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EXTENSION OF DYNAMIC PROGRAMMING TO NONSEPARABLE DYNAMIC OPTIMIZATION PROBLEMS

DUAN LI AND YACOV Y. HAIMES
Department of Systems Engineering, University of Virginia
Charlottesville, Virginia 22903

Abstract—The use of dynamic programming is extended to a general nonseparable class where multiobjective optimization is used as a separation strategy. The original nonseparable dynamic optimization problem is first embedded into a separable, albeit multiobjective, optimization problem where multiobjective dynamic programming using the envelope approach is used as a solution scheme. Under certain conditions, the optimal solution of the original nonseparable problem is proven to be attained by a noninferior solution.

1. INTRODUCTION

If we were to rank the various optimization techniques dealing with dynamic optimization problems on the basis of their overall effectiveness, without doubt dynamic programming (Bellman [1]) would top the list. Based on the principle of optimality, dynamic programming, when applicable, decomposes a multistage optimization problem into a sequence of single-stage optimization problems.

While dynamic programming can be applied to a variety of different problems, both linear and nonlinear, deterministic and stochastic, it has been applicable to problems that satisfy the conditions of separability and monotonicity. There are many practical sequential decision problems, however, that possess nonseparable properties. Recent research has revealed that a class of nonseparable dynamic optimization problems can be embedded into a separable multiobjective optimization setting and can be effectively solved by multiobjective dynamic programming.

Henig [2] and Sniedovich [3] develop solution procedures for nonseparable dynamic programming problems where the overall nonseparable objective is a function of two stagewise additive forms.

Carraway and Morin [4] and Carraway *et al.* [5] propose a generalized dynamic programming for the combinatorial optimization problems where the monotonicity is not satisfied. Local preference relations need to be identified at each state of the recursion in order to make the problem amenable to generalized dynamic programming.

A class of nonseparable dynamic optimization problems has recently been studied within a dynamic programming framework by introducing kth-order separability [6]. The approach uses multiobjective optimization as a separation strategy for nonseparable dynamic problems. Theoretical grounding, on which the optimal solution of the original nonseparable dynamic problem is attained by a noninferior solution of the corresponding multiobjective dynamic programming problem, has been established. The relationship between the overall optimal Lagrangian multipliers and the stage optimal Lagrangian multipliers and the relationship between the overall weighting vector and the stage weighting vector have been explored. This provides the basis for identifying the optimal solution of the original nonseparable problem from among the set of noninferior solutions generated by the envelope approach [7–8]. Nonseparability in stochastic dynamic programming has been addressed in [9].

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The purpose of this paper is to extend the research results in nonseparable dynamic programming by Li and Haimes [6]. In particular, this paper will enlarge the solvable class of nonseparable dynamic optimization problems through the use of multiobjective dynamic programming.

2. MAIN RESULTS

The results presented in [6] will be further extended in this paper to a more general class of nonseparable dynamic problems. Consider the following optimization problem of a discrete-time dynamic system,

$$\min \quad J = J[x(1), x(2), \dots, x(T), u(0), u(1), \dots, u(T-1)], \tag{1a}$$

$$\text{subject to: } x(t+1) = f[x(t), u(t), t], \tag{1b}$$

$$x(0) \text{ given,}$$

where $x(t) \in \mathbf{R}^n$ is the state vector and $u(t) \in \mathbf{R}^p$ is the control vector.

The performance index J is <u>not</u> separable if there do <u>not</u> exist functions $\phi^t(\phi^t : \mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R} \to \mathbf{R}$, $t = 0, 1, \dots, T-1$; $\phi^T : \mathbf{R}^n \to \mathbf{R})$ such that

$$\begin{aligned} J =& \phi^0(x(0), u(0), \phi^1\{x(1), u(1), \phi^2[\dots, \phi^{T-2}(x(T-2), u(T-2), \\ & \phi^{T-1}\{x(T-1), u(T-1), \phi^T[x(T)]\}) \dots]\}). \end{aligned} \tag{2}$$

Classical dynamic programming is not applicable when the overall performance measure J is nonseparable.

The performance index J given in (1) is said to be of the kth-order separability if

(i) the overall performance index J can be expressed by

$$J = J[J_1(x,u), J_2(x,u), \dots, J_k(x,u)], \tag{3}$$

where $x = [x(0), x(1), \dots, x(T)]$ and $u = [u(0), u(1), \dots, u(T-1)]$;

(ii) the overall performance J is a strictly increasing function of J_i $(i = 1, 2, \dots, k)$, i.e.,

$$\frac{\partial J}{\partial J_i} > 0; \tag{4}$$

and

(iii) each J_i $(i = 1, 2, \dots, k)$ is separable; i.e., there exist functions ϕ_i^t $(\phi_i^t : \mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R} \to \mathbf{R}$, $t = 0, 1, \dots, T-1$; $\phi_i^T : \mathbf{R}^n \to \mathbf{R})$ such that

$$\begin{aligned} J_i =& \phi_i^0(x(0), u(0), \phi_i^1\{x(1), u(1), \phi_i^2[\dots, \phi_i^{T-2}(x(T-2), u(T-2), \\ & \phi_i^{T-1}\{x(T-1), u(T-1), \phi_i^T[x(T)]\}) \dots]\}). \end{aligned} \tag{5}$$

Moreover, each J_i is assumed to be backward monotonic.

The original nonseparable dynamic problem with a kth-order separability can be embedded into a corresponding separable, albeit k-objective, multistage optimization problem:

$$\min \begin{bmatrix} J_1 & = & \phi_1^0\{x(0), u(0), \phi_1^1[x(1), u(1), \phi_1^2(\dots, \phi_1^T[x(T)] \dots)]\} \\ J_2 & = & \phi_2^0\{x(0), u(0), \phi_2^1[x(1), u(1), \phi_2^2(\dots, \phi_2^T[x(T)] \dots)]\} \\ & \vdots & \\ J_k & = & \phi_k^0\{x(0), u(0), \phi_k^1[x(1), u(1), \phi_k^2(\dots, \phi_k^T[x(T)] \dots)]\} \end{bmatrix}, \tag{6a}$$

$$\text{subject to} \quad x(t+1) = f[x(t), u(t), t]. \tag{6b}$$

Solving the multiobjective optimization problem posed in Equation (6) entails finding the noninferior solutions. A solution (x^*, u^*) of problem (6) is said to be noninferior if there exists no other feasible (x, u), such that $J_i(x, u) \leq J_i(x^*, u^*)$ for all $i = 1, 2, \dots, k$, with strict inequality for at least one i.

The following theorem reveals that the optimal solution of the original nonseparable problem (1) is among the set of noninferior solutions of the corresponding kth-order separable problem (6). We assume in this paper that every noninferior solution of problem (6) is attainable.

THEOREM 1. *The optimal solution of problem (1) is attained by a noninferior solution of the multiobjective optimization problem given in Equation (6).*

PROOF. If the solution (x^*, u^*) is inferior, there exists a feasible (x, u) such that $J_i(x, u) \leq J_i(x^*, u^*)$ for all $i = 1, 2, \dots, k$, with strict inequality for at least one i. From the assumption that J is a strictly increasing function of J_i's, we have

$$J[J_1(x, u), \dots, J_k(x, u)] < J[J_1(x^*, u^*), \dots, J_k(x^*, u^*)].$$

Therefore, (x^*, u^*) is not the optimal solution of problem (1). ∎

The above investigation has revealed that nonseparability is not an inherent property of dynamic optimization problems; it is rather a representation property. By augmenting the dimension of the objective space, a class of nonseparable dynamic programming problems can be embedded into a separable, albeit multiobjective, dynamic programming problem.

The multiobjective multistage optimization problem given in Equation (6) can be of any separable structure. Neither the weighting approach nor the ϵ-constraint approach [10], however, is suitable to be used as a solution scheme. The weighting sum will often lead back to a nonseparable problem. The ϵ-constraints in most cases are not stagewise separable. New state variables corresponding to J_i's may be introduced. The state equations, however, may not be established, since the assumption of backward separability of J_i may conflict with the requirement of forward separability for state equations. One very important observation is that, in order to seek the optimal solution of a nonseparable dynamic optimization problem, the preference on the multiple-objective functions of the corresponding multiobjective optimization problem must be adjusted along a noninferior trajectory. Therefore, for multiobjective dynamic programming problems with general separable structure, the solution procedure needs to identify the vector-valued cost-to-go recursively at each stage. The dynamic programming using the envelope approach [7–8] seems to be the most suitable method, since it can deal with all types of multiobjective functions that satisfy separability and monotonicity. Furthermore, the envelope approach is a multiobjective dynamic programming method that needs the least number of variables to calculate the vector-valued cost-to-go.

The set of noninferior solutions generated by the envelope approach can be expressed in the objective space as

$$J_1 = J_1(\theta), \tag{7a}$$

$$J_2 = J_2(\theta), \tag{7b}$$

$$\vdots$$

$$J_k = J_k(\theta), \tag{7c}$$

where θ is a $(k-1)$ dimensional parametric vector. Varying θ through all possible values in its defined region, we obtain the whole set of noninferior solutions of problem (6). Substituting Equation (7) into Equation (3), the overall performance index J becomes a function of θ. The optimal solution of the original nonseparable problem (1) can then be found by performing a minimization of J with respect to parametric vector θ. If $J(\theta)$ attains its minimum at an interior point, the optimal value of θ can be obtained by solving

$$\frac{\partial J}{\partial \theta} = 0. \tag{8}$$

3. ILLUSTRATIVE EXAMPLE

Consider the following nonseparable dynamic programming problem:

$$\begin{aligned}\min J = x^2(3)\,[u^2(0)+u^2(1)+u(1)\,u^2(2)]^{1/2} + [u^2(0)+u^2(1)+u(1)\,u^2(2)]^2,\\ \text{subject to}\quad & x(3)=x(2)/u(2),\\ & x(2)=x(1)/u(1),\\ & x(1)=x(0)/u(0),\\ & x(0)=10,\\ & u(0),u(1),u(2)\geq 0.\end{aligned}$$

Let J_1 be $x^2(3)$ and J_2 be $[u^2(0)+u^2(1)+u(1)\,u^2(2)]$. It can be seen that $J = J_1\cdot(J_2)^{0.5}+(J_2)^2$ and the objective function is of second-order separability. The corresponding multiobjective optimization problem is

$$\min \begin{bmatrix} x^2(3) \\ u^2(0)+u^2(1)+u(1)\,u^2(2)\end{bmatrix}$$

$$\begin{aligned}\text{subject to}\quad & x(3)=x(2)/u(2),\\ & x(2)=x(1)/u(1),\\ & x(1)=x(0)/u(0),\\ & x(0)=10,\\ & u(0),u(1),u(2)\geq 0.\end{aligned}$$

The set of noninferior solutions will be first generated using the envelope approach with the ϵ-constraint method adopted at Stage 2. Then the optimal solution of the original nonseparable dynamic programming problem is identified among the members of the noninferior solution set.

Step 1—Find the set of noninferior solutions:

Stage 3: The vector-valued objective function from a given $x(3)$ is

$$\phi_1^3 = x^2(3),$$
$$\phi_2^3 = 0.$$

Stage 2: The vector-valued objective function from a given $x(2)$ is

$$\phi_1^2 = \frac{x^2(2)}{u^2(2)},$$
$$\phi_2^2 = u^2(2).$$

Use the ϵ-constraint method and form the Langrangian

$$L_2 = \frac{x^2(2)}{u^2(2)} + \lambda_{12}(2)\,[u^2(2)-\epsilon_2].$$

Then derive the noninferior solutions by solving $\partial L_2/\partial u(2)=0$, $\lambda_{12}(2)\geq 0$, and $\lambda_{12}(2)\,[\partial L_2/\partial\lambda_{12}(2)]=0$,

$$\begin{aligned} u^*(2) &= \sqrt{\epsilon_2}, && \epsilon_2\geq 0,\\ \lambda_{12}(2) &= \frac{x^2(2)}{\epsilon_2^2}, && \epsilon_2\geq 0,\\ \phi_1^{2*} &= \frac{x^2(2)}{\epsilon_2}, && \epsilon_2\geq 0,\\ \phi_2^{2*} &= \epsilon_2, && \epsilon_2\geq 0.\end{aligned}$$

Here, ϵ_2 is used as the parameter of the noninferior frontier.

Stage 1: For a given $x(1)$, the family of vector-valued objective function is

$$\phi_1^1 = \frac{x^2(1)}{u^2(1)\,\epsilon_2},$$
$$\phi_2^1 = u^2(1) + u(1)\,\epsilon_2,$$

where $u(1)$ is dealt with as the parameter of the family. The envelope of this family can be identified by solving

$$\frac{\partial \phi_2^1}{\partial u(1)} \frac{\partial \phi_1^1}{\partial \epsilon_2} - \frac{\partial \phi_1^1}{\partial u(1)} \frac{\partial \phi_2^1}{\partial \epsilon_2} = 0.$$

That is,

$$[2u(1) + \epsilon_2] \frac{-x^2(1)}{u^2(1)\,\epsilon_2^2} - \frac{-2x^2(1)}{u^3(1)\,\epsilon_2} u(1) = 0.$$

Therefore, on the envelope that represents the noninferior frontier at $x(1)$, we have

$$u^*(1) = \frac{\epsilon_2}{2},$$
$$\phi_1^{1*} = \frac{4x^2(1)}{\epsilon_2^3},$$
$$\phi_2^{1*} = 0.75\,\epsilon_2^2.$$

Stage 0: For a given $x(0) = 10$, the family of vector-valued objective functions is

$$\phi_1^0 = \frac{400}{u^2(0)\,\epsilon_2^3},$$
$$\phi_2^0 = u^2(0) + 0.75\,\epsilon_2^2,$$

where $u(0)$ is dealt with as the parameter of the family. The envelope of this family can be identified by solving

$$\frac{\partial \phi_2^0}{\partial u(0)} \frac{\partial \phi_1^0}{\partial \epsilon_2} - \frac{\partial \phi_1^0}{\partial u(0)} \frac{\partial \phi_2^0}{\partial \epsilon_2} = 0.$$

After some simplification, we have

$$2u^2(0) - \epsilon_2^2 = 0.$$

Therefore, on the envelope that represents the noninferior frontier at $x(0) = 10$, we have

$$u^*(0) = \frac{\epsilon_2}{\sqrt{2}},$$
$$\phi_1^{0*} = \frac{800}{\epsilon_2^5},$$
$$\phi_2^{0*} = 1.25\,\epsilon_2^2.$$

Step 2—Find the solution for the original nonseparable dynamic programming problem:
The performance index J of the original nonseparable problem can now be expressed as a function of ϵ_2, the parameter of the set of noninferior solutions:

$$J = \phi_1^{0*}\,(\,\phi_2^{0*}\,)^{0.5} + (\,\phi_2^{0*}\,)^2 = 400\,\frac{\sqrt{5}}{\epsilon_2^4} + 1.5625\,\epsilon_2^4.$$

Solve

$$\frac{\partial J}{\partial \epsilon_2} = -1600\,\frac{\sqrt{5}}{\epsilon_2^5} + 6.25\,\epsilon_2^3 = 0,$$

we have $\epsilon_2^* = 2.211646$. Substituting ϵ_2^* into the expressions of J, u and x yields the optimal solution of the original nonseparable dynamic programming problem:

$$\begin{aligned} u^*(0) &= 1.5638699, \\ x^*(1) &= 6.3943938, \\ u^*(1) &= 1.105823, \\ x^*(2) &= 5.782475, \\ u^*(2) &= 1.4871604, \\ x^*(3) &= 3.8882658, \\ J^* &= 74.767439. \end{aligned}$$

4. CONCLUSION

This paper extends dynamic programming to a general class of nonseparable dynamic problems through the augmentation of the objective space. Multiobjective optimization constitutes the separation strategy and leads to a multilevel solution scheme. The nonseparable dynamic system's complexity can be greatly reduced through iterative operations on recurrence relations. Another important feature of this approach is the possible simplification of the objective function in the solution process. For example, we can see that the original objective function in our example problem is very complex, while a much simpler form is used in our solution procedure.

The results in this paper can readily be further extended to cases where the overall objective function J is a strictly increasing function for some J_i's and is a strictly decreasing function for others. The requirement of the backward separability of J_i can also be relaxed to the condition of forward and/or backward separability. For those J_i's that are forward separable but not backward separable, certain new state variables can be introduced through the augmentation of the state space.

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