On the Generalized Quasi-variational Inequality Problems

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Submitted by William F. Ames

Received September 29, 1995

THIS PAPER IS DEDICATED TO PROFESSOR KY FAN ON HIS 80TH BIRTHDAY

The purpose of this paper is to study the existence problem of solutions for some kinds of abstract generalized quasi-variational inequality problems by using a new kind of fixed point approach.

0022-247X/96 $18.00

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1. INTRODUCTION

The quasi-variational inequality (QVI) was introduced and studied in Bensoussan and Lions [3, 4] first, which was raised in some problems of random impulse controls, and it has many applications in control and optimization theory, economics and transportation equilibrium, contact problems in elasticity, fluid flow through porous media, game theory, and mathematical programming. In recent years, various extensions of QVI problems have been proposed and analysed (see, for example, [6–8, 12, 15, 18–20]).

In 1982, Chan and Pang [6] and Fang and Peterson [10] considered the following generalized quasi-variational inequality (GQVI) problem: Let $S \subseteq \mathbb{R}^n$ be a nonempty subset and $T: S \to 2^S$ a multifunction. Find $x \in S$, $y \in T(x)$ such that

$$(y, u - x) \geq 0 \quad \text{for all } u \in S.$$  

In 1987, Parida and Sen [14] considered the following GQVI problem: Let $S \subseteq \mathbb{R}^n$ and $C \subseteq \mathbb{R}^p$ be two nonempty subsets, $T: S \to 2^C$ a multifunction, and $M: S \times C \to \mathbb{R}^n$, $\eta: S \times S \to \mathbb{R}^n$ single-valued functions. Find $x \in S$, $y \in T(x)$ such that

$$(M(x, y), \eta(u, x)) \geq 0 \quad \text{for all } u \in S.$$  

Recently, Chang and Shu [7, 8], Kum [12], and Yao and Guo [18–20] considered the following abstract GQVI problem under various conditions: Let $E, F$ be two locally convex Hausdorff topological vector spaces, let $X \subseteq E$, $Y \subseteq F$ be two nonempty subsets, let $S: X \to 2^X$, $T: X \to 2^Y$ be two multifunctions, and $\varphi: X \times Y \times X \to \mathbb{R}$ be a continuous function with $\varphi(x, y, x) \geq 0$ for all $x \in X$, $y \in Y$. Find $x \in S(x)$, $y \in T(x)$ such that

$$\varphi(x, y, u) \geq 0 \quad \text{for all } u \in S(x). \quad (1.1)$$  

The purpose of this paper is continuously to study the existence problem of solutions for GQVI (1.1), under some more general conditions, using a new kind of fixed point approach. Our results presented in this paper contain the main results of [6–8, 13, 16, 18, 19] as their special cases.

For the sake of convenience, we first recall some definitions and notations.

**Definition 1** [2, 11]. An $H$-space is an ordered pair $(X, \{\Gamma_x\})$, where $X$ is a topological space and $\{\Gamma_x\}$ is a given family of nonempty contractible subsets of $X$, indexed by the finite subsets of $X$ such that $A \subseteq B$ implies $\Gamma_A \subseteq \Gamma_B$. 
**Definition 2 [2].** Let \((X, (\Gamma_A))\) be an \(H\)-space, \(D \subset X\) a nonempty subset.

1. \(D\) is said to be \(H\)-convex, if for any finite subset \(A\) of \(D\), \(\Gamma_A \subset D\).
2. \(D\) is said to be weakly \(H\)-convex, if for any finite subset \(A\) of \(D\), \(\Gamma_A \cap D\) is a nonempty contractible set.
3. A subset \(K \subset X\) is said to be a nonempty \(H\)-compact, if for any finite subset \(A\) of \(D\), \(\Gamma_A \subset D\).
4. A multifunction \(F: X \to 2^X\) is called an \(H\)-KKM multifunction, if for any finite subset \(A\) of \(X\), \(\Gamma_A \subset \bigcup_{x \in A} F(x)\).

**Remark.** If \(X\) is a topological vector space, putting \(\Gamma_A = \text{co} A\), the convex hull of \(A\), for any finite subset \(A\) of \(X\), we see that \((X, (\Gamma_A))\) is an \(H\)-space. If \(D\) is a convex subset of \(X\), then it is \(H\)-convex.

**Definition 3.** Let \((X, (\Gamma_A))\) be an \(H\)-space. A functional \(f: X \to \mathbb{R}\) is said to be \(H\)-quasi-convex, if for any \(r \in \mathbb{R}\), the set \(\{x \in X: f(x) < r\}\) is \(H\)-convex.

**Definition 4 [17].** Let \(X\) and \(Y\) be two topological spaces and let \(F: X \to 2^Y\) be a multifunction. 
1. \(F\) is said to be transfer open-valued (resp. transfer closed-valued), if for any \(x \in X\), \(y \in F(x)\) (resp. \(y \notin F(x)\)) there exists an \(x' \in X\) such that \(y \in \text{int}(F(x'))\) (resp. \(y \notin \overline{F(x')}\)). (Throughout this paper we use \(\text{int}(A)\) and \(\overline{A}\) to denote the interior and closure of the set \(A\), respectively).
2. \(F\) is said to be upper semi-continuous in short, u.s.c. at \(x \in X\), if for any neighborhood \(N(F(x))\) of \(F(x)\), there exists a neighborhood \(N(x)\) of \(x\) such that \(x \in N(x)\) implies \(F(x) \subset N(F(x))\). We say that \(F\) is upper semi-continuous on \(X\) if it is u.s.c at every point \(x \in X\).
3. \(F\) is said to be lower semi-continuous in short, l.s.c. at \(x \in X\), if for any \(y \in F(x)\) and any neighborhood \(N(y)\) of \(y\), there exists a neighborhood \(N(x)\) of \(x\) such that \(x \in N(x)\) implies \(F(x) \cap N(y) \neq \emptyset\).
We say that $F$ is lower semi-continuous on $X$ if it is l.s.c. at every point $x \in X$.

(3) $F$ is said to be continuous at $x_0 \in X$ if it is u.s.c. and l.s.c. at $x_0 \in X$.

2. A GENERALIZATION OF FAN–BROWDER’S FIXED POINT THEOREM

In this section we obtain a generalization of Fan–Browder’s fixed point Theorem [5, 9, 21–25].

**Lemma 2.1.** Let $X$ be a nonempty set, let $Y$ be a topological space, and let $G : X \to 2^Y$ be a multifunction.

(1) $G$ is transfer closed-valued if and only if

$$\bigcap_{x \in X} G(x) = \bigcap_{x \in X} \overline{G(x)}. \quad (2.1)$$

(2) $G$ is transfer open-valued if and only if

$$\bigcup_{x \in X} G(x) = \bigcup_{x \in X} \text{int}(G(x)).$$

(3) If $X$ is also a topological space, $G(x)$ is nonempty for each $x \in X$, and $G^{-1}$ is transfer open-valued, then

$$X = \bigcup_{y \in Y} \text{int}(G^{-1}(y)).$$

(4) In addition, if $X$ is a compact topological space, $G$ is nonempty $H$-convex-valued, and $G^{-1}$ is transfer open-valued, then there exists a continuous selection of $G$, i.e., there exists a continuous function $f : X \to Y$ such that

$$f(x) \in G(x) \quad \text{for all } x \in X.$$

**Proof.** (1) Assume that (2.1) is true. For any $x \in X$, if $y \notin G(x)$ then

$$y \notin \bigcap_{x \in X} G(x) = \bigcap_{x \in X} \overline{G(x)},$$

and so there exists $x' \in X$ such that $y \notin \overline{G(x')}$. This implies that $G$ is transfer closed-valued.

Conversely, if $G$ is transfer closed-valued, but

$$\bigcap_{x \in X} G(x) \neq \bigcap_{x \in X} \overline{G(x)},$$
then there exists $y \in \bigcap_{x \in X} G(x)$ such that $y \notin \bigcap_{x \in X} G(x)$. Hence there exists $x \in X$ such that $y \notin G(x)$. Since $G$ is transfer closed-valued, there exists $x' \in X$ such that $y \notin \overline{G(x')}$, and so $y \notin \bigcap_{x \in X} G(x)$. This contradicts $y \in \bigcap_{x \in X} G(x)$. Therefore we have
\[
\bigcap_{x \in X} G(x) \subseteq \bigcap_{x \in X} G(x).
\]
Since $\bigcap_{x \in X} G(x) \subseteq \bigcap_{x \in X} G(x)$, (2.1) is proved.

(2) Define a multifunction $F: X \to 2^Y$ by $F(x) = Y \setminus G(x)$ for each $x \in X$. Then $G$ is transfer open-valued if and only if $F$ is transfer closed-valued. By conclusion (1) we know that $G$ is transfer open-valued if and only if
\[
\bigcap_{x \in X} F(x) = \bigcap_{x \in X} \overline{F(x)}, \text{ i.e., } \bigcup_{x \in X} G(x) = \bigcup_{x \in X} \text{int}(G(x)).
\]

(3) If $G(x)$ is nonempty for each $x \in X$, then there exists a $y \in G(x)$ for each $x \in X$, i.e., $x \in G^{-1}(y)$, so $X = \bigcup_{y \in Y} G^{-1}(y)$. Since $G^{-1}$ is transfer open-valued, by conclusion (2) we have
\[
X = \bigcup_{y \in Y} G^{-1}(y) = \bigcup_{y \in Y} \text{int}(G^{-1}(y)).
\]

(4) By the assumption that $G(x)$ is nonempty for each $x \in X$, and $G^{-1}$ is transfer open-valued, from conclusion (3) we have
\[
X = \bigcup_{y \in Y} \text{int}(G^{-1}(y)).
\]

Besides, since $G$ is $H$-convex-valued, by Tarafdar [16, Theorem 2.2] there exists a continuous selection of $G$. This completes the proof.

**Lemma 2.2.** Let $(X, (\Gamma, \ell))$ be an $H$-space and let $F: X \to 2^Y$ be an $H$-KKM multifunction such that

(i) $F$ is transfer closed-valued;

(ii) there exist a compact subset $L$ of $X$ and an $H$-compact subset $K$ of $X$ such that for each weakly $H$-convex subset $D$ of $X$ with $K \subset D \subset X$ the following holds:
\[
\bigcap_{x \in D} (\overline{F(x)} \cap D) \subseteq L.
\]

Then
\[
\bigcap_{x \in X} F(x) \neq \emptyset.
\]
Proof. Define $\tilde{F}: X \to 2^X$ by $\tilde{F}(x) = \overline{F(x)}$ for each $x \in X$. Since $F$ is an $H$-KKM multifunction, $\tilde{F}$ is also an $H$-KKM multifunction with closed-values. By Bardaro and Ceppitelli [2, Theorem 1]

$$\bigcap_{x \in X} \overline{F(x)} \neq \emptyset.$$ 

Since $F$ is transfer closed-valued, by Lemma 2.1, we have

$$\bigcap_{x \in X} F(x) = \bigcap_{x \in X} \overline{F(x)} \neq \emptyset.$$ 

This completes the proof.

Lemma 2.3. Let $(X, \{\Gamma_A\})$ be an $H$-space, let $Z$ be any set, and let $G$ be a nonempty subset of $Z$. Let $f: X \times X \to Z$ be a function such that

(i) the multifunction $y \mapsto \{x \in X: f(x, y) \in G\}$ is transfer closed-valued;

(ii) for each $x \in X$, the set $\{y \in X: f(x, y) \not\in G\}$ is $H$-convex;

(iii) there exist a compact subset $L$ of $X$ and an $H$-compact subset $K$ of $X$ such that for each weakly $H$-convex subset $D$ of $X$ with $K \subset D \subset X$ the following holds:

$$\bigcap_{y \in D} \left( \{x \in X: f(x, y) \in G\} \cap D \right) \subset L.$$ 

Then one of the following conclusion holds:

1. There exists a $\bar{y} \in X$ such that $f(\bar{y}, \bar{y}) \not\in G$;

2. There exists an $\bar{x} \in X$ such that $f(\bar{x}, y) \in G$ for all $y \in X$.

Proof. Define a multifunction $F: X \to 2^X$ by $F(y) = \{x \in X: f(x, y) \in G\}$ for each $y \in X$. By conditions (i) and (iii), we know that conditions (i) and (ii) in Lemma 2.2 are satisfied. If conclusion (1) does not hold, then

$$f(y, y) \in G \quad \text{for each } y \in X.$$ 

Next we prove that $F$ is an $H$-KKM multifunction. Suppose the contrary. If $F$ is not an $H$-KKM multifunction, then there exists a finite subset $A \subset X$ such that $\Gamma_A \not\subset \bigcup_{y \in A} F(y)$. Hence there exists a $z \in \Gamma_A$, but $z \not\in \bigcup_{y \in A} F(y)$. This means that $f(z, y) \not\in G$ for all $y \in A$, and so

$$A \subset \{y \in X: f(z, y) \not\in G\}.$$ 

By condition (ii) the set $\{y \in X: f(z, y) \not\in G\}$ is $H$-convex; hence we have

$$z \in \Gamma_A \subset \{y \in X: f(z, y) \not\in G\},$$
i.e., \( f(z, z) \notin G \). This contradicts (2.2). Hence \( F \) is an \( H\)-KKM multifunction. By Lemma 2.2 we have
\[
\bigcap_{y \in X} F(y) \neq \emptyset.
\]
Hence there exists an \( \bar{x} \in X \) such that \( \bar{x} \in F(y) \) for all \( y \in X \), i.e., \( f(\bar{x}, y) \notin G \) for all \( y \in X \). This completes the proof.

Using the above results, we are now in a position to give the following fixed point theorem for product \( H\)-spaces.

**Theorem 2.4.** Let \( \{ (X_{\alpha}, \Gamma_{\alpha}) \} : \alpha \in I \) be a family of \( H\)-spaces, \( I \) an index set, \( X = \prod_{\alpha \in I} X_{\alpha} \), and \( \{ T_{\alpha} : \alpha \in I \} \) a family of multifunctions, where \( T_{\alpha} : X \to 2^{X} \) for all \( \alpha \in I \). Suppose that

(i) For any \( x \in X \), and for any \( \alpha \in I \), \( T_{\alpha}(x) \) is a nonempty \( H\)-convex set;

(ii) one of the following conditions holds:
   (a) for each \( \alpha \in I \), \( T_{\alpha}^{-1} : X_{\alpha} \to 2^{X_{\alpha}} \) is transfer open-valued if \( I \) is finite;
   (b) the multifunction \( T^{-1} : X \to 2^{X} \) which is defined by
   \[
   T^{-1}(y) = \bigcap_{\alpha \in I} T_{\alpha}^{-1}(y_{\alpha})
   \]
   for each \( y \in X \),

where \( y = (y_{\alpha})_{\alpha \in I} \) is transfer open-valued, if \( I \) is infinite.

If there exist a compact subset \( L \) of \( X \) and an \( H\)-compact subset \( K \) of \( X \) such that for each weakly \( H\)-convex subset \( D \) of \( X \) with \( K \subset D \subset X \) the following holds,
\[
\bigcap_{y \in D} \left( \left\{ x \in X : y_{\beta} \notin T_{\beta}(x) \text{ for some } \beta \in I \right\} \cap D \right) \subset L,
\]
where \( y_{\beta} \) is the projection of \( y \) onto \( X_{\beta} \), then there exists an \( x_{\ast} \in X \) such that
\[
x_{\ast} \in \prod_{\alpha \in I} T_{\alpha}(x_{\ast}), \text{ i.e., } x_{\ast\alpha} \in T_{\alpha}(x_{\ast}) \text{ for all } \alpha \in I,
\]
where \( x_{\ast\alpha} \) is the projection of \( x_{\ast} \) onto \( X_{\alpha} \) for each \( \alpha \in I \).

**Proof.** Suppose the contrary. Then for any \( x \in X \),
\[
x \notin \prod_{\alpha \in I} T_{\alpha}(x),
\]
i.e., for any given \( x \in X \), there exists some \( \beta \in I \) such that \( x_{\beta} \notin T_{\beta}(x) \).
Next we define a set $G \subset X \times X$ by

$$G = \left\{ (x, y) \in X \times Y : y \not\in \prod_{a \in I} T_a(x) \right\}.$$ 

From (2.4) we know that

$$(x, x) \in G \quad \text{for all } x \in X,$$

i.e., $G$ is nonempty.

Since the product of any number of $H$-convex sets is $H$-convex (see Tarafdar [16, Lemma 1.1]), by condition (i), for any $x \in X$ the set

$$\{ y \in X : (x, y) \not\in G \} = \left\{ y \in X : y \in \prod_{a \in I} T_a(x) \right\} = \prod_{a \in I} T_a(x)$$

is $H$-convex.

On the other hand, we have

$$\{ x \in X : (x, y) \not\in G \} = \left\{ x \in X : y \in \prod_{a \in I} T_a(x) \right\}$$

$$= \{ x \in X : y_a \in T_a(x) \text{ for each } a \in I \}$$

$$= \bigcap_{a \in I} \{ x \in X : x \in T_a^{-1}(y_a) \}$$

$$= \bigcap_{a \in I} T_a^{-1}(y_a). \quad (2.5)$$

If $I$ is infinite, by condition (ii)(b), from (2.5) we know that the multifunction

$$y \mapsto \{ x \in X : (x, y) \not\in G \} \quad (2.6)$$

is transfer open-valued. Next we prove that the multifunction (2.6) is also transfer open-valued if $I$ is finite. In fact, if

$$x \in \bigcap_{a \in I} T_a^{-1}(y_a) = \{ x \in X : (x, y) \not\in G \},$$

then $x \in T_a^{-1}(y_a)$ for $a \in I$. Hence by condition (ii)(a), there exists $y'_a \in X_a$ such that

$$x \in \text{int}(T_a^{-1}(y'_a)) \quad \text{for all } a \in I.$$

Since $I$ is finite, we have

$$x \in \bigcap_{a \in I} \text{int}(T_a^{-1}(y'_a))$$

$$= \text{int}\left[ \bigcap_{a \in I} \text{int}(T_a^{-1}(y'_a)) \right]$$
This shows that the multifunction (2.6) is transfer open-valued. Therefore the multifunction
\[ y \mapsto \{ x \in X : (x, y) \in G \} = X \setminus \{ x \in X : (x, y) \not\in G \} \]
is transfer closed-valued.

Furthermore, it follows from (2.3) that
\[
\bigcap_{y \in D} \left( \{ x \in X : (x, y) \in G \} \cap D \right)
\]
\[= \bigcap_{y \in D} \left( \{ x \in X : y \not\in \prod_{a \in I} T_{a}(x) \} \cap D \right)
\]
\[= \bigcap_{y \in D} \left( \{ x \in X : y_{\beta} \not\in T_{\beta}(x) \text{ for some } \beta \in I \} \cap D \right) \subseteq L.\]

Taking \( Z = X \times X \), and defining \( f : X \times X \to X \times X \) by \( f(x, y) = (x, y) \) for all \((x, y) \in X \times X\) we know that all of the conditions in Lemma 2.3 are satisfied. By Lemma 2.3, there exists an \( \bar{x} \in X \) such that \((\bar{x}, y) \in G\) for all \( y \in X \), i.e., \( y \not\in \prod_{a \in I} T_{a}(\bar{x}) \) for all \( y \in X \). This implies that \( \prod_{a \in I} T_{a}(\bar{x}) \) is empty. Therefore there exists \( \alpha \in I \) such that \( T_{\alpha}(\bar{x}) \) is empty. This contradicts condition (i). Hence there exists an \( x_{*} \in X \) such that
\[ x_{*} \in \prod_{a \in I} T_{a}(x_{*}). \]

This completes the proof.

It should be pointed out that, if \( \{ (X_{\alpha}, \Gamma_{\alpha}) : \alpha \in I \} \) is a family of compact \( H\)-spaces, then condition (2.3) in Theorem 2.4 holds automatically, so we have the following:

**Theorem 2.5.** Let \( (X_{\alpha}, \Gamma_{\alpha}) \) be a family of compact \( H\)-spaces, and let \( X = \prod_{a \in I} X_{a} \) and \( \{ T_{a} : \alpha \in I \} \) be a family of multifunctions, where \( T_{a} : X \to 2^{X_{a}} \) for all \( \alpha \in I \). Suppose further that

(i) for any \( x \in X \), and any \( \alpha \in I \), \( T_{\alpha}(x) \) is a nonempty \( H\)-convex set;

(ii) one of the following condition holds:

(a) for any \( \alpha \in I \), \( T_{\alpha}^{-1} : X_{\alpha} \to 2^{X} \) is transfer open-valued if the index set \( I \) is finite;
(b) the multifunction $T^{-1}: X \rightarrow 2^X$ which is defined as that in
Theorem 2.4, is transfer open-valued, if the index set $I$ is infinite.

Then there exists an $x_\ast \in X$ such that

$$x_\ast \in \prod_{\alpha \in I} T_\alpha(x_\ast).$$

3. GENERALIZED QUASI-VARIATIONAL INEQUALITY
PROBLEMS IN $H$-SPACES

In this section, we shall use the results presented in Section 2 to study
the existence of solutions for the GQVI problem of type (1.1) in the
$H$-space without linear structure.

**Theorem 3.1.** Let $(X, (\Gamma_a))$ be a compact Hausdorff $H$-space, and let
$(Y, (\Gamma_B))$ be a Hausdorff $H$-space. Suppose that

(i) $T: X \rightarrow 2^Y$ is a multifunction with nonempty $H$-convex values and
$T^{-1}: Y \rightarrow 2^X$ is transfer open-valued;

(ii) $S: X \rightarrow 2^X$ is a continuous multifunction with nonempty compact
$H$-convex values and $S^{-1}(x)$ is open for any $x \in X$;

(iii) $\varphi: X \times Y \times X \rightarrow \mathbb{R}$ is a continuous function satisfying

(a) $\varphi(x, y, x) \geq 0$ for all $x \in X$ and all $y \in T(x)$;

(b) the function $z \mapsto \varphi(x, y, z)$ is $H$-quasi-convex.

Then there exist $\bar{x} \in S(\bar{x})$, $\bar{y} \in T(\bar{x})$ such that

$$\varphi(\bar{x}, \bar{y}, x) \geq 0 \quad \text{for all } x \in S(\bar{x}).$$

**Proof.** By (4) in Lemma 2.1 and condition (i), there exists a continuous
selection $f: X \rightarrow Y$ of the multifunction $T$. Next for each $n = 1, 2, \ldots$, we
define a multifunction $F_n: X \rightarrow 2^X$ by

$$F_n(x) = \left\{ z \in S(x): \varphi(x, f(x), z) < \min_{u \in S(x)} \varphi(x, f(x), u) + \frac{1}{n} \right\}$$

for each $x \in X$.

Since $S$ is nonempty, compact-valued, and $H$-convex-valued, by condition
(iii)(b) $F_n(x)$ is nonempty $H$-convex for any $x \in X$. Furthermore, for any
$z \in X$, we have
\[
F^{-1}_n(z) = \{ x \in X : z \in F_n(x) \} \\
= \left\{ x \in X : z \in S(x) \text{ and } \varphi(x, f(x), z) < \min_{u \in S(x)} \varphi(x, f(x), u) + \frac{1}{n} \right\} \\
= S^{-1}(z) \cap \left\{ x \in X : \varphi(x, f(x), z) < \min_{u \in S(x)} \varphi(x, f(x), u) + \frac{1}{n} \right\} \\
= S^{-1}(z) \cap \left\{ x \in X : \varphi(x, f(x), z) + \max_{u \in S(x)} \left[ -\varphi(x, f(x), u) \right] < \frac{1}{n} \right\}.
\]

Since $\varphi$ and $f$ are continuous, and $S$ is compact-valued and continuous, by Aubin and Ekeland [1, p. 119, Proposition 21], the function
\[
x \mapsto \max_{u \in S(x)} \left[ -\varphi(x, f(x), u) \right]
\]
is u.s.c. and so the function
\[
x \mapsto \varphi(x, f(x), z) + \max_{u \in S(x)} \left[ -\varphi(x, f(x), u) \right]
\]
is u.s.c. This implies that
\[
\left\{ x \in X : \varphi(x, f(x), z) + \max_{u \in S(x)} \left[ -\varphi(x, f(x), u) \right] < \frac{1}{n} \right\}
\]
is an open set. On the other hand, $S^{-1}(z)$ is open by condition (ii).

Therefore for any $z \in X$, $F^{-1}_n(z)$ is open. This implies that the multifunction $F^{-1}_n$ is transfer open-valued. By Theorem 2.5 with $I$ being a singleton, there exists an $x_n \in X$ such that
\[
x_n \in F_n(x_n), \quad n = 1, 2 \ldots \tag{3.1}
\]

Since $X$ is compact, without loss of generality, we can assume that $x_n \to \bar{x} \in X$ and so $f(x_n) \to f(\bar{x}) \in T(\bar{x})$. 

On the other hand, from the definition of $F_n$ and (3.1) we have
\[
x_n \in S(x_n), \quad \text{and} \quad \varphi(x_n, f(x_n), x_n) < \min_{u \in S(x_n)} \varphi(x_n, f(x_n), u) + \frac{1}{n},
\]
\[
n = 1, 2, \ldots \quad (3.2)
\]

Since $S$ is compact-valued and continuous, the graph of $S$ is closed, and so $\bar{x} \in S(\bar{x})$. Again, since $S$ is continuous, by Aubin and Ekeland [1, p. 118, Proposition 19], the function
\[
x \mapsto \min_{u \in S(x)} \varphi(x, f(x), u) = - \sup_{u \in S(x)} [-\varphi(x, f(x), u)]
\]
is u.s.c. Therefore from (3.2) we have
\[
\varphi(\bar{x}, f(\bar{x}), \bar{x}) \leq \lim_{n \to \infty} \left[ \min_{u \in S(x_n)} \varphi(x_n, f(x_n), u) + \frac{1}{n} \right]
\]
\[
\leq \min_{u \in S(\bar{x})} \varphi(\bar{x}, f(\bar{x}), u).
\]

Letting $\bar{y} = f(\bar{x})$ and using condition (iii)(a) we have
\[
\varphi(\bar{x}, \bar{y}, x) \geq \min_{u \in S(\bar{x})} \varphi(\bar{x}, \bar{y}, u)
\]
\[
\geq \varphi(\bar{x}, \bar{y}, \bar{x})
\]
\[
\geq 0 \quad \text{for all } x \in S(\bar{x}).
\]

This completes the proof.

From Theorem 3.1 we can obtain the following consequences.

**Corollary 3.2.** Let $(X, (\Gamma_\alpha), (Y, (\Gamma_\beta)), T$, and $\varphi$ be the same as those in Theorem 3.1. Then there exist $\bar{x} \in X$ and $\bar{y} \in T(\bar{x})$ such that
\[
\varphi(\bar{x}, \bar{y}, x) \geq 0 \quad \text{for all } x \in X.
\]

**Proof.** Taking $S(x) = X$ for all $x \in X$, we obtain the conclusion immediately.

**Corollary 3.3.** Let $E, F$ be two Hausdorff topological vector spaces, $X \subset E$ a nonempty compact convex subset, and $Y \subset F$ a nonempty convex subset. Suppose that
\[(i) \quad T: X \to 2^Y \text{ is a multifunction with nonempty convex-values and } T^{-1}: Y \to 2^X \text{ is transfer open-valued};\]
(ii) \( S: X \to 2^X \) is a continuous multifunction with nonempty compact convex values, and \( S^{-1}(x) \) is open for any \( x \in X \);

(iii) \( \varphi: X \times Y \times X \to \mathbb{R} \) is a continuous function such that

(a) \( \varphi(x, y, x) \geq 0 \) for all \( x \in X \) and all \( y \in T(x) \);

(b) \( z \mapsto \varphi(x, y, z) \) is quasi-convex.

Then there exist \( \bar{x} \in S(\bar{x}) \) and \( \bar{y} \in T(\bar{x}) \) such that

\[ \varphi(\bar{x}, \bar{y}, x) \geq 0 \quad \text{for all} \quad x \in S(\bar{x}). \]

**Theorem 3.4.** Let \( (X, (\Gamma_n)) \) be a compact Hausdorff \( H \)-space, and let \( (Y, (\Gamma_n)) \) be a Hausdorff \( H \)-space. Suppose further that

(i) \( T: X \to 2^Y \) is a multifunction with nonempty \( H \)-convex values and \( T^{-1}: Y \to 2^X \) is transfer open-valued.

(ii) \( S: X \to 2^X \) is a l.s.c. multifunction with nonempty \( H \)-convex values and \( S^{-1}: X \to 2^X \) is closed-valued.

(iii) \( \varphi: X \times Y \times X \to \mathbb{R} \) is a continuous function satisfying

(a) \( \varphi(x, y, x) \geq 0 \) for all \( x \in X \) and all \( y \in T(x) \);

(b) for each \( (x, y) \in X \times Y \),

\[ \{ z \in X : \varphi(x, y, z) = \min_{u \in S(z)} \varphi(x, y, u) \} \]

is \( H \)-convex.

(iv) For any continuous function \( f: X \to Y \), there exists a finite subset \( A \subset X \) such that for any \( x \in X \), there exists \( a \in A \) satisfying \( x \in S^{-1}(a) \) and

\[ \varphi(x, f(x), z) = \min_{u \in S(z)} \varphi(x, f(x), u). \]

Then there exists \( \bar{x} \in S(\bar{x}), \bar{y} \in T(\bar{x}) \) such that

\[ \varphi(\bar{x}, \bar{y}, x) \geq 0 \quad \text{for all} \quad x \in S(\bar{x}). \]

**Proof.** From Lemma 2.1(4), there exists a continuous selection \( f: X \to Y \) of \( T \). Next we define a multifunction \( F: X \to 2^X \) by

\[ F(x) = \left\{ z \in S(x) : \varphi(x, f(x), z) = \min_{u \in S(z)} \varphi(x, f(x), u) \right\}. \]

Since \( \varphi \) and \( f \) are continuous, and \( S \) is \( H \)-convex-valued, so \( F \) is nonempty \( H \)-convex valued. Besides, for any \( z \in X \) we have

\[ F^{-1}(z) = \left\{ x \in X : z \in F(x) \right\} \]

\[ = \left\{ x \in X : z \in S(x) \text{ and } \varphi(x, f(x), z) = \min_{u \in S(z)} \varphi(x, f(x), u) \right\} \]

\[ = S^{-1}(z) \cap \left\{ x \in X : \varphi(x, f(x), z) \right. \]

\[ + \sup_{u \in S(z)} [ - \varphi(x, f(x), u)] = 0 \right\}. \]
Since \( \varphi \) is continuous and \( S \) is l.s.c., by Aubin and Ekeland [1, p. 118, Proposition 19], the function

\[
x \mapsto \sup_{u \in S(x)} \left[ -\varphi(x, f(x), u) \right]
\]

is l.s.c., and so the function

\[
x \mapsto \varphi(x, f(x), z) + \sup_{u \in S(x)} \left[ -\varphi(x, f(x), u) \right]
\]

is l.s.c. In view of the closedness of \( S^{-1}(z) \), we know that \( F^{-1}(z) \) is closed for all \( z \in X \).

On the other hand, by condition (iv) there exists a finite subset \( A \subset X \) such that

\[
F^{-1}(A) = \bigcup_{z \in A} F^{-1}(z) = X.
\]

From Corollary 3.6.4 in Chang [7], it follows that there exists \( \bar{x} \in X \) such that \( \bar{x} \in F(\bar{x}) \), i.e.,

\[
\bar{x} \in S(\bar{x}) \quad \text{and} \quad \varphi(\bar{x}, f(\bar{x}), \bar{x}) = \min_{u \in S(\bar{x})} \varphi(\bar{x}, f(\bar{x}), u).
\]

Since \( f \) is a continuous selection of \( T \), \( f(\bar{x}) \in T(\bar{x}) \).

Letting \( \bar{y} = f(\bar{x}) \), we have

\[
\bar{y} \in T(\bar{x}) \quad \text{and} \quad \varphi(\bar{x}, \bar{y}, \bar{x}) = \min_{u \in S(\bar{x})} \varphi(\bar{x}, \bar{y}, u).
\]

By the assumption on \( \varphi \), we have

\[
\varphi(\bar{x}, \bar{y}, x) \geq \min_{u \in S(\bar{x})} \varphi(\bar{x}, \bar{y}, u) = \varphi(\bar{x}, \bar{y}, \bar{x}) \geq 0 \quad \text{for all } x \in S(\bar{x}).
\]

This completes the proof.

In Theorem 3.4, if \( S(x) = X \) for all \( x \in X \), then we have the following corollary.

**Corollary 3.5.** Let \((X, (\Gamma_x)), (Y, (\Gamma_y)), T, \) and \( \varphi \) be the same as those in Theorem 3.4. Suppose that for each continuous function \( f : X \to Y \), there exists a finite subset \( A \subset X \) such that for any \( x \in X \), there exists a \( z \in A \) satisfying

\[
\varphi(x, f(x), z) = \min_{u \in X} \varphi(x, f(x), u).
\]

Then there exist \( \bar{x} \in X \), \( \bar{y} \in T(\bar{x}) \) such that

\[
\varphi(\bar{x}, \bar{y}, x) \geq 0 \quad \text{for all } x \in X.
\]
THEOREM 3.6. Let \((X, \mathcal{G}_x)\) and \((Y, \mathcal{G}_y)\) be two \(H\)-spaces. Suppose that

(i) \(T: X \to 2^Y\) is a multifunction with nonempty \(H\)-convex values and \(T^{-1}: Y \to 2^X\) is transfer open-valued;

(ii) \(S: X \to 2^X\) is a multifunction with nonempty \(H\)-convex values;

(iii) \(\varphi: X \times Y \times X \to \mathbb{R}\) is a function such that

(a) \(\varphi(x, y, x) \geq 0\) for all \(x \in X\) and \(y \in T(x)\);

(b) \(z \mapsto \varphi(x, y, z)\) is l.s.c.;

(c) for any \((x, y) \in X \times Y\),

\[
\{z \in X: \varphi(x, y, z) = \min_{u \in S(x)} \varphi(x, y, u)\}
\]

is \(H\)-convex;

(d) the multifunction

\[
z \mapsto \left\{(x, y) \in S^{-1}(z) \times Y: \varphi(x, y, z) = \min_{u \in S(x)} \varphi(x, y, u)\right\}
\]

is transfer open-valued.

(iv) There exist a compact subset \(L\) of \(X \times Y\) and an \(H\)-compact subset \(K \subset X \times Y\) such that for each weakly \(H\)-convex subset \(D\) of \(X \times Y\) with \(K \subset D \subset X \times Y\), the following holds,

\[
\bigcap_{(u, v) \in D} \left(\{(x, y) \in X \times Y: u \notin F_1(x, y) \text{ or } v \notin T(x)\} \cap D\right) \subset L,
\]

where \(F_1: X \times Y \to 2^X\) is a multifunction defined by

\[
F_1(x, y) = \left\{z \in S(x): \varphi(x, y, z) = \min_{u \in S(x)} \varphi(x, y, u)\right\}
\]

for each \((x, y) \in X \times Y\).

Then there exist \(\bar{x} \in S(\bar{x})\), \(\bar{y} \in T(\bar{x})\) such that

\[
\varphi(\bar{x}, \bar{y}, x) \geq 0 \quad \text{for all } x \in S(\bar{x}).
\]

**Proof.** From the assumption that \(S\) is compact valued and \(H\)-convex valued, we know that the multifunction \(F_1\) is nonempty \(H\)-convex valued. Besides, for any \(z \in X\) we have

\[
F_1^{-1}(z) = \left\{(x, y) \in S^{-1}(z) \times Y: \varphi(x, y, z) = \min_{u \in S(x)} \varphi(x, y, u)\right\}.
\]
Hence by condition (iii)(d), $F_1^{-1}$ is transfer open-valued. On the other hand, if we define a multifunction $F_2: X \times Y \to 2^Y$ by $F_2(x, y) = T_y$, then by condition (i) $F_2$ is nonempty $H$-convex valued and $F_2^{-1}: Y \to 2^{X \times Y}$ is transfer open-valued. Therefore $F_1$ and $F_2$ satisfy all of the conditions in Theorem 2.4 with $I = \{1, 2\}$. Hence there exists a $(\bar{x}, \bar{y}) \in X \times Y$ such that

$$(\bar{x}, \bar{y}) \in F_1(\bar{x}, \bar{y}) \times F_2(\bar{x}, \bar{y}),$$
i.e., $\bar{x} \in F_1(\bar{x}, \bar{y})$ and $\bar{y} \in F_2(\bar{x}, \bar{y})$.

By the definition of $F_1$ and $F_2$ we know that

$$\bar{x} \in S(\bar{x}), \quad \bar{y} \in T(\bar{x}), \quad \varphi(\bar{x}, \bar{y}, \bar{x}) = \min_{u \in S(\bar{x})} \varphi(\bar{x}, \bar{y}, u).$$

Hence by condition (iii)(a), we have

$$\varphi(\bar{x}, \bar{y}, x) \geq \min_{u \in S(\bar{x})} \varphi(\bar{x}, \bar{y}, u) = \varphi(\bar{x}, \bar{y}, \bar{x}) \geq 0 \quad \text{for all } x \in S(\bar{x}).$$

This completes the proof.

**Corollary 3.7.** Let condition (d) in Theorem 3.6 be replaced by the following condition (d'):

(d') For each $z \in X$, the set

$$\left\{(x, y) \in S^{-1}(z) \times Y: \varphi(x, y, z) = \min_{u \in S(x)} \varphi(x, y, u)\right\}$$

contains an open set $O_z$ such that $\bigcup_{z \in X} O_z = X \times Y$ (for some $z \in X$, $O_z$ may be open). Then the conclusion of Theorem 3.6 still holds.

**Proof.** From condition (d'), we know that for each $z \in X$ there exists an open set $O_z \subseteq X \times Y$ such that

$$F_1^{-1}(z) = \left\{(x, y) \in S^{-1}(z) \times Y: \varphi(x, y, z) = \min_{u \in S(x)} \varphi(x, y, u)\right\} \subseteq O_z$$

and

$$\bigcup_{z \in X} O_z = X \times Y.$$ 

Hence for any $(x, y) \in F_1^{-1}(z)$, there exists $z' \in X$ such that $(x, y) \in O_{z'} \subseteq F_1^{-1}(z')$, i.e., $(x, y) \in \text{int}(F_1^{-1}(z'))$. This implies that $F_1^{-1}: X \to 2^{X \times Y}$ is transfer open-valued, and so condition (d) in Theorem 3.6 is satisfied. This completes the proof.

**Theorem 3.8.** Let $E$ be a locally convex Hausdorff topological vector space, and let $F$ be a Fréchet space. Let $X \subseteq E$ be a nonempty compact convex set and $Y \subseteq F$ a nonempty closed convex set. Suppose that
(i) \( T: X \rightarrow 2^Y \) is a u.s.c. multifunction with nonempty compact convex values;

(ii) \( S: X \rightarrow 2^X \) is a continuous multifunction with nonempty closed convex values;

(iii) \( \varphi: X \times Y \times X \rightarrow \mathbb{R} \) is a continuous function such that
   
   (a) \( \varphi(x, y, x) \geq 0 \) for all \( x \in X, y \in Y \);
   
   (b) \( u \mapsto \varphi(x, y, u) \) is quasi-convex.

Then there exist \( \bar{x} \in S(\bar{x}), \bar{y} \in T(\bar{x}) \) such that

\[ \varphi(\bar{x}, \bar{y}, x) \geq 0 \quad \text{for all} \quad x \in S(\bar{x}). \]

**Proof.** Define a multifunction \( \Pi: X \times Y \rightarrow 2^X \) by

\[ \Pi(x, y) = \left\{ z \in S(x) : \varphi(x, y, z) = \min_{u \in S(x)} \varphi(x, y, u) \right\}. \]

Since \( \varphi \) is continuous and the function \( u \mapsto \varphi(x, y, u) \) is quasi-convex, for any \( (x, y) \in X \times Y \), \( \Pi(x, y) \) is a nonempty compact convex set.

Next we prove that \( \Pi: X \times Y \rightarrow 2^X \) is u.s.c. For this purpose, by Proposition 5.1.2 in [7] it suffices to prove that the graph of \( \Pi \) is closed. In fact, let \( (x_a, y_a)_{a \in I} \subset X \times Y \) be any net which converges to \( (x, y) \) and let \( z_a \in \Pi(x_a, y_a) \) be such that \( z_a \rightarrow z \in X \). Since \( z_a \in \Pi(x_a, y_a) \), \( z_a \in S(x_a) \), by the continuity of \( S \), we know that \( z \in S(x) \) and for each \( v \in S(x) \), there exists \( v_a \in S(x_a) \) such that \( v_a \rightarrow v \). Besides, from the fact that \( z_a \in \Pi(x_a, y_a) \) we have

\[ \varphi(x_a, y_a, z_a) \leq \varphi(x_a, y_a, v_a). \]

Furthermore, by the continuity of \( \varphi \), we have

\[ \varphi(x, y, z) = \lim_{a} \varphi(x_a, y_a, z_a) \leq \lim_{a} \varphi(x_a, y_a, v_a) = \varphi(x, y, v). \]  \hspace{1cm} (3.3)

By the arbitrariness of \( v \in S(x) \), from (3.3) we have

\[ \varphi(x, y, z) = \lim_{v \in S(x)} \varphi(x, y, v). \]

This implies that \( z \in \Pi(x, y) \), and so the graph of \( \Pi \) is closed. Hence \( \Pi \) is u.s.c. Since \( T: X \rightarrow 2^Y \) is a u.s.c. multifunction with nonempty compact convex values, \( T(X) \) is compact. Since \( F \) is Fréchet, by Proposition 5.1.3 in
[7], \( \overline{\text{co}} \ T(X) \) is a compact convex set. Let \( H = \overline{\text{co}} \ T(X) \) and define a multifunction \( G: X \times H \to 2^{X \times H} \) by
\[
G(x, y) = \Pi(x, y) \times T(x).
\]
Hence \( G: X \times H \to 2^{X \times H} \) is a u.s.c. multifunction with nonempty compact convex values. By the Kakutani–Fan–Glicksberg fixed point theorem, there exists \((\bar{x}, \bar{y}) \in X \times H\) such that
\[
(\bar{x}, \bar{y}) \in G(\bar{x}, \bar{y}) = \Pi(\bar{x}, \bar{y}) \times T(\bar{x}), \text{ i.e., } \bar{x} \in \Pi(\bar{x}, \bar{y}), \text{ and } \bar{y} \in T(\bar{x}).
\]
Hence we have
\[
\bar{x} \in S(\bar{x}), \quad \bar{y} \in T(\bar{x}), \quad \varphi(\bar{x}, \bar{y}, x) \geq \varphi(\bar{x}, \bar{y}, \bar{x}) \geq 0
\]
for all \( x \in S(\bar{x}) \).

This completes the proof of Theorem 3.8.

**Theorem 3.9.** Let \( E \) be a reflexive Banach space, \( F \) a Fréchet space, \( X \subset E \) and \( Y \subset F \) two nonempty closed convex sets. Suppose that \( T: X \to 2^Y \) is a multifunction with nonempty compact convex values and that it is upper semi-continuous with respect to the weak topology on \( X \) and the topology on \( Y \). Let \( M: X \times Y \to E^* \) be a continuous mapping with respect to the weak topology on \( X \), the topology on \( Y \), and the norm topology on \( E^* \). Let \( \eta: X \times X \to E \) be a weakly continuous mapping (i.e., a continuous mapping with respect to the weak topology on \( X \) and the weak topology on \( E \)). Suppose further that

(i) \( \eta(x, x) = 0 \) for all \( x \in X \);
(ii) the function \( u \mapsto (M(x, y), \eta(u, x)) \) is convex;
(iii) there exists \( \bar{u} \in X \) with \( \|\bar{u}\| < r \) such that for all \( x \in X \) with \( \|x\| = r \)
\[
\max_{y \in T(x)} (M(x, y), \eta(\bar{u}, x)) \leq 0.
\]

Then there exist \( \bar{x} \in X, \bar{y} \in T(\bar{x}) \) such that
\[
(M(\bar{x}, \bar{y}), \eta(x, \bar{x})) \geq 0 \quad \text{for all } x \in X. \tag{3.4}
\]

**Proof.** Taking \( X_\epsilon = X \cap B(r, \theta) \), where \( B(r, \theta) \) is a closed ball centered at \( \theta \) with radius \( r > 0 \), we know that \( X_\epsilon \) is a weakly compact convex subset of \( X \). If we let
\[
\varphi(x, y, u) = (M(x, y), \eta(u, x)),
\]
then \( \varphi: X_\epsilon \times Y \times X_\epsilon \to R \) is a continuous function with respect to the weak topology on \( X_\epsilon \) and the topology on \( Y \). By Theorem 3.8, there exist
\( \bar{x} \in X, \bar{y} \in T(\bar{x}) \) such that
\[
(M(\bar{x}, \bar{y}), \eta(x, \bar{x})) \geq 0 \quad \text{for all } x \in X. \tag{3.5}
\]

Now we consider two cases:

(a) If \( \|\bar{x}\| = r \), by condition (iii) and (3.5) we have
\[
(M(\bar{x}, \bar{y}), \eta(\bar{u}, \bar{x})) = 0. \tag{3.6}
\]

Hence for any \( x \in X \), taking \( \lambda \in (0, 1) \) small enough so that
\[
w = \lambda x + (1 - \lambda)\bar{u} \in X,
\]
by condition (ii) we have
\[
0 \leq (M(\bar{x}, \bar{y}), \eta(w, \bar{x})) \leq \lambda (M(\bar{x}, \bar{y}), \eta(x, \bar{x})) + (1 - \lambda)(M(\bar{x}, \bar{y}), \eta(\bar{u}, \bar{x})) = \lambda (M(\bar{x}, \bar{y}), \eta(x, \bar{x}))
\]
from (3.5) and (3.6). Therefore, in this case, (3.4) is proved.

(b) If \( \|\bar{x}\| < r \), then for any \( x \in X \), taking \( \lambda \in (0, 1) \) such that
\[
z = \lambda x + (1 - \lambda)\bar{x} \in X,
\]
by the same argument given in case (a), we can prove that (3.4) is true. This completes the proof.

**Definition 3.1.** Let \( E \) be a Hausdorff topological space, \( X \subset E \), \( \eta: X \times X \to E \) a function with \( \eta(x, x) = 0 \) for all \( x \in X \). Let \( G: X \to 2^E \) be a multifunction with nonempty values. \( G \) is said to be \( \eta \)-monotone, if for any \( x, u \in X \) and for any \( y \in G(x) \), \( v \in G(u) \), we have
\[
(y, \eta(u, x)) + (v, \eta(x, u)) \leq 0.
\]

**Theorem 3.10.** Let \( E, X, F, Y, T, \) and \( M \) be the same as those in Theorem 3.9. Let \( \eta: X \times X \to E \) be a weakly continuous function satisfying the following conditions:

(i) \( \eta(x, x) = 0 \) for all \( x \in X \);

(ii) \( \eta \) the function \( u \mapsto (M(x, y), \eta(u, x)) \) is convex and the multifunction \( G: X \to 2^E \) define by
\[
G(x) = \{M(x, y): y \in T(x)\}
\]
is \( \eta \)-monotone;

(iii) there exist \( \bar{u} \in X \) and \( \bar{v} \in T(\bar{u}) \) such that
\[
\lim_{\|x\| \to \infty, x \in X} (M(\bar{u}, \bar{v}), \eta(x, \bar{u})) > 0. \tag{3.7}
\]
Then there exist \( \bar{x} \in X, \bar{y} \in T(\bar{x}) \) such that

\[
(M(\bar{x}, \bar{y}), \eta(x, \bar{x})) \geq 0 \quad \text{for all } x \in X.
\]

**Proof.** It follows from (3.7) that there exists \( r > \|u\| \) such that for all \( x \in X \) with \( \|x\| = r \), we have

\[
(M(u, v), \eta(x, u)) > 0.
\]

Since \( G \) is \( \eta \)-monotone, for any \( x \in X \) with \( \|x\| = r \) and for any \( y \in T(x) \) we have

\[
(M(x, y), \eta(\bar{u}, x)) \leq - (M(u, v), \eta(x, u)) < 0.
\]

Therefore all of the conditions in Theorems 3.9 are satisfied. The conclusion of Theorem 3.10 follows from Theorem 3.9 immediately.

### 4. SOME PARTICULAR FORMS

From the results presented in the preceding sections we can obtain the following results.

**Theorem 4.1** (Yao and Guo [20, Theorem 3.1]). Let \( X \) be a nonempty compact convex subset of \( \mathbb{R}^n \) and \( f: X \rightarrow \mathbb{R}^n \). Suppose that for every \( y \in X \), the set

\[
\{ x \in X: (f(x), x - y) \leq 0 \}
\]

is closed. Then the variational inequality

\[
(f(x), u - x) \geq 0 \quad \text{for all } u \in X
\]

has a solution in \( X \).

**Proof.** Define a multifunction \( T: X \rightarrow 2^X \) by

\[
T(x) = \{ u \in X: (f(x), x - u) > 0 \} \quad \text{for each } x \in X.
\]

It is easy to see that \( T(x) \) is convex for each \( x \in X \), and by (4.1)

\[
T^{-1}(u) = \{ x \in X: u \in T(x) \}
\]

\[
= \{ x \in X: (f(x), x - u) > 0 \}
\]

\[
= X \setminus \{ x \in X: (f(x), x - u) \leq 0 \}
\]

is open for each \( u \in X \), and so \( T^{-1} \) is transfer open-valued. If for any \( x \in X \), \( T(x) \) is nonempty, then by Theorem 2.5 in the case of \( I \) being a
singleton there exists \( x_\# \in X \) such that \( x_\# \in T(x_\#) \). But this contradicts (4.2). Therefore there must exist an \( \bar{x} \in X \) such that \( T(\bar{x}) \) is empty, i.e.,

\[
(f(\bar{x}), \bar{x} - u) \leq 0 \quad \text{for all } u \in X.
\]

Hence

\[
(f(\bar{x}), u - \bar{x}) \geq 0 \quad \text{for all } u \in X.
\]

By the same argument we can prove the following result.

**Theorem 4.2** (Yao and Guo [20, Theorem 3.6]). Let \( X \) be a nonempty closed convex subset of \( \mathbb{R}^n \) and \( f: X \to \mathbb{R}^n \). Suppose that

(i) For each \( y \in X \), the set \( \{x \in X : (f(x), x - y) \leq 0\} \) is closed;

(ii) there exists a nonempty bounded subset \( D \) of \( X \) such that for each \( x \in X \setminus D \), there exists a \( y \in D \) with

\[
(f(x), x - y) > 0.
\]

Then there exists \( \bar{x} \in X \) such that

\[
(f(\bar{x}), u - \bar{x}) \geq 0 \quad \text{for all } u \in X.
\]

**Proof.** Let \( X_r = X \cap B_r \), which is chosen sufficiently large so that \( D \subseteq \text{int}_X(X_r) \). Since \( X_r \) is nonempty compact and convex, by Theorem 4.1 there exists \( \bar{x} \in X_r \) such that

\[
(f(\bar{x}), u - \bar{x}) \geq 0 \quad \text{for all } u \in X_r.
\]  

(4.3)

On the other hand, by condition (ii) it is obvious that \( \|x\| < r \). Hence for any \( u \in X \setminus X_r \), there exists a \( \lambda \in (0, 1) \) such that \( \lambda u + (1 - \lambda)\bar{x} \in X_r \). By (4.3) we have

\[
0 \leq (f(\bar{x}), \lambda u + (1 - \lambda)\bar{x} - \bar{x}) = (f(\bar{x}), \lambda(u - \bar{x})) \]

\[
= \lambda (f(\bar{x}), u - \bar{x}),
\]

and so

\[
(f(\bar{x}), u - \bar{x}) \geq 0 \quad \text{for all } u \in X.
\]

This completes the proof.

5. **Applications**

(1) **Minimization Problem**

**Definition 5.1.** Let \( \Omega \) be an open subset of \( \mathbb{R}^n \), and let \( f: \Omega \to \mathbb{R} \) be a Gâteaux differentiable function. The function \( f \) is pseudoconvex on \( \Omega \) if,
for every distinct point \( x, u \in \Omega \), we have

\[
(\nabla f(x), u - x) \geq 0 \quad \text{implies} \quad f(u) \geq f(x).
\]

**Theorem 5.1.** Let \( X \) be a closed convex subset of \( \mathbb{R}^n \), and let \( f \) be a Gateaux differentiable function from an open convex set \( \Omega \) containing \( X \) into \( \mathbb{R} \). Suppose that \( f \) is pseudoconvex on \( \Omega \) and the Gateaux differential \( \nabla f \) satisfies the following conditions:

(i) For any \( y \in X \), the set

\[
\{ x \in X : (\nabla f(x), x - y) \leq 0 \}
\]

is closed.

(ii) There exists \( x_0 \in X \) such that

\[
\lim_{\|x\| \to \infty, x \in X} (\nabla f(x), x - x_0) > 0.
\]

(5.1)

Then there exists \( \bar{x} \in X \) such that

\[
f(\bar{x}) = \min_{x \in X} f(x).
\]

**Proof.** By condition (5.1), there exists \( r > 0 \) such that for any \( x \in X \) with \( \|x\| > r \),

\[
(\nabla f(x), x - x_0) > 0.
\]

Taking \( m = \max(r, \|x_0\|) \), we have

\[
(\nabla f(x), x - x_0) > 0 \quad \text{for all} \quad x \in X \setminus B_m.
\]

By Theorem 4.2, there exists \( \bar{x} \in X \) such that

\[
(\nabla f(\bar{x}), u - \bar{x}) \geq 0 \quad \text{for all} \quad u \in X.
\]

Since \( f \) is pseudoconvex on \( X \), from the preceding inequality we have

\[
f(x) - f(\bar{x}) \geq 0 \quad \text{for all} \quad x \in X.
\]

Hence we have

\[
f(\bar{x}) = \min_{x \in X} f(x).
\]

**Remark.** This theorem generalizes Theorem 7.3 in Yao and Guo [20].

(11) **Nonlinear Programming and Saddle Point Problem**

Let \( E \) be a reflexive Banach space, \( F \) a Fréchet space, \( X \subseteq E \) and \( Y \subseteq F \) two nonempty subsets, and \( L : X \times Y \to \mathbb{R} \) a function.
DEFINITION 5.2. The first kind of nonlinear programming problem is to find $(\bar{x}, \bar{y}) \in U$ such that

$$(P_1)^{(\bar{x}, \bar{y})} L(\bar{x}, \bar{y}) = \min_{(x, y) \in U} L(x, y),$$

where

$$U = \{(x, y) \in X \times Y: L(x, y) = \max_{u \in Y} L(x, u)\}.$$

The second kind of nonlinear programming problem is to find $(\bar{x}, \bar{y}) \in W$ such that

$$(P_2)^{(\bar{x}, \bar{y})} L(\bar{x}, \bar{y}) = \max_{(x, y) \in W} L(x, y),$$

where

$$W = \{(x, y) \in X \times Y: L(x, y) = \min_{u \in X} L(u, y)\}.$$

The saddle point problem is to find $\bar{x} \in X, \bar{y} \in Y$ such that

$$(SPP) L(\bar{x}, y) \leq L(\bar{x}, \bar{y}) \leq L(x, \bar{y}) \quad \text{for all } x \in X, y \in Y.$$  

LEMMA 5.2 [13]. A point $(x_*, y_*) \in X \times Y$ is a solution of (SPP) if and only if it is a solution of both (P₁) and (P₂).

DEFINITION 5.3. Let $E$ be a Banach space, $\Omega \subset E$ a nonempty open subset, $\psi: \Omega \to \mathbb{R}$ a Gâteaux differentiable functional, and $\eta: \Omega \times \Omega \to E$ a function such that

(i) $\eta(x, x) = 0$ for $x \in \Omega$,

(ii) $\psi(x) - \psi(u) \geq (\nabla \psi(u), \eta(x, u))$ for all $x, u \in \Omega$,

where $\nabla \psi(u)$ is the Gâteaux derivative of $\psi$ at $u \in \Omega$. Then $\psi$ is said to be $\eta$-convex.

LEMMA 5.3. Let $E$ be a reflexive Banach space, $F$ a Fréchet space, $X \subset E$ a nonempty closed convex set, $Y \subset F$ a nonempty compact convex set, and $L: \Omega \times Y \to \mathbb{R}$ a function such that the function $x \mapsto L(x, y)$ is $\eta$-convex on an open set $\Omega$ in $E$ containing $X$. If $x^* \in X$ and $y^* \in Y$ satisfies

(i) $L(x^*, y^*) = \max_{y \in Y} L(x^*, y)$

(ii) $(\nabla_x L(x^*, y^*), \eta(x, x^*)) \geq 0$ for all $x \in X$,

where $\nabla_x L$ is the Gâteaux derivative of $L$ with respect to $x$, then $(x^*, y^*)$ is a solution of (SPP).
Proof. By Definition 5.3 and condition (ii) we have

\[ L(x, y^*) - L(x^*, y^*) \geq (\nabla_x L(x^*, y^*), \eta(x, x^*)) \geq 0 \]

for all \( x \in X \). This implies that

\[ L(x^*, y^*) \leq L(x, y^*) \quad \text{for all} \quad x \in X. \quad (5.2) \]

By condition (i) we have

\[ L(x^*, y) \leq L(x^*, y^*) \quad \text{for all} \quad y \in Y. \quad (5.3) \]

From (5.2) and (5.3), the conclusion is proved.

**Theorem 5.4.** Let \( E, F, X, Y, \) and \( \Omega \) be the same as those in Lemma 5.3. Let \( L: \Omega \times Y \to \mathbb{R} \) be a function and let \( \eta: \Omega \times \Omega \to E \) be a weakly continuous function satisfying the following conditions:

(i) the function \( x \to L(x, y) \) is \( \eta \)-convex and weakly continuous;

(ii) the function \( y \to L(x, y) \) is concave and continuous,

(iii) \( \nabla_x L(u, v): X \times Y \to E^* \) is continuous with respect to the weak topology on \( X \), the topology on \( Y \), and the norm topology on \( E^* \).

If there exist \( \bar{u} \in X \) and \( \bar{v} \in Y \) such that

\[ L(\bar{u}, \bar{v}) = \max_{v \in Y} L(\bar{u}, v) \quad \text{and} \quad \lim_{\|x\| \to \infty} (\nabla_x L(\bar{u}, \bar{v}), \eta(x, \bar{u})) > 0, \]

then the (SPP) has a solution, and so both the first and second nonlinear programming problems have a solution.

Proof. Since \( Y \) is a compact convex set in \( F \) and the function \( y \to L(x, y) \) is concave and continuous, it is easy to see that for each \( x \in X \) the set

\[ T(x) = \left\{ y \in Y: L(x, y) = \max_{v \in Y} L(x, v) \right\} \]

is a nonempty compact convex subset of \( Y \), and the multifunction \( T: X \to 2^Y \) is weakly u.s.c.

Moreover by condition (i) the function \( x \to L(x, y) \) is \( \eta \)-convex; hence for any \( x, u \in X, y \in T(x), \) and \( v \in T(u) \) we have

\[ L(x, y) - L(u, v) \geq L(x, v) - L(u, v) \geq (\nabla_x L(u, v), \eta(x, u)), \]

\[ L(u, v) - L(x, y) \geq L(u, y) - L(x, y) \geq (\nabla_x L(x, y), \eta(u, x)). \]
This implies that
\[
(\nabla_x L(u, v), \eta(x, u)) + (\nabla_x L(x, y), \eta(u, x)) \leq 0. \tag{5.5}
\]
Therefore the multifunction \(G: X \to \mathcal{P}^{\ast}(\mathbb{R}^n)\) defined by
\[
G(x) = \{\nabla_x L(x, y) : y \in T(x)\}
\]
is \(\eta\)-monotone. By Theorem 3.10, there exist \(\bar{x} \in X, \bar{y} \in T(\bar{x})\) such that
\[
(\nabla_x L(\bar{x}, \bar{y}), \eta(x, \bar{x})) \geq 0 \quad \text{for all } x \in X.
\]
This argument shows that all of the conditions in Lemma 5.3 are satisfied. Thus by Lemma 5.3 the conclusion is proved.

ACKNOWLEDGMENTS

The first author was supported in part by National Natural Science Foundation of China. The second author was supported in part by the Basic Science Research Institute Program, Ministry of Education, 1994, Project No. BSRI-94-1405, and Nondirected Research Fund, Korea Research Foundation, 1994.

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