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Q-Deformed KZB Heat Equation: Completeness, Modular Properties and SL(3, $\mathbb{Z}$)

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We study the properties of one-dimensional hypergeometric integral solutions of the q-difference ("quantum") analogue of the Knizhnik–Zamolodchikov–Bernard equations on tori. We show that they also obey a difference KZB heat equation in the modular parameter, give formulae for modular transformations, and prove a completeness result, by showing that the associated Fourier transform is invertible. These results are based on SL$(3, \mathbb{Z})$ transformation properties parallel to those of elliptic gamma functions.

1. INTRODUCTION

In this paper we continue the study of the q-analogue of the Knizhnik–Zamolodchikov–Bernard (qKZB) equations on elliptic curves and their solutions initiated in [FTV1, FTV2, FV1].

In [FTV1], hypergeometric solutions of qKZB equations were introduced. In [FTV2], the monodromy of hypergeometric solutions was calculated, and a symmetry between equations and monodromies was discovered: the equations giving the monodromy are again qKZB equations but with modular parameter and step of the difference equations exchanged. In [FV1], we introduced the q-analogue of the KZB heat equation, which

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governs the change in the modular parameter of the elliptic curve, proved that it is compatible with the other equations and explained how to recover the KZB heat equation in the semiclassical limit.

In this paper, we prove three results about our hypergeometric solutions in the case where the sum of highest weights is two: we show that the hypergeometric solutions also obeys the qKZB heat equation of [FV1], see Theorem 2.1. We give a formula (Theorem 3.3) for the transformation properties of the hypergeometric solutions under the modular group SL(2, $\mathbb{Z}$). We prove a completeness result, Corollary 4.6, by showing that the associated "Fourier transform" is invertible.

Then we show that these results are part of a bigger picture, in which the modular group combines with the transformation defined by the qKZB heat equation to give a set of quadratic identities for our generalized hypergeometric integrals. In fact, this picture can already be seen in a simpler situation, in which the elliptic gamma function [FV2, R] plays the role of the hypergeometric integral. The elliptic gamma function is a function $\Gamma(z, \tau, p)$ of three complex variables obeying identities [FV2] involving its values at points related by an action of $\mathrm{SL}(3, \mathbb{Z}) \ltimes \mathbb{Z}^3$. These identities mean that Γ is a "degree 1" generalization of a Jacobi modular function, see [FV2].

These identities are a scalar version of the identities obeyed by the hypergeometric solutions of the qKZB equation. Their meaning is that the hypergeometric integrals define a discrete projectively flat $\mathrm{SL}(3, \mathbb{Z})$-connection (i.e., a lift of the action to the projectivization) on a vector bundle over "regular" orbits of $\mathrm{SL}(3, \mathbb{Z})$ acting on the variety of pairs (point in $\mathbb{C}^3 - \{0\}$, plane through 0 containing it). This is the content of Theorem 6.8. The results on the elliptic gamma function are used in the proof, since the "phase function" which appears in the integrand of hypergeometric solutions is a ratio of gamma functions. In fact, we see "experimentally" that there seems to be a principle stating that to every identity obeyed by the gamma function, there corresponds an identity for hypergeometric integrals. The proofs of the identities consist in applying the gamma function identity to the phase function in the integrand, and then use a version of Stokes' theorem to relate the integrals. The second step is relatively simple in the case of the one-dimensional integrals to which we restrict ourselves here, but becomes exceedingly involved in the higher dimensional case. Proving our identities in the higher dimensional case, i.e., if the sum of highest weights is larger than two, remains a challenging open problem. An alternative approach to this problem is based on representation theory: in [EV] a representation–theoretic interpretation of degenerate version of the qKZB equations was established. It was shown that traces of intertwining operators for quantum groups satisfy a version of the qKZB equations and are eigenfunctions of analogues of Macdonald operators. This fact indicates that our Theorem 2.1 is an elliptic analogue of the Macdonald–Mehta identities proved by Cherednik [C] and Etingof and Kirillov [EK], and the present work is a glimpse into an elliptic version of Macdonald's theory.

In fact, Theorem 6.8 concerns the case of generic parameters, where an infinite-dimensional vector bundle is preserved by the projectively flat connection. In a next paper, we will restrict our attention to special $\mathrm{SL}(3,\mathbb{Z})$-orbits. The projectively flat connection can be then defined on a finite-dimensional vector bundle of theta-functions which are a q-deformed version of the space of conformal blocks in conformal field theory. In this setting the analogy with Macdonald's theory will appear more explicitly. Another degeneration of the $\mathrm{SL}(3,\mathbb{Z})$ symmetry of our hypergeometric integrals are the $\mathrm{SL}(3,\mathbb{Z})$ symmetries of the ordinary Fourier transform indicated in [FV4].

2. HYPERGEOMETRIC SOLUTIONS OF THE QKZB EQUATIONS

We use the definitions and notations of [FV1]. The elliptic sl_2 qKZB equations are a compatible system of difference equations for a function $v(\vec{z},\lambda,\tau)$ of $\vec{z}\in\mathbb{C}^n$, $\lambda\in\mathbb{C}$ and τ, $\operatorname{Im}\tau>0$, taking values in the zero weight space $V_{\mathbf{\Lambda}}[0]$ of a tensor product of $E_{\tau,\eta}(sl_2)$ evaluation modules. This space comes with a basis of eigenvectors of commuting operators $h^{(i)}$, $i=1,\dots,n$ and depends on parameters $\mathbf{\Lambda}=(\Lambda_1,\dots,\Lambda_n)\in\mathbb{C}^n$. The qKZB equations have the form

$$v(\vec{z}+p\delta_i,\tau)=K_i(\vec{z},\tau,p,\eta)v(\vec{z},\tau),\quad i=1,\dots,n. \tag{1}$$

They are supplemented by the qKZB heat equation

$$v(\vec{z},\tau)=T(\vec{z},\tau,p,\eta)v(\vec{z},\tau+p). \tag{2}$$

The step p is a complex parameter and $(\delta_i)_{i=1,\dots,n}$ is the standard basis of $\mathbb{C}^n$. Here v is viewed as a function of $\vec{z}$ and τ with values in the space $\mathscr{F}(V_{\mathbf{\Lambda}}[0])$ of meromorphic $V_{\mathbf{\Lambda}}[0]$-valued functions of a complex variable λ. The qKZB operators $K_i(\vec{z},\tau,p,\eta)$ are difference operators in λ and can be expressed in terms of a product of (dynamical) R-matrices. The last equation is the qKZB heat equation and involves the integral operator $T(\vec{z},\tau,p,\eta)$. The latter is expressed in terms of hypergeometric integrals: let $u(\vec{z},\lambda,\mu,\tau,p,\eta)\in V_{\mathbf{\Lambda}}[0]\otimes V_{\mathbf{\Lambda}}[0]$ be the universal hypergeometric function as in [FV1]. We may view it as a function $u(\vec{z},\tau,p,\eta)$ taking its values in the $V_{\mathbf{\Lambda}}[0]\otimes V_{\mathbf{\Lambda}}[0]$-valued functions of the two "dynamical variables" λ and μ. Then, it is a projective solution (i.e., a solution up to a constant factor) of the qKZB equations in the first factor, and of the mirror qKZB equations in the second:

$$
\begin{aligned}
u(\vec{z}+\delta_j p,\tau,p,\eta) &= K_j(\vec{z},\tau,p,\eta)\otimes D_j\, u(\vec{z},\tau,p,\eta),\\
u(\vec{z}+\delta_j \tau,\tau,p,\eta) &= D_j^\vee \otimes K_j^\vee(\vec{z},p,\tau,\eta)\, u(\vec{z},\tau,p,\eta),\\
u(\vec{z}+\delta_j,\tau,p,\eta) &= u(\vec{z},\tau,p,\eta).
\end{aligned}
\tag{3}
$$

The mirror qKZB operators $K_i^\vee(\vec{z},p,\tau,\eta)$ are obtained from the qKZB operators by exchanging τ and p and "reversing the order of factors", see [FV1]. The operators D_j, $D_j^\vee$ are operators of multiplication by certain functions of the $h^{(i)}$ and the dynamical variable μ and λ, respectively: in terms of the function

$$\alpha(\lambda) = \exp(-\pi i\lambda^2/4\eta),$$

we have, for $j=1,\dots,n$,

$$
\begin{aligned}
D_j(\mu) &= \frac{\alpha(\mu-2\eta(h^{(j+1)}+\cdots+h^{(n)}))}{\alpha(\mu-2\eta(h^{(j)}+\cdots+h^{(n)}))}\, e^{\pi i\eta\Lambda_j(\sum_{l=1}^{j-1}\Lambda_l-\sum_{l=j+1}^{n}\Lambda_l)},\\
D_j^\vee(\lambda) &= \frac{\alpha(\lambda-2\eta(h^{(1)}+\cdots+h^{(j-1)}))}{\alpha(\lambda-2\eta(h^{(1)}+\cdots+h^{(j)}))}\, e^{-\pi i\eta\Lambda_j(\sum_{l=1}^{j-1}\Lambda_l-\sum_{l=j+1}^{n}\Lambda_l)}.
\end{aligned}
$$

The integral operator $T(\vec{z},\tau,p,\eta)$ is then

$$T(\vec{z},\tau,p,\eta)v = (\alpha\otimes Q_{\tau+p})u(\vec{z},\tau,\tau+p,\eta)\otimes v.$$

α is the operator of multiplication by the function $\alpha(\lambda)$ and $Q_{\tau+p}$ is a bilinear form on $V_{\mathbf{\Lambda}}[0]$-valued functions, whose kernel is the Shapovalov bilinear form on $V_{\mathbf{\Lambda}}[0]$:

$$Q_\sigma(f\otimes g) = \int Q(\mu,\sigma,\eta)(f(\mu),g(-\mu))\alpha(\mu)\,d\mu.$$

If $\Lambda_1+\cdots+\Lambda_n=2$, the universal hypergeometric function is given in terms of Jacobi's first theta function θ, and the phase function, see Appendix A, by the following formulae: $V_{\mathbf{\Lambda}}[0]$ has a basis $\varepsilon_j = e_0\otimes\cdots\otimes e_1\otimes\cdots\otimes e_0$, with e_1 in the jth factor ($j=1,\dots,n$) and $h^{(i)}\varepsilon_j = (\Lambda_j - 2\delta_{ij})\varepsilon_j$. Then the *weight functions* are $\omega(t,\vec{z},\lambda,\tau,\eta)=\sum_{a=1}^{n}\omega_a(t,\vec{z},\lambda,\tau,\eta)\varepsilon_a$, with

$$\omega_a(t,\vec{z},\lambda,\tau,\eta)=\frac{\theta(\lambda+2\eta+t-z_a-\eta\Lambda_a-2\eta\sum_{j<a}\Lambda_j,\tau)}{\theta(t-z_a-\eta\Lambda_a,\tau)}\times\prod_{j=1}^{a-1}\frac{\theta(t-z_j+\eta\Lambda_j,\tau)}{\theta(t-z_j-\eta\Lambda_j,\tau)}.$$

On the other hand, $\omega^\vee(t,\vec{z},\mu,p,\eta)=\sum\omega_b^\vee(t,\vec{z},\mu,p,\eta)\varepsilon_b$, where

$$\omega_b^\vee(t,\vec{z},\mu,p,\eta)=\frac{\theta(\mu+2\eta+t-z_b-\eta\Lambda_b-2\eta\sum_{j>b}\Lambda_j,p)}{\theta(t-z_b-\eta\Lambda_b,p)}\prod_{j=b+1}^{n}\frac{\theta(t-z_j+\eta\Lambda_j,p)}{\theta(t-z_j-\eta\Lambda_j,p)}.$$

The universal hypergeometric function u is expressed in terms of these weight functions and the phase function

$$\Omega_a(z,\tau,p)=\prod_{j,k=0}^{\infty}\frac{(1-e^{2\pi i(z-a+j\tau+kp)})(1-e^{2\pi i(-z-a+(j+1)\tau+(k+1)p)})}{(1-e^{2\pi i(z+a+j\tau+kp)})(1-e^{2\pi i(-z+a+(j+1)\tau+(k+1)p)})}.$$

We then have $u(\vec{z},\lambda,\mu,\tau,p,\eta)=\sum_{a,b}u_{a,b}(\vec{z},\lambda,\mu,\tau,p,\eta)\varepsilon_a\otimes\varepsilon_b$, with

$$u_{a,b}(\vec{z},\lambda,\mu,\tau,p,\eta)=e^{-\frac{i\pi\lambda\mu}{2\eta}}\int_\gamma\prod_{j=1}^{n}\Omega_{\eta\Lambda_j}(t-z_j,\tau,p)\omega_a(t,\vec{z},\lambda,\tau,\eta)\omega_b^\vee(t,\vec{z},\mu,p,\eta)\,dt. \quad (4)$$

The integral is defined as the analytic continuation from the region where $\mathrm{Re}(\Lambda_i)<0$, $z_j\in\mathbb{R}$ and $\mathrm{Im}(\eta)<0$. In this region, the integration cycle is just the interval $[0,1]$. After the analytic continuation, the cycle is deformed to go above the pole at $z_j-\eta\Lambda_j$ and below the pole at $z_j+\eta\Lambda_j$, as in Fig. 1.

The Shapovalov form is $Q(\lambda,\tau,\eta)(\varepsilon_a,\varepsilon_b)=\delta_{a,b}Q_a(\lambda,\tau,\eta)$ with

$$Q_a(\lambda,\tau,\eta)=\frac{\theta(2\eta\Lambda_a,\tau)\theta'(0,\tau)}{\theta(\lambda-2\eta+2\eta\sum_{j<a}\Lambda_j,\tau)\theta(\lambda-2\eta+2\eta\sum_{j\leqslant a}\Lambda_j,\tau)}.$$

Our first result is that our hypergeometric projective solutions of the qKZB equations are also projective solutions of the quantum heat equation:

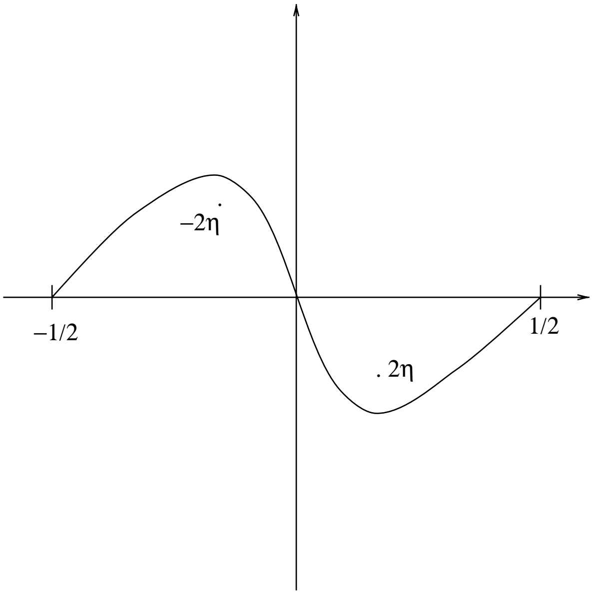

FIG. 1. The integration cycle.

THEOREM 2.1. *Suppose that* $\Lambda_1 + \cdots + \Lambda_n = 2$. *Let us view* $u(\vec{z}, \lambda, \mu, \tau, p, \eta)$ *as a function* $u(\vec{z}, \tau, p, \eta)$ *with values in the space of* $V_{\mathbf{\Lambda}}[0] \otimes V_{\mathbf{\Lambda}}[0]$-*valued functions of* λ *and* μ. *Then* u *is a projective solution of the qKZB heat equation*

$$u(\vec{z}, \tau, p, \eta) = C\, T(\vec{z}, \tau, p, \eta) \otimes D_0\, u(\vec{z}, \tau + p, p, \eta),$$

where

$$C = -\frac{e^{4\pi i \eta}}{2\pi\sqrt{4i\eta}}.$$

Here $T(\vec{z}, \tau, p, \eta)$ *acts on the first dynamical variable* λ *and* D_0 *is the operator of multiplication by the function* $e^{-\pi i \mu^2/4\eta}$ *of the second dynamical variable* μ.

It is then easy to construct true solutions to system (1), (2) from these projective solutions: for any μ, $1 \otimes \prod_j D_j(\mu)^{-z_j/p} D_0(\mu)^{\tau/p} u(\vec{z}, \lambda, \mu, \tau, p, \eta) C^{\tau/p}$, viewed as a function of $\vec{z}, \lambda, \tau$, obeys (1) and (2) in the first factor.

In more explicit terms, we have the following statement.

THEOREM 2.2. *Let for* $a = 1, \ldots, n$

$$\rho_a(\lambda, \vec{z}, \tau, \eta) = e^{\frac{2\pi i\eta}{\tau}(\sum_1^{a-1} z_j\Lambda_j - \sum_a^n z_j\Lambda_j) - \frac{i\pi}{\tau}(\lambda+2\eta-2\eta\sum_1^a \Lambda_j - 2z_a)(\lambda+2\eta-2\eta\sum_1^{a-1}\Lambda_j) - \frac{i\pi}{\tau}\lambda\sum_1^n z_j\Lambda_j},$$

and let the fundamental hypergeometric solution $\bar{u} = \sum \bar{u}_{a,b}\varepsilon_a \otimes \varepsilon_b$ *be defined by*

$$\bar{u}_{a,b}(\vec{z}, \lambda, \mu, \tau, p, \eta) = e^{-\frac{i\pi\mu^2\tau}{4\eta p}} \rho_b(-\mu, \vec{z}, p, \eta) u_{a,b}(\vec{z}, \lambda, \mu, \tau, p, \eta).$$

Then, for any μ, b, *the* $V_{\mathbf{\Lambda}}[0]$*-valued function* $v(z, \lambda, \tau) = \sum_a \bar{u}_{a,b}(\vec{z}, \lambda, \mu, \tau, p, \eta)\varepsilon_a$ *is a solution of the system*

$$v(\vec{z} + p\delta_i, \tau) = K_i(\vec{z}, \tau, p, \eta)v(\vec{z}, \tau), \qquad i = 1, \ldots, n,$$

$$v(\vec{z}, \tau) = C\, T(\vec{z}, \tau, p, \eta)v(\vec{z}, \tau + p),$$

where $C = -e^{4\pi i\eta}/2\pi\sqrt{4i\eta}$.

Proof. A consequence of (3), Theorem 2.1 and the identity

$$\prod_{i=1}^n D_i(\mu)^{-\frac{z_i}{p}}\varepsilon_b = \rho_b(-\mu, \vec{z}, p, \eta)e^{\frac{i\pi}{p}(\mu-2\eta+2\eta\sum_1^b \Lambda_j)(\mu-2\eta+2\eta\sum_1^{b-1}\Lambda_j)}\varepsilon_b. \qquad (5)$$

The exponential function on the right-hand side is independent of $\vec{z}, \lambda, \tau$ and therefore does not affect the statement of the theorem. ∎

Remarks. (1) The constant C could also be eliminated by including a factor $C^{\tau/p}$ in $\bar{u}$, but it is simpler to consider C as part of the heat equation.

(2) In the definition of the fundamental hypergeometric solution above, we have included an additional factor, see (5), with respect to the obvious choice. This leads to simpler formulae in the next section, since ρ_b appears in the modular transformations of the qKZB operators.

3. MODULAR TRANSFORMATIONS

The coefficients of the qKZB equations are quasi-periodic functions of $z_1, \ldots, z_n, \lambda$ with periods 1 and τ. It is therefore natural to consider the qKZB equations as equations for sections of a certain vector bundle over a Cartesian power of the elliptic curves $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$.

Since, for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$, the elliptic curves with modulus τ and $(a\tau + b)/(c\tau + d)$ are isomorphic, the corresponding qKZB equations are related. For generators, the formulae relating solutions are the following. Let for $a = 1, \dots, n$

$$B_a(\vec{z}, \lambda, \tau, p, \eta) = e^{-\frac{\pi i \eta}{\tau p}\sum_{j<k}(z_j - z_k)^2 \Lambda_j \Lambda_k + \frac{\pi i p}{4\eta\tau}\lambda^2} \rho_a(\lambda, \vec{z}, \tau, \eta).$$

Then we have:

PROPOSITION 3.1. (i) *Suppose that* $v(\vec{z}, \lambda) = \sum_{a=1}^n v_a(\vec{z}, \lambda)\varepsilon_a$ *is a solution of the qKZB equations with parameters* $\tau + 1, p, \eta$. *Then* $v(\vec{z}, \lambda)$ *is a solution of the qKZB equations with parameters* τ, p, η.

(ii) *Suppose that* $v(\vec{z}, \lambda) = \sum_{a=1}^n v_a(\vec{z}, \lambda)\varepsilon_a$ *is a solution of the qKZB equations with parameters* $-1/\tau, p/\tau, \eta/\tau$. *Then*

$$\tilde{v}(\vec{z}, \lambda) = \sum_{a=1}^n B_a(\vec{z}, \lambda, \tau, p, \eta) v_a(\vec{z}/\tau, \lambda/\tau)\varepsilon_a$$

is a solution of the qKZB equations with parameters τ, p, η.

The proof of this proposition is based on the formulae for transformation properties of the qKZB equations under modular transformations, see Appendix B.

Now the question about monodromy is well-posed: can one express the fundamental solution at the transformed values of the parameters in terms of the fundamental solution at the original values? The answer is provided by the following result.

THEOREM 3.2. (i) $\bar{u}_{a,b}(\vec{z}, \lambda, \mu, \tau + 1, p, \eta) = e^{-\frac{i\pi\mu^2}{4\eta p}} \bar{u}_{a,b}(\vec{z}, \lambda, \mu, \tau, p, \eta)$.

(ii) *Suppose that* $\mathrm{Im}(\eta\tau/p) < 0$, $\mathrm{Im}(p/\tau) > 0$.

$$B_a(\vec{z}, \lambda, \tau, p, \eta)\bar{u}_{a,b}\left(\frac{\vec{z}}{\tau}, \frac{\lambda}{\tau}, \frac{\nu}{\tau}, -\frac{1}{\tau}, \frac{p}{\tau}, \frac{\eta}{\tau}\right)$$
$$= \sum_{c=1}^n \int \bar{u}_{a,c}(\vec{z}, \lambda, \mu, \tau, p, \eta) M_{c,b}(\vec{z}, \mu, \nu, \tau, p, \eta)\, d\mu.$$

The monodromy matrix $(M_{c,b})$ *is*

$$M_{c,b}(\vec{z}, \mu, \nu, \tau, p)$$
$$= e^{\frac{2\pi i \eta}{3\tau p}(\eta^2 \sum_1^n \Lambda_j^3 + \tau^2 + p^2 - 3p + 3\tau + 3\tau p + 1) - \frac{2\pi i}{p\tau}(\nu - 2\eta + 2\eta \sum_1^{b-1} \Lambda_j)(\nu - 2\eta + 2\eta \sum_1^b \Lambda_j)}$$
$$\times \frac{1}{2\pi i}\sqrt{\frac{ip}{4\eta\tau}}\, Q_c(\mu, p, \eta) u_{c,b}\left(\frac{\vec{z}}{p}, -\frac{\mu}{p}, \frac{\nu}{p}, -\frac{1}{p}, -\frac{\tau}{p}, \frac{\eta}{p}\right).$$

The integration over μ is over the path $x \mapsto x\eta + \varepsilon$, $x \in \mathbb{R}$ for any generic real ε.

The first statement of the theorem is trivial to check. The second is the first of the two *modular relations*. Introduce functions $\rho, \rho^\vee$:

$$\rho_a(\lambda, \vec{z}, \tau, \eta) = e^{\frac{2\pi i\eta}{\tau}(\sum_1^{a-1} z_j\Lambda_j - \sum_a^n z_j\Lambda_j) - \frac{i\pi}{\tau}(\lambda+2\eta-2\eta\sum_1^a \Lambda_j - 2z_a)(\lambda+2\eta-2\eta\sum_1^{a-1}\Lambda_j) - \frac{i\pi}{\tau}\lambda\sum_1^n z_j\Lambda_j},$$

$$\rho_b^\vee(\lambda, \vec{z}, \tau, \eta) = e^{\frac{2\pi i\eta}{\tau}(\sum_{b+1}^{n} z_j\Lambda_j - \sum_1^b z_j\Lambda_j) - \frac{i\pi}{\tau}(\lambda+2\eta-2\eta\sum_b^n \Lambda_j - 2z_b)(\lambda+2\eta-2\eta\sum_{b+1}^{n}\Lambda_j) - \frac{i\pi}{\tau}\lambda\sum_1^n z_j\Lambda_j}.$$

Then the modular relations are the identities:

THEOREM 3.3. (i) *Suppose that* $\mathrm{Im}(\eta\tau/p) < 0$, $\mathrm{Im}(p/\tau) > 0$. *Then the universal hypergeometric function u satisfies the following relation*:

$$\sum_{c=1}^n \int u_{a,c}(\vec{z}, \lambda, \mu, \tau, p, \eta) u_{c,b}\left(\frac{\vec{z}}{p}, -\frac{\mu}{p}, \frac{\nu}{p}, -\frac{1}{p}, -\frac{\tau}{p}, \frac{\eta}{p}\right)$$
$$\times Q_c(\mu, p, \eta)\rho_c(-\mu, \vec{z}, p, \eta) e^{-\frac{\pi i\tau\mu^2}{4\eta p}}\, d\mu$$
$$= 2\pi i\sqrt{\frac{4\eta\tau}{ip}}\,\rho_a(\lambda, \vec{z}, \tau, \eta)\rho_b^\vee\left(\frac{\nu}{p}, \frac{\vec{z}}{p}, -\frac{\tau}{p}, \frac{\eta}{p}\right) e^{\frac{i\pi p}{4\eta\tau}(\lambda^2+(\nu/p)^2)}$$
$$\times u_{a,b}\left(\frac{\vec{z}}{\tau}, \frac{\lambda}{\tau}, \frac{\nu}{\tau}, -\frac{1}{\tau}, \frac{p}{\tau}, \frac{\eta}{\tau}\right) e^{-\frac{\pi i\eta}{3p\tau}\psi}, \tag{6}$$

$$\psi = 3\sum_{j<k} \Lambda_j\Lambda_k(z_j - z_k)^2 + 2\left(\sum_{j=1}^n \eta^2\Lambda_j^3 + \tau^2 + p^2 - 3p + 3\tau + 3\tau p + 1\right).$$

The integration over μ is over the path $x \mapsto x\eta + \varepsilon$, $x \in \mathbb{R}$ for any generic real ε.

(ii) *Suppose that* $\mathrm{Im}(\eta p/\tau) < 0$, $\mathrm{Im}(\tau/p) > 0$. *Then the universal hypergeometric function u satisfies the following relation*:

$$\sum_{c=1}^n \int u_{a,c}\left(\frac{\vec{z}}{\tau}, \frac{\lambda}{\tau}, \frac{\mu}{\tau}, -\frac{p}{\tau}, -\frac{1}{\tau}, \frac{\eta}{\tau}\right) u_{c,b}(\vec{z}, -\mu, \nu, \tau, p, \eta)$$
$$\times Q_c(\mu, \tau, \eta)\rho_c^\vee(\mu, \vec{z}, \tau, \eta) e^{-\frac{i\pi p}{4\eta\tau}\mu^2}\, d\mu$$

$$= 2\pi i \sqrt{\frac{4\eta p}{i\tau}}\, \rho_a\left(\frac{\lambda}{p}, \frac{\vec{z}}{p}, \frac{\tau}{p}, \frac{\eta}{p}\right)^{-1} \rho_b^{\vee}(v, \vec{z}, p, \eta)$$

$$\times\, u_{a,b}\left(\frac{\vec{z}}{p}, \frac{\lambda}{p}, \frac{v}{p}, \frac{\tau}{p}, -\frac{1}{p}, \frac{\eta}{p}\right) e^{\frac{i\pi\tau}{4\eta p}((\lambda/\tau)^2+v^2)} e^{-\frac{\pi i\eta}{3p\tau}\psi},$$

$$\psi = 3\sum_{j<k} \Lambda_j \Lambda_k (z_j - z_k)^2 + 2\left(\sum_{j=1}^{n} \eta^2 \Lambda_j^3 + \tau^2 + p^2 + 3p - 3\tau + 3\tau p + 1\right).$$

The integration over μ is over the path $x \mapsto x\eta + \varepsilon$, $x \in \mathbb{R}$ for any generic real ε.

This theorem is proved in Sections 5.2 and 5.3.

4. THE INTEGRAL TRANSFORMATION

Throughout this section, we assume, for definiteness, that $\operatorname{Im}\eta < 0$, and that η is sufficiently small. The results hold for a more general range of parameters by analytic continuation.

4.1. A Space of Functions on which the Integral Transform is Defined

The qKZB heat equation is based on an integral transformation. In this section, we give a space on which this integral transformation is defined and invertible.

Let us fix our parameters Λ_i, η, τ. Then the Shapovalov form has poles at the points $-\eta\sigma_j$, where

$$\sigma_j = 2\left(\sum_{k\leqslant j} \Lambda_k - 1\right), \qquad j = 0, \ldots, n, \tag{7}$$

as well as at the translates of this points by the lattice of periods.

At these points, the hypergeometric integrals obey the following "resonance relations" [FV3].

PROPOSITION 4.1. *Let r, s, $1 \leqslant a, b \leqslant n$ be integers and let $\sigma_j = 2\sum_{k\leqslant j} \Lambda_k - 2$. Then*:

(i) *If $a < n$, then*

$$u_{a+1,b}(\vec{z}, \eta\sigma_a + r + s\tau, \mu, \tau, p, \eta)$$
$$= e^{2\pi i s(z_{a+1} - z_a + \eta\Lambda_{a+1} + \eta\Lambda_a)} u_{a,b}(\vec{z}, \eta\sigma_a + r + s\tau, \mu, \tau, p, \eta).$$

(ii) $u_{1,b}(\vec{z}, -2\eta + r + s\tau, \mu, \tau, p, \eta)$
$= e^{2\pi is(z_1 - z_n + \eta\Lambda_1 + \eta\Lambda_n - p)} u_{n,b}(\vec{z}, 2\eta + r + s\tau, \mu, \tau, p, \eta).$

(iii) *If* $b<n$, *then*

$$u_{a,b+1}(\vec{z}, \lambda, -\eta\sigma_b + r + sp, \tau, p, \eta)$$
$$= e^{2\pi is(z_{b+1} - z_b - \eta\Lambda_{b+1} - \eta\Lambda_b)} u_{a,b}(\vec{z}, \lambda, -\eta\sigma_b + r + sp, \tau, p, \eta).$$

(iv) $u_{a,1}(\vec{z}, \lambda, 2\eta + r + sp, \tau, p, \eta)$
$= e^{2\pi is(z_1 - z_n - \eta\Lambda_1 - \eta\Lambda_n + \tau)} u_{a,n}(\vec{z}, \lambda, -2\eta + r + sp, \tau, p, \eta).$

Proof. (i) Using the functional relation $\theta(x + r + s\tau, \tau) = (-1)^{r+s} \exp(-\pi is(2x + s\tau))\theta(x, \tau)$, we see that

$$\omega_a(t, \vec{z}, \eta\sigma_a + r + s\tau, \tau, \eta) = (-1)^{r+s} e^{-2\pi is(t - z_a + \eta\Lambda_a) - \pi is^2\tau} \prod_{j=1}^{a} \frac{\theta(t - z_j + \eta\Lambda_j)}{\theta(t - z_j - \eta\Lambda_j)}$$

and

$$\omega_{a+1}(t, \vec{z}, \eta\sigma_a + r + s\tau, \tau, \eta) = (-1)^{r+s} e^{-2\pi is(t - z_{a+1} - \eta\Lambda_{a+1}) - \pi is^2\tau} \prod_{j=1}^{a} \frac{\theta(t - z_j + \eta\Lambda_j)}{\theta(t - z_j - \eta\Lambda_j)}.$$

Thus, we have equality at the level of integrands.

(ii) Similarly, we have

$$\omega_1(t, \vec{z}, -2\eta + r + s\tau, \tau, \eta) = (-1)^{r+s} e^{-2\pi is(t - z_1 - \eta\Lambda_1) - \pi is^2\tau},$$

and using the relation $\sum \Lambda_i = 2$, we obtain

$$\omega_n(t, \vec{z}, 2\eta + r + s\tau, \tau, \eta) = (-1)^{r+s} e^{-2\pi is(t - z_n + \eta\Lambda_n) - \pi is^2\tau} \prod_{j=1}^{n} \frac{\theta(t - z_j + \eta\Lambda_j)}{\theta(t - z_j - \eta\Lambda_j)}.$$

The last product of ratios of theta functions may be absorbed using the functional relation (A.5) for Ω. This gives

$$u_{n,b}(\vec{z}, 2\eta + r + s\tau, \mu, \tau, p, \eta)$$
$$= e^{-\frac{i\pi}{2\eta}(2\eta + r + s\tau)\mu} \int e^{-4\pi i\eta} \prod_j \Omega_{\eta\Lambda_j}(t - z_j + p, \tau, p)(-1)^{r+s}$$
$$\times e^{-2\pi is(t - z_n + \eta\Lambda_n) - \pi is^2\tau} \omega_b^\vee(t, \vec{z}, \mu, p, \eta)\, dt.$$

The result is then obtained by shifting the integration variable t by $-p$ and using the relation

$$\omega_b^\vee(t-p,\vec{z},\mu,p,\eta) = e^{2\pi i(\mu+2\eta)}\omega_b^\vee(t,\vec{z},\mu,p,\eta).$$

The remaining claims (iii), (iv) are proved in a similar way. ∎

Definition. Let $E^0(\vec{z},\tau,c;\eta)$ be the space of holomorphic functions $\varphi:\mathbb{C}\to V_{\mathbf{\Lambda}}[0]$, $\varphi(\lambda)=\sum \varphi_a(\lambda)\varepsilon_a$ such that

(i) If $a<n$, then $\varphi_{a+1}(\eta\sigma_a+r+s\tau) = e^{2\pi is(z_{a+1}-z_a+\eta\Lambda_{a+1}+\eta\Lambda_a)}\varphi_a(\eta\sigma_a+r+s\tau)$.

(ii) $\varphi_1(-2\eta+r+s\tau) = e^{2\pi is(z_1-z_n+\eta\Lambda_1+\eta\Lambda_n+c)}\varphi_n(2\eta+r+s\tau)$.

Let $E(\vec{z},\tau,c;\eta)$ be the space of functions $\varphi\in E^0(\vec{z},\tau,c;\eta)$ such that

(iii) There exist constants $C_1, C_2>0$ (depending on φ) such that

$$|\varphi_a(\lambda)|\leqslant C_1\exp\left(\pi\frac{(\operatorname{Im}\lambda)^2}{\operatorname{Im}\tau}+C_2|\lambda|\right),$$

for all $a=1,\dots,n$.

Examples of functions in $E(\vec{z},\tau,c;\eta)$ can be constructed using the universal hypergeometric function:

Proposition 4.2. *If* $\operatorname{Im} c<0$, *then the function*

$$\lambda\mapsto\sum_a u_{a,b}(\vec{z},\lambda,\mu,\tau,-c,\eta)\varepsilon_a,$$

belongs to $E(\vec{z},\tau,c;\eta)$ *for all values of the remaining parameters. If* $\operatorname{Im} c>0$, *then the function*

$$\mu\mapsto\sum_b u_{a,b}(\vec{z},\lambda,\mu,c,\tau,-\eta)\varepsilon_b,$$

belongs to $E(\vec{z},\tau,c;\eta)$ *for all values of the remaining parameters.*

Proof. Proposition 4.1 implies that these functions belong to $E^0(\vec{z},\tau,c;\eta)$. The bound (iii) follows from Lemma 4.7 below. ∎

Lemma 4.3. $\varphi(\lambda)\in E^0(\vec{z},\tau,c;\eta)$ *if and only if* $e^{\pi i\lambda^2/4\eta}\varphi(\lambda)\in E^0(\vec{z},\tau,c-\tau;\eta)$.

Proof. It is clear that $e^{\frac{i\pi\lambda^2}{4\eta}}\phi(\lambda)$ obeys (i) in the definition of $E^0(\vec{z},\tau, c-\tau;\eta)$. Property (ii) follows from the identity

$$e^{\frac{\pi i}{4\eta}(-2\eta+r+s\tau)^2} = e^{-2\pi i s\tau}e^{\frac{\pi i}{4\eta}(2\eta+r+s\tau)^2},$$

for $r, s \in \mathbb{Z}$. ∎

For the construction of the integral transform, we will need the following variant of Proposition 4.2.

COROLLARY 4.4. *The function*

$$\lambda \mapsto e^{-\pi i\frac{\lambda^2+\mu^2}{4\eta}} \sum_a u_{a,b}(\vec{z},\lambda,\mu,\tau,p,\eta)\varepsilon_a,$$

belongs to $E^0(\vec{z},\tau,\tau-p;\eta)$ *for all* μ. *The function*

$$\mu \mapsto e^{\pi i\frac{\lambda^2+\mu^2}{4\eta}} \sum_b u_{a,b}(\vec{z},\lambda,\mu,\tau,p,-\eta)\varepsilon_b,$$

belongs to $E^0(\vec{z},p,\tau-p;\eta)$ *for all* λ.

Proof. An immediate consequence of Proposition 4.2. and Lemma 4.3. ∎

4.2. The Definition of the Integral Transform

We define two integral transforms $\mathscr{F}_{\vec{z},p,\tau}$ and $\tilde{\mathscr{F}}_{\vec{z},\tau,p}$, then show that they are inverse to each other. The first is $\varphi \mapsto \mathscr{F}_{\vec{z},p,\tau}(\varphi) = \hat{\varphi}$ with

$$\hat{\varphi}_c(\mu) = \frac{e^{\pi i\frac{\mu^2}{4\eta}}}{16\pi^2\eta}\int_{\mathbb{R}+i\sigma}\sum_{b=1}^{n}\varphi_b(-\nu)Q_b(\nu,\tau,\eta)u_{b,c}(\vec{z},\nu,\mu,\tau,p,-\eta)e^{\pi i\frac{\nu^2}{4\eta}}\,d\nu. \tag{8}$$

The second is $\psi \mapsto \tilde{\mathscr{F}}_{\vec{z},\tau,p}(\psi) = \check{\psi}$ with

$$\check{\psi}_a(\lambda) = e^{-\pi i\frac{\lambda^2}{4\eta}}\int_{\eta\mathbb{R}+\tilde{\sigma}}\sum_{c=1}^{n}u_{a,c}(\vec{z},\lambda,\mu,\tau,p,\eta)Q_c(\mu,p,\eta)\psi_c(-\mu)e^{-\pi i\frac{\mu^2}{4\eta}}\,d\mu. \tag{9}$$

These transformations depend on the choice of a real parameter σ or $\tilde{\sigma}$. We say that σ is an *admissible shift* for $\mathscr{F}$ if it does not lie in any interval $[\mathrm{Im}(2\eta+sp), \mathrm{Im}(-2\eta+sp)]$, $s \in \mathbb{Z}$.

We say that $\tilde{\sigma}$ is an *admissible shift* for $\tilde{\mathscr{F}}$ if it is a generic real number.

THEOREM 4.5. *Suppose that* $\operatorname{Im}\eta<0$. *If* $\varphi\in E(\vec z,\tau,\tau-p;\eta)$ *then integral* (8) *is absolutely convergent, independent of the choice of admissible shift and defines a linear map* $\mathscr{F}_{\vec z,p,\tau}: E(\vec z,\tau,\tau-p;\eta)\to E(\vec z,p,\tau-p;\eta)$. *If* $\psi\in E(\vec z,p,\tau-p;\eta)$ *then integral* (9) *is absolutely convergent, independent of the choice of admissible shift and defines a linear map* $\tilde{\mathscr{F}}_{\vec z,\tau,p}: E(\vec z,p,\tau-p;\eta)\to E(\vec z,\tau,\tau-p;\eta)$. *Moreover,*

$$\mathscr{F}_{\vec z,p,\tau}\circ\tilde{\mathscr{F}}_{\vec z,\tau,p}=\mathrm{Id}_{E(\vec z,p,\tau-p;\eta)},\qquad \tilde{\mathscr{F}}_{\vec z,\tau,p}\circ\mathscr{F}_{\vec z,p,\tau}=\mathrm{Id}_{E(\vec z,\tau,\tau-p;\eta)}.$$

The proof of the first part of the theorem is contained in the next subsection. The proof of the inversion formula is deferred to Section 5.4.

This theorem implies a completeness result for the qKZB equations. For generic $\vec z^0\in\mathbb{C}^n$, we may consider the qKZB equations on the set $\vec z^0+(p\mathbb{Z})^n$. Any solution is uniquely determined by its initial condition at $\vec z^0$. A class of solutions is given by taking linear combinations of components of the fundamental hypergeometric solution: Let us say that a solution v is *of hypergeometric type* if it is of the form

$$v_a(z,\lambda)=\int_\gamma\sum_b\bar u_{ab}(z,\lambda,\mu,\tau,p,\eta)F_b(\mu)\,d\mu,$$

for some functions $F_b(\mu)$ and some cycle γ.

COROLLARY 4.6. *Any solution* $v(z,\lambda)$ *with initial condition in* $e^{\frac{\pi i\lambda^2}{4\eta}}E(z,\tau,\tau-p;\eta)$ *is of hypergeometric type.*

More precisely, let $\vec z^0\in\mathbb{C}^n$ *be generic. Suppose* $\varphi\in E(\vec z^0,\tau,\tau-p;\eta)$ *and let* $\hat\varphi(\mu)=\mathscr{F}_{\vec z^0,p,\tau}(\varphi)(\mu)$. *Then, for all* $\vec z\in\vec z^0+(p\mathbb{Z})^n$, $\psi(\vec z,\mu)=\prod_{i=1}^n D_i(-\mu)^{-(z_i-z_i^0)/p}\hat\varphi(\mu)$, *viewed as a function of* μ, *belongs to* $E(\vec z,p,\tau-p;\eta)$ *and the function* $v(\vec z,\lambda)=\sum v_a(\vec z,\lambda)\varepsilon_a$, *with*

$$v_a(\vec z,\lambda)=\int_{\eta\mathbb{R}+\tilde\sigma}\sum_b u_{a,b}(\vec z,\lambda,\mu,\tau,p,\eta)Q_b(\mu,p,\eta)e^{-\pi i\frac{\mu^2}{4\eta}}\psi_b(\vec z,-\mu)\,d\mu,$$

is, for any generic $\tilde\sigma\in\mathbb{R}$, *a solution of hypergeometric type of the qKZB equations with modulus* τ, *step* p *and initial condition* $v(\vec z^0,\lambda)=e^{\pi i\frac{\lambda^2}{4\eta}}\varphi(\lambda)$. *Moreover, for all* $\vec z\in\vec z^0+(p\mathbb{Z})^n$, $v(\vec z,\lambda)$ *is independent of the choice of* $\tilde\sigma$, and $e^{-i\pi\lambda^2/4\eta}v(\vec z,\lambda)$ *belongs to* $E(\vec z,\tau,\tau-p;\eta)$.

Proof. Let $d_{j,a}(\mu)$ denote the eigenvalues of $D_j(\mu)$:

$$D_j(\mu)\varepsilon_a=d_{j,a}(\mu)\varepsilon_a.$$

They grow at most exponentially as $\mu \to \infty$. The coordinates of ψ are then

$$\psi_a(\vec{z}, \mu) = \prod_{j=1}^{n} d_{j,a}(-\mu)^{-(z_i - z_i^0)/p} \hat{\varphi}_a(\mu).$$

By the theorem, $\hat{\varphi} \in E(\vec{z}^0, p, \tau - p; \eta)$. Then ψ clearly obeys bound (iii) of the definition of $E(\vec{z}, p, \tau - p; \eta)$. The resonance relations (i), (ii) are checked by inserting the definitions. For example, for $a<n$ we have (using the fact that $(\vec{z} - \vec{z}^0)/p$ has integral coordinates)

$$\psi_{a+1}(\vec{z}, \eta\sigma_a + r + sp) = \psi_a(\vec{z}, \eta\sigma_a + r + sp) e^{2\pi i s(z_{a+1}^0 - z_a^0 + \eta\Lambda_{a+1} + \eta\Lambda_a)}$$

$$\times \prod_j \left\{ \frac{d_{j,a+1}(-\eta\sigma_a - r - sp)}{d_{j,a}(-\eta\sigma_a - r - sp)} \right\}^{-\frac{z_j - z_j^0}{p}}.$$

On the other hand, the expression in curly brackets is equal to $e^{2\pi i s p}$ if $j = a$, to $e^{-2\pi i s p}$ if $j = a + 1$, and to 1 otherwise. It then follows by Theorem 4.5 that $\check{\psi}(\vec{z}, \lambda) = e^{-i\pi\lambda^2/4\eta} v(\vec{z}, \lambda)$ is in $E(\vec{z}, \tau, \tau - p; \eta)$ and that the initial condition $\check{\psi}(\vec{z}^0) = \varphi(\lambda)$ is satisfied. v is a solution of hypergeometric type since the $\vec{z}, \lambda$-dependent part of the kernel of integration is $1 \otimes \prod_i D_i(\mu)^{-z_i/p} u$ which is equal to the fundamental solution $\bar{u}$ up to $\vec{z}, \lambda$-independent factors, cf. (5). ∎

Remarks. (1) The result may be expressed in the following terms: our generalized Fourier transform maps the qKZB equations to the difference equations

$$\psi(\vec{z} + p\delta_i, \mu) = D_i(-\mu)^{-1} \psi(\vec{z}, \mu),$$

which are easily solved, since D_i is a diagonal multiplication operator. So the solution given in the corollary is the Fourier transform of the solution of this simple system of equations with initial condition given by the inverse Fourier transform of the given initial condition.

(2) An initial condition at $\vec{z}^0$ uniquely determines a solution only on the set $\Gamma = \vec{z}^0 + (p\mathbb{Z})^n$, since solutions defined for all $\vec{z}$ can always be changed by multiplying by p-periodic functions without changing the initial condition. However, the formula in Corollary 4.6 gives a solution for all $\vec{z}$. On the other hand, it is only for $\vec{z} \in \vec{z}^0 + (p\mathbb{Z})^n$ that we know that the integral is independent of the choice of $\tilde{\sigma}$. For general $\vec{z}$, there is no cancellation of residues at pairs of poles that would allow us to move the integration contours, and one should expect that the solution depends on $\tilde{\sigma}$.

In fact, if we move $\tilde{\sigma}$ just a little bit, then the integration contour crosses infinitely many poles (for generic η), so one would expect that the solution depends discontinuously on $\tilde{\sigma}$. It would be interesting to understand this dependence in more detail.

4.3. *Proof of Theorem 4.5*

We prove here that the integral transformations are well defined on the considered spaces.

We start by estimating the integrands.

LEMMA 4.7.

$$
\begin{aligned}
&|e^{\pi i\frac{\lambda\mu}{2\eta}}u_{a,b}(\vec{z},\lambda,\mu,\tau,p,\eta)| \\
&\qquad \leqslant C_1 \exp\left(\pi\frac{(\operatorname{Im}\lambda)^2}{\operatorname{Im}\tau}+C_2|\operatorname{Im}\lambda|+\pi\frac{(\operatorname{Im}\mu)^2}{\operatorname{Im}p}+C_2|\operatorname{Im}\mu|\right),
\end{aligned}
$$

for some $C_1, C_2 > 0$ *depending on* $\vec{z},\tau,p,\eta,\mathbf{\Lambda}$.

For every $\varepsilon > 0, \vec{z},\tau,p,\eta,\mathbf{\Lambda}$ *there exist constants* $C_3, C_4 > 0$ *such that if the distance between* λ *and the singularities of* Q_a *is at least* ε, *then*

$$
|Q_a(\lambda,\tau,\eta)| \leqslant C_3 \exp\left(-2\pi\frac{(\operatorname{Im}\lambda)^2}{\operatorname{Im}\tau}+C_4|\operatorname{Im}\lambda|\right).
$$

Proof. The first bound is obtained by applying the estimates of Lemma C.1 to integral (4). The second follows using the lower bound in Lemma C.1. ∎

In order to show that the integral is independent of the choice of admissible shift and to bound the integral transform, we will need to shift the integration contour. The next result shows that this is possible since poles occur in pairs with opposite residues.

LEMMA 4.8. (i) *Suppose* $\psi = \sum \psi_a\varepsilon_a \in E(\vec{z},p,\tau-p;\eta)$. *Then for all* $r,s \in \mathbb{Z}$, $c = 1,\dots,n$ *and* $a = 1,\dots,n-1$,

$$
\begin{aligned}
&\operatorname{res}_{\mu=-\eta\sigma_a+r+sp}\, e^{-\pi i\frac{\lambda^2+\mu^2}{4\eta}}u_{c,a}(\vec{z},\lambda,\mu,\tau,p,\eta)Q_a(\mu,p,\eta)\psi_a(-\mu) \\
&\quad = -\operatorname{res}_{\mu=-\eta\sigma_a+r+sp}\, e^{-\pi i\frac{\lambda^2+\mu^2}{4\eta}}u_{c,a+1}(\vec{z},\lambda,\mu,\tau,p,\eta)Q_{a+1}(\mu,p,\eta)\psi_{a+1}(-\mu),
\end{aligned}
$$

$$\operatorname{res}_{\mu=-2\eta+r+sp} e^{-\pi i\frac{\lambda^2+\mu^2}{4\eta}} u_{c,n}(\vec{z},\lambda,\mu,\tau,p,\eta)Q_n(\mu,p,\eta)\psi_n(-\mu)$$

$$= -\operatorname{res}_{\mu=2\eta+r+sp} e^{-\pi i\frac{\lambda^2+\mu^2}{4\eta}} u_{c,1}(\vec{z},\lambda,\mu,\tau,p,\eta)Q_1(\mu,p,\eta)\psi_1(-\mu).$$

(ii) *Suppose* $\varphi = \sum \varphi_a \varepsilon_a \in E(\vec{z},\tau,\tau-p;\eta)$. *Then for all* $r,s \in \mathbb{Z}$, $c = 1,\dots,n$ *and* $a = 1,\dots,n-1$,

$$\operatorname{res}_{\lambda=-\eta\sigma_a+r+s\tau} e^{\pi i\frac{\lambda^2+\mu^2}{4\eta}} u_{a,c}(\vec{z},\lambda,\mu,\tau,p,-\eta)Q_a(\lambda,\tau,\eta)\varphi_a(-\lambda)$$

$$= -\operatorname{res}_{\lambda=-\eta\sigma_a+r+s\tau} e^{\pi i\frac{\lambda^2+\mu^2}{4\eta}} u_{a+1,c}(\vec{z},\lambda,\mu,\tau,p,-\eta)Q_{a+1}(\lambda,\tau,\eta)\varphi_{a+1}(-\lambda),$$

$$\operatorname{res}_{\lambda=-2\eta+r+s\tau} e^{\pi i\frac{\lambda^2+\mu^2}{4\eta}} u_{n,c}(\vec{z},\lambda,\mu,\tau,p,-\eta)Q_n(\lambda,\tau,\eta)\varphi_n(-\lambda)$$

$$= -\operatorname{res}_{\lambda=2\eta+r+s\tau} e^{\pi i\frac{\lambda^2+\mu^2}{4\eta}} u_{1,c}(\vec{z},\lambda,\mu,\tau,p,-\eta)Q_1(\lambda,p,\eta)\varphi_1(-\lambda).$$

Proof. A straightforward calculation. ∎

Proof of the first part of Theorem 4.5. It follows from Proposition 4.2 that the integrand in (8) has only simple poles at $\pm 2\eta + \mathbb{Z} + \tau\mathbb{Z}$. If σ is an admissible shift then the integration contour stays away from the singularities and we can use the estimates of Lemma 4.7. The integrand is then the product of a function growing at most like $e^{C|v|}$ times $e^{\pi i v^2/4\eta}$ which for $\operatorname{Im}\eta < 0$ converges very rapidly to zero in the real direction. So the integrand is an L^1 function. Moreover by Lemma 4.8, the residue at $2\eta + r + s\tau$ is opposite to the residue at $-2\eta + r + s\tau$. It follows that the integration contour can me moved across each of these pairs of singularities without changing the value of the integral. This shows that the integral is independent of the choice of admissible shift.

Let us next show that $\hat{\varphi}$ defined by (8) belongs to $E(\vec{z},p,\tau-p;\eta)$. Properties (i) and (ii) follow from Corollary 4.4, and we are left with the proof of (iii). For this we shift variables: let $u^0_{a,b}(\vec{z},\lambda,\mu,\tau,p,\eta) = e^{i\pi\frac{\lambda\mu}{4\eta}} u_{a,b}(\vec{z},\lambda,\mu,\tau,p,\eta)$. Then,

$$\hat{\varphi}_c(\mu) = \frac{1}{16\pi^2\eta}\int_{\mathbb{R}+i\sigma} \sum_{b=1}^{n} \varphi_b(-v+\mu)Q_b(v-\mu,\tau,\eta) \times u^0_{b,c}(\vec{z},v-\mu,\mu,\tau,p,-\eta)e^{\pi i\frac{v^2}{4\eta}}\,dv.$$

The integration contour is moved by this change of variable, but using the independence on the choice of σ we may choose it to lie in the interval $[0, \operatorname{Im}\tau]$ say.

Then Lemma 4.7 yields

$$|\hat{\varphi}_c(\mu)| \leqslant \frac{e^{\pi \frac{(\operatorname{Im}\mu)^2}{\operatorname{Im}\tau}}}{16\pi^2|\eta|} \int_{\mathbb{R}+i\sigma} \sum_{b=1}^{n} Ce^{C'|\mu-\nu|} |e^{\pi i \frac{\nu^2}{4\eta}}| \, d\nu.$$

The triangle inequality $|\mu - \nu| \leqslant |\mu| + |\nu|$ and the Gaussian integral over ν give estimate (iii) in the definition of $E(\vec{z}, p, \tau - p; \eta)$.

The proof that (9) defines a function in $E(\vec{z}, \tau, \tau - p; \eta)$ is analogous with a little difference. Now the integration contour is parallel to the line $2\eta\mathbb{R}$. So if the shift $\tilde{\sigma}$ is generic it will not meet the singular set $S = \{\pm 2\eta\} + \mathbb{Z} + p\mathbb{Z}$ of the integrand. However, it will come arbitrarily close to these singularities. Still, the integral is absolutely convergent, since the distance to the singular set decreases polynomially: $\operatorname{dist}(\mu, S) \geqslant \operatorname{const}(1 + |\mu|)^{-\alpha}$ for some $\alpha > 0$. This implies that the divergence coming from the poles close to the integration contour is at most polynomial. This does not spoil the integrability which is due to the exponential decay of $\exp(-i\pi\mu^2/4\eta)$. ∎

5. CALCULATIONS

5.1. The Heat Equation

Here we prove Theorem 2.1.

The statement of the theorem is

$$\begin{aligned} &u_{a,b}(\vec{z}, \lambda, \nu, \tau, p, \eta) \\ &\quad = -\frac{e^{4\pi i\eta}}{2\pi\sqrt{4i\eta}} e^{-i\frac{\pi}{4\eta}(\lambda^2+\nu^2)} \int_{\eta\mathbb{R}+\tilde{\sigma}} \sum_{c=1}^{n} u_{a,c}(\vec{z}, \lambda, \mu, \tau, p+\tau, \eta) \\ &\qquad \times Q_c(\mu, \tau+p, \eta) e^{-\frac{i\pi\mu^2}{4\eta}} u_{c,b}(\vec{z}, -\mu, \nu, \tau+p, p, \eta) \, d\mu. \qquad (10) \end{aligned}$$

We proceed to evaluate the right-hand side. The μ-dependent part may be simplified by using the following identity:

LEMMA 5.1.

$$\sum_{c=1}^{n} \omega_c^{\vee}(t, \vec{z}, \mu, \tau+p, \eta) Q_c(\mu, \tau+p, \eta) \omega_c(s, \vec{z}, -\mu, \tau+p, \eta)$$

$$= \frac{\theta(\mu - 2\eta + t - s, \tau + p)\theta'(0, \tau + p)}{\theta(\mu - 2\eta, \tau + p)\theta(t - s, \tau + p)} \prod_{j=1}^{n} \frac{\theta(t - z_j + \eta \Lambda_j, \tau + p)}{\theta(t - z_j - \eta \Lambda_j, \tau + p)}$$
$$- \frac{\theta(\mu + 2\eta + t - s, \tau + p)\theta'(0, \tau + p)}{\theta(\mu + 2\eta, \tau + p)\theta(t - s, \tau + p)} \prod_{j=1}^{n} \frac{\theta(s - z_j + \eta \Lambda_j, \tau + p)}{\theta(s - z_j - \eta \Lambda_j, \tau + p)}.$$

Proof. Consider each term in the sum on the left as a function of μ: it is periodic with period one and as μ is replaced by $\mu + \tau + p$, it is multiplied by $\exp(-2\pi i(t - s))$. The poles of the term labeled by c are at $\mu = \mu_c = 2\eta - 2\eta \sum_{j<c} \Lambda_j$ and at $\mu = \mu_c - 2\eta\Lambda_c = \mu_{c+1}$ modulo $\mathbb{Z} + (\tau + p)\mathbb{Z}$. The most general form of a meromorphic function of μ with these properties is

$$A_c \frac{\theta(\mu - \mu_c + t - s, \tau + p)\theta'(0, \tau + p)}{\theta(\mu - \mu_c, \tau + p)\theta(t - s, \tau + p)}$$
$$+ B_c \frac{\theta(\mu - \mu_{c+1} + t - s, \tau + p)\theta'(0, \tau + p)}{\theta(\mu - \mu_{c+1}, \tau + p)\theta(t - s, \tau + p)}.$$

The coefficients are determined by comparing the residues at the poles:

$$A_c = \omega_c^{\vee}(t, \vec{z}, \mu_c, \tau + p, \eta)\omega_c(s, \vec{z}, -\mu_c, \tau + p, \eta)$$
$$= \prod_{j=c}^{n} \frac{\theta(t - z_j + \eta\Lambda_j, \tau + p)}{\theta(t - z_j - \eta\Lambda_j, \tau + p)} \prod_{j=1}^{c-1} \frac{\theta(s - z_j + \eta\Lambda_j, \tau + p)}{\theta(s - z_j - \eta\Lambda_j, \tau + p)}.$$

Similarly, one determines B_c which turns out to be equal to $-A_{c+1}$. It follows that in the sum over c only two terms are not canceled and we obtain our claim. ∎

The products of ratios of theta functions in the above identity may be absorbed into the phase functions $\Omega_{\eta\Lambda_j}$ by means of their functional relation (A.6). The right-hand side of (10) is then

$$\frac{-e^{-\frac{i\pi}{2\eta}\lambda\nu}}{2\pi\sqrt{4i\eta}} \int e^{-\frac{i\pi}{4\eta}(\lambda - \nu + \mu)^2} \omega_a(t, \vec{z}, \lambda, \tau, \eta)\omega_b^{\vee}(s, \vec{z}, \nu, p, \eta)$$
$$\times \left[\prod_{j=1}^{n} (\Omega_{\eta\Lambda_j}(t - z_j + \tau, \tau, \tau + p)\Omega_{\eta\Lambda_j}(s - z_j, \tau + p, p)) \right.$$
$$\times \frac{\theta(\mu - 2\eta + t - s, \tau + p)\theta'(0, \tau + p)}{\theta(\mu - 2\eta, \tau + p)\theta(t - s, \tau + p)}$$

$$- \prod_{j=1}^{n} (\Omega_{\eta\Lambda_j}(t - z_j, \tau, \tau + p)\Omega_{\eta\Lambda_j}(s - z_j + p, \tau + p, p))$$
$$\times \frac{\theta(\mu + 2\eta + t - s, \tau + p)\theta'(0, \tau + p)}{\theta(\mu + 2\eta, \tau + p)\theta(t - s, \tau + p)}\Bigg] \, dt \, ds \, d\mu. \tag{11}$$

The next thing to note is that if we change variables in the first of these two terms by replacing t by $t - \tau$, s by $s + p$ and μ by $\mu + 4\eta$, we obtain exactly the second term up to a sign! Indeed, the shift of t in ω_a produces a factor $e^{2\pi i(\lambda+2\eta)}$, the shift of s in $\omega_b^\vee$ produces a factor $e^{-2\pi i(\nu+2\eta)}$ and we get an additional $e^{2\pi i\mu}$ from shifting the argument $t - s$ in the ratio of theta function in the square bracket. These factors are canceled by the shift of μ in the exponential function.

To compute this integral we therefore have to carefully consider the deformation of integration contours involved in the change of variables. As we shall see, this deformation produces a residue at the pole $t = s$.

For these considerations we assume that $\mathrm{Im}(\eta\Lambda_j) > N$, $j = 1, \ldots, n$, for some $N > 0$ large compared to τ, p, and that the points z_j are on the real axis. The general case can then be obtained by analytic continuation. In this range of parameters, the integration cycles for the t and s integration is the interval $[0, 1]$. The integrand, viewed as a function of t or s, is then regular in the strip $\mathrm{Im}(t)$, $\mathrm{Im}(s) \in (-N, N)$. The first term in (11), however, has additional poles at $t = s + \alpha + \beta(\tau + p)$, $(\alpha, \beta \in \mathbb{Z})$. To deal with these poles we move slightly the s-integration cycle into the upper half plane. After the change of variable, in the first term in (11), $\bar{t} = t + \tau$, $\bar{s} = s - p$, the new variables are integrated over $\bar{t} \in \tau + [0, 1]$, $\bar{s} \in -p + i\varepsilon + [0, 1]$ for some small $\varepsilon > 0$. The first term becomes then equal to the minus the second term in (11) after deforming the integration cycles to the original position, but during this deformation we encounter the pole at $t = s$. Therefore, we obtain a residue

$$(11) = \frac{i}{\sqrt{4i\eta}} e^{-\frac{i\pi}{2\eta}\lambda\nu} \int e^{-\frac{i\pi}{4\eta}(\lambda-\nu+\mu)^2} \omega_a(s, \vec{z}, \lambda, \tau, \eta)\omega_b^\vee(s, \vec{z}, \nu, p, \eta)$$
$$\times \prod_{j=1}^{n} \Omega_{\eta\Lambda_j}(s - z_j, \tau, \tau + p)\Omega_{\eta\Lambda_j}(s - z_j + p, \tau + p, p) \, ds \, d\mu.$$

Using identity (A.6) and

$$\frac{i}{\sqrt{4i\eta}} \int_{\eta\mathbb{R}} e^{-\frac{i\pi}{4\eta}(\lambda-\nu+\mu)^2} \, d\mu = 1, \qquad \mathrm{Im}\, \eta < 0,$$

we see that this expression reduces to $u(\vec{z}, \lambda, \nu, \tau, p, \eta)$, completing the proof of Theorem 2.1.

5.2. First Modular Relation

Here we present the proof of Theorem 3.3(i). The assumption that $\mathrm{Im}(\eta\tau/p) < 0$ implies that the integration over μ on the path $x \mapsto \eta x + \varepsilon$, $x \in \mathbb{R}$, converges absolutely: the singularities of Q are avoided for generic ε, and at infinity the factor $\exp(-i\pi\tau\mu^2/4\eta p)$ converges very fast to zero. To get the identity as stated in the theorem, one uses the following simple properties of ρ_a:

$$\rho_a\left(\frac{\lambda}{\tau}, \frac{\vec{z}}{\tau}, -\frac{1}{\tau}, \frac{\eta}{\tau}\right) = \rho_a(\lambda, \vec{z}, \tau, \eta)^{-1},$$
$$\rho_b^\vee\left(\frac{\nu}{p}, \frac{\vec{z}}{p}, -\frac{\tau}{p}, \frac{\eta}{p}\right)\rho_b\left(-\frac{\nu}{p}, \frac{\vec{z}}{p}, -\frac{\tau}{p}, \frac{\eta}{p}\right) = e^{\frac{2\pi i}{p\tau}(\nu - 2\eta + 2\eta\sum_1^{b-1}\Lambda_j)(\nu - 2\eta + 2\eta\sum_1^{b}\Lambda_j)}.$$

Let us proceed with the proof. We insert on the left-hand side the hypergeometric integral for $u_{a,c}$ and $u_{c,b}$ and call the integration variables t and s, respectively. It will be convenient to make the change of variables $s \to s/p$. To apply Lemma 5.1, we use the transformation properties of weight functions:

$$\begin{aligned}\omega_c&\left(\frac{s}{p}, \frac{\vec{z}}{p}, -\frac{\mu}{p}, -\frac{1}{p}, \frac{\eta}{p}\right)\\ &= \rho_c(-\mu, \vec{z}, p, \eta)^{-1} e^{-\frac{\pi i}{p}(2s - \sum_j z_j\Lambda_j)(\mu - 2\eta)}\omega_c(s, \vec{z}, -\mu, p, \eta). \qquad (12)\end{aligned}$$

A similar formula involving $\rho_b^\vee$ gives the transformation behavior of $\omega_b^\vee$. Using Lemma 5.1 and the functional relation (A.5) of Ω_a, we get

$$\begin{aligned}\sum_{c=1}^{n} &\int u_{a,c}(\vec{z}, \lambda, \mu, \tau, p, \eta) u_{c,b}\left(\frac{\vec{z}}{p}, -\frac{\mu}{p}, \frac{\nu}{p}, -\frac{1}{p}, -\frac{\tau}{p}, \frac{\eta}{p}\right)\\ &\times Q_c(\mu, p, \eta)\rho_c(-\mu, \vec{z}, p, \eta) e^{-\frac{\pi i\tau\mu^2}{4\eta p}}\, d\mu\\ &\quad = \frac{1}{p}\int \omega_a(t, \vec{z}, \lambda, \tau, \eta)\omega_b^\vee(s/p, \vec{z}/p, \nu/p, -\tau/p, \eta/p) e^{-\frac{i\pi}{2\eta}(\lambda - \nu/p)\mu}\\ &\quad\times \Bigg[\Phi(\mu, t, s) e^{-4\pi i\eta} - \Phi(\mu + 4\eta, t - \tau, s - \tau)\\ &\quad\times e^{-\frac{2\pi i\tau}{p}(\mu + 2\eta) - \frac{4\pi i\eta}{p}}\Bigg]\, dt\, ds\, e^{-\frac{\pi i\tau}{4\eta p}\mu^2}\, d\mu, \qquad (13)\end{aligned}$$

where

$$\Phi(\mu, t, s) = \frac{\theta(\mu - 2\eta + t - s, p)\theta'(0, p)}{\theta(\mu - 2\eta, p)\theta(t - s, p)} e^{-\frac{\pi i}{p}(2s - \sum_{j=1}^{n} z_j \Lambda_j)(\mu - 2\eta)} \times \prod_{j=1}^{n} \Omega_{\eta\Lambda_j}(t - z_j + \tau, p, \tau)\Omega_{\frac{\eta}{p}\Lambda_j}\left(\frac{s - z_j}{p}, -\frac{1}{p}, -\frac{\tau}{p}\right).$$

As in the proof of the heat equation, the two terms in this equation cancel after formally changing variables $t \mapsto t - \tau$, $s \mapsto s - \tau$, $\mu \mapsto \mu + 4\eta$ in the first term. However, we have to carefully see what happens to the integration cycles after this change of variables. We consider the region of parameters where $z_j \in \mathbb{R}$, $\mathrm{Im}(\eta\Lambda_j) \gg 0$ and $\pi/2 \gg \arg(p) > \arg(\tau) > 0$.

Then the original integration contours may be chosen to be the interval $[0, 1]$. For $z \in \mathbb{C}$, let us denote by γ_z the path

$$r \mapsto rz, \qquad r \in [0, 1].$$

After the change of variable $s \to s/p$, the s integration contour becomes the path γ_p.

Lemma 5.2. *Suppose that p, τ are complex numbers in the upper half plane. Let $f(t, s)$ be a meromorphic function of two variables such that $f(t + 1, s) = f(t, s + p) = f(t, s)$ and such that $\alpha(t, s) = (t - s)f(t, s)$ is regular in a domain containing*

$$\{(t + r\tau, sp + r\tau) \mid t, r, s \in [0, 1]\}.$$

Then

$$\int_{\gamma_1 \times \gamma_p} f(t + \tau, s + \tau)\, dt \wedge ds = \int_{\gamma_1 \times \gamma_p} f(t, s)\, dt \wedge ds + 2\pi i \int_{\gamma_\tau} \alpha(t, t)\, dt,$$

and all integrals are absolutely convergent.

Proof. Since p is not real, the integration contours intersect transversally, so that $f(t, s)$ (which has at most a simple pole on the diagonal) is absolutely integrable. More generally, with our assumption on the regularity of f, the integral

$$I_r = \int_{\gamma_1 \times \gamma_p} f(t + r\tau, s + r\tau)\, dt \wedge ds$$

is absolutely convergent for all $r \in [0, 1]$. Let, for $\varepsilon > 0$, D_ε be the integration domain obtained from $\gamma_1 \times \gamma_p$ by removing the points where

$|t-s|\leqslant\varepsilon$ and set

$$I_r(\varepsilon)=\int_{D_\varepsilon} f(t+r\tau,s+r\tau)\,dt\wedge ds.$$

Then $I_1-I_0=\lim_{\varepsilon\to 0}\int_0^1\frac{d}{dr}I_r(\varepsilon)\,dr$. We have

$$\begin{aligned}\frac{d}{dr}I_r(\varepsilon)&=-\tau\int_{D_\varepsilon} d(f(t+r\tau,s+r\tau)(dt-ds))\\&=-\tau\int_{\partial D_\varepsilon} f(t+r\tau,s+r\tau)(dt-ds)\\&=-\tau\int_{\partial D_\varepsilon}\alpha(t+r\tau,s+r\tau)\frac{dt-ds}{t-s}\\&=\tau\int_{|u|=\varepsilon}\alpha(u+r\tau,r\tau)\frac{du}{u}+O(\varepsilon)\\&=2\pi i\tau\alpha(r\tau,r\tau),\end{aligned}$$

from which the claim follows immediately. In the application of Stokes' theorem above, the other boundary components of ∂D_ε do not contribute since the function is periodic. The remaining component is homotopic to a circle with negative orientation, which explains the change of sign. ∎

Applying this lemma to our situation after shifting μ by 4η in the first term yields

$$\begin{aligned}(13)=&\frac{1}{p}\int_{-\eta\infty}^{\eta\infty}\int_{\gamma_\tau}\omega_a(t,\vec{z},\lambda,\tau,\eta)\omega_b^\vee(t/p,\vec{z}/p,\nu/p,-\tau/p,\eta/p)e^{-\frac{i\pi}{2\eta}(\lambda-\nu/p)\mu-\frac{4\pi i\eta}{p}}\\&\times e^{-\frac{\pi i}{p}(2t-\sum_j z_j\Lambda_j)(\mu+2\eta)}\prod_{j=1}^n\Omega_{\eta\Lambda_j}(t-z_j,p,\tau)\Omega_{\frac{\eta}{p}\Lambda_j}\left(\frac{t-z_j-\tau}{p},-\frac{1}{p},-\frac{\tau}{p}\right)dt\\&\times e^{-\frac{\pi i\tau}{4\eta p}\mu^2}\,d\mu.\end{aligned}$$

Now the Gaussian integral over μ along $\eta\mathbb{R}$ may be performed explicitly, and our claim follows from (A.13) and (12).

5.3. Second Modular Relation

Here we present the proof of Theorem 3.3(ii) which is parallel to the proof of Theorem 3.3(i).

The assumption that $\operatorname{Im}(\eta p/\tau)<0$ implies that the integration over μ over the path $x\mapsto\eta x+\varepsilon,\ x\in\mathbb{R}$, converges absolutely: the singularities of Q are avoided for generic ε, and at infinity the factor $\exp(-i\pi p\mu^2/4\eta\tau)$ converges very fast to zero.

We insert on the left-hand side the hypergeometric integral for $u_{a,c}$ and $u_{c,b}$ and call the integration variables t and s, respectively. It will be convenient to make the change of variables $t \to t/\tau$. To apply Lemma 5.1, we use the transformation properties of weight functions:

$$\omega_c^{\vee}\left(\frac{t}{\tau},\frac{\vec{z}}{\tau},\frac{\mu}{\tau},-\frac{1}{\tau},\frac{\eta}{\tau}\right) = \rho_c^{\vee}(\mu,\vec{z},\tau,\eta)^{-1} e^{\frac{\pi i}{\tau}(2t-\sum_j z_j\Lambda_j)(\mu+2\eta)}\omega_c^{\vee}(t,\vec{z},\mu,\tau,\eta). \tag{14}$$

We have

$$\begin{aligned}
&\sum_{c=1}^{n}\int u_{a,c}\left(\frac{\vec{z}}{\tau},\frac{\lambda}{\tau},\frac{\mu}{\tau},-\frac{p}{\tau},-\frac{1}{\tau},\frac{\eta}{\tau}\right) u_{c,b}(\vec{z},-\mu,\nu,\tau,p,\eta)\\
&\quad\times Q_c(\mu,\tau,\eta)\rho_c^{\vee}(\mu,\vec{z},\tau,\eta)\, e^{-\frac{i\pi p}{4\eta\tau}\mu^2}\,d\mu\\
&\quad=\frac{1}{\tau}\int \omega_a(t/\tau,\vec{z}/\tau,\lambda/\tau,-p/\tau,\eta/\tau)\omega_b^{\vee}(s,\vec{z},\nu,p,\eta)e^{-\frac{i\pi}{2\eta}(\lambda/\tau-\nu)\mu}\\
&\quad\times[\Phi(\mu,t,s)e^{-\frac{4\pi i\eta}{\tau}}-\Phi(\mu+4\eta,t+p,s+p)e^{-\frac{2\pi ip}{\tau}(\mu-2\eta)-4\pi i\eta}]\,dt\,ds\,e^{-\frac{\pi ip}{4\eta\tau}\mu^2}\,d\mu,
\end{aligned}$$

where

$$\begin{aligned}
\Phi(\mu,t,s) =&\frac{\theta(\mu-2\eta+t-s,\tau)\theta'(0,\tau)}{\theta(\mu-2\eta,\tau)\theta(t-s,\tau)}e^{\frac{\pi i}{\tau}(2t-\sum_{j=1}^{n}z_j\Lambda_j)(\mu+2\eta)}\\
&\times\prod_{j=1}^{n}\Omega_{\frac{\eta}{\tau}\Lambda_j}\left(\frac{t-z_j-p}{\tau},-\frac{p}{\tau},-\frac{1}{\tau}\right)\Omega_{\eta\Lambda_j}(s-z_j,\tau,p).
\end{aligned}$$

The two terms on the right-hand side of this equation cancel after formally changing variables $t\mapsto t-p$, $s\mapsto s-p$, $\mu\mapsto\mu-4\eta$ in the second term. However, we have to carefully see what happens to the (t,s) integration cycle after this change of variables.

We consider the region of parameters where $z_j\in\mathbb{R}$, $\mathrm{Im}(\eta\Lambda_j)\gg 0$ and $\pi/2\gg\arg(\tau)>\arg(p)>0$. In this case, the (t,s) integration cycle is the product of paths $\gamma_\tau\times\gamma_1$.

Applying to this situation Lemma 5.2 after decreasing μ by 4η, we see that the left-hand side of the new modular relation is equal to

$$\begin{aligned}
&\frac{2\pi i}{\tau}\int_{-\eta\infty}^{\eta\infty}\int_{\gamma_p}\omega_a(t/\tau,\vec{z}/\tau,\lambda/\tau,-p/\tau,\eta/\tau)\omega_b^{\vee}(t,\vec{z},\nu,p,\eta)e^{-\frac{i\pi}{2\eta}(\lambda/\tau-\nu)\mu}e^{-\frac{\pi ip}{4\eta\tau}\mu^2}\\
&\times e^{-\frac{4\pi i\eta}{\tau}}e^{\frac{\pi i}{\tau}(2t-\sum_{j=1}^{n}z_j\Lambda_j)(\mu+2\eta)}\prod_{j=1}^{n}\Omega_{\frac{\eta}{\tau}\Lambda_j}\left(\frac{t-z_j-p}{\tau},-\frac{p}{\tau},-\frac{1}{\tau}\right)\Omega_{\eta\Lambda_j}(s-z_j,\tau,p).
\end{aligned}$$

Now the Gaussian integral over μ along $\eta\mathbb{R}$ may be performed explicitly, and our claim follows from (A.13) and (12).

5.4. *The Inversion Formulae*

Here we conclude the proof of Theorem 4.5. We have to prove the identities

$$\frac{1}{16\pi^2\eta}\int_{\mathbb{R}+i\sigma}\sum_{c=1}^{n} u_{c,a}(\vec{z},-\mu,\lambda,p,\tau,-\eta)u_{c,b}(\vec{z},\mu,v,p,\tau,\eta)Q_c(-\mu,p,\eta)\,d\mu$$
$$=\frac{\delta_{ab}\delta(\lambda+v)}{Q_b(v,\tau,\eta)}, \tag{15}$$

$$\frac{1}{16\pi^2\eta}\int_{\eta\mathbb{R}+\tilde{\sigma}}\sum_{c=1}^{n} u_{a,c}(\vec{z},\lambda,\mu,\tau,p,\eta)u_{b,c}(\vec{z},v,-\mu,\tau,p,-\eta)Q_c(\mu,p,\eta)\,d\mu$$
$$=\frac{\delta_{ab}\delta(\lambda+v)}{Q_b(v,\tau,\eta)}. \tag{16}$$

We give a proof of the first of these identities. The second is treated in a similar way.

As in the proof of the heat equation, the μ-dependent part of the integrand may be simplified by collecting the terms with the same poles, and using the functional relation of Ω_a, with the result:

$$\frac{1}{16\pi^2\eta}\int\sum_{c=1}^{n} u_{c,a}(\vec{z},-\mu,\lambda,p,\tau,-\eta)u_{c,b}(\vec{z},\mu,v,p,\tau,\eta)Q_c(-\mu,p,\eta)\,d\mu$$
$$=\int \omega_a^{\vee}(t,\vec{z},\lambda,\tau,-\eta)\omega_b^{\vee}(s,\vec{z},v,\tau,\eta)e^{-\frac{i\pi}{2\eta}(\lambda+v)\mu}$$
$$\times\Bigg[-\frac{\theta(\mu+2\eta+s-t,p)\theta'(0,p)}{\theta(\mu+2\eta,p)\theta(s-t,p)}\prod_{j=1}^{n}\Omega_{-\eta\Lambda_j}(t-z_j,p,\tau)\Omega_{\eta\Lambda_j}(s-z_j,p,\tau)$$
$$+\frac{\theta(\mu-2\eta+s-t,p)\theta'(0,p)}{\theta(\mu-2\eta,p)\theta(s-t,p)}\prod_{j=1}^{n}\Omega_{-\eta\Lambda_j}(t-z_j+\tau,p,\tau)$$
$$\times\,\Omega_{\eta\Lambda_j}(s-z_j+\tau,p,\tau)\Bigg]\,dt\,ds\,d\mu. \tag{17}$$

As in the proof of the heat equation, the two terms in this equation cancel after formally changing variables $t\mapsto t-\tau$, $s\mapsto s-\tau$, $\mu\mapsto\mu+4\eta$ in the second term. However, we have to carefully see what happens to the integration cycles after this change of variables. We consider a region of parameters where $z_j\in\mathbb{R}$ and η,τ are small compared to 1, $\mathrm{Im}(p)$ and to the distance between the z_j. We study the singularities of the integrand and the form of the integration cycles in the vicinity of z_j. The poles of the integrand of $u(\vec{z}, \mu, v, p, \tau, \eta)$ are at $s = z_j - \eta\Lambda_j - k\tau - lp$ and at $s = z_j + \eta\Lambda_j + k\tau + lp$ $(k, l \in \mathbb{Z}_{\geqslant 0})$. The cycle for the integration over s goes below $z_j + \eta\Lambda_j$ and above $z_j - \eta\Lambda_j$. The poles of the integrand of $u(\vec{z}, -\mu, \lambda, p, \tau, -\eta)$ in the vicinity of z_j are at $t = z_j + \eta\Lambda_j - k\tau - lp$ and at $t = z_j - \eta\Lambda_j + k\tau + lp$ $(k, l \in \mathbb{Z}_{\geqslant 0})$. The t integration cycle goes above $z_j + \eta\Lambda_j$ and below $z_j - \eta\Lambda_j$. These integration contours are depicted in Fig. 2.

To treat the two terms on the right-hand side of (17) separately, we split the contour for the t integration into two pieces, as shown in Fig. 3. Then the integration cycle does not meet the additional poles at $t = s$ of both terms.

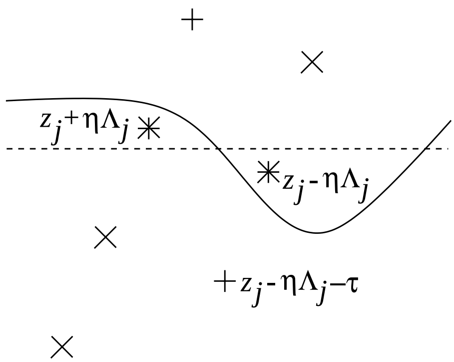

FIG. 2. The integration cycle. The solid line is the contour for the integration over t and the dashed line is the contour of integration for s. They are oriented from left to right. The symbols $\times$ and $+$ indicate the poles of the integrand as a function of t and as a function of s, respectively. The points $z_j \pm \eta\Lambda_j$ are poles for both variables.

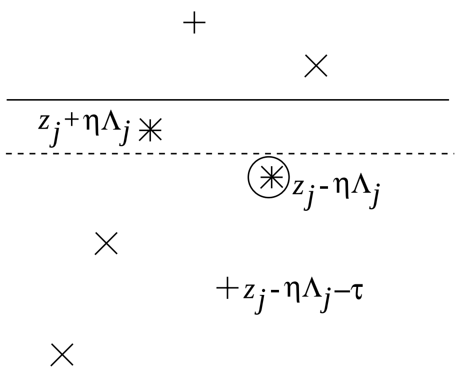

FIG. 3. The integration cycle of Fig. 2 can be deformed to this cycle. The circle around $z_j - \eta\Lambda_j$ is oriented counterclockwise.

Let us first consider the first term in (17). As a function of t it is actually regular at $z_j - \eta\Lambda_j$ so that the circle around this point does not contribute. In Fig. 4 the remaining integration cycle is shown, along with the position of the poles for t and s.

After changing variables $t \mapsto t - \tau$, $s \mapsto s - \tau$, $\mu \mapsto \mu + 4\eta$ in the second term of (17) the integrand of the second term becomes equal to the first one, but the integration contour are shifted: the μ-integration cycle can be deformed back to the original one without encountering singularities. The cycle for the t and s integration is depicted in Fig. 5.

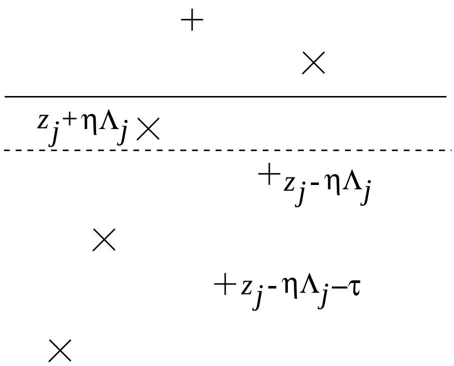

FIG. 4. In the first term on the right-hand side of (17) the integrand is more regular and the cycle may be replaced by this one.

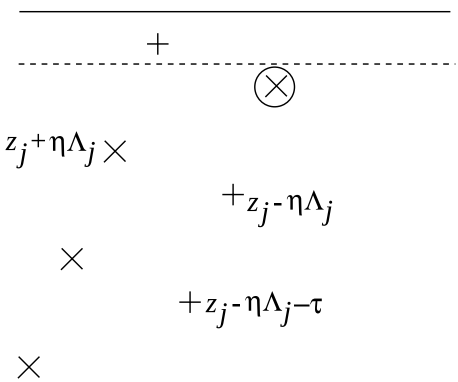

FIG. 5. This is the integration cycle after the shift of the variables t and s in the second term. The poles of the result are indicated by the symbols $\times, +$ as in the preceding figures.

We have to deform the integration cycle to the position shown in Fig. 4. In so doing, we pick the residue at $t = z_j - \eta\Lambda_j + \tau$ of the residue at $s = t$. At this point, we may use identity (A.14). The result is

$$(17) = \frac{-(2\pi i)^2}{16\pi^2\eta}\,\mathrm{res}_{t=z_j-\eta\Lambda_j+\tau}\,\omega_a^\vee(t,\vec{z},\lambda,\tau,-\eta)\omega_b^\vee(t,\vec{z},\nu,\tau,\eta)\int e^{-\frac{i\pi}{2\eta}(\lambda+\nu)\mu}\,d\mu$$

$$= \mathrm{res}_{t=z_j-\eta\Lambda_j}\,\omega_a^\vee(t,\vec{z},\lambda,\tau,-\eta)\omega_b^\vee(t,\vec{z},\nu,\tau,\eta)\delta(\lambda+\nu).$$

The residue may be computed at $z_j - \eta\Lambda_j$ since its argument is τ-periodic. Our claim follows then from the

Lemma 5.3.

$$\sum_{j=1}^{n} \mathrm{res}_{t=z_j-\eta\Lambda_j}\omega_a^\vee(t,\vec{z},\lambda,\tau,-\eta)\omega_b^\vee(t,\vec{z},-\lambda,\tau,\eta) = \delta_{ab}Q_a(-\lambda,\tau,\eta)^{-1}.$$

Proof. This formula can be deduced from the more general result in [TV], formula (C.4). For the sake of completeness, we include the simple proof in this special case. The product of weight functions of which we compute the residue is an elliptic function of t whose simple poles in a fundamental domain are in the set $\{z_j \pm \eta\Lambda_j,\ j = 1,\dots,n\}$.

If $a > b$, then the function is regular at $z_j - \eta\Lambda_j$, $j = 1,\dots,n$, so the residues vanish. If $a<b$, then the function is regular at $z_j + \eta\Lambda_j$. Therefore, the sum of the residues in the claim is the sum over the residues at *all* poles and vanishes by the residue theorem.

If $a = b$, only the residue at $t = z_a - \eta\Lambda_a$ gives a non-vanishing contribution, which is easily evaluated, and gives $Q_a(-\lambda,\tau,\eta)^{-1}$. ∎

6. SL$(3,\mathbb{Z})$-IDENTITIES FOR HYPERGEOMETRIC INTEGRALS

In this section, we recast our results into a form which shows the analogy with the identities discovered in [FV2] for the elliptic gamma function. In [FV2] we showed that the elliptic gamma function is a "degree 1" generalized Jacobi modular function for the group $G = \mathrm{SL}(3,\mathbb{Z})\ltimes\mathbb{Z}^3$. This amounts to the identities of Theorem 6.1 below. The hypergeometric integral u obeys a non-Abelian version of these identities. We also discuss the geometric interpretation of these identities: they can be interpreted as the projective flatness of a discrete connection on a vector bundle over a suitable G-space, see also [FV4]. As in the case of Gamma functions these properties are more transparent if we rewrite the identities, which we formulate in 6.2

in homogeneous variables x defined by $\tau = x_1/x_3,\ p = x_2/x_3$. The G-action is then linear. The next step, as in the case of Γ, is to extend the range of parameter to a G-space, as the condition of positivity of imaginary parts is not preserved by the action. This can be done in a surprisingly easy way by reflection arguments. We do it in the simplest case of one tensor factor in 6.5. After this is done, we rewrite the identities as the projective flatness of an $\mathrm{SL}(3,\mathbb{Z})$-Connection in 6.6.

6.1. Identities for the Elliptic Gamma Function

We first recall the properties of the elliptic gamma function that are relevant here. More details are included in Appendix A.

THEOREM 6.1 (Felder and Varchenko [FV1]). *The elliptic gamma function, defined by the formula*

$$\Gamma(z,\tau,p) = \prod_{j,k=0}^{\infty} \frac{1 - e^{2\pi i((j+1)\tau + (k+1)p - z)}}{1 - e^{2\pi i(j\tau + kp + z)}},$$

for $\mathrm{Im}(\tau), \mathrm{Im}(p) > 0$, *obeys the following identities*:

$$\Gamma(z+1,\tau,p) = \Gamma(z,\tau,p), \tag{18}$$

$$\Gamma(z+p,\tau,p) = \theta_0(z,\tau)\Gamma(z,\tau,p), \tag{19}$$

$$\Gamma(z+\tau,\tau,p) = \theta_0(z,p)\Gamma(z,\tau,p), \tag{20}$$

$$\Gamma(z,\tau+1,p) = \Gamma(z,\tau,p+1) = \Gamma(z,\tau,p), \tag{21}$$

$$\Gamma(z,\tau+p,p) = \frac{\Gamma(z,\tau,p)}{\Gamma(z+\tau,\tau,p+\tau)}, \tag{22}$$

$$\Gamma(z/\tau,-1/\tau,p/\tau) = e^{i\pi Q(z;\tau,p)}\Gamma((z-\tau)/p,-\tau/p,-1/p)\Gamma(z,\tau,p), \tag{23}$$

for some $Q \in \mathbb{Q}(\tau,p)[z]$ *given in Appendix* A.

In fact, Γ is also defined for negative imaginary parts of τ and p by a reflection procedure, see Section A.5, and the above identities continue to hold in this wider range of parameters.

6.2. Identities for Hypergeometric Integrals

Let $E_\delta(\vec{z},\tau,c;\eta)$ be the space of functions $\varphi\in E^0(\vec{z},\tau,c;\eta)$ such that there exist constants $C_1, C_2>0$ (depending on φ) such that

$$|e^{\frac{\pi i\delta\lambda^2}{4\eta}}\varphi_a(\lambda)|\leqslant C_1\exp\left(\pi\frac{(\operatorname{Im}\lambda)^2}{\operatorname{Im}\tau}+C_2|\lambda|\right),$$

for all $a=1,\ldots,n$.

For $\delta=0$ we have constructed examples of functions in E_δ in Proposition 4.2.

PROPOSITION 6.2. *Let $\Phi_S(\vec{z},\tau,c,\eta)$, $\Phi_T(\eta)$ be the operators acting on $V_\Lambda[0]$-valued functions $\varphi(\lambda)=\sum_{a=1}^n\varphi_a(\lambda)\varepsilon_a$* as

$$(\Phi_S(\vec{z},\tau,c,\eta)\varphi)_a(\lambda)=e^{-\frac{\pi ic\lambda^2}{4\eta\tau}}\rho_a(\lambda,\vec{z},\tau,\eta)\varphi_a\left(\frac{\lambda}{\tau}\right),$$

$$(\Phi_T(\eta)\varphi)_a(\lambda)=e^{-\frac{\pi i\lambda^2}{4\eta}}\varphi_a(\lambda),$$

$$(\Phi_C\varphi)_a(\lambda)=\varphi_a(-\lambda).$$

Then these operators restrict to isomorphisms

$$\Phi_S(\vec{z},\tau,c,\eta):E_\delta\left(\frac{\vec{z}}{\tau},-\frac{1}{\tau},\frac{c}{\tau},\frac{\eta}{\tau}\right)\to E_{\frac{\delta+c}{\tau}}(\vec{z},\tau,c,\eta),$$

$$\Phi_T(\eta):E_\delta(\vec{z},\tau,c,\eta)\to E_{\delta+1}(\vec{z},\tau,c+\tau,\eta),$$

$$\Phi_C:E_\delta(-\vec{z},\tau,-c,-\eta)\to E_{-\delta}(\vec{z},\tau,c,\eta).$$

Then Theorem 4.5 about the Fourier transform may be reformulated, in a slightly generalized form, as follows:

PROPOSITION 6.3. *The operator*

$$(V(\vec{z},\tau,p,\eta)\varphi)_a(\lambda)=\int\sum_{b=1}^n u_{ab}(\vec{z},\lambda,\mu,\tau,p,\eta)Q_b(\mu,p,\eta)\varphi_b(-\mu)\,d\mu,$$

is an invertible linear map from $E_\delta(\vec{z},p,\tau,\eta)$ onto $E_{-1/\delta}(\vec{z},\tau,-p,\eta)$.

The properties of the universal hypergeometric function may then be expressed as relations for these operators: first of all the analogue of identities (18)–(20) are the qKZB and mirror qKZB equations (3). The remaining identities are given by the following result.

THEOREM 6.4. (i) *$E_\delta(\vec z, \tau + r, p + s, \eta) = E_\delta(\vec z, \tau, p, \eta)$, if $r, s \in \mathbb{Z}$. The identities*

$$V(\vec z, \tau + 1, p, \eta) = V(\vec z, \tau, p + 1, \eta) = V(\vec z, \tau, p, \eta)$$

hold in $\mathrm{Hom}(E_\delta(\vec z, p, -\tau, \eta), E_{-1/\delta}(\vec z, \tau, p, \eta))$.

(ii) *The identity*

$$V(\vec z, \tau, p, \eta) = c_T \Phi_T(\eta) V(\vec z, \tau, \tau + p, \eta) \Phi_T(\eta)\, V(\vec z, \tau + p, p, \eta)\, \Phi_T(\eta)$$

holds in $\mathrm{Hom}(E_\delta(\vec z, p, \tau, \eta), E_{-1/\delta}(\vec z, \tau, -p, \eta))$ *with*

$$c_T = -\frac{e^{4\pi i\eta}}{2\pi\sqrt{4i\eta}}.$$

(iii) *The identity*

$$\begin{aligned}\Phi_S(\vec z, \tau, -p, \eta)\, V\left(\frac{\vec z}{\tau}, -\frac{1}{\tau}, \frac{p}{\tau}, \frac{\eta}{\tau}\right) \Phi_S\left(\frac{\vec z}{\tau}, \frac{p}{\tau}, -\frac{1}{\tau}, \frac{\eta}{\tau}\right) \\ = c_S V(\vec z, \tau, p, \eta) \Phi_S(\vec z, p, \tau, \eta)\, V\left(\frac{\vec z}{p}, -\frac{1}{p}, -\frac{\tau}{p}, \frac{\eta}{p}\right)\end{aligned}$$

holds in $\mathrm{Hom}(E_\delta(\vec z/p, -\tau/p, -1/p, \eta/p), E_{\delta'}(\vec z, \tau, -p, \eta))$ *with* $\delta' = p\delta/(1 - \tau\delta)$ *and*

$$c_S = \sqrt{\frac{ip}{4\eta\tau}}\, e^{\frac{\pi i\eta}{3p\tau}\left(3\sum_{j<k} \Lambda_j\Lambda_k(z_j - z_k)^2 + 2\left(\sum_{j=1}^n \eta^2\Lambda_j^3 + \tau^2 + p^2 - 3p + 3\tau + 3\tau p + 1\right)\right)}.$$

Proof. Statement (i) is trivial. Statement (ii) is a reformulation of Theorem 2.1 and (iii) is a reformulation of Theorem 3.3. ∎

6.3. The Universal Hypergeometric Function for One Tensor Factor

For simplicity, we assume from now on that the tensor product V_Λ of sl_2 representations consists of one factor which is the three-dimensional representation V_2. In this case, the universal hypergeometric function does not depend on z_1 and is a scalar function,

$$u(\lambda, \mu, \tau, p, \eta) = e^{-\frac{i\pi\lambda\mu}{2\eta}} \int_C \Omega_{2\eta}(t, \tau, p) \frac{\theta(\lambda + t, \tau)\theta(\mu + t, p)}{\theta(t - 2\eta, \tau)\theta(t - 2\eta, p)}\, dt. \qquad (24)$$

The integration cycle C is as in Section 2: in this case it can be defined to be the interval $[0, 1]$ if η has positive imaginary part. For general η, the integral is defined by analytic continuation, with the effect of deforming the contour as in Fig. 1. The resulting function is a meromorphic function on $\mathbb{C}\times\mathbb{C}\times H_+\times H_+\times\mathbb{C}$, where H_+ denotes the open upper half-plane. This function is regular as long as the integration cycle is not pinched between poles of the integrand. As these poles are at $\pm(2\eta+r\tau+sp+t)$, $r,s\in\mathbb{Z}_{\geqslant 0}, t\in\mathbb{Z}$, u is regular if 4η does not belong to the set $\mathbb{Z}\tau+\mathbb{Z}p+\mathbb{Z}$.

The following is an interesting alternative formula for the function u:

$$\begin{aligned} u(\lambda,\mu,\tau,p,\eta) = &-\frac{\Gamma(4\eta,\tau,p)}{\prod_{j=1}^{\infty}[(1-e^{2\pi ij\tau})(1-e^{2\pi ijp})]}e^{-\frac{\pi i}{2\eta}(\lambda+2\eta)(\mu+2\eta)} \\ &\times\sum_{j,k=0}^{\infty} e^{-2\pi i(j\lambda+k\mu)+(2j+1)(2k+1)\eta} \\ &\times\theta_0(\lambda+2\eta+kp,\tau)\theta_0(\mu+2\eta+j\tau,p) \\ &\times\prod_{l=0}^{k-1}\frac{\theta_0(lp+4\eta,\tau)}{\theta_0((l+1)p,\tau)}\prod_{l=0}^{j-1}\frac{\theta_0(l\tau+4\eta,p)}{\theta_0((l+1)\tau,p)}, \end{aligned}$$

where $\theta_0(z,\tau)=\prod_{j=0}^{\infty}(1-e^{2\pi i(j\tau+z)})(1-e^{2\pi i((j+1)\tau-z)})$, see Appendix A. This formula can be proved by moving C to infinity and picking up the residues at the poles, and is valid in a certain region of parameter space. We will not use this formula in this paper.

The Shapovalov form is

$$Q(\lambda,\tau,\eta)=\frac{\theta(4\eta,\tau)\theta'(0,\tau)}{\theta(\lambda-2\eta,\tau)\theta(\lambda+2\eta,\tau)}.$$

6.4. *Spaces and Operators*

Consider $\mathbb{C}^3$ and the projectivization of the dual space, $P(\mathbb{C}^3)^*$. Consider $W\subset(\mathbb{C}-0)^3\times P(\mathbb{C}^3)^*$ where

$$W=\{((x_1,x_2,x_3),(y_1:y_2:y_3))\mid x_1y_1+x_2y_2+x_3y_3=0\}.$$

The natural projection $W\to\mathbb{C}^3-0$ is a projective line bundle. The group $SL(3,\mathbb{Z})$ acts on W, $g:(x,y)\mapsto(gx,(g^t)^{-1}y)$ for $g\in SL(3,\mathbb{Z})$.

For $(x;y)\in W$ such that $\operatorname{Im}x_1/x_3\neq 0, y_2\neq 0$ introduce the space of functions

$$E(x;y)=\begin{cases}\{v(\lambda)\mid v(x_3\lambda)\in E_{\delta=-y_1/y_2}(\frac{x_1}{x_3},\frac{x_2}{x_3},\frac{1}{2x_3})\}, & \text{if } \operatorname{Im}\frac{x_1}{x_3}>0,\\ \{v(\lambda)\mid v(x_3\lambda)\in E_{\delta=-y_1/y_2}(-\frac{x_1}{x_3},-\frac{x_2}{x_3},\frac{1}{2x_3})\}, & \text{if } \operatorname{Im}\frac{x_1}{x_3}<0.\end{cases}$$

Thus, $E(x;y)$ consists of entire holomorphic functions $v(\lambda)$ obeying the resonance relation $v(-1+rx_3+sx_1)=e^{2\pi is(2\sigma+x_2)/x_3}v(1+rx_3+sx_1)$, where

σ is the sign of $\operatorname{Im} x_1/x_3$, and the bound

$$|v(\lambda)e^{-\frac{\pi i y_1 \lambda^2}{2x_3 y_2}}| \leqslant C_1 \exp\left(\pi \frac{(\operatorname{Im}\frac{\lambda}{x_3})^2}{|\operatorname{Im}\frac{x_1}{x_3}|} + C_2|\lambda|\right), \tag{25}$$

for some constants $C_1, C_2 > 0$. Note that

$$E(x_1, x_2, x_3; y_1, y_2, y_3) = E(-x_1, x_2, -x_3; -y_1, y_2, -y_3).$$

For $\operatorname{Im}\frac{x_1}{x_3} > 0, \operatorname{Im}\frac{x_2}{x_3} > 0$, introduce an integral operator $U(x_1, x_2, x_3)$ acting on functions in $E(x_2, x_1, x_3; y_2, y_1, y_3)$ as

$$U(x_1, x_2, x_3)v(\lambda) = \frac{e^{\frac{\pi i}{x_3}}}{2\pi\sqrt{2ix_3}} \int u\left(\frac{\lambda}{x_3}, \frac{\mu}{x_3}, \frac{x_1}{x_3}, \frac{x_2}{x_3}, \frac{1}{2x_3}\right) Q\left(\frac{\mu}{x_3}, \frac{x_2}{x_3}, \frac{1}{2x_3}\right) v(-\mu)\, d\mu.$$

The integration path is given as follows. Suppose first that $|x_2|, |x_3| \gg 1$. Then the integration is over a straight line which does not intersect the segments joining $-1 + rx_2 + sx_3$ to $1 + rx_2 + sx_3$ $(r, s \in \mathbb{Z})$ and such that the integrand decays exponentially at infinity along it. Such a path exists as the integrand behaves as $\exp(-\text{const}\,\mu^2)$ at infinity, which decays for μ in a cone. The resonance relation implies that the integrand has opposite residue at the pairs $\pm 1 + rx_2 + sx_3$, so that, up to sign, the integral is independent of the choice of path. For x_2, x_3 general, the integral is defined by analytic continuation. The operator is defined up to multiplication by ± 1 since one needs to choose a square root and an orientation of the path.

We assume that

$$2 \notin \mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}x_3.$$

This ensures that $U(x_1, x_2, x_3)$ is well defined. Indeed, the integration kernel (24) defining U is regular at points obeying this condition by the discussion of Section 6.3 above. Moreover, this condition implies that in the integral over μ defining U, the integration contour is not pinched between poles.

Proposition 6.3 states that $U(x_1, x_2, x_3)$ is an invertible linear map

$$U(x_1, x_2, x_3) : E(x_2, x_1, x_3; y_2, y_1, y_3) \to E(x_1, -x_2, x_3; y_1, -y_2, y_3),$$

if $\operatorname{Im}\frac{x_1}{x_3} > 0, \operatorname{Im}\frac{x_2}{x_3} > 0$.

Introduce operators $\alpha(x_3)$, $\beta(x_1, x_2, x_3)$ where

$$\alpha(x_3)v(\lambda) = e^{-\frac{i\pi}{x_3}(\frac{1}{2}\lambda^2 - \frac{1}{3})}v(\lambda),$$

$$\beta(x_1,x_2,x_3)v(\lambda) = e^{-\frac{\pi i}{x_2x_3}(x_1(\frac{1}{2}\lambda^2-\frac{1}{3})+\lambda^2-1)+\frac{2\pi i}{9x_1x_2x_3}}v(\lambda) \qquad \text{if } \operatorname{Im}\frac{x_2}{x_3}>0,$$

$$\beta(x_1,x_2,x_3)v(\lambda) = e^{-\frac{\pi i}{x_2x_3}(x_1(\frac{1}{2}\lambda^2-\frac{1}{3})-\lambda^2+1)+\frac{2\pi i}{9x_1x_2x_3}}v(\lambda) \qquad \text{if } \operatorname{Im}\frac{x_2}{x_3}<0.$$

The λ-independent terms in α and β were added to simplify relations (1)–(4) below, which would otherwise only hold up to factors depending on x.

PROPOSITION 6.5. *The operators α and β are isomorphisms*

$$\alpha(x_3) : E(x_2,x_1,x_3;y_2,y_1,y_3) \to E(x_2,x_1+x_2,x_3;y_2-y_1,y_1,y_3),$$

$$\beta(x_1,x_2,x_3) : E(x_3,x_1,-x_2;y_3,y_1,-y_2) \to E(x_2,x_1,x_3,y_2,y_1,y_3),$$

for all $(x,y)\in W$ for which they are defined.

Proof. The statement for α is easily checked inserting the definitions. The fact that β respects the resonance relation is also easily checked. We are left to prove that v obeys bound (25) if and only if βv does. Let us prove the only if part in the case where x_2/x_3 has positive imaginary part. The other cases are proved in the same way. So we assume that $|\exp(\pi i y_3\lambda^2/2x_2y_1)v(\lambda)|\leqslant C_1\exp(\pi\frac{(\operatorname{Im}\lambda/x_2)^2}{\operatorname{Im}(-x_3/x_2)}+C_2|\lambda|)$. It follows using the relation $\sum x_iy_i=0$ that $v'=\beta(x_1,x_2,x_3)v$ obeys the bound $|\exp(-\pi i y_2\lambda^2/2x_3y_1+\pi i\lambda^2/x_2x_3)v'(\lambda)|\leqslant C_1'\exp(\pi\frac{(\operatorname{Im}\lambda/x_2)^2}{\operatorname{Im}(-x_3/x_2)}+C_2|\lambda|)$. The claim then follows from the identity

$$\operatorname{Im}\frac{\lambda^2}{x_2x_3} = \frac{(\operatorname{Im}\frac{\lambda}{x_3})^2}{\operatorname{Im}\frac{x_2}{x_3}} - \frac{(\operatorname{Im}\frac{\lambda}{x_2})^2}{\operatorname{Im}-\frac{x_3}{x_2}}. \quad \blacksquare$$

Theorems 2.2, 3.3 and 4.5 take the form of identities:

(1) The *q-heat equation*,

$$\alpha(x_3)\,U(x_1,x_1+x_2,x_3)\,\alpha(x_3)U(x_1+x_2,x_2,x_3)\,\alpha(x_3) \;=\; U(x_1,x_2,x_3),$$

holds on $E(x_2,x_1,x_3;y_2,y_1,y_3)$, if $\operatorname{Im}\frac{x_1}{x_3}>0, \operatorname{Im}\frac{x_2}{x_3}>0$.

(2) The *first modular equation*,

$$U(x_1,x_2,x_3)\beta(x_1,x_2,x_3)\,U(-x_3,-x_1,x_2)$$
$$=\beta(-x_2,x_1,x_3)\,U(-x_3,x_2,x_1)\,\beta(-x_3,x_2,x_1),$$

holds on $E(-x_1,-x_3,x_2;-y_1,-y_3,y_2)$ if $\operatorname{Im}\frac{x_1}{x_3}>0, \operatorname{Im}\frac{x_2}{x_3}>0, \operatorname{Im}\frac{x_2}{x_1}>0$.

(3) The *second modular equation*,

$$U(-x_2,-x_3,x_1)\beta(-x_2,-x_3,x_1)\,U(x_1,x_2,x_3)$$
$$=\beta(x_3,x_2,-x_1)\,U(x_1,-x_3,x_2)\,\beta(x_1,x_3,-x_2),$$

holds on $E(x_2,x_1,x_3;y_2,y_1,y_3)$ if $\mathrm{Im}\frac{x_1}{x_3}>0,\mathrm{Im}\frac{x_2}{x_3}>0,\mathrm{Im}\frac{x_1}{x_2}>0$.

(4) The *inversion relation*,

$$U(x_1,x_2,x_3)\,U(-x_2,-x_1,-x_3)=1,$$

holds on $E(-x_1,-x_2,-x_3;-y_1,-y_2,-y_3)$ if $\mathrm{Im}\frac{x_1}{x_3}>0,\mathrm{Im}\frac{x_2}{x_3}>0$.

Note that each of the operators U in these identities is defined up to multiplication by ± 1, so the right-hand side of each of the identities is equal to the left-hand side up to multiplication by ± 1.

6.5. New Range of Parameters

Extend the definition of the operator $U(x_1,x_2,x_3)$ from the domain $\mathrm{Im}\frac{x_1}{x_3}>0,\mathrm{Im}\frac{x_2}{x_3}>0$ to the domain $x_1/x_3,x_2/x_3\in\mathbb{C}-\mathbb{R}$ by the formulas

$$U(x_1,x_2,x_3)=U(x_2,-x_1,x_3)^{-1}\qquad \text{if } \mathrm{Im}\frac{x_1}{x_3}<0,\ \mathrm{Im}\frac{x_2}{x_3}>0,$$

$$U(x_1,x_2,x_3)=U(-x_2,x_1,x_3)^{-1}\qquad \text{if } \mathrm{Im}\frac{x_1}{x_3}>0,\ \mathrm{Im}\frac{x_2}{x_3}<0,$$

$$U(x_1,x_2,x_3)=U(-x_1,-x_2,x_3)\qquad \text{if } \mathrm{Im}\frac{x_1}{x_3}<0,\ \mathrm{Im}\frac{x_2}{x_3}<0.$$

THEOREM 6.6. (i) *For any* $(x;y)\in W$ *such that* $x_1/x_3,x_2/x_3\in\mathbb{C}-\mathbb{R}$ *and* $2\notin\mathbb{Z}x_1+\mathbb{Z}x_2+\mathbb{Z}x_3$, *the operator* $U(x)$ *defines an invertible linear map* $E(x_2,x_1,x_3;y_2,y_1,y_3)\to E(x_1,-x_2,x_3;y_1,-y_2,y_3)$.

(ii) *Moreover, the q-heat equation, first and second modular equations, the inversion equation hold for* x_1,x_2,x_3 *such that* $x_1/x_2,x_1/x_3,x_2/x_3\in\mathbb{C}-\mathbb{R}$ *and* $2\notin\mathbb{Z}x_1+\mathbb{Z}x_2+\mathbb{Z}x_3$.

The six-term relations of Theorem 6.6 are the commutativity of the following diagrams.

The *q-heat hexagon*:

$$
\begin{tikzcd}
& E(x_2,x_1,x_3;y_2,y_1,y_3) \arrow[dl, "\alpha(x_3)"'] \arrow[dr, "U(x_1,x_2,x_3)"] & \\
E(x_2,x_1+x_2,x_3;y_2-y_1,y_1,y_3) \arrow[d, "U(x_1+x_2,x_2,x_3)"'] & & E(x_1,-x_2,x_3;y_1,-y_2,y_3) \\
E(x_1+x_2,-x_2,x_3;y_1,y_1-y_2,y_3) \arrow[dr, "\alpha(x_3)"'] & & E(x_1,-x_1-x_2,x_3;y_1-y_2,-y_2,y_3) \arrow[u, "\alpha(x_3)"'] \\
& E(x_1+x_2,x_2,x_3;y_2,y_1-y_2,y_3) \arrow[ur, "U(x_1,x_1+x_2,x_3)"'] &
\end{tikzcd}
$$

The *first modular hexagon*:

$$
\begin{tikzcd}
& E(-x_1,-x_3,x_2;-y_1,-y_3,y_2) \arrow[dl, "\beta(-x_3,x_2,x_1)"'] \arrow[dr, "U(-x_3,-x_1,x_2)"] & \\
E(x_2,-x_3,x_1;y_2,-y_3,y_1) \arrow[d, "U(-x_3,x_2,x_1)"'] & & E(-x_3,x_1,x_2;-y_3,y_1,y_2) \arrow[d, "\beta(x_1,x_2,x_3)"] \\
E(-x_3,-x_2,x_1;-y_3,-y_2,y_1) \arrow[dr, "\beta(-x_2,x_1,x_3)"'] & & E(x_2,x_1,x_3;y_2,y_1,y_3) \arrow[dl, "U(x_1,x_2,x_3)"] \\
& E(x_1,-x_2,x_3;y_1,-y_2,y_3) &
\end{tikzcd}
$$

The *second modular hexagon*:

$$
\begin{tikzcd}
& E(x_2,x_1,x_3;y_2,y_1,y_3) \arrow[dl, "\beta(x_1,x_3,-x_2)"'] \arrow[dr, "U(x_1,x_2,x_3)"] & \\
E(-x_3,x_1,x_2;-y_3,y_1,y_2) \arrow[d, "U(x_1,-x_3,x_2)"'] & & E(x_1,-x_2,x_3;y_1,-y_2,y_3) \arrow[d, "\beta(-x_2,x_3,x_1)"] \\
E(x_1,x_3,x_2;y_1,y_3,y_2) \arrow[dr, "\beta(x_3,x_2,-x_1)"'] & & E(-x_3,-x_2,x_1;-y_3,-y_2,y_1) \arrow[dl, "U(-x_2,-x_3,x_1)"] \\
& E(-x_2,x_3,x_1;-y_2,y_3,y_1) &
\end{tikzcd}
$$

The proof of Theorem 6.6 is done by reduction to the case where the imaginary parts of x_1/x_3 and x_2/x_3 are positive. In fact, it is straightforward to see that in all cases the identities for general imaginary parts can be rewritten using the definitions as identities at some other values of x, y where the imaginary parts are positive.

6.6. Projectively Flat $\mathrm{SL}(3,\mathbb{Z})$ *Connections*

We give here an interpretation of our relations in terms of projectively flat discrete $\mathrm{SL}(3,\mathbb{Z})$-connections.

We start by introducing the notion of discrete connections. Let X be a G-space with a fixed presentation by generators e_j, $j=1,\dots,k$ and relations $R_i=1$, $i=1,\dots,r$. A *discrete G-connection* on a complex vector bundle $\pi: F\to X$, with fibers $F(x)=\pi^{-1}(x)$, $x\in X$, assigns to each generator e_j a collection of linear isomorphisms $\phi_{e_j}(x): F(e_j^{-1}x)\to F(x)$, $x\in X$. The *parallel translation* along an element w of the free group Free_k generated by the e_j's is the collection of maps $\phi_w(x): F(\bar{w}^{-1}x)\to F(x)$ uniquely defined by the properties $\phi_1(x)=\mathrm{Id}_{F(x)}$, $\phi_{ww'}(x)=\phi_w(x)\circ\phi_{w'}(\bar{w}^{-1}x)$. Here $w\mapsto\bar{w}$ is the canonical projection $\mathrm{Free}_k\to G$. The *curvature* of a discrete G-connection is the collection of parallel translations $\phi_{R_i}(x)\in\mathrm{End}(F(x))$ along the relations. A connection is called *projectively flat* if $\phi_{R_i}(x)\in\mathbb{C}\,\mathrm{Id}$ for all i and x.

Let $\pi: F\to Y$ be a vector bundle over a subset Y of a G-space X. Then a *discrete connection defined on* Y assigns to each generator e_j a collection of linear isomorphisms $\phi_{e_j}(x): F(e_j^{-1}x)\to F(x)$, for all those $x\in Y$ such that $e_j^{-1}x\in Y$. The parallel translation $\phi_w(x)$ is then defined on some subset $Y_w\subset Y$. A discrete connection defined on Y is projectively flat if $\phi_{R_i}(x)\in\mathbb{C}\,\mathrm{Id}$ for all $i=1,\dots,r$ and $x\in Y_{R_i}$.

In our case, $G=\mathrm{SL}(3,\mathbb{Z})$ and for X we take certain orbits in W.

The group $\mathrm{SL}(3,\mathbb{Z})$ is generated by the elementary matrices e_{ij}, $(1\leqslant i,j\leqslant 3, i\neq j)$ with ones in the diagonal and at (i,j) and zeros everywhere else. The relations can be chosen [M] to be

$$\begin{aligned} e_{ij}e_{kl}&=e_{kl}e_{ij}, \qquad i\neq l,\quad j\neq k,\\ e_{ij}e_{jk}&=e_{ik}e_{jk}e_{ij}, \qquad i,j,k \text{ distinct},\\ (e_{13}\,e_{31}^{-1}e_{13})^4&=1. \end{aligned} \tag{26}$$

An $\mathrm{SL}(3,\mathbb{Z})$-orbit X in W is called *regular* if for all $(x,y)\in X$

(1) $x_1/x_3\in\mathbb{C}-\mathbb{R}$ and $x_2/x_3\in\mathbb{C}-\mathbb{R}$ and
(2) $2\notin\mathbb{Z}x_1+\mathbb{Z}x_2+\mathbb{Z}x_3$.

Let X be a regular orbit in W. For $(x,y)\in Y^{\psi}=\{(x,y)\in X\mid y_1\neq 0\}$, set

$$F^{\psi}(x;y)=E(x_2,x_1,x_3;y_2,y_1,y_3).$$

For $(x,y) \in Y^\phi = \{(x,y) \in X \mid y_2 \neq 0\}$, set

$$F^\phi(x;y) = E(x_1,-x_2,x_3;y_1,-y_2,y_3).$$

PROPOSITION 6.7. *Let $X \subset W$ be a regular orbit.*

(i) *The assignment $e_{ij} \mapsto \phi_{ij}(x)$ with*

$$\phi_{12}(x) = \alpha(x_3)^{-1}U(x_1-x_2,x_1,x_3)^{-1}\alpha(x_3)^{-1},$$

$$\phi_{13}(x) = \phi_{23}(x) = 1,$$

$$\phi_{21}(x) = \alpha(x_3)^{-1},$$

$$\phi_{31}(x) = \beta(-x_2,x_1-x_3,x_3),$$

$$\phi_{32}(x) = \beta(x_3-x_2,x_3,-x_1)^{-1}U(x_3-x_2,x_3,-x_1)^{-1}\beta(-x_3,x_1,x_3-x_2)^{-1},$$

defines a projectively flat $\mathrm{SL}(3,\mathbb{Z})$-connection defined on $Y^\phi \subset X$.

(ii) *The assignment $e_{ij} \mapsto \psi_{ij}(x)$ with*

$$\psi_{12}(x) = \alpha(x_3),$$

$$\psi_{13}(x) = \psi_{23}(x) = 1,$$

$$\psi_{21}(x) = \alpha(x_3)U(x_2,x_2-x_1,x_3)\alpha(x_3),$$

$$\psi_{31}(x) = \beta(x_1-x_3,-x_3,x_2)^{-1}U(x_1-x_3,-x_3,x_2)^{-1}\beta(x_3,x_2,x_3-x_1)^{-1},$$

$$\psi_{32}(x) = \beta(x_1,x_2-x_3,x_3),$$

defines a projectively flat $\mathrm{SL}(3,\mathbb{Z})$-connection defined on $Y^\psi \subset X$.

(iii) *For all $i \neq j$ and all $x \in Y^\phi \cap Y^\psi$ such that $e_{ij}^{-1}x \in Y^\phi \cap Y^\psi$, we have*

$$U(x)\psi_{ij}(x) = \phi_{ij}(x)U(e_{ij}^{-1}x).$$

Proof. It first follows from Theorem 6.6(i) and Proposition 6.5 that $\phi_{ij}(x)$ is in all cases a well-defined isomorphism from $F^\phi(e_{ij}^{-1}(x,y))$ to $F^\phi(x,y)$ and similarly for ψ. The other claims are then simple consequences of the q-heat, modular and inversion relations (Theorem 6.6(ii)), and the relations

$$U(x_1,x_2+x_3,x_3) = U(x_1+x_3,x_2,x_3) = U(x_1,x_2,x_3),$$

which follow from the fact that $u(\lambda,\mu,\tau,p,\eta)$ is 1-periodic in τ and p.

The easiest way to do the computations is to first check (iii), which can be easily deduced from our three term relations. This identity implies that if the curvature associated to a relation is scalar for one of the connections ϕ or ψ, then it is scalar (and equal) also for the other connection. Then the curvature can be computed using ϕ or ψ, whichever is simpler. For example, to compute the curvature $C_{12}^{32} = \phi_R$ associated to the relation $R = e_{12}e_{32}e_{12}^{-1}e_{32}^{-1}$, it is better to use the connection ψ. One gets

$$\psi_{12}(x)\psi_{32}(e_{12}^{-1}x) = C_{12}^{32}(x)\psi_{32}(x)\psi_{12}(e_{32}^{-1}x),$$

where

$$C_{12}^{32}(x) = \exp\frac{2\pi i\, x_2}{9\, x_1x_3(x_1 - x_2)(x_2 - x_3)}. \quad \blacksquare$$

In particular, if an orbit does not contain any point with $y_1 = 0$ or $y_2 = 0$, which is true for generic orbits, we have $Y^\phi = Y^\psi = X$ and the connections are defined everywhere. The trouble is that we do not know if the spaces $F^\phi,\ F^\psi$ are non-trivial for these orbits. By contrast, we have infinitely many examples of linearly independent functions in $F^\psi(x, y)$ with $y_2 = 0$ or $F^\phi(x, y)$ with $y_1 = 0$. Indeed, these spaces are isomorphic to spaces $E_\delta(\tau, -p, \eta)$ with $\delta = 0$, which contain the functions $u_\mu : \lambda \mapsto u(\lambda, \mu, \tau, p, \eta)$ with any fixed μ, see Proposition 4.2. These functions are non-zero for generic μ since they are the meromorphic kernel of an invertible integral operator. There are infinitely many linearly independent functions among them since they behave differently under shifts by 1: $u_\mu(\lambda + 1) = -e^{-\pi i\mu/2\eta}u_\mu(\lambda)$. In this case, we may construct a connection defined everywhere by gluing the two partially defined connections:

THEOREM 6.8. *Let X be a regular orbit containing a point (x, y) with $y_1 = 0$. Let $Y^\psi,\ Y^\phi \subset X$ be the subsets as above, on which the connections $\phi,\ \psi$ are defined. We have $Y^\phi \cup Y^\psi = X$. Let F be the vector bundle on X obtained from F^ψ and F^ϕ by identifying the fibers over $Y^\psi \cap Y^\phi$ via $U(x) : F^\psi(x, y) \to F^\phi(x, y)$. Let a connection $\chi_{ij}(x)$ be defined as $\phi_{ij}(x)$, if both (x, y) and $e_{ij}^{-1}(x, y)$ are in Y^ϕ, and as $\psi_{ij}(x)$ if both (x, y) and $e_{ij}^{-1}(x, y)$ are in Y^ψ. Then χ is a well-defined projectively flat connection on X.*

Proof. Since any two points $(x, y), e_{ij}^{-1}(x, y) \in X$ related by a generator belong both to Y^ϕ or both to Y^ψ, the connection χ_{ij} is defined in all cases. Moreover, by Proposition 6.7(iii), χ is well defined on F, at least up to sign. It is easy to check that that the curvatures along the relations involve products of isomorphisms $\chi_{ij}(x)$ mapping between fibers at points (x, y) which are all in Y^ϕ or all in Y^ψ. Thus, the claim that χ is a projectively flat connection follows from the projective flatness of ϕ and ψ. $\blacksquare$

6.7. Formula for the Curvature

We give here a formula for the curvature of the discrete connection ϕ (and ψ). It is convenient to do a gauge transformation $\bar{\phi}_{ij}(x) = \phi_{ij}(x)g_{ij}(x)$, with

$$g_{12}(x) = \exp\left(\frac{2\pi i}{9x_1(x_1-x_2)x_3}\right), \qquad g_{13}(x) = \exp\left(\frac{2\pi i}{9x_1(x_1-x_3)x_2}\right),$$

$$g_{21}(x) = \exp\left(\frac{2\pi i}{9x_2(x_2-x_1)x_3}\right), \qquad g_{23}(x) = \exp\left(\frac{2\pi i}{9x_2(x_2-x_3)x_1}\right),$$

$$g_{31}(x) = 1, \qquad g_{32}(x) = \exp\left(-\frac{2\pi i}{9x_3(x_3-x_2)x_1}\right).$$

Then the curvature of the connection $\bar{\phi}$ has components $\bar{C}_{ij}^{kl}(x)$, $\bar{C}(x)$ associated to relations (26). By definition, they are given by

$$\bar{\phi}_{ij}(x)\bar{\phi}_{kl}(e_{ij}^{-1}x) = \bar{C}_{ij}^{kl}(x)\bar{\phi}_{kl}(x)\bar{\phi}_{ij}(e_{kl}^{-1}x), \qquad i\neq l, \quad j\neq k,$$

$$\bar{\phi}_{ij}(x)\bar{\phi}_{jk}(e_{ij}^{-1}x) = \bar{C}_{ij}^{jk}(x)\bar{\phi}_{ik}(x)\bar{\phi}_{jk}(e_{ik}^{-1}x)\bar{\phi}_{ij}(e_{jk}^{-1}e_{ik}^{-1}x), \qquad i,j,k \text{ distinct},$$

$$\bar{\phi}_s(x)\bar{\phi}_s(s^{-1}x)\bar{\phi}_s(s^{-2}x)\bar{\phi}_s(s^{-3}x) = \bar{C}(x),$$

where $sx = (-x_3, x_2, x_1)$ and

$$\bar{\phi}_s(x) = \bar{\phi}_{13}(x)\bar{\phi}_{31}(e_{31}e_{13}^{-1}x)^{-1}\bar{\phi}_{13}(e_{13}^{-1}e_{31}x).$$

We have

$$\bar{C}_{12}^{32}(x) = \bar{C}_{32}^{12}(x)^{-1} = \exp\left(\frac{2\pi i x_2}{3x_1x_3(x_2-x_3)(x_1-x_2)}\right),$$

$$\bar{C}_{13}^{32}(x) = \exp\left(\frac{2\pi i}{3x_3(x_1-x_2)(x_3-x_1)}\right), \quad \bar{C}_{31}^{12}(x) = \exp\left(\frac{2\pi i}{3x_1(x_3-x_2)(x_3-x_1)}\right),$$

and all other $\bar{C}_{ij}^{kl}(x)$ as well as $\bar{C}(x)$ are equal to 1.

6.8. Comparison with the Elliptic Gamma Cocycle [FV2]

In [FV2], a non-trivial 2-cocycle of $\mathrm{SL}(3,\mathbb{Z})$ with values in $\exp(2\pi i\mathbb{Q}(x_j/x_k)[z/x_3])$ was obtained by a similar construction involving the elliptic gamma function instead of U. In the language of discrete connections we use here, one defines an $\mathrm{SL}(3,\mathbb{Z})$ connection on the trivial line bundle over a dense subset of $\mathbb{C}^3$ by setting

$$\phi_{\Gamma,1,2}(x,z) = \Gamma\left(\frac{z-x_2}{x_3}, \frac{x_1-x_2}{x_3}, -\frac{x_1}{x_3}\right)^{-1},$$
$$\phi_{\Gamma,3,2}(x,z) = \Gamma\left(\frac{z}{x_1}, \frac{x_2-x_3}{x_1}, \frac{x_3}{x_1}\right),$$
$$\phi_{\Gamma,i,j}(x,z) = 1, \qquad j \neq 2.$$

For our purpose, z may be considered here as a complex parameter on which $\mathrm{SL}(3,\mathbb{Z})$ acts trivially. The curvature of this connection was computed in [FV2]. It has the form $C^{kl}_{\Gamma,ij} = \exp(\pi i L^{kl}_{ij}(z,x))$, $C_\Gamma(x) = 1$, for some cubic polynomials $L^{kl}_{ij}(x,z)$ in z with coefficients in $\mathbb{Q}(x_1,x_2,x_3)$. The relation to the curvature of $\bar{\phi}$ is

$$\bar{C}^{kl}_{ij}(x) = \exp(-2\pi i \times \text{Coefficient of } z^3 \text{ in } L^{kl}_{ij}(x,z)).$$

The curvature of a projectively flat connection defines an extension of the group and thus a characteristic class, see [FV2]. Conjecturally, $\bar{C}^{kl}_{ij}$ defines a non-trivial class in the group cohomology $H^2(\mathrm{SL}(3,\mathbb{Z}), \exp(\mathbb{C}(x_1,x_2,x_3)))$. This means (if the conjecture is true) that there is no gauge transformation given by exponentials of rational functions that can make the curvature trivial.

APPENDIX A. THETA FUNCTIONS AND ELLIPTIC GAMMA FUNCTIONS

We summarize some formulae about theta functions, gamma functions and phase functions, see [FV2] for more details.

A.1. THE THETA FUNCTION

Jacobi's first theta function is defined by the series

$$\theta(z,\tau) = -\sum_{j\in\mathbb{Z}} e^{i\pi\tau(j+1/2)^2 + 2\pi i(j+1/2)(z+1/2)}, \qquad z,\tau \in \mathbb{C}, \qquad \operatorname{Im}\tau > 0.$$

It is an entire holomorphic odd function such that

$$\theta(z+n+m\tau,\tau) = (-1)^{m+n} e^{-\pi i m^2\tau - 2\pi i m z}\theta(z,\tau), \qquad m,n\in\mathbb{Z}, \tag{A.1}$$

and obeys the heat equation

$$4\pi i \frac{\partial}{\partial\tau}\theta(z,\tau) = \theta''(z,\tau).$$

Its transformation properties with respect to $\mathrm{SL}(2,\mathbb{Z})$ are described in terms of generators by the identities:

$$\theta(-z,\tau) = -\theta(z,\tau), \quad \theta(z,\tau+1) = e^{\frac{i\pi}{4}}\theta(z,\tau), \quad \theta\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = i\sqrt{-i\tau}e^{\frac{i\pi z^2}{\tau}}\theta(z,\tau).$$

We choose the square root in the right half plane.

A.2. INFINITE PRODUCTS

Let $x, q \in \mathbb{C}$ with $|q|<1$. The function

$$(x;q) = \prod_{j=0}^{\infty}(1 - xq^j)$$

is a solution of the functional equation

$$(qx;q) = \frac{1}{1-x}(x;q).$$

Let $x = e^{2\pi iz}$ and $q = e^{2\pi i\tau}$. Then

$$\theta(z,\tau) = ie^{\pi i(\tau/4 - z)}(x;q)(q/x;q)(q;q). \tag{A.2}$$

We will also need the following variant of the theta function θ:

$$\theta_0(z,\tau) = (x;q)(q/x;q) = -i\frac{e^{\pi i(z-\tau/4)}}{(q;q)}\theta(z,\tau).$$

This function obeys

$$\begin{aligned}
\theta_0(z+1,\tau) &= \theta_0(z,\tau),\\
\theta_0(z+\tau,\tau) &= -e^{-2\pi iz}\theta_0(z,\tau),\\
\theta_0(\tau - z,\tau) &= \theta_0(z,\tau).
\end{aligned} \tag{A.3}$$

Its modular properties are $\theta_0(z,\tau+1) = \theta_0(z,\tau)$, and if $z' = z/\tau$, $\tau' = -1/\tau$,

$$e^{\pi i(\tau/6 - z)}\theta_0(z,\tau) = ie^{\pi i(-zz' + \tau'/6 - z')}\theta_0(z',\tau').$$

A.3. ELLIPTIC GAMMA FUNCTIONS

Here we consider two parameters τ and p in the upper half plane, and set $q = e^{2\pi i\tau}$, $r = e^{2\pi ip}$, and consider the function of $x = e^{2\pi iz}$,

$$(x; q, r) = \prod_{j,k=0}^{\infty} (1 - xq^j r^k) = (x; r, q).$$

It is a solution of the functional equations

$$(qx; q, r) = \frac{(x; q, r)}{(x; r)}, \qquad (rx; q, r) = \frac{(x; q, r)}{(x; q)}.$$

The *elliptic gamma function* [FV2, R] is

$$\Gamma(z, \tau, p) = \frac{(qr/x; q, r)}{(x; q, r)} = \Gamma(z, p, \tau).$$

It obeys the identities

$$\Gamma(z + 1, \tau, p) = \Gamma(z, \tau, p), \qquad \Gamma(z + p, \tau, p) = \theta_0(z, \tau)\Gamma(z, \tau, p), \tag{A.4}$$

and is normalized by $\Gamma((\tau + p + 1)/2, \tau, p) = 1$. The zeros of $(x; q, r)$ are at $x = q^{-j}r^{-k}$, $j, k = 0, 1, 2, \ldots$. They are all simple. Thus, Γ has only simple zeros and simple poles. The zeros are at $z = (j + 1)\tau + (k + 1)p + l$, and the poles are at $z = -j\tau - kp + l$. Here j, k run over non-negative integers and l over all integers.

In fact, $\Gamma(z, \tau, p)$ is, up to normalization, the unique 1-periodic meromorphic solution of $u(z + p) = \theta_0(z, \tau)u(z)$ holomorphic in the upper half plane.

A.4. MODULAR PROPERTIES

We consider the transformation properties of the elliptic gamma function under modular transformations of p and τ. We have the identities

$$\begin{aligned}
\Gamma(z, \tau, p) &= \Gamma(z, p, \tau), \\
\Gamma(z, \tau + 1, p) &= \Gamma(z, \tau, p), \\
\Gamma(z, \tau + p, p) &= \frac{\Gamma(z, \tau, p)}{\Gamma(z + \tau, \tau, p + \tau)}, \\
\Gamma(z/p, \tau/p, -1/p) &= e^{i\pi Q(z;\tau,p)}\Gamma((z - p)/\tau, -1/\tau, -p/\tau)\Gamma(z, \tau, p),
\end{aligned}$$

$$Q(z;\tau,p) = \frac{z^3}{3\tau p} - \frac{\tau+p-1}{2\tau p}z^2 + \frac{\tau^2+p^2+3\tau p-3\tau-3p+1}{6\tau p}z + \frac{1}{12}(\tau+p-1)(\tau^{-1}+p^{-1}-1).$$

A.5. EXTENDING THE RANGE OF PARAMETERS

Since many operations we perform do not preserve the upper half plane, it is important to extend the range of values τ and p can take. We set

$$(x;q^{-1}) = \frac{1}{(qx;q)}, \qquad (x;q^{-1},r) = \frac{1}{(qx;q,r)}, \qquad (x;q,r^{-1}) = \frac{1}{(rx;q,r)}.$$

These formulae define an extension of the functions $(x;q),(x;q,r)$ to meromorphic functions on $\{(x,q,r)||q|\neq 1\neq |r|\}$. It is clear that the functional relations

$$(qx,q) = \frac{1}{1-x}(x,q), \qquad (qx;q,r) = \frac{1}{(x;r)}(x;q,r),$$

still hold in this larger domain. Correspondingly, we extend the definition of θ_0 and the elliptic gamma function by using the same formulae in terms of the infinite products. We obtain

$$\theta_0(z,-\tau) = \frac{1}{\theta_0(z+\tau,\tau)}, \qquad \Gamma(z,-\tau,p) = \frac{1}{\Gamma(z+\tau,\tau,p)},$$
$$\Gamma(z,\tau,-p) = \frac{1}{\Gamma(z+p,\tau,p)}.$$

An easy check gives the following result:

PROPOSITION A.1. *All identities for Γ and θ_0 of the preceding subsections continue to hold for all z,τ,p such that $\tau,p\notin\mathbb{R}$.*

However, the statements about the position of zeros and poles are no longer valid.

A.6. THE PHASE FUNCTION

We keep the notation of the previous subsection and introduce a new variable a, and set $\alpha = e^{2\pi i a}$. The phase function is

$$\Omega_a(z,\tau,p) = \frac{\Gamma(z+a,\tau,p)}{\Gamma(z-a,\tau,p)} = \frac{(qr/x\alpha;q,r)(x/\alpha;q,r)}{(x\alpha;q,r)(qr\alpha/x;q,r)}.$$

We have

$$\Omega_a(z+p,\tau,p) = \frac{\theta_0(z+a,\tau)}{\theta_0(z-a,\tau)}\Omega_a(z,\tau,p) = e^{2\pi i a}\frac{\theta(z+a,\tau)}{\theta(z-a,\tau)}\Omega_a(z,\tau,p). \quad (A.5)$$

The properties of this function follow from those of the gamma function:

Pproposition A.2. *The function $\Omega_a(z,\tau,p)$ obeys the identities*

$$\Omega_a(z+p,\tau,p) = e^{2\pi i a}\frac{\theta(z+a,\tau)}{\theta(z-a,\tau)}\Omega_a(z,\tau,p), \quad (A.6)$$

$$\Omega_a(z+\tau,\tau,p) = e^{2\pi i a}\frac{\theta(z+a,p)}{\theta(z-a,p)}\Omega_a(z,\tau,p), \quad (A.7)$$

$$\Omega_a(z+1,\tau,p) = \Omega_a(z,\tau,p), \quad (A.8)$$

$$\Omega_a(z,\tau,p) = \Omega_a(z,p,\tau), \quad (A.9)$$

$$\Omega_a(z,\tau,p) = \Omega_a(z,\tau,\tau+p)\Omega_a(z+p,\tau+p,p), \quad (A.10)$$

$$\Omega_a(z,\tau,p) = \Omega_a(z+\tau,\tau,\tau+p)\Omega_a(z,\tau+p,p), \quad (A.11)$$

$$\Omega_a(z,\tau+1,p) = \Omega_a(z,\tau,p+1) = \Omega_a(z,\tau,p), \quad (A.12)$$

$$\begin{aligned}\Omega_{a/\tau}\left(\frac{z}{\tau},-\frac{1}{\tau},\frac{p}{\tau}\right) &= e^{\pi i S_a(z;\tau,p)}\Omega_a(z,\tau,p)\Omega_{a/p}\left(\frac{z-\tau}{p},-\frac{1}{p},-\frac{\tau}{p}\right),\\ S_a(z;\tau,p) &= \frac{a}{3\tau p}(6z^2-6(\tau+p-1)z\\ &\quad +2a^2+\tau^2+p^2+3\tau p-3\tau-3p+1),\end{aligned} \quad (A.13)$$

$$\Omega_a(z,\tau,p)\Omega_{-a}(z,\tau,p) = 1, \quad (A.14)$$

$$\Omega_a(-z,\tau,p) = \Omega_a(z,\tau,p)\frac{\theta(z+a,\tau)\theta(z+a,p)}{\theta(z-a,\tau)\theta(z-a,p)}. \quad (A.15)$$

APPENDIX B. MODULAR PROPERTIES OF R-MATRICES AND QKZB OPERATORS

Here we summarize some formulae giving the transformation properties of the R-matrix. Fix $\Lambda_1, \Lambda_2 \in \mathbb{C}$ and let $R(z_1 - z_2, \lambda, \tau, \eta)$ be the R-matrix of $E_{\tau,\eta}(sl_2)$ associated to the evaluation Verma modules $V_{\Lambda_1}(z_1),\ V_{\Lambda_2}(z_2)$. We have

$$R(z_1 - z_2, \lambda, \tau + 1, \eta) = R(z_1 - z_2, \lambda, \tau, \eta)$$

and

$$\begin{aligned} R\left(\frac{z}{\tau}, \frac{\lambda}{\tau}, -\frac{1}{\tau}, \frac{\eta}{\tau}\right) &= e^{(z_1 - z_2)\frac{2\pi i \eta \Lambda_1 \Lambda_2}{\tau}} A_1(\lambda - 2\eta h^{(2)}) A_2(\lambda) \\ &\quad \times R(z_1 - z_2, \lambda, \tau, \eta) A_1(\lambda)^{-1} A_2(\lambda - 2\eta h^{(1)})^{-1}. \end{aligned}$$

Here $A_i(\lambda) = A(z_i, h^{(i)}, \Lambda_i, \lambda, \eta)$ with

$$A(z, h, \Lambda, \lambda, \eta) = \exp \frac{i\pi}{\tau}\left[z(h\lambda - \eta h^2) + \frac{1}{2}(\Lambda - h)(\lambda + \eta\Lambda - \eta h)(\lambda - \eta\Lambda - \eta h)\right].$$

These formulae can be deduced from the functional realization of representations [FTV1]: the R-matrix may be defined as the unique linear map such that

$$R(z_1 - z_2, \lambda, \tau, \eta)\omega(z_1, z_2, \lambda, \tau, \eta) = \omega^{\vee}(z_1, z_2, \lambda, \tau, \eta).$$

The weight function $\omega = \omega_{ij} e_i^{\Lambda_1} \otimes e_j^{\Lambda_2}$ and the mirror weight function $\omega^{\vee}$ take values in the tensor product $V_{\Lambda_1} \otimes V_{\lambda_2}$ of evaluation Verma modules. They are given explicitly in terms of ratios of theta functions, see [FTV1, FV1].

From the above formulae, we deduce the transformation properties of the qKZB operators. One finds

$$\begin{aligned} &K_i\left(\frac{z}{\tau}, \frac{\lambda}{\tau}, -\frac{1}{\tau}, \frac{p}{\tau}, \frac{\eta}{\tau}\right) \\ &\quad = e^{\frac{2\pi i \eta}{\tau}\sum_{j=1}^{n} \Lambda_i \Lambda_j (z_i - z_j + p\theta_{ij})} \prod_{j=1}^{n} A(z_j - p\theta_{ij}, h^{(j)}, \Lambda_j, \lambda - 2\eta \sum_{l=1}^{j-1} h^{(l)}, \eta) \\ &\qquad \times K_i(z, \lambda, \tau, p, \eta) \prod_{j=1}^{n} A\left(z_j - p\theta_{ij}, h^{(j)}, \Lambda_j, \lambda - 2\eta \sum_{l=1}^{j-1} h^{(l)}, \eta\right)^{-1}, \end{aligned}$$

where we set $\theta_{ij} = 1$ if $i > j$ and $\theta_{ij} = 0$ if $i \leqslant j$. In this somewhat simplified notation, the operators A depend on λ and are to be viewed as multiplication operator by these functions of λ; K_i taken at λ/τ really means that we conjugate K_i by the operator of dilation of λ by a factor $1/\tau$.

APPENDIX C. ESTIMATES FOR THETA FUNCTIONS

Lemma C.1. (i) *For all* τ *in the upper half plane, there exists a constant* $C_1(\tau) > 0$ *such that for all* $\lambda \in \mathbb{C}$,

$$|\theta(\lambda,\tau)| \leqslant C_1(\tau) \exp\left(\pi \frac{(\mathrm{Im}\,\lambda)^2}{\mathrm{Im}\,\tau}\right).$$

(ii) *For all* $\varepsilon > 0$ *and* τ *in the upper half plane, there exists a constant* $C_2(\tau) > 0$ *such that for all* λ *such that* $\min\{|\lambda - r - \tau s|, r, s \in \mathbb{Z}\} \geqslant \varepsilon$,

$$|\theta(\lambda,\tau)| \geqslant C_2(\tau)\varepsilon \exp\left(\pi \frac{(\mathrm{Im}\,\lambda)^2}{\mathrm{Im}\,\tau}\right).$$

Proof. Let $S(\tau)$ be the compact set $S(\tau) = \{\lambda \in \mathbb{C} \,|\, |\mathrm{Re}(\lambda)| \leqslant 1/2, |\mathrm{Im}\,\lambda| \leqslant \mathrm{Im}\,\tau/2\}$. Then $\theta(\lambda,\tau)$ vanishes in $S(\tau)$ only for $\lambda = 0$.

We use the functional relation (A.1) of the theta function. If $\lambda = \lambda_0 + m' + m\tau$, with $m', m \in \mathbb{Z}$ and $\lambda_0 \in S(\tau)$ then

$$\begin{aligned}
|\theta(\lambda,\tau)| &= e^{\pi m(2\mathrm{Im}\,\lambda_0 + m\,\mathrm{Im}\,\tau)}|\theta(\lambda_0,\tau)| \\
&= e^{\pi \frac{\mathrm{Im}\,\lambda - \mathrm{Im}\,\lambda_0}{\mathrm{Im}\,\tau}(\mathrm{Im}\,\lambda_0 + \mathrm{Im}\,\lambda)}|\theta(\lambda_0,\tau)| \\
&= e^{\pi \frac{(\mathrm{Im}\,\lambda)^2}{\mathrm{Im}\,\tau}} e^{-\pi \frac{(\mathrm{Im}\,\lambda_0)^2}{\mathrm{Im}\,\tau}} |\theta(\lambda_0,\tau)|.
\end{aligned}$$

Then the claim follows by setting $C_1(\tau) = \max\{e^{-\pi \frac{(\mathrm{Im}\,\lambda_0)^2}{\mathrm{Im}\,\tau}}|\theta(\lambda_0,\tau)|, \lambda_0 \in S(\tau)\}$ and $C_2(\tau) = \min\{e^{-\pi \frac{(\mathrm{Im}\,\lambda_0)^2}{\mathrm{Im}\,\tau}}|\theta(\lambda_0,\tau)/\lambda_0|, \lambda_0 \in S(\tau)\}$, so that when $|\lambda_0| \geqslant \varepsilon$, we have

$$\theta(\lambda_0,\tau) \geqslant \theta(\lambda_0,\tau)\frac{\varepsilon}{|\lambda_0|} \geqslant C_2(\tau)\varepsilon. \quad \blacksquare$$

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