a graph $G$ has an $N_r^k$-coloring if it has a no-hole, $k$-tuple, $(r + 1)$-distant coloring.


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The *span* of a coloring is the difference between the largest and smallest colors actually used to color any vertex. The minimum span over all $k$-tuple $T$-colorings of a graph $G$ is denoted by $\mathrm{sp}_T^k(G)$, or by $\mathrm{sp}_r^k(G)$ if $T = \{0, \ldots, r\}$. In general, more colors may be required by a no-hole coloring of $G$ than by a coloring which leaves some gaps. Troxell defined a *near-optimal* $N_r^k$-coloring of $G$ as one whose span is at most $\mathrm{sp}_r^k(G) + r$.

## 2. Results

There are graphs with $N_r^k$-colorings but without near-optimal ones. Troxell [4] constructed graphs that are "arbitrarily bad" in this regard. She also showed that paths, unit interval graphs and some cycles do have near-optimal colorings whenever they have $N_r^k$-colorings.

Tesman [3] and Furedi et al. [1] proved the following.

**Proposition 1.** $\mathrm{sp}_r^k(G) \geqslant (r + 1)(\chi(G) - 1) + 2(k - 1)$.

Using this bound, Troxell showed the following.

**Proposition 2** (Troxell [4]). *If $n$ is even and $C_n$ has an $N_r^k$-coloring, then it has a near-optimal one. If $n$ is odd, $k < 2(r + 1)$ and $C_n$ has an $N_r^k$-coloring, then it has a near-optimal one.*

Left open was the general case for $n$ odd, which we now proceed to settle. We will need one simple result about the fractional chromatic number. (There are a few equivalent definitions of the fractional chromatic number in wide use; the following definition is from [2].)

**Definition.** A graph $G$ has an $(a, b)$-coloring if each vertex can be assigned a $b$-subset of $\{0, \ldots, a - 1\}$ so that adjacent vertices are assigned disjoint subsets. The *fractional chromatic number* of $G$ is $\chi_F(G) = \inf\{a/b \mid G \text{ can be } (a,b)\text{-colored}\}$.

The next proposition is well known; we include a proof for completeness.

**Proposition 3.** $\chi_F(C_{2m+1}) = 2 + 1/m$.

**Proof.** Suppose that we have an $(a, b)$-coloring of $C_{2m+1}$. Each of the $a$ colors can be assigned to at most $\alpha(C_{2m+1}) = m$ vertices of $C_{2m+1}$, where $\alpha(G)$ denotes the size of a largest independent set of vertices in $G$. Hence, $2m + 1 \leqslant am/b$, or $(2m + 1)/m \leqslant a/b$, so that $(2m + 1)/m \leqslant \chi_F(C_{2m+1})$. It now suffices to exhibit a $(2m + 1, m)$-coloring of $C_{2m+1}$. Suppose that $C_{2m+1}$ has vertex set $V = \{v_1, \ldots, v_{2m+1}\}$ and edges $\{v_1, v_{2m+1}\}$ and $\{v_i, v_{i+1}\}$ for $1 \leqslant i < 2m + 1$. Assign the colors $\{(i - 1)m, \ldots, im - 1\}$ to $v_i$ (interpreting the colors mod $2m + 1$). The only condition not immediately obvious is that the sets assigned to $v_1$ and $v_n$ are disjoint. The set assigned to $v_n$ is

$\{(2m)m, \ldots, (2m+1)m-1\}$, which is $\{m+1, \ldots, 2m\}$, clearly disjoint from $\{0, \ldots, m-1\}$. $\square$

**Lemma.** *Suppose that* $n = 2l + 1$, $l \geqslant 2$. *Then* $\mathrm{sp}_r^k(C_n) \geqslant 2k + r + \lceil (k+r)/l \rceil - 1$.

**Proof.** Let $C_n$ be a cycle, with vertex set $V = \{v_1, \ldots, v_n\}$ and edges $\{v_1, v_n\}$ and $\{v_i, v_{i+1}\}$ for $1 \leqslant i < n$. Suppose we have a $k$-tuple, $(r+1)$-distant coloring of $C_n$, $f\colon V \to \binom{S}{k}$, $\boldsymbol{S} = \{0, \ldots, s-1\}$. Then we claim that there is a $(k+r)$-tuple coloring using colors drawn from $\{0, \ldots, s+r-1\}$. Accepting the claim, and using Proposition 3, we see that

$$\frac{s+r}{k+r} \geqslant 2 + \frac{1}{l},$$

which (by simple algebraic manipulation) implies that $s - 1 \geqslant 2k + r + (k+r)/l - 1$. Since $s$ is an integer, this means that $s - 1 \geqslant 2k + r + \lceil (k+r)/l \rceil - 1$, as desired.

Now we prove the claim. By Proposition 1, $s > 2k + 2r$. Since $f$ is an $(r+1)$-distant coloring, we can find sets of colors $D_1 = \{a_0, \ldots, a_{r-1}\} \subset \boldsymbol{S}$ and $D_2 = \{b_0, \ldots, b_{r-1}\} \subset \boldsymbol{S}$ such that for all $j$, $1 \leqslant \mathrm{ds}(f(v_1), a_j) \leqslant r$, $1 \leqslant \mathrm{ds}(f(v_2), b_j) \leqslant r$, and $D_1 \cap D_2 = \emptyset$, as follows.

The set $\boldsymbol{S} - f(v_1) - f(v_2)$ consists of a collection of *gaps*, that is, of maximal sets of consecutive integers. Without loss of generality, suppose that the smallest element of $f(v_1) \cup f(v_2)$ is in $f(v_1)$. Then at least one gap consists of at least $r$ numbers, and has its lower endpoint adjacent to an element of $f(v_1)$ and its upper endpoint adjacent to an element of $f(v_2)$. If this gap contains at least $2r$ numbers, or if there is a second gap of size at least $r$, then it is clear that $D_1$ and $D_2$ may be chosen as claimed. Otherwise, suppose the unique gap of size at least $r$ is $\{j, j+1, \ldots, j+2r-i-1\}$. The numbers in each of the remaining gaps may be assigned to $D_1$ or $D_2$, but not both. Assign numbers from these gaps to $D_1$ and $D_2$ in any way, until a total of $i$ numbers have been assigned, $i_1$ of them to $D_1$ and $i_2$ to $D_2$. (Note that $\boldsymbol{S} - f(v_1) - f(v_2) - \{j, j+1, \ldots, j+2r-i-1\}$ has $s - (2k+2r-i) > i$ elements.) Assign the numbers $\{j, \ldots, j+r-i_1-1\}$ to $D_1$ and $\{j+r-i+i_2, \ldots, j+2r-i-1\}$ to $D_2$, completing the construction.

Let $D_{2i} \subset \boldsymbol{S}$, $2 \leqslant i \leqslant l$, be a set of $r$ colors such that $1 \leqslant \mathrm{ds}(f(v_{2i}), x) \leqslant r$, for every $x \in D_{2i}$. Finally, let $D = D_{2i+1}$, $1 \leqslant i \leqslant l$, be a set of $r$ new colors. Define $g(v_i) = f(v_i) \cup D_i$, $1 \leqslant i \leqslant n$. It follows easily from the definitions of the $D_i$ and the fact that $f$ is a $T$-coloring, that $g$ is a $(k+r)$-tuple coloring of $C_n$. $\square$

**Remark.** The bound provided by this lemma is better than the bound in Proposition 1 only if $\lceil (k+r)/l \rceil - 1 > r$.

**Proposition 4.** *Suppose that* $n = 2l + 1$, $l \geqslant 2$, *and* $r < k$. *Then* $C_n$ *has a near-optimal* $N_r^k$*-coloring.*

**Proof.** Let $m = \lceil (k+r)/l \rceil$, $a = l\lceil (k+r)/l \rceil - k - r$, and $s = 2k + 2r + m$. The following is an $N_r^k$-coloring of $C_n$, and is near optimal by the lemma. We use colors $\{0, \dots, s-1\}$; colors as listed should be interpreted mod $s$. Each vertex is assigned a set of $k$ consecutive colors, where we consider $s - 1$ to be adjacent to 0, i.e., "consecutive" means "consecutive mod $s$". Note that the gaps between color sets alternately contain $r$ and $r + 1$ colors up to $v_{1+2a}$, after which all gaps contain $r$ colors.

$$
\begin{array}{ll}
v_1 & 0 \dots k-1 \\
v_2 & k+r \dots 2k+r-1 \\
v_3 & 2k+2r+1 \dots 3k+2r \\
 & \vdots \\
v_{1+2a} & a(2k+2r+1) \dots (k-1)+a(2k+2r+1) \\
v_{2+2a} & a(2k+2r+1)+(k+r) \dots (k-1)+a(2k+2r+1)+(k+r) \\
v_{3+2a} & a(2k+2r+1)+2(k+r) \dots (k-1)+a(2k+2r+1)+2(k+r) \\
 & \vdots \\
v_n & a(2k+2r+1)+(l-a)(2k+2r) \dots \\
 & \quad (k-1)+a(2k+2r+1)+(l-a)(2k+2r)
\end{array}
$$

It is easy to see that this is a $k$-tuple, $(r + 1)$-distant coloring, except that it may not be obvious that the sets coloring $v_1$ and $v_n$ are at distance $r + 1$. A little algebraic manipulation shows that the set assigned to $v_n$ is $\{k + r + m, \dots, 2k + r + m - 1\}$, so the sets are indeed at distance $(r + 1)$. (This is actually true in a stronger sense than needed here. Since $(2k + r + m - 1) + (r + 1) = 2k + 2r + m \equiv 0 \pmod s$, the sets of colors are at distance $r + 1$ even when distance is measured mod $s$.)

It remains to be seen that this coloring has no holes; we need to show that the colors $\{k, \dots, k + r - 1\}$ and $(2k + r, \dots, 2k + 2r\}$ get used. It is helpful to list the colors assigned to the even numbered vertices (that is, the real colors, in the range $\{0, \dots, s - 1\}$).

$$
\begin{array}{ll}
v_2 & k+r \dots 2k+r-1 \\
v_4 & k+r-(m-1) \dots 2k+r-m \\
 & \vdots \\
v_{2+2a} & k+r-a(m-1) \dots 2k+r-a(m-1)-1 \\
v_{4+2a} & k+r-a(m-1)-m \dots 2k+r-a(m-1)-m-1 \\
v_{6+2a} & k+r-a(m-1)-2m \dots 2k+r-a(m-1)-2m-1 \\
 & \vdots
\end{array}
$$

$$\begin{array}{ll} v_{2i+2a} & k+r-a(m-1)-(i-1)m \ldots 2k+r-a(m-1)-(i-1)m-1 \\ & \vdots \\ v_{2l} & m \ldots k+m-1 \end{array}$$

Perhaps the easiest way to visualize this is to think of a block of $k$ colors sliding down from $k + r$ to $m$ in increments of $m - 1$ or $m$. Since $m \leqslant k$, the block does not slide far enough from one step to the next to leave any gaps, so all of the colors between $k$ and $k + r - 1$ do get used.

Similarly, we list the colors assigned to the odd numbered vertices; for convenience, we list only the starting color. There are two cases; if $m = 1$, then the starting color for $v_3$ through $v_{1+2a}$ is $2k + 2r + 1 = 0 \pmod s$.

$$\begin{array}{ll} v_3 & 0 \ldots \\ & \vdots \\ v_{1+2a} & 0 \ldots \\ v_{3+2a} & 2k+2r+1-(a-1)(m-1)-m \ldots \\ v_{5+2a} & 2k+2r+1-(a-1)(m-1)-2m \ldots \\ & \vdots \\ v_{2i+1+2a} & 2k+2r+1-(a-1)(m-1)-im \ldots \\ & \vdots \\ v_{1+2l} & k+r+m \ldots \end{array}$$

If $m > 1$ the list is

$$\begin{array}{ll} v_3 & 2k+2r+1 \ldots \\ v_5 & 2k+2r+1-(m-1) \ldots \\ & \vdots \\ v_{1+2a} & 2k+2r+1-(a-1)(m-1) \ldots \\ v_{3+2a} & 2k+2r+1-(a-1)(m-1)-m \ldots \\ v_{5+2a} & 2k+2r+1-(a-1)(m-1)-2m \ldots \\ & \vdots \\ v_{2i+1+2a} & 2k+2r+1-(a-1)(m-1)-im \ldots \\ & \vdots \\ v_{1+2l} & k+r+m \ldots \end{array}$$

In either case, since $k + r + m \leqslant 2k + r$, the colors in $\{2k + r, \ldots, 2k + 2r\}$ are used. $\square$

Finally, combining this result with Troxell's results, we get the following theorem.

**Theorem.** *$C_n$ has an $N_r^k$-coloring if and only if*

$$n \geqslant 2(\lceil r/k \rceil + 1) + 1.$$

*If $C_n$ has an $N_r^k$-coloring then it has a near-optimal one.*

## References

[1] Z. Furedi, J.R. Griggs and D.J. Kleitman, Pair labellings with given distance, SIAM J. Discrete Math. 2 (1989) 491–499.

[2] M. Larsen, J. Propp and D. Ullman, The fractional chromatic number of a graph and construction of Mycielski, preprint.

[3] B.A. Tesman, $T$-colorings, list $T$-colorings and set $T$-colorings of graphs, Ph.D. Thesis, Department of Mathematics, Rutgers University New Brunswick, NJ (1989).

[4] D. Sakai Troxell, No-hole $k$-tuple $(r + 1)$-distant colorings, Discrete Appl. Math. 64 (1996) 67–85.