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On the Degree of Approximation of Functions in $C^1_{2\pi}$ with Operators of the Jackson Type

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Let $C^1_{2\pi}$ be the class of real functions of a real variable that are 2π-periodic and have a continuous derivative. The positive linear operators of the Jackson type are denoted by $L_{n,p}$ ($n \in \mathbb{N}$), where p is a fixed positive integer. The object of this paper is to determine the exact degree of approximation when approximating functions $f \in C^1_{2\pi}$ with the operators $L_{n,p}$. The value of $\max_x |L_{n,p}(f; x)-f(x)|$ is estimated in terms of $\omega_1(f; \delta)$, the modulus of continuity of f', with $\delta = \pi/n$. Exact constants of approximation are obtained for the operators $L_{n,p}$ ($n \in \mathbb{N}$, $p \geqslant 2$) and for the Fejér operators $L_{n,1}$ ($n \in \mathbb{N}$). Furthermore, the limiting behaviour of these constants is investigated as $n \to \infty$ and $p \to \infty$, separately or simultaneously.

1. INTRODUCTION AND SUMMARY

1.1. The class of real, continuous, 2π-periodic functions of a real variable is denoted by $C_{2\pi}$. Assume $f \in C_{2\pi}$ and let p be a positive integer. The positive linear operators $L_{n,p}$ are then defined by

$$L_{n,p}(f; x) = \int_{-\pi}^{\pi} f(x+t)\, k_{n,p}(t)\, dt \qquad (n \in \mathbb{N}) \tag{1.1}$$

where

$$k_{n,p}(t) = A_{n,p}^{-1} \left(\frac{\sin nt/2}{\sin t/2}\right)^{2p}, \tag{1.2}$$

with $A_{n,p}$ such that $\int_{-\pi}^{\pi} k_{n,p}(t)\, dt = 1$.

For $p = 1$ we obtain the Fejér operators, while the name of Jackson is associated with $L_{n,2}$. Approximation properties of the operators $L_{n,p}$, in particular those of $L_{n,1}$ and $L_{n,2}$, have been extensively studied; cf. Görlich

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and Stark's survey paper [3], Matsuoka [5] and Schurer and Steutel [6]. Assuming f to be nonconstant, Wang Hsing-Hua [9] proved

$$\sup_{n+1\in\mathbb{N}} \sup_{f\in C_{2\pi}} \frac{\max_x |L_{N,2}(f;x)-f(x)|}{\omega(f;\pi/(n+1))} = \frac{3}{2} \qquad \left(N = \left[\frac{n}{2}\right]+1\right)^1,$$

where ω denotes the modulus of continuity of f. A similar result for the Jackson operators in two dimensions was obtained by Bugaets and Martynyuk [2].

1.2. In this paper the setting is the class $C^1_{2\pi}$ of real functions of a real variable that are 2π-periodic and have a continuous derivative. The degree of approximation is measured in terms of ω_1, the modulus of continuity of f'. In particular, we shall deal with the problem of determining the exact constants of approximation for the operators $L_{n,p}$. For $n \in \mathbb{N}$, $p-1 \in \mathbb{N}$ fixed, the exact constant of approximation for the operator $L_{n,p}$ is defined by

$$c_{n,p} := np^{1/2} \sup \left\{\frac{|L_{n,p}(f;x)-f(x)|}{\omega_1(f;\pi/n)}; x\in\mathbb{R}, f\in C^1_{2\pi}, \omega_1\left(f;\frac{\pi}{n}\right)>0\right\}, \tag{1.3}$$

whereas for the Fejér operators ($p = 1$) the definition reads

$$c_{n,1} := \sup \left\{\frac{|L_{n,1}(f;x)-f(x)|}{\omega_1(f;\pi/n)}; x\in\mathbb{R}, f\in C^1_{2\pi}, \omega_1\left(f;\frac{\pi}{n}\right)>0\right\}. \tag{1.4}$$

The norming is prescribed by the limiting behaviour. In order to keep the constants $c_{1,p}$ also bounded, definition (1.3) in the case $n = 1$ is replaced by

$$c_{1,p} := c_{1,2} \qquad (p = 3, 4,...). \tag{1.5}$$

For fixed $p \in \mathbb{N}$ the exact constant of approximation for the sequence of operators $\{L_{n,p}; n \in \mathbb{N}\}$ is then defined by

$$c^{(p)} := \sup_{n\in\mathbb{N}} c_{n,p}. \tag{1.6}$$

Finally, the exact constant of approximation for the whole class of operators $\{L_{n,p}; n\in\mathbb{N}, p \geqslant 2\}$ is defined by

$$c := \sup_{p\geqslant 2} c^{(p)}. \tag{1.7}$$

[1] $[a]$ denotes the largest integer not exceeding a.

1.3. We give a brief sketch of the contents of the various sections. Section 2 contains a number of preliminary lemmas. In Section 3 the so-called extremal functions are introduced; just as in the investigation of the Bernstein polynomials (cf. [7]), they play a crucial role in determining the constants $c_{n,p}$. The pattern of deducing the extremal functions is in part similar to the procedure given in [7]; a serious complication however is caused by the constraint of periodicity. The main result of Section 4 is the proof of $c = 2\pi^{-1/2} = 1.12837917$[2]. We also determine the exact constant of approximation for the Fejér operators; it is shown that $c^{(1)} = \pi/4 = 0.78539816$. In Section 5 the limiting behaviour of $c_{n,p}$ is considered as $n \to \infty$ or/and $p \to \infty$. A separate discussion is devoted to the behaviour of $c_{n,1}$ as $n \to \infty$.

2. Preliminary Results

2.1. Approximation properties of the operators $L_{n,p}$ were investigated in [6]. There the following lemma is proved.

LEMMA 2.1. *The coefficients $\mu_k^{(n,p)}$ in the expansion*

$$\left(\frac{\sin nt/2}{\sin t/2}\right)^{2p} = \mu_0^{(n,p)} + 2 \sum_{k=1}^{np-p} \mu_k^{(n,p)} \cos kt \tag{2.1}$$

are given by

$$\mu_k^{(n,p)} = \sum_{j=0}^{2p} (-1)^j \binom{2p}{j} \binom{np+p-k-nj-1}{2p-1}, \tag{2.2}$$

with the usual convention that $\binom{a}{b} = 0$ if $a < b$.

2.2. We proceed with a few inequalities that will be used in Section 4 to estimate integrals over the kernel (2.1).

LEMMA 2.2. *For $n \in \mathbb{N}$ one has*

$$\frac{\sin t}{n \sin t/n} \geqslant \exp\left(-\frac{1}{2} at^2\right) \qquad \left(0 < t \leqslant \frac{\pi}{2}\right),$$

where

$$a = \frac{8}{\pi^2} \log\left(\frac{\pi}{2}\right) = 0.366039. \tag{2.3}$$

[2] Here and elsewhere numbers are rounded to the last digit shown.

Proof. As $x^{-1}\sin x$ is decreasing on $(0, \pi)$, we have

$$n \sin \frac{t}{n} \leqslant (n+1) \sin \frac{t}{n+1} \qquad \left(0 < t \leqslant \frac{\pi}{2}\right). \tag{2.4}$$

It is therefore sufficient to show that

$$t^{-1} \sin t \geqslant \exp\left(-\frac{1}{2} a t^2\right) \qquad \left(0 < t \leqslant \frac{\pi}{2}\right).$$

Put

$$q(t) := t^{-1} \sin t - \exp(-\tfrac{1}{2} a t^2).$$

Taylor series expansion shows that q is positive on the interval $(0, 3\pi/8]$. Furthermore, q is decreasing on $[3\pi/8, \pi/2]$, while $q(\pi/2) = 0$. This proves the lemma. ∎

Remark. If $n = 3$ Lemma 2.2 can be sharpened as follows (cf. [8, p. 6]):

$$\frac{\sin t}{3 \sin t/3} \geqslant \exp\left(-\frac{1}{2} b t^2\right) \qquad \left(0 < t \leqslant \frac{\pi}{2}\right), \tag{2.5}$$

where

$$b = \frac{8}{\pi^2} \log\left(\frac{3}{2}\right) = 0.328658.$$

LEMMA 2.3. *For $n \in \mathbb{N}$ one has*

$$\frac{\sin t}{n \sin t/n} \leqslant \exp\left(-\frac{1}{6}\left(1 - \frac{1}{n^2}\right) t^2\right) \qquad (0 < t < \pi). \tag{2.6}$$

Proof. Inequality (2.6) can be rewritten in the form

$$\frac{\sin t}{t} \exp\left(\frac{1}{6} t^2\right) \leqslant \frac{\sin t/n}{t/n} \exp\left(\frac{1}{6}\left(\frac{t}{n}\right)^2\right) \qquad (n \in \mathbb{N}).$$

Therefore it is sufficient to show that $r(t) := t^{-1} \sin t \exp(t^2/6)$ is decreasing on $(0, \pi)$. One has

$$r'(t) = \tfrac{1}{3} t^{-2} \exp(\tfrac{1}{6} t^2)\{-3 \sin t + 3t \cos t + t^2 \sin t\},$$

which is easily seen to be negative on $(0, \pi)$. ∎

LEMMA 2.4. *For $0 \leqslant t \leqslant \pi/2$ one has*

$$\sin t + \frac{1}{6}\sin^3 t + \frac{3}{40}\sin^5 t$$
$$\leqslant t \leqslant \sin t + \frac{1}{6}\sin^3 t + \frac{3}{40}\sin^5 t + \left(\frac{\pi}{2} - \frac{149}{120}\right)\sin^7 t.$$

Proof. This follows from the expansion of $t = \arcsin(\sin t)$ for $t \in [0, \pi/2]$. For details see [8, p. 7]. ∎

Finally, we note that (cf. [4, p. 97])

$$\int_0^{\pi/2} (\sin t)^{2p}\, dt = \int_0^{\pi/2} (\cos t)^{2p}\, dt = \frac{\pi(2p)!}{2^{2p+1}(p!)^2},$$

a result that will be frequently used in Section 4.

3. THE EXTREMAL FUNCTIONS

3.1. As in (1.1) let

$$L_{n,p}(f; x) = \int_{-\pi}^{\pi} f(x+t)\, k_{n,p}(t)\, dt,$$

where $k_{n,p}$ is given by (1.2). Assuming $n \in \mathbb{N}$ and $p \in \mathbb{N}$ fixed, we shall determine $d_{n,p}$ defined by

$$d_{n,p} = \sup\{|\Delta_{n,p}(f; x)|; x \in \mathbb{R}, f \in \mathscr{F}_n\}, \tag{3.1}$$

where

$$\Delta_{n,p}(f; x) := L_{n,p}(f; x) - f(x)$$

and $\mathscr{F}_n := \mathscr{F}$ is defined by

$$\mathscr{F} = \left\{f\colon [-\pi, \pi] \to \mathbb{R}; f \in C^1_{2\pi},\ \omega_1\left(f; \frac{\pi}{n}\right) \leqslant 1\right\}.$$

LEMMA 3.1.

$$d_{n,p} = \sup_{f \in \mathscr{F}_0} |\Delta_{n,p} f|,$$

where, defining $\bar{f}$ by $\bar{f}(t) = f(-t)$,

$$\mathscr{F}_0 = \{f \in \mathscr{F}; \bar{f} = f, f(0) = 0, f'(t) \geqslant 0 \text{ for } t \in [0, \pi]\}, \tag{3.2}$$

and $\Delta_{n,p} f$ is defined, for $f \in \mathscr{F}_0$, by

$$\Delta_{n,p} f = \Delta_{n,p}(f; 0) = \int_{-\pi}^{\pi} f(t)\, k_{n,p}(t)\, dt. \tag{3.3}$$

Proof. As for $x \in \mathbb{R}$ and $f \in \mathscr{F}$ also $f_x \in \mathscr{F}$, where f_x is defined by $f_x(t) = f(t + x)$, we have $L_{n,p}(f; x) = L_{n,p}(f_x ; 0)$. Hence it is no restriction to take $x = 0$. As $L_{n,p}$ is linear and $f - f(0) \in \mathscr{F}$ if $f \in \mathscr{F}$, it is no restriction to take $f(0) = 0$. Furthermore, as $\bar{k}_{n,p} = k_{n,p}$ we have $\Delta_{n,p}\bar{f} = \Delta_{n,p} f = \Delta_{n,p}(f + \bar{f})/2$; hence it is no restriction to take f such that $f = \bar{f}$. Finally, it is no restriction to assume that $\Delta_{n,p} f \geqslant 0$, as for $f \in \mathscr{F}$ we have $-f \in \mathscr{F}$. It follows that for even $f \in \mathscr{F}$ with $f(0) = 0$ we have $f \in \mathscr{F}_0$ and $\hat{f} \geqslant f$ if we define $\hat{f}$ by $\hat{f}(0) = 0$ and $\hat{f}'(t) = \max(0, f'(t))$ for $t \in [0, \pi]$, and by symmetry on $[-\pi, 0]$. As $\hat{f}(t) \geqslant f(t)$ for all t it follows from (3.3) that $\Delta_{n,p}\hat{f} \geqslant \Delta_{n,p} f$. This proves Lemma 3.1. ∎

We now have to maximize

$$\Delta_{n,p} f = \int_{-\pi}^{\pi} f(t)\, k_{n,p}(t)\, dt = \int_{-\pi/2}^{3\pi/2} f(t)\, k_{n,p}(t)\, dt$$

for $f \in \mathscr{F}_0$. We first prove two general lemmas.

LEMMA 3.2. *Let K be a finite, nondecreasing function on $[-\frac{1}{2}, \frac{1}{2}]$ and for fixed $n \in \mathbb{N}$ let $\mathscr{G}_1 \equiv \mathscr{G}_{1,n}$ be defined by*

$$\mathscr{G}_1 = \left\{ g\colon \left[-\frac{1}{2}, \frac{1}{2}\right] \to \mathbb{R};\ \bar{g} = g,\ g(0) = 0,\ g' \text{ continuous},\ \omega_1\left(g; \frac{1}{n}\right) \leqslant 1 \right\}.$$

Then

$$\sup_{g \in \mathscr{G}_1} \int_{-1/2}^{1/2} g(t)\, dK(t) = \int_{-1/2}^{1/2} \tilde{g}_1(t)\, dK(t),$$

where $\tilde{g}_1 \equiv \tilde{g}_{1,n}$ is defined by $\tilde{g}_1(0) = 0$ and

$$\tilde{g}_1'(t) = j + \frac{1}{2} \qquad \left(\frac{j}{n} < t < \frac{j+1}{n},\ j = 0, \pm 1, \pm 2, \ldots\right).$$

Proof. The proof of this lemma involves exactly the same steps as the proof of Theorem 3.1 in [7]. This is apparent if we write the Bernstein polynomial as $B_n(f; x) = \int_0^1 f(t)\, dK(x, t)$. ∎

LEMMA 3.3. *Let K be a finite, nondecreasing function on $[-\frac{1}{2}, \frac{1}{2}]$ and for fixed $n \in \mathbb{N}$ let $\mathscr{G}_2 \equiv \mathscr{G}_{2,n}$ be defined by*

$$\mathscr{G}_2 = \left\{g\colon \left[-\frac{1}{2}, \frac{1}{2}\right] \to \mathbb{R};\ \bar{g} = g,\ g\left(\frac{1}{2}\right) = 0,\ g' \text{ continuous},\ \omega_1\left(g; \frac{1}{n}\right) \leqslant 1\right\}.$$

Then

$$\sup_{g \in \mathscr{G}_2} \int_{-1/2}^{1/2} g(t)\, dK(t) = \int_{-1/2}^{1/2} \tilde{g}_2(t)\, dK(t),$$

where $\tilde{g}_2 \equiv \tilde{g}_{2,n}$ is defined by $\tilde{g}_2(\frac{1}{2}) = 0$ and

$$\begin{aligned} \tilde{g}_2'(t) &= -j \qquad \left(\frac{2j-1}{4m} < t < \frac{2j+1}{4m},\ j = 0, \pm 1, \pm 2,\ldots\right) \\ &\qquad\qquad \text{if} \quad n = 2m, \\ \tilde{g}_2'(t) &= -\left(j + \frac{1}{2}\right) \qquad \left(\frac{j}{2m+1} < t < \frac{j+1}{2m+1},\ j = 0, \pm 1, \pm 2,\ldots\right) \\ &\qquad\qquad \text{if} \quad n = 2m+1. \end{aligned} \tag{3.4}$$

Proof. For $g \in \mathscr{G}_2$ we have, using integration by parts,

$$Dg := \int_{-1/2}^{1/2} g(t)\, dK(t) = \int_{-1/2}^{1/2} g'(t) \int_t^{1/2} dK(u)\, dt. \tag{3.5}$$

We first state and prove three propositions.

PROPOSITION (i). *It is no restriction to take g concave, i.e. to take g' nonincreasing.*

Proof. For $g \in \mathscr{G}_2$ we define an even function $\breve{g}$ by $\breve{g}(\frac{1}{2}) = 0$, and

$$\breve{g}'(t) = \sup_{t \leqslant s \leqslant 0} g'(s) \qquad (-\tfrac{1}{2} \leqslant t \leqslant 0).$$

It is easily verified that $\breve{g}'$ is nonincreasing and that $\breve{g} \in \mathscr{G}_2$. As $\breve{g}'(-t) - g'(-t) = g'(t) - \breve{g}'(t) \geqslant 0$ on $[0, \frac{1}{2}]$ and $\int_t^{1/2} dK(u)$ is nonincreasing, it follows from (3.5) that $D\breve{g} \geqslant Dg$.

PROPOSITION (ii). *Let*

$$\mathscr{G}_2^* = \left\{g\colon \left[-\frac{1}{2}, \frac{1}{2}\right] \to \mathbb{R};\ \bar{g} = g,\ g\left(\frac{1}{2}\right) = 0,\ g' \text{ nonincreasing},\ g' \text{ continuous} \right.$$
$$\left. \text{except for finitely many jumps};\ \omega_1\left(g; \frac{1}{n}\right) \leqslant 1\right\}.$$

Then

$$\sup_{g \in \mathscr{G}_2} Dg = \sup_{g \in \mathscr{G}_2^*} Dg.$$

Proof. By Proposition (i) g' may be taken to be nonincreasing. Furthermore, any g' with $g \in \mathscr{G}_2^*$ is the pointwise limit of functions g_n' with $g_n \in \mathscr{G}_2$ and having the same ω_1 value. By (3.5) this proves Proposition (ii).

PROPOSITION (iii). *It is no restriction to assume that $g \in \mathscr{G}_2^*$ satisfies*

$$g'(t) = g'\left(t - \frac{1}{n}\right) - 1 \qquad \left(-\frac{1}{2} + \frac{1}{n} \leqslant t \leqslant \frac{1}{2}\right). \tag{3.6}$$

Proof. If for $g \in \mathscr{G}_2^*$ condition (3.6) is violated for $t = t_0$ with $t_0 \in [-\frac{1}{2} + 1/n, 1/2n)$, then g can be replaced by $g_0 \in \mathscr{G}_2^*$ as indicated in figure 3.1 below, where the graphs of g' and g_0' are shown. Here g_0' is obtained from g' for $t < 0$ as follows:

$$g_0'(t) = \begin{cases} g'(t_0) + 1 & \left(t \leqslant t_0 - \frac{1}{n} \text{ and } g'(t) < g'(t_0) + 1\right) \\ g'(t) & (\text{otherwise}), \end{cases}$$

and by symmetry for $t > 0$.

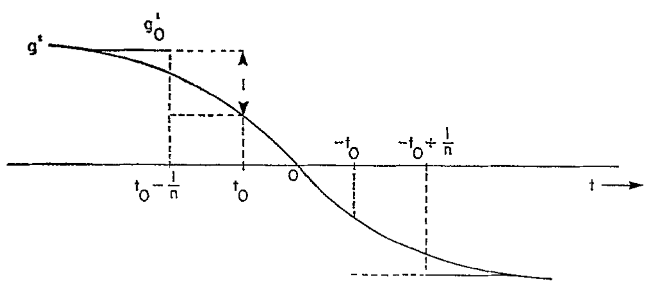

FIGURE 3.1

Clearly, $Dg_0 \geqslant Dg$ (cf. the proof of Proposition (i)), and $g_0 \in \mathscr{G}_2^*$ if $g \in \mathscr{G}_2^*$. Hence attention may be restricted to functions g satisfying (3.6), with the possible exception of the point $t = 1/2n$, which does not affect the value of Dg. This proves the proposition.

Now let $g \in \mathscr{G}_2^*$ satisfy (3.6) and let $g'(t) = (g'(t - 0) + g'(t + 0))/2$ be defined for t in $[-\frac{1}{2}, \frac{1}{2}]$; redefining g' in this sense at discontinuity points does not affect (3.6). Then, as we have $g'(0) = 0$ and $g'(-1/2n) = -g'(1/2n) = \frac{1}{2}$, it follows in view of (3.6) that for $j = 0, 1,..., n$ we have

$$g'\left(-\frac{1}{2} + \frac{j}{n}\right) = \frac{n}{2} - j. \tag{3.7}$$

We now replace g by $\tilde{g} \in \mathscr{G}_2^*$ obtained by joining the straight lines tangent to the graph of g at the points $(-\frac{1}{2} + j/n, g(-\frac{1}{2} + j/n))$, i.e., with tangents given by (3.7). As g is concave we have $\tilde{g} \geqslant g$ and hence $D\tilde{g} \geqslant Dg$. Finally, we show that $\tilde{g} \in \mathscr{G}_2^*$. Writing $\gamma_j = g(-\frac{1}{2} + j/n)$ for $g \in \mathscr{G}_2^*$ satisfying (3.6), we have by integration

$$\gamma_{j+1} - 2\gamma_j + \gamma_{j-1} = -\frac{1}{n}.$$

with $\gamma_0 = \gamma_n = 0$. It follows that all functions in $\mathscr{G}_2^*$ satisfying (3.6) have graphs that pass through the points $(-\frac{1}{2} + j/n, j(1 - j/n)/2)$ for $j = 0, 1,..., n$. Thus the graph of $\tilde{g}$ passes through these points, and hence $\tilde{g}$ is identical with the function $\tilde{g}_2$ as defined in (3.4). Clearly, from the previous propositions and the construction of $\tilde{g}_2$ it follows that $D\tilde{g}_2 \geqslant Dg$ for $g \in \mathscr{G}_2$. This proves Lemma 3.3. ∎

3.2. We are now in a position to prove the main result of this section.

THEOREM 3.1. *Let $d_{n,p}$ be defined by* (3.1). *Then*

$$d_{n,p} = \int_{-\pi}^{\pi} \tilde{f}_n(t)\, k_{n,p}(t)\, dt, \tag{3.8}$$

where $\tilde{f}_n$ is defined by $\tilde{f}_n(0) = 0$, $\tilde{f}_n$ is even, and

$$\tilde{f}'_{2m}(t) = \begin{cases} j + \frac{1}{2} & \left(\frac{j\pi}{2m} < t < \frac{(j+1)\pi}{2m}, j = 0, 1,..., m-1\right) \\ 2m - j & \left(\frac{\pi}{2} < \frac{(2j-1)\pi}{4m} < t < \frac{(2j+1)\pi}{4m} < \pi, \right. \\ & \left. j = m, m+1,..., 2m\right) \end{cases} \qquad (m = 1, 2,...) \tag{3.9}$$

$$\tilde{f}'_{2m+1}(t) = \begin{cases} j + \frac{1}{2} & \left(\frac{j\pi}{2m+1} < t < \frac{(j+1)\pi}{2m+1}, j = 0, 1,..., m\right) \\ 2m - j + \frac{1}{2} & \left(\frac{j\pi}{2m+1} < t < \frac{(j+1)\pi}{2m+1}, \right. \\ & \left. j = m+1, m+2,..., 2m\right) \end{cases} \qquad (m = 0, 1,...) \tag{3.10}$$

Proof. The function $\tilde{f}_n$, except for a linear transformation, consists of the functions $\tilde{g}_{1,n}$ and $\tilde{g}_{2,n}$ put together. To be precise we have

$$\tilde{f}_n(t) = \begin{cases} \pi \tilde{g}_{1,n}\left(\frac{t}{\pi}\right) & \left(0 \leqslant t \leqslant \frac{\pi}{2}\right), \\ \tilde{f}_n\left(\frac{\pi}{2}\right) + \pi \tilde{g}_{2,n}\left(\frac{t}{\pi} - 1\right) & \left(\frac{\pi}{2} \leqslant t \leqslant \pi\right) \end{cases}$$

for $t \in [0, \pi]$, and by symmetry elsewhere. One easily verifies that (by good luck) the jumps at (or close to) $t = \pi/2$ of $\tilde{f}_n'$ are such that $\omega_1(\tilde{f}_n; \pi/n) = 1$, and hence that $\tilde{f}_n'$ is the pointwise limit of derivatives of functions in $\mathscr{F}_0$ (cf. (3.2)). Finally we have

$$\sup_{f \in \mathscr{F}_0} \int_0^\pi f(t)\, k_{n,p}(t)\, dt$$

$$\leqslant \sup_{f_1 \in \mathscr{F}_0} \left\{ \int_0^{\pi/2} f_1(t)\, k_{n,p}(t)\, dt + f_1\left(\frac{\pi}{2}\right) \int_{\pi/2}^{\pi} k_{n,p}(t)\, dt \right\}$$

$$+ \sup_{f_2 \in \mathscr{F}_0, f_2(\pi/2)=0} \int_{\pi/2}^{\pi} f_2(t)\, k_{n,p}(t)\, dt$$

$$= \pi \int_0^{\pi/2} \tilde{g}_{1,n}\left(\frac{t}{\pi}\right) k_{n,p}(t)\, dt + \pi \tilde{g}_{1,n}\left(\frac{1}{2}\right) \int_{\pi/2}^{\pi} k_{n,p}(t)\, dt$$

$$+ \pi \int_{\pi/2}^{\pi} \tilde{g}_{2,n}\left(\frac{t}{\pi} - 1\right) k_{n,p}(t)\, dt$$

$$= \int_{-\pi}^{\pi} \tilde{f}_n(t)\, k_{n,p}(t)\, dt.$$

This proves the theorem. ∎

COROLLARY. *The (extremal) functions $\tilde{f}_n$ are given by*

$$\tilde{f}_{2m}(t) = \frac{1}{2}|t| + \sum_{j=1}^{m-1}\left(|t| - \frac{j\pi}{2m}\right)_+ + \frac{1}{2}\left(|t| - \frac{\pi}{2}\right)_+$$
$$- \sum_{j=m}^{2m-1}\left(|t| - \frac{(2j+1)\pi}{4m}\right)_+ \qquad (m = 1, 2,...), \tag{3.11}$$

$$\tilde{f}_{2m+1}(t) = \frac{1}{2}|t| + \sum_{j=1}^{m}\left(|t| - \frac{j\pi}{2m+1}\right)_+$$
$$- \sum_{j=m+1}^{2m}\left(|t| - \frac{j\pi}{2m+1}\right)_+ \qquad (m = 0, 1, 2,...), \tag{3.12}$$

where $a_+ := \max(0, a)$.

4. The Exact Constants of Approximation $c_{n,p}$ for the Operators $L_{n,p}$

4.1. Case $p \geqslant 2$. The main object of this section is to determine the exact constant of approximation for the class of operators $\{L_{n,p}\,;\, n \in \mathbb{N}; p-1 \in \mathbb{N}\}$, i.e. to determine (cf. (1.7))

$$c := \sup_{p \geqslant 2} \sup_{n \in \mathbb{N}} c_{n,p}\,,$$

where $c_{n,p}$ is defined by (1.3).

Both in (1.3) and in the definition of $d_{n,p}$ in (3.1), it is not an essential restriction to take ω_1 in fact equal to one. It then follows from Theorem 3.1 that

$$c_{n,p} = np^{1/2} d_{n,p} = np^{1/2} \int_{-\pi}^{\pi} \tilde{f}_n(t)\, k_{n,p}(t)\, dt \qquad (n, p = 2, 3, \ldots) \quad (4.1)$$

and

$$c_{1,2} = 2^{-3/2} \pi^{-1} \int_{-\pi}^{\pi} |\, t\,|\, dt = \pi\, 2^{-3/2} = 1.1107.$$

According to definition (1.5) one has

$$c_{1,p} := c_{1,2} = 1.1107 \qquad (p = 3, 4, \ldots). \quad (4.2)$$

Defining

$$\tilde{f}_n(t) = \tfrac{1}{2}\,|\, t\,| + h_n(t), \quad (4.3)$$

we conclude from (4.1) that

$$c_{n,p} = S_1(n, p) + S_2(n, p), \quad (4.4)$$

where

$$\begin{aligned} S_1(n, p) &= np^{1/2} \int_{-\pi}^{\pi} \tfrac{1}{2}\,|\, t\,|\, k_{n,p}(t)\, dt, \\ S_2(n, p) &= np^{1/2} \int_{-\pi}^{\pi} h_n(t)\, k_{n,p}(t)\, dt. \end{aligned} \quad (4.5)$$

Using (1.2) and taking into account that $h_n(t) \equiv 0$ if $|\, t\,| \leqslant \pi/n$, we easily find

$$\begin{aligned} S_1(n, p) &= \left(\int_0^{n\pi p^{1/2}/2} \left(\frac{\sin tp^{-1/2}}{n \sin tp^{-1/2} n^{-1}} \right)^{2p} dt \right)^{-1} \\ &\quad \times \left(\int_0^{n\pi p^{1/2}/2} t \left(\frac{\sin tp^{-1/2}}{n \sin tp^{-1/2} n^{-1}} \right)^{2p} dt \right) \\ &=: N^{-1}(n, p)\, T_1(n, p), \end{aligned} \quad (4.6)$$

$$\begin{aligned} S_2(n, p) &= \left(\int_0^{n\pi p^{1/2}/2} \left(\frac{\sin tp^{-1/2}}{n \sin tp^{-1/2}\, n^{-1}} \right)^{2p} dt \right)^{-1} \\ &\quad \times \left(p \int_{\pi/2}^{n\pi/2} nh_n \left(\frac{2t}{n} \right) \left(\frac{\sin t}{n \sin t/n} \right)^{2p} dt \right) \\ &=: N^{-1}(n, p)\, T_2(n, p). \end{aligned} \quad (4.7)$$

We observe (cf. Lemma 2.1) that

$$N(n, p) = \frac{\pi \mu_0^{(n,p)} p^{1/2}}{2n^{2p-1}}. \tag{4.8}$$

4.1.1. We proceed with giving a lower bound for $N(n, p)$ and upper bounds for $T_1(n, p)$ and $T_2(n, p)$.

LEMMA 4.1.

$$N(n, p) \geqslant \frac{1}{2}\left(\frac{\pi}{a}\right)^{1/2} \operatorname{erf}\left(\frac{\pi}{2}(pa)^{1/2}\right) \qquad (n, p \in \mathbb{N}), \tag{4.9}$$

where (*cf.* (2.3))

$$a = \frac{8}{\pi^2} \log\left(\frac{\pi}{2}\right) = 0.366039.$$

Proof. In view of (4.6) and Lemma 2.2 we have

$$N(n, p) \geqslant \int_0^{\pi p^{1/2}/2} \left(\frac{\sin tp^{-1/2}}{n \sin tp^{-1/2}n^{-1}}\right)^{2p} dt$$

$$\geqslant \int_0^{\pi p^{1/2}/2} \exp(-at^2)\, dt = \frac{1}{2}\left(\frac{\pi}{a}\right)^{1/2} \operatorname{erf}\left(\frac{\pi}{2}(pa)^{1/2}\right). \quad ∎$$

This bound for $N(n, p)$ increases with p. Numerical values can be obtained from [1, p. 311].

LEMMA 4.2.

$$T_1(n, p) \leqslant \frac{3}{2}\left(1 - \frac{1}{n^2}\right)^{-1} \left\{1 - \exp\left(-\frac{1}{3}\left(1 - \frac{1}{n^2}\right)\pi^2 p\right)\right\}$$

$$+ \frac{p}{p-1}\pi^2\, 2^{-(2p+1)} \qquad (n, p = 2, 3, \ldots). \tag{4.10}$$

Proof. Assuming $n \geqslant 2$, $p \geqslant 2$ and taking into account (4.6) one has

$$T_1(n, p) = \int_0^{\pi p^{1/2}} t \left(\frac{\sin tp^{-1/2}}{n \sin tp^{-1/2}n^{-1}}\right)^{2p} dt + R.$$

By means of the inequality

$$\sin x \geqslant \frac{2}{\pi} x \qquad \left(0 \leqslant x \leqslant \frac{\pi}{2}\right) \tag{4.11}$$

it is easily shown that

$$R < \frac{p}{p-1}\,\pi^2\,2^{-(2p+1)}.$$

Furthermore, an application of Lemma 2.3 gives

$$\int_0^{\pi p^{1/2}} t\left(\frac{\sin tp^{-1/2}}{n\sin tp^{-1/2}n^{-1}}\right)^{2p} dt$$

$$\leqslant \int_0^{\pi p^{1/2}} t\exp\left(-\frac{1}{3}\left(1-\frac{1}{n^2}\right)t^2\right) dt$$

$$= \frac{3}{2}\left(1-\frac{1}{n^2}\right)^{-1}\left\{1-\exp\left(-\frac{1}{3}\left(1-\frac{1}{n^2}\right)\pi^2 p\right)\right\}. \quad \blacksquare$$

LEMMA 4.3.

$$T_2(n,p) \leqslant \left\{\frac{4}{9}\left(1-\frac{1}{n^2}\right)^{-1} + \frac{16}{3\pi^2 p}\left(1-\frac{1}{n^2}\right)^{-2}\right\}$$

$$\times \exp\left(-\frac{1}{12}\left(1-\frac{1}{n^2}\right)\pi^2 p\right) + \frac{\pi^2 p}{27(p-2)}\,2^{-2p+4}$$

$$(n \geqslant 2,\ p \geqslant 3). \tag{4.12}$$

Proof. One easily verifies that

$$h_n(t) \leqslant \frac{4n^2}{27\pi^2}\,t^3 \qquad (t \geqslant 0). \tag{4.13}$$

Taking into account (4.7) and using (4.13) one has

$$T_2(n,p) \leqslant \frac{32}{27\pi^2}\left\{\frac{1}{p}\int_{\pi p^{1/2}/2}^{\pi p^{1/2}} t^3\left(\frac{\sin tp^{-1/2}}{n\sin tp^{-1/2}n^{-1}}\right)^{2p} dt\right.$$

$$\left. + \int_{\pi}^{n\pi/2} pt^3\left(\frac{\sin t}{n\sin t/n}\right)^{2p} dt\right\}. \tag{4.14}$$

The first integral in the right-hand side of (4.14) is taken care of by an application of Lemma 2.3, whereas $\pi^2 p 2^{-2p+4}/(27(p-2))$ is a crude estimate for the second integral in (4.14). For details we refer to [8, pp. 19–20]. $\blacksquare$

Lemmas 4.1, 4.2 and 4.3 will be used to estimate the constants $c_{n,p}$ if n and p are not too small. For very small values of n and p a different approach will be needed (cf. Sections 4.1.5, 4.1.6).

4.1.2. Case $n = 2$, $p \geqslant 2$. In this section we consider the behaviour of the sequence $\{c_{2,p}\}_2^\infty$. According to formulae (4.1), (3.11), (4.3) and (2.7) we have

$$c_{2,p} = 2p^{1/2} \int_{-\pi}^{\pi} \tilde{f}_2(t)\, k_{2,p}(t)\, dt$$

$$= \frac{2^{2p+2}(p!)^2\, p^{1/2}}{\pi(2p)!} \left\{ \int_0^{\pi/2} t(\cos t)^{2p}\, dt + \int_0^{\pi/2} h_2(2t)(\cos t)^{2p}\, dt \right\}, \qquad (4.15)$$

where

$$h_2(2t) = \begin{cases} 0 & \left(0 \leqslant t \leqslant \frac{\pi}{4}\right) \\ t - \frac{\pi}{4} & \left(\frac{\pi}{4} \leqslant t \leqslant \frac{3\pi}{8}\right) \\ -t + \frac{\pi}{2} & \left(\frac{3\pi}{8} \leqslant t \leqslant \frac{\pi}{2}\right). \end{cases}$$

LEMMA 4.4. *The sequence* $\{c_{2,p}\}_2^\infty$ *is increasing and*

$$\lim_{p\to\infty} c_{2,p} = 2\pi^{-1/2} = 1.12837917. \qquad (4.16)$$

Proof. Only a sketch of the proof will be given; for details the reader is referred to [8, pp. 21–23]. By use of Lemmas 2.4 and 2.3 an elementary but rather tedious computation shows that $c_{2,p} < c_{2,p+1}$ when $p \geqslant 8$. The constants $c_{2,p}$ $(p = 2, 3,..., 8)$ can be evaluated explicitly (cf. Table 4.1 of Section 4.1.7). Taking these data into account the first assertion of the lemma follows. As for (4.16), it is easily seen that the limiting behaviour of $c_{2,p}$ is governed by the first integral in the right-hand side of (4.15). We have

$$p \int_0^{\pi/2} t(\cos t)^{2p}\, dt = \int_0^{\pi p^{1/2}/2} u(\cos up^{-1/2})^{2p}\, du$$

$$\sim \int_0^\infty u \exp(-u^2)\, du = \tfrac{1}{2} \qquad (p \to \infty).$$

This, together with an application of Stirling's formula proves (4.16). ∎

The remaining part of Section 4.1 will be devoted to showing that $c = 2\pi^{-1/2}$.

4.1.3. Case $n \geqslant 4$, $p \geqslant 5$. Here we shall be concerned with estimating the constants $c_{n,p}$ if $n \geqslant 4$ and $p \geqslant 5$. In view of (4.9) and using [1, p. 311] we find that

$$N(n, p) \geqslant \frac{1}{2}\left(\frac{\pi}{a}\right)^{1/2} \operatorname{erf}\left(\frac{\pi}{2}(5a)^{1/2}\right) > 1.4579 \qquad (n \in \mathbb{N}, p \geqslant 5). \tag{4.17}$$

An application of formulae (4.10) and (4.12) yields, for $n \geqslant 4$ and $p \geqslant 5$,

$$T_1(n, p) \leqslant \frac{8}{5}\left(1 - \exp\left(-\frac{25\pi^2}{16}\right)\right) + 5\pi^2\, 2^{-13} < 1.6061, \tag{4.18}$$

$$T_2(n, p) \leqslant \left(\frac{64}{135} + \frac{4096}{3375\pi^2}\right) \exp\left(-\frac{25\pi^2}{64}\right) + \frac{5\pi^2}{5184} < 0.0222. \tag{4.19}$$

By (4.4), (4.6) and (4.7) it follows from (4.17), (4.18), (4.19) that

$$c_{n,p} < 1.1169 < 2\pi^{-1/2} \qquad (n \geqslant 4, p \geqslant 5). \tag{4.20}$$

4.1.4. Case $n = 3$, $p \geqslant 5$. Inequality (4.9) is not sharp enough to take care of this case. However, proceeding as in the proof of Lemma 4.1 and applying (2.5) and (4.6) we obtain the following improvement on (4.9):

$$N(3, p) > \frac{1}{2}\left(\frac{\pi}{b}\right)^{1/2} \operatorname{erf}\left(\frac{\pi}{2}(pb)^{1/2}\right) \qquad (p \in \mathbb{N}),$$

where

$$b = \frac{8}{\pi^2} \log\left(\frac{3}{2}\right) = 0.328658.$$

Hence

$$N(3, p) > 1.5386 \qquad (p \geqslant 5).$$

Using this estimate together with an application of Lemmas 4.2 and 4.3 one easily obtains

$$c_{3,p} < 1.1177 < 2\pi^{-1/2} \qquad (p \geqslant 5). \tag{4.21}$$

4.1.5. In view of the results already obtained, the cases $p = 2$, $p = 3$ and $p = 4$ remain to be considered. The bounds provided by Lemmas 4.1, 4.2 and 4.3 are now not accurate enough to show that $c_{n,p} < 2\pi^{-1/2}$. We shall do this in a different way. We first note that it is easy to get good bounds on $N(n, 2)$, $N(n, 3)$ and $N(n, 4)$. By means of (4.8) and (2.2) we have for instance

$$N(n, 3) = \frac{\pi\, 3^{1/2}(11n^5 + 5n^3 + 4n)}{40n^5} > \frac{11}{40}\,\pi\, 3^{1/2} > 1.4963 \quad (n \in \mathbb{N}). \quad (4.22)$$

We now consider $T_1(n, p)$, which, apart from a factor p, is identical with $\int_0^{n\pi/2} t((\sin t)/(n \sin t/n))^{2p}\, dt$. Using (2.4) we deduce that

$$I_{n+1,p} := \int_0^{(n+1)\pi/2} t\left(\frac{\sin t}{(n+1)\sin t/(n+1)}\right)^{2p} dt$$

$$\leqslant \int_0^{n\pi/2} t\left(\frac{\sin t}{n \sin t/n}\right)^{2p} dt + \int_{n\pi/2}^{(n+1)\pi/2} t\left(\frac{\sin t}{(n+1)\sin t/(n+1)}\right)^{2p} dt$$

$$\leqslant I_{n,p} + \frac{\pi^2(2p)!}{2^{2p+2}(p!)^2}\,(n+1)^{-2p+1}\left(\sin\left(\frac{n\pi}{2(n+1)}\right)\right)^{-2p}, \quad (4.23)$$

where we have used (2.7).

Repeated application of (4.23) for a fixed $n_0 \in \mathbb{N}$ gives

$$I_{n_0+s,p} \leqslant I_{n_0,p} + \frac{\pi^2(2p)!}{2^{2p+2}(p!)^2}\left(\sin\frac{n_0\pi}{2(n_0+1)}\right)^{-2p} \sum_{j=n_0+1}^{\infty} j^{-2p+1} \qquad (s \in \mathbb{N}). \quad (4.24)$$

A similar procedure will be used to obtain an estimate for $T_2(n, p)$, which, apart from a factor p, is identical with (cf. (4.7)) the integral

$$\int_{\pi/2}^{n\pi/2} nh_n\left(\frac{2t}{n}\right)\left(\frac{\sin t}{n \sin t/n}\right)^{2p} dt. \quad (4.25)$$

In order to do this we need the following result, for the proof of which we refer to [8, pp. 27–28].

Lemma 4.5. *Let the function h_n be defined by* (4.3). *Then for $n \in \mathbb{N}$ one has*

$$nh_n\left(\frac{2t}{n}\right) \leqslant 0.26\pi n^2 \sin^2\frac{t}{n} \qquad \left(|t| \leqslant \frac{n\pi}{2}\right). \quad (4.26)$$

In order to estimate the integral (4.25) it is convenient to have the graphs of the functions $nh_n(2t/n)$ available for the first few values of n. These graphs are shown in figure 4.1, in the construction of which formulae (3.11), (3.12) have been used.

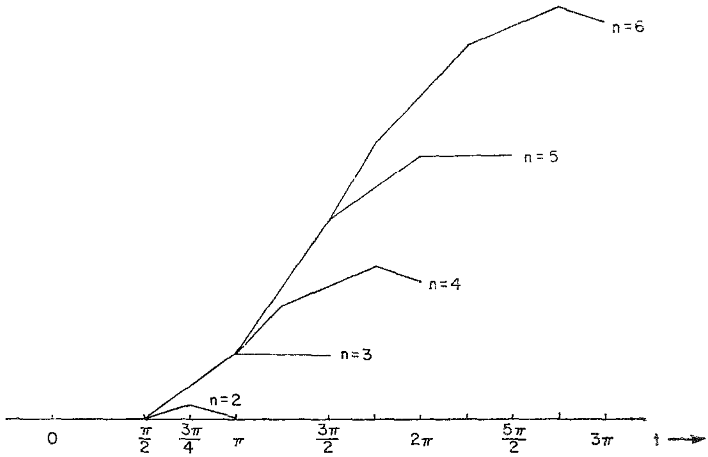

FIGURE 4.1

Using Lemma 4.5 and taking into account (2.4), (2.7) and (4.11), we have for $n \geqslant 5$

$$\begin{aligned} J_{n,p} &:= \int_{\pi/2}^{n\pi/2} nh_n\left(\frac{2t}{n}\right)\left(\frac{\sin t}{n\sin t/n}\right)^{2p} dt = \int_{\pi/2}^{3\pi/2} + \int_{3\pi/2}^{n\pi/2} \\ &\leqslant \int_{\pi/2}^{3\pi/2} 5h_5\left(\frac{2t}{5}\right)\left(\frac{\sin t}{5\sin t/5}\right)^{2p} dt + 0.26\pi \int_{3\pi/2}^{n\pi/2} \frac{(\sin t)^{2p}}{(n\sin t/n)^{2p-2}}\, dt \\ &< J_{5,p} + 0.26\pi \sum_{j=3}^{n-1} \left(n \sin\frac{j\pi}{2n}\right)^{-2p+2} \int_{j\pi/2}^{(j+1)\pi/2} (\sin t)^{2p}\, dt \\ &\leqslant J_{5,p} + \frac{0.26\pi^2(2p)!}{2^{2p+1}(p!)^2} \sum_{j=3}^{n-1} j^{-2p+2} < J_{5,p} + \frac{0.26\pi^2(2p)!}{2^{2p+1}(p!)^2} \sum_{j=3}^{\infty} j^{-2p+2}. \end{aligned} \tag{4.27}$$

Inequalities (4.22), (4.24) and (4.27) will be used to dispose of the case $p = 3$ and n sufficiently large.

4.1.6. Case $n \geqslant 3$, $2 \leqslant p \leqslant 4$. In Section 4.1.7 we shall give a set of formulae with which the constants $c_{n,p}$ may be computed explicitly. These formulae yield (cf. (4.5))

$$\begin{aligned} S_1(5, 3) &= \frac{5}{4}\pi\, 3^{1/2} - \frac{10(3)^{1/2}}{1751\pi}\left(1686 + \frac{1246}{9} + \frac{666}{25} + \frac{246}{49} + \frac{56}{81} + \frac{6}{121}\right) \\ &= 0.955187, \end{aligned}$$

$$\begin{aligned} S_2(5, 3) &= 2\pi\, 3^{1/2} - \frac{20(3)^{1/2}}{1751\pi}\left(2\cos^2\frac{\pi}{5} + \cos\frac{\pi}{5} - 1\right) \\ &\quad \times \left(1686 - \frac{1246}{9} - \frac{246}{49} + \frac{56}{81} + \frac{6}{121}\right) = 0.017208. \end{aligned}$$

Accordingly $c_{5,3} = 0.972395$.

Using these data, formula (4.22) and the definitions of $I_{n,p}$ and $J_{n,p}$ we obtain

$$I_{5,3} = 0.485382, \qquad J_{5,3} = 0.008744.$$

Applying (4.24) and (4.27) in the case $p = 3$ gives for $n > 5$

$$I_{n,3} \leqslant I_{5,3} + \frac{5\pi^2}{64}\left(\sin\frac{5\pi}{12}\right)^{-6} \sum_{j=6}^{\infty} j^{-5} < 0.485635,$$

$$J_{n,3} \leqslant J_{5,3} + \frac{13\pi^2}{320} \sum_{j=3}^{\infty} j^{-4} < 0.016693.$$

From these results and (4.22) it follows that

$$S_1(n, 3) < 0.9737, \qquad S_2(n, 3) < 0.0335 \qquad (n > 5)$$

and hence

$$c_{n,3} < 1.0072 < 2\pi^{-1/2} \qquad (n > 5). \tag{4.28}$$

The constants $c_{3,3}$ and $c_{4,3}$ can be computed explicitly; their values are contained in Table 4.1 of Section 4.1.7.

The cases $n \geqslant 3$, $p = 2$ and $n \geqslant 3$, $p = 4$ are treated in a similar way. Using the values of $S_1(19, 2)$ and $S_2(19, 2)$ from Table 4.1 we obtain

$$c_{n,2} < 1.1256 < 2\pi^{-1/2} \qquad (n > 19). \tag{4.29}$$

Similarly one finds

$$c_{n,4} < 1.0107 < 2\pi^{-1/2} \qquad (n > 4). \tag{4.30}$$

For computational details the reader is referred to [8, pp. 31–32].

4.1.7. As is apparent from the preceding sections, $c_{n,p}$ has to be computed explicitly for a few particular n and p. The values of $c_{2,p}$ ($p = 2, 3, \ldots, 8$) (cf. Section 4.1.2), $c_{3,3}$ and $c_{4,3}$, $c_{n,2}$ ($n = 2, \ldots, 19$), $c_{3,4}$ and $c_{4,4}$ (cf. Section 4.1.6) are contained in Table 4.1. To obtain these numbers we use (4.5), together with Lemma 2.1 and formulae (3.11), (3.12). It can then be shown that one has

$$S_1(n, p) = \frac{np^{1/2}}{4\pi}\left\{\pi^2 - 8 \sum_{k=1}^{[(np-p+1)/2]} \frac{\mu_{2k-1}}{\mu_0(2k-1)^2}\right\}, \tag{4.31}$$

where the coefficients $\mu_k := \mu_k^{(n,p)}$ are given by (2.2).

If $n = 2m + 1$ the expression for $S_2(n, p)$ reads

$$S_2(n, p) = \frac{1}{2}\pi m^2 p^{1/2} + \frac{2np^{1/2}}{\pi\mu_0} \sum_{j=1}^{m} \left[\left(\sum_{k=1}^{np-p} \frac{\mu_k \cos kt}{k^2}\right)\right]_{j\pi/n}^{(m+j)\pi/n} \tag{4.32}$$

When $n = 2m$ a similar formula for $S_2(n, p)$ can be derived, which we refrain from giving here. For details we refer to [8, pp. 32–36].

By means of (4.31) and (4.32) the data of Table 4.1 was computed on the Burroughs 7700 of the Computing Centre of the Eindhoven University of Technology.

TABLE 4.1

n	p	$S_1(n,p)$	$S_2(n,p)$	$c_{n,p}$	n	p	$S_1(n,p)$	$S_2(n,p)$	$c_{n,p}$
2	2	1.0210	0.0291	1.0501	6	2	0.9571	0.1085	1.0655
2	3	1.0545	0.0123	1.0668	7	2	0.9528	0.1114	1.0642
2	4	1.0721	0.0053	1.0773	8	2	0.9495	0.1127	1.0622
2	5	1.0829	0.0023	1.0852	9	2	0.9472	0.1143	1.0615
2	6	1.0903	0.0010	1.0913	10	2	0.9454	0.1144	1.0599
2	7	1.0956	0.0005	1.0960	11	2	0.9441	0.1149	1.0590
2	8	1.0996	0.0002	1.0998	12	2	0.9430	0.1149	1.0579
3	2	0.9945	0.0733	1.0678	13	2	0.9422	0.1152	1.0573
3	3	0.9890	0.0197	1.0087	14	2	0.9414	0.1150	1.0565
3	4	0.9974	0.0063	1.0037	15	2	0.9409	0.1150	1.0559
4	2	0.9740	0.0917	1.0657	16	2	0.9404	0.1149	1.0553
4	3	0.9656	0.0181	0.9837	17	2	0.9400	0.1149	1.0549
4	4	0.9742	0.0051	0.9793	18	2	0.9396	0.1148	1.0543
5	2	0.9641	0.1059	1.0701	19	2	0.9393	0.1147	1.0540

4.1.8. Taking into account (4.16) and (4.20), (4.21), (4.28), (4.29), (4.30), together with the contents of Table 4.1 and formula (4.2), we have the following theorem.

THEOREM 4.1. *Let $c_{n,p}$ be the exact constant of approximation for the operator $L_{n,p}$ as defined in* (1.3). *Then*

$$c := \sup_{p\geq 2} \sup_{n\in\mathbb{N}} c_{n,p} = \lim_{p\to\infty} c_{2,p} = 2\pi^{-1/2} = 1.12837917.$$

COROLLARY. *Let $f \in C^1_{2\pi}$ and let ω_1 be the modulus of continuity of f', then for $n \in \mathbb{N}$ and $p = 2, 3,\ldots$ one has*

$$\max_x | L_{n,p}(f; x) - f(x)| \leqslant \frac{2\pi^{-1/2}}{np^{1/2}}\, \omega_1\left(f; \frac{\pi}{n}\right).$$

where the value $2\pi^{-1/2}$ is best possible.

4.2. Case $p = 1$. Taking into account definition (1.4) and the form of the Fejér operators, together with the results on extremal functions of Section 3, we have

$$c_{n,1} = \frac{1}{2\pi n} \int_{-\pi}^{\pi} \tilde{f}_n(t) \left(\frac{\sin nt/2}{\sin t/2}\right)^2 dt \qquad (n \in \mathbb{N}). \tag{4.33}$$

We write (cf. (4.3) and (4.4))

$$c_{n,1} = S_1(n, 1) + S_2(n, 1),$$

where

$$S_1(n, 1) = \frac{1}{2\pi n} \int_0^{\pi} t \left(\frac{\sin nt/2}{\sin t/2}\right)^2 dt,$$

$$S_2(n, 1) = \frac{1}{\pi n} \int_0^{\pi} h_n(t) \left(\frac{\sin nt/2}{\sin t/2}\right)^2 dt.$$

By means of Lemma 2.1 it is easily shown that $S_1(n, 1)$ decreases monotonically to zero as $n \to \infty$. In order to determine $c^{(1)} := \sup_{n\in\mathbb{N}} c_{n,1}$ we need an upper bound for $S_2(n, 1)$. As $h_n(t) \equiv 0$ if $|t| \leqslant \pi/n$ one has

$$S_2(n, 1) = \frac{1}{\pi n} \int_{\pi/n}^{\pi} h_n(t) \left(\frac{\sin nt/2}{\sin t/2}\right)^2 dt = \frac{2}{\pi n^2} \int_{\pi/2}^{n\pi/2} h_n\left(\frac{2t}{n}\right)\left(\frac{\sin t}{\sin t/n}\right)^2 dt$$

$$\leqslant \frac{0.52}{n} \int_{\pi/2}^{n\pi/2} \sin^2 t \, dt < 0.13\pi = 0.4084 \qquad (n \in \mathbb{N}), \tag{4.34}$$

by an application of formula (4.26). Furthermore, one easily verifies that

$$c_{1,1} = \frac{\pi}{4} = 0.7854, \qquad c_{2,1} = \frac{9\pi}{32} - \frac{1}{2\pi}(2^{1/2} + 1) = 0.4993 \tag{4.35}$$

and

$$S_1(3, 1) = \frac{\pi}{4} - \frac{4}{3\pi} = 0.3610. \tag{4.36}$$

As $S_1(n, 1)$ is decreasing, formulae (4.34), (4.36), together with (4.35) imply the following theorem.

THEOREM 4.2. *Let $c_{n,1}$ be the exact constant of approximation for the operator $L_{n,1}$ as defined in* (1.4). *Then* (*cf.* (1.6))

$$c^{(1)} := \sup_{n\in\mathbb{N}} c_{n,1} = c_{1,1} = \frac{\pi}{4} = 0.78539816.$$

COROLLARY. *Let $f \in C^1_{2\pi}$ and let ω_1 be the modulus of continuity of f', then for $n \in \mathbb{N}$ one has for the Fejér operators $L_{n,1}$*

$$\max_x |L_{n,1}(f; x) - f(x)| \leqslant \frac{\pi}{4}\, \omega_1\left(f; \frac{\pi}{n}\right),$$

where the value $\pi/4$ is best possible.

5. THE LIMITING BEHAVIOUR OF THE CONSTANTS $c_{n,p}$

5.1. In this section we investigate the limiting behaviour as $n \to \infty$ or/and $p \to \infty$ of the exact constants of approximation $c_{n,p}$. There are four cases to be considered, viz., $n \to \infty$, $p \geqslant 2$; $n \geqslant 2$, $p \to \infty$; $n \to \infty$, $p \to \infty$; $n \to \infty$, $p = 1$, the last case corresponding to the Fejér operators. It turns out that $c_{n,1}$ has an asymptotic behaviour that is different from that of $c_{n,p}$ for $p \geqslant 2$.

5.2. Case $n \to \infty$, $p \geqslant 2$. Let $d_{n,p}$ be given by (2.8), i.e. let

$$d_{n,p} = \int_{-\pi}^{\pi} \tilde{f}_n(t)\, k_{n,p}(t)\, dt \qquad (p \geqslant 2).$$

As a guide to norming we regard $k_{n,p}$ as the probability density of a random variable (r.v.) $T_{n,p}$. For the expectation $ET_{n,p}$ and variance var $T_{n,p}$ we have

$$ET_{n,p} = 0, \qquad \operatorname{var} T_{n,p} = ET^2_{n,p} = \int_{-\pi}^{\pi} t^2 k_{n,p}(t)\, dt.$$

It is easily verified (cf. [6, Lemmas 2 and 4]) that

$$\operatorname{var} T_{n,p} \sim \frac{6}{n^2 p} \qquad (n \to \infty,\ p \to \infty).$$

Denoting the probability density of a r.v. X by g_X we generally have for $a > 0$

$$g_{aX}(t) = \frac{1}{a}\, g_X\left(\frac{t}{a}\right),$$

and therefore, letting $n \to \infty$,

$$\begin{aligned} g_{(n/2)T_{n,p}}(t) &= \frac{2}{n}\, g_{T_{n,p}}\left(\frac{2t}{n}\right) \\ &= \frac{2}{n}\, k_{n,p}\left(\frac{2t}{n}\right) \to \left(\int_{-\infty}^{\infty} \left(\frac{\sin t}{t}\right)^{2p} dt\right)^{-1} \left(\frac{\sin t}{t}\right)^{2p} =: g_p(t). \quad (5.1) \end{aligned}$$

It follows by dominated convergence ($\tilde{f}_n(t) \leqslant a \mid t \mid + bnt^2$; cf. (5.3)) that for $n \to \infty$ one has

$$\frac{n}{2} \int_{-\pi}^{\pi} \tilde{f}_n(t)\, k_{n,p}(t)\, dt = \int_{-n\pi/2}^{n\pi/2} \tilde{f}_n\left(\frac{2t}{n}\right) k_{n,p}\left(\frac{2t}{n}\right) dt \to \int_{-\infty}^{\infty} f^*(t)\, g_p(t)\, dt,$$

where (cf. formulae (3.11) and (3.12))

$$f^*(t) := \lim_{n\to\infty} \frac{n}{2} \tilde{f}_n\left(\frac{2t}{n}\right) = \frac{1}{2} \mid t \mid + \sum_{j=1}^{\infty} \left(\mid t \mid - \frac{\pi j}{2}\right)_+ .$$

Summing up we have the following theorem.

THEOREM 5.1. *For $p \geqslant 2$*

$$c_p := \lim_{n\to\infty} c_{n,p} = \lim_{n\to\infty} np^{1/2} \int_{-\pi}^{\pi} \tilde{f}_n(t)\, k_{n,p}(t)\, dt$$

$$= p^{1/2} \int_{-\infty}^{\infty} \mid t \mid g_p(t)\, dt + 4p^{1/2} \sum_{j=1}^{\infty} \int_{\pi j/2}^{\infty} \left(t - \frac{\pi j}{2}\right) g_p(t)\, dt,$$

where g_p is defined by (5.1).

Table 5.1 contains the values of c_p for $p = 2(1)10;\ 20(10)40$. For details concerning the numerical evaluation of these numbers we refer to [8, pp. 41–42].

TABLE 5.1

p	c_p	p	c_p	p	c_p
2	1.04547748	6	0.95729217	10	0.96499488
3	0.95048119	7	0.95991291	20	0.97109617
4	0.95047131	8	0.96199158	30	0.97313267
5	0.95404617	9	0.96365107	40	0.97415086

5.3. Case $n \geqslant 2$, $p \to \infty$. We begin by considering $cnp^{1/2}T_{n,p}$, where c will be given a convenient value. It is easily verified that

$$g_{cnp^{1/2}T_{n,p}}(t) = \frac{1}{cnp^{1/2}}\, k_{n,p}\left(\frac{t}{cnp^{1/2}}\right) \to g_{(n)}(t) \qquad (p \to \infty),$$

where

$$g_{(n)}(t) = \left(\int_{-\infty}^{\infty} \exp\left(-\frac{t^2(n^2-1)}{12n^2c^2}\right) dt\right)^{-1} \exp\left(-\frac{t^2(n^2-1)}{12n^2c^2}\right).$$

If we take $(n^2 - 1)/(6n^2c^2) = 1$ it follows that $(p(n^2 - 1)/6)^{1/2}\, T_{n,p}$ is asymptotically standard normal. By dominated convergence this means that

$$a_{n,p} \int_{-\pi}^{\pi} \tilde{f}_n(t)\, k_{n,p}(t)\, dt = \int_{-\pi a_{n,p}}^{\pi a_{n,p}} \tilde{f}_n\left(\frac{t}{a_{n,p}}\right) k_{n,p}\left(\frac{t}{a_{n,p}}\right) dt$$

$$\to \frac{(2\pi)^{-1/2}}{2} \int_{-\infty}^{\infty} |\, t\,|\, \exp\left(\frac{-t^2}{2}\right) dt$$

$$= (2\pi)^{-1/2},$$

where $a_{n,p} = (p(n^2 - 1)/6)^{1/2}$.

We have now proved the following theorem.

THEOREM 5.2. *For* $n \geqslant 2$

$$\lim_{p\to\infty} c_{n,p} = \lim_{p\to\infty} np^{1/2} \int_{-\pi}^{\pi} \tilde{f}_n(t)\, k_{n,p}(t)\, dt = \left(\frac{3}{\pi}\right)^{1/2} \left(1 - \frac{1}{n^2}\right)^{-1/2}.$$

We note that if $n = 2$ we have $\lim_{p\to\infty} c_{2,p} = 2\pi^{-1/2}$ (cf. (4.16)). Furthermore, it follows that $\{\lim_{p\to\infty} c_{n,p}\}_2^\infty$ is a decreasing sequence.

5.4. Case $n \to \infty$, $p \to \infty$. From Theorems 5.1 and 5.2 we obtain (compare Table 5.1)

THEOREM 5.3.

$$\lim_{\substack{n\to\infty\\ p\to\infty}} c_{n,p} = \lim_{\substack{n\to\infty\\ p\to\infty}} np^{1/2} \int_{-\pi}^{\pi} \tilde{f}_n(t)\, k_{n,p}(t)\, dt = \left(\frac{3}{\pi}\right)^{1/2} = 0.97720502,$$

where the limits may be taken in either order.

Proof. It is an immediate consequence of Theorem 5.2 that $\lim_{n\to\infty} \lim_{p\to\infty} c_{n,p} = (3/\pi)^{1/2}$. On the other hand we obtain from Theorem 5.1, using dominated convergence for integrals and sum,

$$\lim_{p\to\infty} c_p = \lim_{p\to\infty} A_p^{-1} \Bigg[\int_0^\infty u \left(\frac{\sin up^{-1/2}}{up^{-1/2}}\right)^{2p} du$$

$$+ 2 \sum_{j=1}^{\infty} \int_{\pi j p^{1/2}/2}^{\infty} \left(u - \frac{\pi j p^{1/2}}{2}\right)\left(\frac{\sin up^{-1/2}}{up^{-1/2}}\right)^{2p} du\Bigg]$$

$$= \left(\int_0^\infty \exp\left(-\frac{u^2}{3}\right) du\right)^{-1} \int_0^\infty u \exp\left(-\frac{u^2}{3}\right) du = \left(\frac{3}{\pi}\right)^{1/2},$$

where we have written

$$\int_0^\infty \left(\frac{\sin up^{-1/2}}{up^{-1/2}}\right)^{2p} du = A_p . \quad \blacksquare$$

5.5. Case $n \to \infty$, $p = 1$. The behaviour of $c_{n,1}$ (cf. (1.4)) differs from that of $c_{n,p}$ for $p \geqslant 2$. This is due to the easily verified fact that

$$\operatorname{var} T_{n,1} = ET_{n,1}^2 = \int_{-\pi}^{\pi} t^2 k_{n,1}(t)\, dt = O\left(\frac{1}{n}\right) \qquad (n \to \infty), \tag{5.2}$$

whereas for $p \geqslant 2$ we have $\operatorname{var} T_{n,p} = O(n^{-2})$. In view of (3.9) and (3.10) we deduce that

$$\tilde{f}_n(t) = \begin{cases} \dfrac{nt^2}{2\pi} + tO(1) & \left(0 < t < \dfrac{\pi}{2}\right) \\ \dfrac{n\pi}{4} - n\,\dfrac{(\pi - t)^2}{2\pi} + tO(1) & \left(\dfrac{\pi}{2} < t < \pi\right) \end{cases} \qquad (n \to \infty). \tag{5.3}$$

From (5.2) it follows that $E \mid T_{n,1} \mid = O(n^{-1/2})$. Using this, (5.3) and formula (4.33), one has by dominated convergence

$$c_{n,1} = \frac{1}{2\pi n} \int_{-\pi}^{\pi} \tilde{f}_n(t) \left(\frac{\sin nt/2}{\sin t/2}\right)^2 dt$$

$$\sim \frac{1}{2\pi^2} \left[\int_0^{\pi/2} t^2 \left(\frac{\sin nt/2}{\sin t/2}\right)^2 dt\right.$$

$$\left. + \int_{\pi/2}^{\pi} \left\{\frac{\pi^2}{2} - (\pi - t)^2\right\} \left(\frac{\sin nt/2}{\sin t/2}\right)^2 dt\right]$$

$$\to \frac{1}{4\pi^2} \int_0^{\pi/2} \frac{t^2}{\sin^2 t/2}\, dt + \frac{1}{4\pi^2} \int_{\pi/2}^{\pi} \left\{\frac{\pi^2}{2} - (\pi - t)^2\right\} \frac{1}{\sin^2 t/2}\, dt.$$

Here, the limit is obtained by the Riemann–Lebesgue lemma. By simple transformations it follows that (cf. [4, p. 123])

$$\lim_{n \to \infty} c_{n,1} = \frac{2}{\pi^2} \int_0^{\pi/4} \frac{t^2}{\sin^2 t}\, dt - \frac{2}{\pi^2} \int_0^{\pi/4} \frac{t^2}{\cos^2 t}\, dt + \frac{1}{4}$$

$$= \frac{2}{\pi^2} \int_0^{\pi/2} \frac{t}{\sin t}\, dt = \frac{4\theta}{\pi^2},$$

where θ denotes Catalan's constant: $\theta = \sum_{j=0}^{\infty} (-1)^j (2j+1)^{-2} = 0.91596559$. This proves the following theorem.

THEOREM 5.4.

$$\lim_{n\to\infty} c_{n,1} = \lim_{n\to\infty} \int_{-\pi}^{\pi} \tilde{f}_n(t)\, k_{n,1}(t)\, dt = \frac{4\theta}{\pi^2} = 0.37122687.$$

COROLLARY. *Let $f \in C_{2\pi}^1$ and let ω_1 be the modulus of continuity of f', then for $n \in \mathbb{N}$ one has the following inequality for the Fejér operators $L_{n,1}$*

$$\max_x |L_{n,1}(f; x) - f(x)| \leqslant c_{n,1}\omega_1\left(f; \frac{\pi}{n}\right),$$

with $\lim_{n\to\infty} c_{n,1} = 4\theta/\pi^2 = 0.37122687$.

Remark. With a little more effort it can be proved that $c_{n,1} = 4\theta/\pi^2 + O(n^{-1})$ as $n \to \infty$.

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