JOURNAL OF ALGEBRA 17, 369–388 (1971)

Strong Radical Properties of Alternative and Associative Rings

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Communicated by A. W. Goldie

Received January 3, 1970

INTRODUCTION

We study strong radicals, i.e. radical properties that contain all onesided radical ideals. For alternative rings we show that the lower strong radical can be constructed beginning with any homomorphically closed class. For associative rings we show (Theorem 6) that this construction stops at ω_0 if the beginning class is hereditary and contains all zero rings. If the beginning class has two other conditions we can show (Theorem 3) that the construction stops at 2. We obtain a characterization of the lower strong radical class (Theorem 5) which is different from a more natural description since the natural one is too large (Example 5).

For alternative rings we get a condition for upper radicals to be strong (Theorem 10) but this is not an if and only if condition. Finally we give an example (6) to show that not every upper radical is strong.

1. LOWER RADICAL PROPERTIES

Let $\mathfrak{K}$ be a nonempty class of not necessarily associative rings and let it be universal, i.e. with the property that subrings and homomorphic images of

* On National Research Council visitor grant at U.B.C.

rings in $\mathfrak{K}$ are again in $\mathfrak{K}$. Let $\mathfrak{N}$ be a nonempty subclass of $\mathfrak{K}$ and suppose that $\mathfrak{N}$ is homomorphically closed [i.e. if R is in $\mathfrak{N}$ then so is every homomorphic image of R]. Define $\mathfrak{N}_1 = \mathfrak{N}$ and define

$\mathfrak{N}_\alpha =$ the class of all rings R in $\mathfrak{K}$ which have the property that every nonzero homomorphic image of R contains a nonzero onesided ideal in $\mathfrak{N}_\beta$ for some $\beta < \alpha$.

Note that $\{0\}$ belongs to $\mathfrak{N}_\alpha$ for every ordinal α. Rings in $\mathfrak{N}_\alpha$ will be said to be of degree α over $\mathfrak{N}$, or $\mathfrak{N}_\alpha$ rings.

PROPOSITION 1.

i) *$\mathfrak{N}_\alpha$ is homomorphically closed, for every ordinal α.*

ii) *If $\beta < \alpha$ then $\mathfrak{N}_\beta \subseteq \mathfrak{N}_\alpha$.*

Proof. Take R in $\mathfrak{N}_\alpha$ and let R' be any homomorphic image of R. If $R' = 0$ then it is in $\mathfrak{N}_\alpha$. If $R' \neq 0$ then any nonzero homomorphic image of R' is also a homomorphic image of R. Thus it contains a nonzero onesided ideal in $\mathfrak{N}_\beta$ for some $\beta < \alpha$ and therefore R' is in $\mathfrak{N}_\alpha$.

If R is in $\mathfrak{N}_\beta$ then every nonzero homomorphic image of R is also in $\mathfrak{N}_\beta$, by (i), and thus contains a nonzero onesided ideal, namely itself, in $\mathfrak{N}_\beta$. Therefore if $\beta < \alpha$, R is in $\mathfrak{N}_\alpha$ by definition. Thus $\mathfrak{N}_\beta \subseteq \mathfrak{N}_\alpha$.

This proposition holds for any universal class defined above. Now we turn our attention to the case when $\mathfrak{K}$ contains only alternative rings.

LEMMA 1. *If R is an alternative ring and I is a left ideal of R then $I^* = I + IR$ is a two sided ideal of R.*

Proof. Consider $RI^* = R(I + IR) \subseteq RI + R \cdot IR \subseteq I + R \cdot IR$. We must therefore show that $R \cdot IR \subseteq I^*$. Now $R \cdot IR \subseteq RI \cdot R + (R, I, R)$ where $(R, I, R) =$ the set of all finite sums of elements of the form $x \cdot ay - xa \cdot y$, where a is in I and x, y are in R. Since $RI \subseteq I$, $RI \cdot R \subseteq IR \subseteq I^*$. Also $(R, I, R) = (R, R, I) \subseteq RR \cdot I + R \cdot RI \subseteq I \subseteq I^*$. Thus $RI^* \subseteq I^*$ and I^* is a left ideal of R.

Furthermore $I^*R = (I + IR)R \subseteq IR + IR \cdot R \subseteq I^* + IR \cdot R$. We must therefore show that $IR \cdot R \subseteq I^*$. Now $IR \cdot R \subseteq I \cdot RR + (I, R, R)$. Clearly $I \cdot RR \subseteq IR \subseteq I^*$. Also $(I, R, R) = (R, R, I) \subseteq RR \cdot I + R \cdot RI \subseteq RI \subseteq I \subseteq I^*$. Therefore I^* is a right ideal of R and the lemma is proved.

LEMMA 2. *Let $\mathfrak{N}$ be a homomorphically closed class of alternative rings. Consider any alternative ring R with a nonzero left ideal I which belongs to $\mathfrak{N}$. Then R contains a nonzero two sided ideal in $\mathfrak{N}_2$.*

Proof. Consider $I^* = I + IR$. This is a nonzero two sided ideal of R

(Lemma 1) and we shall show it is in $\mathfrak{N}_2$. Take any nonzero homomorphic image I^*/W of I^*. If $I \not\subseteq W$ then

$$\frac{I+W}{W} \cong \frac{I}{I \cap W}$$

is a nonzero left ideal of I^*/W and it is in $\mathfrak{N}$ since it is a homomorphic image of I.

If $I \subseteq W$ there must exist an element x in R such that $Ix \not\subseteq W$, otherwise $IR \subseteq W$ and $W = I^*$. Consider $W + Ix$. This is a left ideal of I^* because

$$I^* \cdot (W + Ix) \subseteq I^*W + I^* \cdot Ix;$$
$$I^*W \subseteq W;$$
$$I^* \cdot Ix = (I + IR)\, Ix \subseteq I \cdot Ix + IR \cdot Ix;$$

since

$$I \subseteq W, \qquad I \cdot Ix \subseteq W \cdot Ix \subseteq W \cdot I^* \subseteq W;$$

finally

$$IR \cdot Ix \subseteq (IR \cdot I)x + (IR, I, x);$$

since

$$IR \cdot I \subseteq R \cdot I \subseteq I, \qquad (IR \cdot I)x \subseteq Ix \subseteq W + Ix;$$

and

$$(IR \cdot I, x) = (IR, x, I) = (IR \cdot x)I + IR \cdot xI,$$

then

$$(IR \cdot x)I \subseteq RI \subseteq I \subseteq W \subseteq W + Ix$$

and

$$IR \cdot xI \subseteq IR \cdot I \subseteq RI \subseteq I \subseteq W + Ix.$$

Therefore everything is in $W + Ix$ and it is a left ideal of I^*. Thus $(W + Ix)/W$ is a nonzero left ideal of I^*/W. We want to show it is in $\mathfrak{N}$.

Define θ: $I \to (W + Ix)/W$ by θ: $a \to ax + W$. Clearly θ preserves addition. To see that it also preserves multiplication we consider θ: $ab \to ab \cdot x + W$.

Now $ab \cdot x = a \cdot bx + (a, b, x)$. Since a is in $I \subseteq W$, $a \cdot b\,x \in W \cdot I^* \subseteq W$. Also $(a, b, x) = (x, a, b) = x \cdot ab - xa \cdot b$. Now $x \cdot ab$ is in $xI \subseteq I \subseteq W$, and $xa \cdot b$ is in $R \cdot I \subseteq I \subseteq W$. Therefore $ab \cdot x + W = 0 + W$. We must therefore show that $(ax + W)(bx + W) = 0 + W$. Now $(ax + W)(bx + W) = ax \cdot bx + W$. Since $ax \cdot bx = a(x \cdot bx) + (a, x, bx)$, and bx is in I^*, then $x \cdot bx$ is in $RI^* \subseteq I^*$, and $a(x \cdot bx) \in I \cdot I^* \subseteq W \cdot I^* \subseteq W$. Finally $(a, x, bx) = (x, bx, a) = x(x, a, b)$ and $(x, a, b) = x \cdot ab - xa \cdot b \in I$, thus $x(x, a, b)$ is in $RI \subseteq I \subseteq W$. Therefore $(ax + W)(bx + W) = 0 + W = ab \cdot x + W$, and θ preserves multiplication. Thus θ is a homomorphism and since I is in $\mathfrak{N}$, so is $(W + Ix)/W$. Thus every nonzero homomorphic image of I^* contains a nonzero left ideal in $\mathfrak{N}$ and by definition I^* is in $\mathfrak{N}_2$.

COROLLARY *If an alternative ring R has a nonzero right ideal which belongs to a homomorphically closed class $\mathfrak{N}$, then R has a nonzero two sided ideal in $\mathfrak{N}_2$.*

Let $\mathfrak{K}$ again be an arbitrary nonempty universal class of rings (i.e. not necessarily alternative). Then a nonempty subclass $\mathfrak{S}$ of $\mathfrak{K}$ will be called a *strong radical class* if it satisfies:

1. $\mathfrak{S}$ is homomorphically closed.

2. Every ring R in $\mathfrak{K}$ has an ideal $S(R)$ which is in $\mathfrak{S}$ and which contains all onesided ideals of R which are in $\mathfrak{S}$.

3. For every ring R in $\mathfrak{K}$, $S(R/S(R)) = 0$.

Thus a strong radical is an ordinary radical with the extra condition that onesided radical ideals are contained in it. We begin with a result of Amitsur [1, page 108].

PROPOSITION 2. *A radical property $\mathfrak{S}$ is a strong radical property if and only if every $\mathfrak{S}$-semi-simple ring has no nonzero onesided ideals in $\mathfrak{S}$.*

Proof. If $\mathfrak{S}$ is a strong radical property, if I is a onesided ideal of R, and if I is in $\mathfrak{S}$ then $I \subseteq S(R)$. And if R is $\mathfrak{S}$-semi-simple, $S(R) = 0$ and $I = 0$. Conversely if I is in $\mathfrak{S}$, is a onesided ideal of R and $I \not\subseteq S(R)$, then $(I + S(R))/S(R) \cong I/(I \cap S(R)) \neq 0$ is a onesided ideal of $R/S(R)$ and it is in $\mathfrak{S}$. Thus if every $\mathfrak{S}$-semi-simple ring has no nonzero onesided ideals in $\mathfrak{S}$, we must have $I \subseteq S(R)$ and $\mathfrak{S}$ is a strong radical property.

THEOREM 1. *Let $\mathfrak{K}$ be a nonempty universal class of alternative rings. A nonempty subclass $\mathfrak{S}$ of $\mathfrak{K}$ is a strong radical class if and only if $\mathfrak{S}$ satisfies:*

(a) *$\mathfrak{S}$ is homomorphically closed*

(b) *If every nonzero homomorphic image of a ring R in $\mathfrak{K}$, contains a nonzero onesided ideal in $\mathfrak{S}$, then R itself is in $\mathfrak{S}$.*

Proof. If $\mathfrak{S}$ is strong it certainly has (a). To establish (b), let R be any ring of $\mathfrak{K}$ not in $\mathfrak{S}$. Then $R/S(R)$ is a nonzero $\mathfrak{S}$-semi-simple ring and thus it has no nonzero onesided ideals in $\mathfrak{S}$ (Proposition 2). This gives us (b). Note that this half of the theorem does not make use of alternativity.

To prove the converse, suppose $\mathfrak{S}$ satisfies (a) and (b). Then it is clear that $\mathfrak{S}$ is at least an ordinary radical class for (b) implies that a ring R is in $\mathfrak{S}$ if every nonzero homomorphic image of R contains a nonzero ideal in $\mathfrak{S}$ (see for example [8, Theorem 1, page 4,]—the proof does not depend on associativity). To show that $\mathfrak{S}$ is strong, let I be a onesided ideal of R and assume I is in $\mathfrak{S}$. We will show that $I \subseteq S(R)$.

By Lemma 2 or by the corollary to Lemma 2, R contains a nonzero two sided ideal J in $\mathfrak{S}_2$ with $I \subseteq J$. However (b) implies that $\mathfrak{S}_2 = \mathfrak{S}$ and thus $I \subseteq J \subseteq S(R)$ since J is in $\mathfrak{S}$.

The construction of lower strong radicals does not present any serious difficulties for alternative rings. If we begin with any nonempty homomorphically closed subclass $\mathfrak{N}$ of a nonempty universal class $\mathfrak{K}$ of alternative rings, we consider the chain $\mathfrak{N} = \mathfrak{N}_1 \subseteq \mathfrak{N}_2 \subseteq \cdots \subseteq \mathfrak{N}_\alpha \subseteq \cdots$ and let

$$LS(\mathfrak{N}) = \bigcup_\alpha \mathfrak{N}_\alpha = \{R \text{ in } \mathfrak{K} : R \text{ is of some degree over } \mathfrak{N}\}.$$

THEOREM 2. *$LS(\mathfrak{N})$ is a strong radical and it is the smallest strong radical containing $\mathfrak{N}$. We call it the lower strong radical determined by $\mathfrak{N}$.*

Proof. By Proposition 1, $LS(\mathfrak{N})$ has (a). To establish (b) let R in $\mathfrak{K}$ have the property that every nonzero homomorphic image R' of R has a nonzero onesided ideal I' in $LS(\mathfrak{N})$. Then I' is in some $\mathfrak{N}_\sigma$. For various homomorphic images we obtain various σ's. Let α be an ordinal which is bigger than or equal to all of these σ's. Then for every R', I' is in $\mathfrak{N}_\sigma \subseteq \mathfrak{N}_\alpha$. Therefore R is in $\mathfrak{N}_{\alpha+1}$ and thus R is in $LS(\mathfrak{N})$. This establishes (b) and shows, by Theorem 1, that $LS(\mathfrak{N})$ is a strong radical.

To show that $LS(\mathfrak{N})$ is the smallest strong radical containing $\mathfrak{N}$, let $\mathfrak{Q}$ be a strong radical $\supseteq \mathfrak{N}$. We must show that $LS(\mathfrak{N}) \subseteq \mathfrak{Q}$. We proceed by induction and assume that $\mathfrak{N}_\gamma \subseteq \mathfrak{Q}$ for every $\gamma < \alpha$. Let R be in $\mathfrak{N}_\alpha$. Then every nonzero homomorphic image of R contains a nonzero onesided ideal in $\mathfrak{N}_\gamma$ for some $\gamma < \alpha$. These onesided ideals are thus all in $\mathfrak{Q}$. However $\mathfrak{Q}$ is strong and has (b) and thus R must be in $\mathfrak{Q}$. Therefore $\mathfrak{N}_\alpha \subseteq \mathfrak{Q}$. Since $\mathfrak{N} = \mathfrak{N}_1 \subseteq \mathfrak{Q}$, we have $LS(\mathfrak{N}) = \bigcup \mathfrak{N}_\alpha \subseteq \mathfrak{Q}$.

Remarks. For a given nonempty homomorphically closed class $\mathfrak{N}$ we can construct $L(\mathfrak{N})$ = the ordinary lower radical determined by $\mathfrak{N}$. Since $L(\mathfrak{N})$ is contained in any radical class that contains $\mathfrak{N}$, we must have

$$\mathfrak{N} \subseteq L(\mathfrak{N}) \subseteq LS(\mathfrak{N}).$$

For some $\mathfrak{N}$'s, $L(\mathfrak{N}) = LS(\mathfrak{N})$. For example if $\mathfrak{N}$ is the class of all zero rings in the class $\mathfrak{K}$ of all associative rings, then $L(\mathfrak{N})$ is the Baer lower radical. It is known [5, Theorem 2] that every Baer radical ring R is in $\mathfrak{N}_2$ (two sided construction), i.e. every nonzero homomorphic image of R has a nonzero zero ideal. To prove that the Baer lower radical is strong we shall use Proposition 2.

Assume then that T is a Baer semi-simple ring and that I is a nonzero right ideal of T and a Baer radical ring. Then I has some nonzero zero ideals. By Zorn's Lemma select a nonzero maximal zero ideal W of I. This can be done since the $\bigcup_i Z_i$ when $Z_1 \subset Z_2 \subset \cdots$ is again a zero ideal if each of the Z_i is a zero ideal. Now WI is a right ideal of T, it is in W and therefore it is a zero ring. If $WI \neq 0$ then $WI + TWI$ is a two-sided ideal of T and $(WI + TWI)^2 = 0$. This is impossible since T is Baer semi-simple. Thus $WI = 0$.

Consider I/W. Since $I^2 \neq 0$ (if $I^2 = 0, I + TI$ would be nonzero zero ideal of T), $I/W \neq 0$. It is again a Baer radical ring and thus has a nonzero zero ideal Q/W. Thus $Q^2 \subseteq W$. Then $QI \cdot QI \subseteq QQI \subseteq WI = 0$. Thus $QI = 0$ else T has a nonzero zero ideal $QI + TQI$. Therefore $Q^2 = 0$. Since $Q \supsetneqq W$ this contradicts the maximality of W as a zero ideal of I. Therefore no such I exists and the Baer lower radical is strong.

In general, $L(\mathfrak{N}) \subsetneqq LS(\mathfrak{N})$. We shall show later that for example the Brown-McCoy radical in associative rings is not strong.

The question of when the construction of $L(\mathfrak{N})$ stops, was examined in [5]. It was shown that for associative rings, $L(\mathfrak{N}) = \mathfrak{N}_{\omega_0}$ and for alternative rings, $L(\mathfrak{N}) = \mathfrak{N}_{\omega_0^2}$. Rubikin has shown [11] that for general nonassociative classes, there is no upper bound. In the associative case, if $\mathfrak{N}$ contains all zero rings and is hereditary (i.e. ideals of rings in $\mathfrak{N}$ are also in $\mathfrak{N}$) [5], or if $\mathfrak{N}$ contains all zero rings and if R/I is a zero ring and I is in $\mathfrak{N}$ implies R is in $\mathfrak{N}$ [7], then $L(\mathfrak{N}) = \mathfrak{N}_2$. With just heredity $L(\mathfrak{N}) = \mathfrak{N}_3$, see [6] and [12]. However Heinicke [9] showed that there are associative classes $\mathfrak{N}$ for which $L(\mathfrak{N})$ does not stop until $\mathfrak{N}_{\omega_0}$. The subscripts in this paragraph refer to the two sided construction.

The construction of $LS(\mathfrak{N})$ goes further out than $L(\mathfrak{N})$ but the steps are larger than the corresponding two sided steps for $L(\mathfrak{N})$.

THEOREM 3. *Let $\mathfrak{K}$ be the class of all associative rings and let $\mathfrak{N}$ be a non-empty homomorphically closed subclass of $\mathfrak{K}$, satisfying*:

1. *$\mathfrak{N}$ contains all zero rings*;
2. *$\mathfrak{N}$ is hereditary*;
3. *If a ring R has a nonzero right* (*left*) *ideal in $\mathfrak{N}$ then R also has a nonzero left* (*right*) *ideal in $\mathfrak{N}$*;
4. *If I is an ideal of R, if $I^2 = 0$ and if R/I is in $\mathfrak{N}$, then R is in $\mathfrak{N}$*;

then

$$LS(\mathfrak{N}) = \mathfrak{N}_2$$
$$= \{R\text{: every nonzero homomorphic image of } R \text{ contains a nonzero onesided ideal in } \mathfrak{N}\}.$$

[Note that 4 can of course be replaced by 4′: If I and R/I are in $\mathfrak{N}$ then R is in $\mathfrak{N}$. 4′ is stronger than 4 but perhaps more elegant].

Proof. We will show that $\mathfrak{N}_3 = \mathfrak{N}_2$ and this yields $LS(\mathfrak{N}) = \mathfrak{N}_2$. Take R in $\mathfrak{N}_3$ and let R' be any nonzero homomorphic image of R. Then R' has a nonzero onesided ideal I in $\mathfrak{N}_2$. Suppose without loss of generality that I is a right ideal. Now I must contain a nonzero onesided ideal J in $\mathfrak{N}_1 = \mathfrak{N}$ and by 3. we may take J to be a right ideal of I.

Consider JI. It is a right ideal of R', it is contained in J and it is a two sided ideal of J. Since J is in $\mathfrak{N}$, by 2., we have JI in $\mathfrak{N}$. Thus if $JI \neq 0$, R' contains a nonzero onesided ideal in $\mathfrak{N}$.

Assume then that $JI = 0$. Then $J \subseteq \ell(I) = \{x \text{ in } I: xI = 0\}$. Now $\ell(I)$is an ideal of I. If $\ell(I) = I$ then $I^2 = 0$ and I is in $\mathfrak{N}$ by 1. Again, then, R' contains a nonzero onesided ideal in $\mathfrak{N}$. So assume that $\ell(I) \neq I$. Then $I/\ell(I) \neq 0$. It is in $\mathfrak{N}_2$ since I is in $\mathfrak{N}_2$. Thus $I/\ell(I)$ contains a nonzero onesided ideal $P/\ell(I)$ in $\mathfrak{N}$. We may take this to be a right ideal, by 3. By 4., P itself must be in $\mathfrak{N}$ since $\ell(I)^2 = 0$. Now $PI \neq 0$ since $P \not\subseteq \ell(I)$. Furthermore PI is a right ideal of R', it is contained in P and it is a two sided ideal of P. Thus PI is in $\mathfrak{N}$ by 2. Thus even in this case R' has a nonzero onesided ideal in $\mathfrak{N}$, and therefore R is in $\mathfrak{N}_2$.

Remark. The theory of strong radicals can be weakened to consider right strong radicals (or left); i.e. radicals $\mathfrak{S}$ which contain all right $\mathfrak{S}$ ideals. Then beginning with a nonempty homomorphically closed class $\mathfrak{M} = \mathfrak{M}_1$, one considers $\mathfrak{M}_\alpha$ as the class of all rings R such that every nonzero homomorphic image of R contains a nonzero right ideal in $\mathfrak{M}_\gamma$ for $\gamma < \alpha$. The $\bigcup \mathfrak{M}_\alpha$ is then the lower right strong radical.

The proof of Theorem 3 then goes through without using the condition 3. and one has:

COROLLARY *If a nonempty homomorphically closed class $\mathfrak{M}$ satisfies:*

1. *$\mathfrak{M}$ contains all zero rings,*
2. *$\mathfrak{M}$ is hereditary,*
4. *If I is an ideal of R, $I^2 = 0$ and if R/I is in $\mathfrak{M}$ then R is in $\mathfrak{M}$;*

then the lower right strong radical determined by $\mathfrak{M}$ is $\mathfrak{M}_2$, and the lower left strong radical determined by $\mathfrak{M}$ is $\mathfrak{M}_2$.

There exist nonempty homomorphically closed classes $\mathfrak{N}$ satisfying the four conditions of Theorem 3.

EXAMPLE 1. Let $\mathfrak{N}$ be the class of all associative rings which satisfy a polynomial identity. Then it is clear that $\mathfrak{N}$ is homomorphically closed, contains all zero rings and is hereditary. To establish 4., suppose that I is an ideal of R, $I^2 = 0$ and R/I is in $\mathfrak{N}$. Let $f(r_1, r_2, ..., r_n)$ in I for every $r_1, ..., r_n$ in R be the polynomial identity associated with R/I. Define

$$g(x_1, ..., x_n) .\equiv. f(x_1, ..., x_n) \cdot f(x_1, ..., x_n).$$

Then for any $r_1, ..., r_n$ in R we have $g(r_1, ..., r_n) = 0$ because $I^2 = 0$. Thus R itself satisfies a polynomial identity and R is in $\mathfrak{N}$.

Finally, to establish 3., suppose J is a nonzero right ideal of R, J in $\mathfrak{N}$, and let $f(t_1, ..., t_n) = 0$ for any $t_1, ..., t_n$ in J. We may assume that f is multilinear because any ring that satisfies a polynomial identity, also satisfies a multilinear identity [10, page 224].

If $J^2 = 0$ then $\bar{J}^2 = 0$ where $\bar{J}$ is the two sided ideal of R generated by J. Then $\bar{J}$ is in $\mathfrak{N}$ and R has a nonzero left ideal in $\mathfrak{N}$. If $J^2 \neq 0$ then there exists an element t in J such that $Rt \neq 0$. We shall prove that this left ideal Rt is in $\mathfrak{N}$. Define

$$g(x_1, ..., x_n, x_{n+1}) .\equiv. x_{n+1} \cdot f(x_1, ..., x_n).$$

This is not identically zero since f is not identically zero. Take any elements $r_1, ..., r_n, r_{n+1}$ in Rt. Then $r_i = y_i t$ with the y_i in R, for $i = 1, ..., n + 1$. Then

$$\begin{aligned} g(r_1, ..., r_n, r_{n+1}) &= r_{n+1} \cdot f(r_1, ..., r_n) \\ &= y_{n+1}t \cdot f(y_1t, ..., y_nt) \\ &= y_{n+1} \cdot f(ty_1, ..., ty_n) \cdot t \end{aligned}$$

since f is multilinear. Furthermore ty_i is in J for $i = 1, ..., n$, since t is in J and J and J is a right ideal. Thus $f(ty_1, ..., ty_n) = 0$ and $g(r_1, ..., r_n, r_{n+1}) = 0$. Therefore Rt satisfies a polynomial identity, is in $\mathfrak{N}$ and 3. is established.

Remark. For this class $\mathfrak{N}$ of polynomial identity rings it turns out that $L(\mathfrak{N}) \subsetneqq LS(\mathfrak{N})$. To see this we adapt an example of Amitsur [2, page 133].

Let $\mathfrak{K}$ be the class of all associative algebras over a fixed field ϕ. Let $\mathfrak{N}$ be the class of all algebras in $\mathfrak{K}$ which satisfy a polynomial identity with coefficients in ϕ. It is clear that $\mathfrak{N}$ is homomorphically closed, contains all zero algebras over ϕ and is hereditary. Therefore by [5, Theorem 2, page 420], $L(\mathfrak{N}) = \{R$: every nonzero homomorphic image of R has a nonzero ideal in $\mathfrak{N}\}$.

Let M^* be the algebra of all infinite matrices of finite rank over ϕ. Then M^* is simple and does not satisfy any polynomial identity. Therefore M^* is $L(\mathfrak{N})$ semi-simple. However M^* has a nonzero right ideal $e_{11} \cdot M^*$, the set of all one-rowed matrices. This right ideal is in $\mathfrak{N}$ for it satisfies the identity

$$(x_1x_2 - x_2x_1)\, x_3 = 0.$$

Thus M^* is an $LS(\mathfrak{N})$ algebra.

EXAMPLE 2. Let $\mathfrak{N}$ be the class of all associative nilrings. Clearly this is homomorphically closed, contains all zero rings, and is hereditary. Also if R/I is nil and $I^2 = 0$ then R is nil. Finally if J is a nonzero right nilideal of R we will construct a nonzero left nilideal in R. If $J^2 = 0$ then $J + RJ$ is an ideal of R and $(J + RJ)^2 = 0$. If $J^2 \neq 0$, there exists an element t in J such that $Rt \neq 0$. For any element rt in Rt,

$$(rt)^{n+1} = r(tr)^n\, t.$$

Since tr is in J, $(tr)^n = 0$ for some n and thus rt is nilpotent, Rt is a nil left ideal of R.

Thus this $\mathfrak{N}$ is another example of a class satisfying the four conditions of Theorem 3.

We have already pointed out that the Baer Lower radical is strong. It is also well known that both the Jacobson and Levitzki radicals are strong. Thus it is a bit surprising that the Brown-McCoy radical is not.

EXAMPLE 3. [8, Page 55]. Let S be the set of all polynomials in x, $\alpha_0 + x\alpha_1 + \cdots + x^m\alpha_m$, where the coefficients α_i are rational functions in y with real coefficients. Addition is defined in the usual way, and multiplication is defined normally except that $\alpha x = x\alpha + \alpha'$ where α' is the derivative of α re y. Then S is a simple associative ring with unity. It is a principal ideal domain and has the ascending chain condition on both right and left ideals.

Every onesided ideal of S is also simple. To see this let I be a nonzero right ideal of S. Then $I = f(x) \cdot S$. Suppose T is a nonzero ideal of I. Then $ITI = f(x)\, STf(x)S \subseteq T$. However $STf(x)S$ being an ideal of S is either 0 or S. If it is S then $ITI = f(x)S = I \subseteq T$ and thus $T = I$. If $STf(x)S = 0$ then $Tf(x) = 0$ since S has a unity, and then $T = 0$ since there are no zerodivisors in S. Thus I is simple. A similar argument holds for left ideals of S.

Since the degree of $f(x) \cdot g(x) =$ degree of $f(x) +$ degree of $g(x)$, no proper onesided ideals of S have a unity element. In fact there is only one idempotent in S, namely 1.

Thus S is Brown-McCoy semi-simple but every proper onesided ideal of S is Brown-McCoy radical (they *are* Jacobson semi-simple). Therefore the Brown-McCoy radical does not contain all the onesided Brown-McCoy ideals.

Returning to Example 2, where $\mathfrak{N}$ = all nil rings, we know this is an ordinary radical property and by Theorem 3, $LS(\mathfrak{N}) = \mathfrak{N}_2$. Thus

$$\mathfrak{N} = L(\mathfrak{N}) \subseteq LS(\mathfrak{N}) = \mathfrak{N}_2 .$$

It is an open question whether $\mathfrak{N}$ is a strong radical, equal to $LS(\mathfrak{N})$, or not. This open question is equivalent to Koethe's famous conjecture about the existence of rings with nonzero onesided nilideals but without twosided nilideals. Restating it in terms of $\mathfrak{N}_2$ we can pose Koethe's conjecture as follows:

If every nonzero homomorphic image of a ring R contains a nonzero onesided nilideal, must R be a nil ring?

Another easier question is whether there exists a simple ring which is not nil and which has a nonzero nil onesided ideal. Amitsur [3, Theorem 10, page 47] has proved that for algebras over nondenumerable fields $\mathfrak{N} = LS(\mathfrak{N})$.

It is natural to try and extend Theorem 3 and one hope would be to show that for any nonempty homomorphically closed class $\mathfrak{N}$ of associative rings, $LS(\mathfrak{N}) = \mathfrak{N}_{\omega_0}$. The proof of this theorem ($L(\mathfrak{N}) = \mathfrak{N}_{\omega_0}$) in the two sided case is based on two lemmas [5, page 418]:

LEMMA 1. *If $R \neq 0$ is in $L(\mathfrak{N})$ then R has a nonzero accessible subring B, with B in $\mathfrak{N}$.*

LEMMA 2. *If B in $\mathfrak{N}$ is a nonzero accessible subring of R then the ideal in R generated by B is in $\mathfrak{N}_q$ for some finite q.*

There is no trouble with the first result, in fact almost the same proof holds.

LEMMA 3. *If $R \neq 0$ is in $LS(\mathfrak{N})$ then R has a nonzero subring B, B in $\mathfrak{N}$ and B is onesided accessible to R, i.e. $B = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n = R$ where A_i is a onesided ideal in A_{i+1} for every $i = 0, 1, \ldots, n-1$.*

However the second result is simply false.

EXAMPLE 4. Let R be the set of all 2×2 matrices over the reals. Let $A = \{\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)\}$, let $B = \{\left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right)\}$. Then B is an ideal of A, A is a right ideal of R and R is simple with unity.

Let $\mathfrak{N}$ be the class of all zero rings (associative). Then B is in $\mathfrak{N}$ and B is accessible to R. The right ideal of R generated by B is A itself. Now is A in $\mathfrak{N}_q$ with q finite? The ring A can be mapped onto the reals $\{\left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)\}$ and this has no nonzero onesided ideals except itself and therefore no nonzero onesided accessible subring in $\mathfrak{N}$. Thus A is not in $\mathfrak{N}_q$ for any q.

In the two sided situation we have $L(\mathfrak{N}) = \{R$: every nonzero homomorphic image of R contains a nonzero subring B in $\mathfrak{N}$ which is accessible to $R\}$. In the onesided case all we have is

$LS(\mathfrak{N}) \subsetneqq \{R$: every nonzero homomorphic image of R contains a nonzero subring B in $\mathfrak{N}$ which is onesided accessible to $R\} .\equiv. B(\mathfrak{N})$.

The inclusion follows from Lemma 3, and the strictness follows from Example 4, because the ring of 2×2 matrices belongs to the class $B(\mathfrak{N})$ but not to $LS(\mathfrak{N})$. This is so because $LS(\mathfrak{N}) \subseteq$ class of all Jacobson radical rings (which is a strong radical) and the ring of 2×2 matrices is a Jacobson semisimple ring.

The problem then is to get some useful characterization of $LS(\mathfrak{N})$ and the seemingly natural one $B(\mathfrak{N})$ turns out to be too large.

A second candidate is

$V(\mathfrak{N}) .\equiv.$ the class of all rings R such that every nonzero homomorphic image R' of R contains a nonzero subring B in $\mathfrak{N}$ such that $B = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = R'$ where each A_i is a onesided ideal of A_{i+1} for $i = 0, 1, \ldots, n - 1$ and for every i, every nonzero homomorphic image A'_i of A_i contains a nonzero subring in $\mathfrak{N}$ which is onesided accessible to A'_i.

Then clearly $V(\mathfrak{N}) \subsetneqq B(\mathfrak{N})$ because Example 4 is in $B(\mathfrak{N})$ but not in $V(\mathfrak{N})$. We also have $LS(\mathfrak{N}) \subseteq V(\mathfrak{N})$ because of

THEOREM 4. *$V(\mathfrak{N})$ is a strong radical class containing $\mathfrak{N}$.*

Proof. It is clear that $V(\mathfrak{N})$ contains $\mathfrak{N}$ and that it is homomorphically closed. If every nonzero homomorphic image R' of a ring R contains a nonzero onesided ideal I in $V(\mathfrak{N})$ then

$$R' \supseteq I = A_n \supseteq A_{n-1} \supseteq \cdots \supseteq A_1 \supseteq A_0 = B \neq 0 \text{ in } \mathfrak{N}$$

where A_i is a onesided ideal of A_{i+1} for $i = 0, \ldots, n - 1$, and every nonzero homomorphic image A'_i of A_i contains a nonzero subring in $\mathfrak{N}$ which is onesided accessible to A'_i for $i = 0, \ldots, n$. It is clear then that R itself is in $V(\mathfrak{N})$. By Theorem 1, $V(\mathfrak{N})$ is a strong radical.

It turns out that even $V(\mathfrak{N})$ is too large and

$$LS(\mathfrak{N}) \subsetneqq V(\mathfrak{N}) \subsetneqq B(\mathfrak{N}).$$

EXAMPLE 5. Let R be the ring of all infinite matrices over the reals, which are column finite and such that each matrix has only a finite number of nonzero rows. It is well known that R is simple and Jacobson semi-simple. Thus R is not in $LS(\mathfrak{N})$ when $\mathfrak{N}$ is the class of all zero rings (associative).

Let A be the subset of all matrices in R which have zeros in the first column and let Z be the subset of all matrices in A which have nonzero entries only in the top row. Now $Z^2 = 0$ and thus Z is in $\mathfrak{N}$. Also Z is an ideal of A and A is a left ideal of R. We will show that every nonzero homomorphic image of A contains a zero subring which is onesided accessible. This will prove that R is in $V(\mathfrak{N})$ and give us the strict inclusion we seek.

First we note that A/Z consists of classes represented by matrices in R with zeros in the first row and first column. Then it is clear that $A/Z \cong R$, for if M is any matrix in R then we associate the class in A/Z represented by

$$\begin{pmatrix} 0 & 000 \cdots 0 \cdots \\ 0 & \\ \vdots & M \end{pmatrix}$$

and this map is a homomorphism. Since R is simple, $A/Z \cong R$. Thus Z is a maximal ideal in A.

Let Q be any nonzero ideal in A. If $Q \not\subseteq Z$ then there is a matrix N in Q with a nonzero entry x_{ij} in the ijth position with $i > 1$, $j > 1$. Let C be the matrix (in A) with 1 in the first row, ith column and zeros elsewhere. Then

$$CN = \begin{pmatrix} 0 & x_{i2} & x_{i3} & \cdots & x_{ij}^{\neq 0} & \cdots \\ & & 0 & & & \end{pmatrix}.$$

Let D be the matrix (in A) with x_{ij}^{-1} in the jth row, kth column (any $k > 1$) and zeros elsewhere. Then CND has a 1 in the first row kth column, and zeros elsewhere. This is possible for any $k > 1$ and all the CND are in Q. Thus Q contains all of Z and if $Q \neq Z$ then $Q = A$. Thus if Q is a proper ideal of A, $Q \subseteq Z$.

Now let us consider any nonzero homomorphic image of A. It is either $A/Z \cong R$ which has a onesided accessible subring in $\mathfrak{N}$ or it is A/Q which contains Z/Q as an ideal and this ideal is in $\mathfrak{N}$. Therefore R is in $V(\mathfrak{N})$.

We must therefore find a characterization of $LS(\mathfrak{N})$ in a different way. To this end we make the following definitions:

A *path* of a ring R is a sequence $(R = R_0, R_1, R_2, \ldots, R_i, R_{i+1}, \ldots)$ where each R_{i+1} is either a nonzero proper homomorphic image of R_i, or is a nonzero proper onesided ideal of R_i, with the proviso that for any i, R_i and R_{i+1} are not both homomorphic images (R_i of R_{i-1} and R_{i+1} of R_i).

In general, paths are infinite sequences, but if R_i has no proper nonzero onesided ideals, then the path is finite.

Two paths are said to be *similar* if the first place where they differ is a place where the entries are different onesided ideals of the previous entry, i.e. if $P_1 = (R_0, R_1, ..., R_n, I, ...)$ and $P_2 = (R_0, R_1, ..., R_n, J, ...)$ where $I \neq J$, both onesided ideals of R_n.

A *clan* of R is a class of paths of R, no two of which are similar, and such that every path of R is similar to at least one path in the class.

If M is any nonempty class of rings, a clan of a ring R will be said *to meet* M if every path in the clan meets M after a finite number of steps, i.e. for every path $P = (R_0, R_1, ..., R_n, ...)$ there exists an integer t such that R_t is in M.

Now let $\mathfrak{N}$ be any nonempty homomorphically closed class of rings (associative) and define

$$Z(\mathfrak{N}) .\equiv. \{R\text{: there exists a clan of } R \text{ which meets } \mathfrak{N}\}.$$

THEOREM 5. $LS(\mathfrak{N}) = Z(\mathfrak{N})$.

Proof. It is clear that $\mathfrak{N} \subseteq Z(\mathfrak{N})$ for if R is in $\mathfrak{N}$ then every path of R begins with $R = R_0$ in $\mathfrak{N}$ and thus meets $\mathfrak{N}$ immediately. Thus any clan of R will do.

To see that $Z(\mathfrak{N})$ is homomorphically closed, take R in $Z(\mathfrak{N})$ and let R' be any nonzero homomorphic image of R. Take a clan W for R that meets $\mathfrak{N}$ and take all paths in W that begin with $(R, R', ...)$. If we remove the first R from these paths we obtain a class C of paths for R'. Clearly no two paths in C are similar. Furthermore if P is any path of R', $P = (R', S, ...)$ where S is a onesided ideal of R', then the extended path $(R, R', S, ...)$ is a path for R and thus must be similar to a path in W. Since similar paths cannot differ at a homomorphic step, the path in W must begin with $(R, R', ...)$. Dropping the first R we thus get a path for R' which is in C and which is similar to $P = (R', S, ...)$. If on the other hand $P = (R', S, ...)$ where S is a homomorphic image of R', then $(R, S, ...)$ is a path for R. It must be similar to a path in W and since S is a homomorphic image of R, the path in W must begin with $(R, S, ...)$. Replacing R by R' we get a path $(R', S, ...)$ of R' which is similar to P. If we adjoin, to C, all such paths of R' obtained by replacing R' by R, picking a similar path in W and replacing R by R', we then obtain a clan W' for R'. This clan meets $\mathfrak{N}$ for W meets $\mathfrak{N}$, and paths in W' are the same as paths in W from the second entry on. If a path in W meets $\mathfrak{N}$ in the first entry, then R is in $\mathfrak{N}$ and then R' is in $\mathfrak{N}$.

To see that $Z(\mathfrak{N})$ is a strong radical property, suppose that every nonzero homomorphic image T' of a ring T, has a nonzero onesided ideal I' in $Z(\mathfrak{N})$. We will show that T is in $Z(\mathfrak{N})$ by constructing a clan for T that meets $Z(\mathfrak{N})$.

For every $T' \neq 0$, with onesided ideal $I' \neq 0$ in $Z(\mathfrak{N})$, take a clan B for I' that meets $\mathfrak{N}$. Extend each path in B of the form $(I', I_1, I_2, ...)$ to $(T, T', I', I_1, I_2, ...)$ when $T \neq T'$ and to $(T, I', I_1, I_2, ...)$ when $T = T'$. Let V be the class of all such extended paths. Clearly no two are similar. Let

$P = (T, T_1, T_2, \ldots)$ be any path for T. If T_1 is a onesided ideal of T then P is similar to some one of the paths $(T, I', I_1, \ldots)$. If however, T_1 is a homomorphic image of T then T_2 must be a onesided ideal of T_1 and P is similar to some one of the $(T, T', I', \ldots)$ where $T' = T_1$. Therefore V is a clan for T. Also V meets $\mathfrak{N}$. Therefore T is in $Z(\mathfrak{N})$ and by Theorem 1, $Z(\mathfrak{N})$ is a strong radical. Therefore $LS(\mathfrak{N}) \subseteq Z(\mathfrak{N})$.

To see that $Z(\mathfrak{N}) \subseteq LS(\mathfrak{N})$ take R not in $LS(\mathfrak{N})$. Then there exists a nonzero homomorphic image R' of R, which is $LS(\mathfrak{N})$ semi-simple. Thus R' has no nonzero onesided ideals in $LS(\mathfrak{N})$. Take any one, say I_1. Then I_1 is not in $LS(\mathfrak{N})$ and it has a nonzero homomorphic image I_1' which is $LS(\mathfrak{N})$ semi-simple. Continuing in this way we obtain a path $(R, R', I_1, I_1', I_2, I_2', \ldots)$ where the I_n are nonzero onesided ideals of the previous entry and the I_n' are nonzero homomorphic images of the corresponding I_n's and these I_n' are all $LS(\mathfrak{N})$ semi-simple (they are of course obtained by factoring out the $LS(\mathfrak{N})$ radical of I_n). By choosing all possible I_1's and for each I_1, all possible I_2's etc. we obtain a class of such paths and it is clear that none of them meet $\mathfrak{N}$. We propose to show that for any clan C of R, C must contain at least one of these paths that does not meet $\mathfrak{N}$. Then C does not meet $\mathfrak{N}$ and R is not in $Z(\mathfrak{N})$.

Assume then that we are given a clan C of R. Every path of R is similar to some path in C. Thus there must exist paths in C that begin with R, R'. Since no two distinct paths in C are similar, all paths in C that begin with R, R' must continue with the same one-sided ideal J_1 of R'. Thus they all start with R, R', J_1. We take J_1' to be J_1 modulo its $LS(\mathfrak{N})$ radical. Then there must exist paths in C which are similar to $R, R', J_1, J_1', \ldots$. However, any path in C that is so similar must in fact begin with the same four first evtries because similar paths can differ at first only at a one-sided ideal step, that is from R' to J_1. But all paths in C that begin R, R' continue on to J_1.

We continue this construction, at each stage choosing the one-sided ideal J_{n+1} of J_n' which appears in every path of C that begins with $R, R', J_1, J_1', \ldots, J_n, J_n'$; and then observing that for J_{n+1}' equal to J_{n+1} modulo its $LS(\mathfrak{N})$ radical, there must exist a path in C which is similar to $R, R', J_1, J_1', \ldots, J_{n+1}, J_{n+1}', \ldots$ and identical with it up to that point. In this way we obtain a path in C which does not meet $\mathfrak{N}$.

Therefore R cannot have a clan that meets $\mathfrak{N}$, R is not in $Z(\mathfrak{N})$ and therefore $LS(\mathfrak{N}) = Z(\mathfrak{N})$.

Remarks. It is clear that Example 5 does not have a clan that meets the class of zerorings.

This clan approach should also give us a representation of $L(\mathfrak{N})$, using of course, two sided ideals.

We return now to our analysis of the construction of $LS(\mathfrak{N}) = \bigcup_\alpha \mathfrak{N}_\alpha$, where $\mathfrak{N}$ is a nonempty homomorphically closed class of associative rings.

LEMMA 4. *If R has a nonzero onesided ideal in $\mathfrak{N}_\alpha$ then R has a nonzero ideal in $\mathfrak{N}_{\alpha+1}$.*

Proof. The same as Lemma 2, together with the fact that $\mathfrak{N}_\alpha$ is homomorphically closed (Proposition 1).

COROLLARY 1. *If R is in $\mathfrak{N}_{\omega_0}$ then every nonzero homomorphic image of R has a nonzero ideal in $\mathfrak{N}_n$, n finite.*

COROLLARY 2. *If R has a nonzero left (right) ideal in $\mathfrak{N}_n$, n finite, then it has a nonzero right (left) ideal in $\mathfrak{N}_{n+1}$.*

THEOREM 6. *If $\mathfrak{N}$ is a nonempty homomorphically closed subclass of the class of all associative rings, and if $\mathfrak{N}$ is hereditary and contains all zero rings then $LS(\mathfrak{N}) = \mathfrak{N}_{\omega_0}$.*

Proof. Let $M = \bigcup_{n \text{ finite}} \mathfrak{N}_n$. We will show that M satisfies the four conditions of Theorem 3. Then $LS(\mathfrak{N}) = LS(M) = M_2 = \mathfrak{N}_{\omega_0}$. It is clear that M contains all zero rings. If R has a onesided ideal I in M then I is in $\mathfrak{N}_n$ for some n. By Corollary 2, Lemma 4, R has a onesided ideal of the other side in $\mathfrak{N}_{n+1} \subseteq M$. This is condition 3. Next, if R/I is in M and $I^2 = 0$, then we show that R is in M. Since R/I is in M, it is in $\mathfrak{N}_n$ for some finite n. The ideal I of course is in $\mathfrak{N}_1$. Then R is in $\mathfrak{N}_{n+1}$. To prove this let R/K be any nonzero homomorphic image of R. If $I \nsubseteq K$ then R/K contains the nonzero ideal $(I + K)/K \cong I/(I \cap K)$ in $\mathfrak{N}_1 \subseteq \mathfrak{N}_n$. If however $I \subseteq K$ then $R/I/K/I \cong R/K$. Now R/I is in $\mathfrak{N}_n$ and this is homomorphically closed (Proposition 1) and therefore R/K is itself in $\mathfrak{N}_n$. Thus every nonzero homomorphic image of R contains a nonzero ideal in $\mathfrak{N}_n$ and therefore R is in $\mathfrak{N}_{n+1} \subseteq M$. This is condition 4.

Finally, to show that M is hereditary, we proceed by induction. Assume that for every $m < n$, if R is in $\mathfrak{N}_m$ and I is an ideal of R, then I is in $\mathfrak{N}_m$. Take R in $\mathfrak{N}_n$, I an ideal of R, and let I/J be any nonzero homomorphic image of I. Let $\bar{J}$ be the ideal of R generated by J. If $\bar{J} \supsetneq J$ then [8, page 107, Lemma 61], $(\bar{J}/J)^3 = 0$. If $(\bar{J}^2 + J)/J \neq 0$ it is a zero ring and thus in $\mathfrak{N}_1$. If $\bar{J}^2 \subseteq J$ then $\bar{J}/J$ is a zero ring. In either case then, I/J has a nonzero onesided ideal in $\mathfrak{N}_1$.

If however $\bar{J} = J$ then we consider R/J. It is in $\mathfrak{N}_n$ and thus has a nonzero onesided ideal Q/J in $\mathfrak{N}_m$ for some $m < n$. If $(I \cap Q)/J \neq 0$, then it is in $\mathfrak{N}_m$ since it is an ideal of Q/J. It is also a onesided ideal of I/J.

If however $I \cap Q = J$ then we consider $\bar{Q}$, the ideal of R generated by Q.

If $\bar{Q} = R$, suppose that Q is a left ideal of R. Then $\bar{Q} = Q + QR = R$. Also $IQ \subseteq I \cap Q = J$. Then $IR = IQ + IQR \subseteq J + JR \subseteq J$. Thus $II \subseteq J$ and $(I/J)^2 = 0$. Thus I/J is itself in $\mathfrak{N}_1$. Similarly if Q is a right ideal of R we get I/J in $\mathfrak{N}_1$.

If however $\bar{Q} \subsetneqq R$, we consider $\bar{Q} \cap I$. This contains J. If $\bar{Q} \cap I \supsetneqq J$ then $(\bar{Q} \cap I)/J$ is a nonzero ideal of I/J. Suppose Q is a left ideal of R. Then $IQ \subseteq I \cap Q \subseteq J$. Also $\bar{Q} = Q + QR$. Thus $I\bar{Q} = IQ + IQR \subseteq J + JR \subseteq J$. Therefore $(\bar{Q} \cap I) \cdot (\bar{Q} \cap I) \subseteq I\bar{Q} \subseteq J$. Thus $(\bar{Q} \cap I)/J$ is a zero ring and is in $\mathfrak{N}_1$. Similarly if Q is a right ideal of R we get $(\bar{Q} \cap I)/J$ in $\mathfrak{N}_1$.

If however $\bar{Q} \cap I = J$, we consider all ideals of R which contain Q, and which intersect I in J. This set is not vacuous for it contains $\bar{Q}$. By Zorn's lemma, choose K to be maximal in this set. Then K is an ideal of R, it contains both Q and $\bar{Q}$, and $K \cap I = J$. And K is maximal of this type. We consider R/K. It is nonzero and it is in $\mathfrak{N}_n$. Then it has a nonzero onesided ideal P/K in $\mathfrak{N}_m$ with $m < n$. If $(P \cap I)/J \neq 0$ then

$$\frac{P \cap I}{J} = \frac{P \cap I}{P \cap I \cap K} \cong \frac{(P \cap I) + K}{K}$$

and this is in $\mathfrak{N}_m$ since it is an ideal of P/K. Also it is a onesided ideal of I/J.

If however $P \cap I = J$, we consider $\bar{P}$. As for $\bar{Q}$, if $\bar{P} = R$, then I/J is a zero ring and is in $\mathfrak{N}_1$. If however $\bar{P} \subsetneqq R$, then at last we can be certain that $\bar{P} \cap I \supsetneqq J$ because K is maximal of this type. Then $(\bar{P} \cap I)/J$ is a nonzero ideal of I/J. It is a zero ring because $(\bar{P} \cap I) \cdot (\bar{P} \cap I) \subseteq I\bar{P} \subseteq J$. Thus it is in $\mathfrak{N}_1$.

Therefore in every case, I/J has a nonzero onesided ideal which is either in $\mathfrak{N}_1$ or in $\mathfrak{N}_m$. Therefore I is $\mathfrak{N}_n$. This completes the induction and proves the theorem.

Remark. We have actually proved that if a class $\mathfrak{N}_1$ is hereditary and contains all zero rings, then $\mathfrak{N}_m$ is hereditary for every positive integer m.

If $\mathfrak{N}$ is not hereditary or if $\mathfrak{N}$ does not contain all zero rings, or if $\mathfrak{N}$ lacks both of these conditions, then it is unclear when the lower strong radical construction stops. For this general case we can prove:

THEOREM 7. *$\mathfrak{N}_{\omega_0}$ is a radical class. ($\mathfrak{N}$ is an arbitrary nonempty homomorphically closed class of associative rings.)*

Remarks. The subscript here denotes the strong radical construction. In general it seems that $\mathfrak{N}_{\omega_0}$ is not all of $LS(\mathfrak{N})$. What we can prove is that it is at least an ordinary radical property. Then of course we have

$$\mathfrak{N} \subseteq L(\mathfrak{N}) \subseteq \mathfrak{N}_{\omega_0} \subseteq LS(\mathfrak{N}).$$

Proof. $\mathfrak{N}_{\omega_0}$ is homomorphically closed. Furthermore, if every nonzero homomorphic image R of R' contains a nonzero ideal I in $\mathfrak{N}_{\omega_0}$, then we shall prove that R' is in $\mathfrak{N}_{\omega_0}$. Since I is in $\mathfrak{N}_{\omega_0}$ it must have a nonzero onesided ideal in $\mathfrak{N}_m$ and therefore a nonzero two sided ideal J in $\mathfrak{N}_{m+1}$ (Corollary 2, Lemma 4). Consider $J + JR$. It is in $\mathfrak{N}_{m+2}$ since every nonzero homomorphic image $(J + JR)/K$ has a nonzero one-sided ideal in $\mathfrak{N}_{m+1}$. To see this one can use the argument of Lemma 2 (simplified to the associative case) or see [5]. Thus every nonzero homomorphic image of R' has a nonzero right ideal in $\mathfrak{N}_{m+2}$, and therefore R' is in $\mathfrak{N}_{\omega_0}$.

Remark. A similar argument proves that $\mathfrak{N}_\alpha$ is a radical class, for every limit ordinal α.

2. Upper Radical Properties

Let $\mathfrak{K}$ be a nonempty universal class of not necessarily associative rings. Let $\mathfrak{M}$ be a nonempty subclass of $\mathfrak{K}$.
We shall say that

$\mathfrak{M}$ is *regular* if every nonzero ideal of a ring in $\mathfrak{M}$ can be homomorphically mapped onto a nonzero ring in $\mathfrak{M}$.

$\mathfrak{M}$ is *onesided regular* if every nonzero onesided ideal of a ring in $\mathfrak{M}$ can be homomorphically mapped onto a nonzero ring in $\mathfrak{M}$.

$\bigcup_{\mathfrak{M}}$ is defined as the class of all rings in $\mathfrak{K}$ which cannot be homomorphically mapped onto a nonzero ring in $\mathfrak{M}$.

Then the following theorem is well known [8];

THEOREM 8. *If $\mathfrak{M}$ is regular then $\bigcup_{\mathfrak{M}}$ is the upper radical property determined by $\mathfrak{M}$.*

If $\mathfrak{K}$ contains only alternative rings we have

THEOREM 9. *If $\mathfrak{M}$ is onesided regular then $\bigcup_{\mathfrak{M}}$ is the upper strong radical property determined by $\mathfrak{M}$, i.e. it is the largest strong radical property for which all rings in $\mathfrak{M}$ are semi-simple.*

Proof. To prove $\bigcup_{\mathfrak{M}}$ is strong we shall use Theorem 1. It is clear the $\bigcup_{\mathfrak{M}}$ is homomorphically closed. To establish the other condition of Theorem 1, take R not in $\bigcup_{\mathfrak{M}}$. Then R can be mapped onto a nonzero ring R' in $\mathfrak{M}$. Then R' has no nonzero onesided ideals in $\bigcup_{\mathfrak{M}}$ since $\mathfrak{M}$ is onesided regular. This proves that if every nonzero homomorphic image of R has a nonzero onesided ideal in $\bigcup_{\mathfrak{M}}$ then R is itself in $\bigcup_{\mathfrak{M}}$. Thus by Theorem 1, $\bigcup_{\mathfrak{M}}$ is a strong radical property.

Since $\mathfrak{M}$ is also regular, by the previous theorem we have that $\bigcup_{\mathfrak{M}}$ is the upper radical re $\mathfrak{M}$ and therefore it is the upper strong radical re $\mathfrak{M}$.

Continuing with an alternative universal class $\mathfrak{K}$, we define:

A radical property $\mathfrak{S}$ to be *semi-simple hereditary* or s.s. hereditary, if every ideal of an $\mathfrak{S}$ semi-simple ring is also $\mathfrak{S}$ semi-simple. It is known [4, Corollary 2, page 602] that for alternative rings every radical property is s.s. hereditary.

A radical property $\mathfrak{S}$ is *onesided s.s. hereditary* if every onesided ideal of an $\mathfrak{S}$ semi-simple ring is also $\mathfrak{S}$ semi-simple.

THEOREM 10. *If a radical property $\mathfrak{S}$ is onesided s.s. hereditary then it is a strong radical.*

Proof. Let I be a nonzero onesided$\mathfrak{S}$-ideal of a ring R. Let $S(R)$ be the $\mathfrak{S}$ radical of R. Then consider $(I + S(R)/S(R) \cong I/(I \cap S(R))$.

If I is not in $S(R)$ then $I/I \cap S(R)$ is nonzero and is in $\mathfrak{S}$. Then the $\mathfrak{S}$ semi-simple ring $R/S(R)$ has a nonzero onesided $\mathfrak{S}$-ideal. This contradicts the onesided s.s. hereditary assumption. Thus $I \subseteq S(R)$ and $\mathfrak{S}$ is a strong radical.

Remark. The converse of this theorem is false because the Baer Lower radical is a strong radical, yet Example 4 shows it is not onesided s.s. hereditary.

Not every $\bigcup_{\mathfrak{M}}$, with $\mathfrak{M}$ regular, is a strong radical.

EXAMPLE 6. Let $\mathfrak{K}$ be the class of all finite dimensional algebras over an algebraically closed field ϕ. Then the only nontrivial simple algebras in $\mathfrak{K}$ are the matrix algebras ϕ_n, for $n = 1, 2, 3,\ldots$. Let $\mathfrak{M}$ be a nonempty set of some (or all) of these ϕ_n's. Then $\mathfrak{M}$ is regular and $\bigcup_{\mathfrak{M}}$ is a radical property. However not all such $\bigcup_{\mathfrak{M}}$'s are strong radicals. To see this let $\mathfrak{M} = \{\phi_2\}$. Now ϕ_2 is $\bigcup_{\mathfrak{M}}$ semi-simple, and it contains the onesided ideal $I = \{(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix})\}$. But I cannot be mapped onto ϕ_2 and thus I is a a nonzero onesided $\bigcup_{\mathfrak{M}}$ ideal of ϕ_2. Thus $\bigcup_{\mathfrak{M}}$ is not strong.

What we can say in connection with this example is:

THEOREM 11. *$\bigcup_{\mathfrak{M}}$ is a strong radical if and only if $\phi_n \in \mathfrak{M} \rightarrow \phi_m \in \mathfrak{M}$ for every m such that $1 \leqslant m \leqslant n$.*

Proof. First we observe that any $\bigcup_{\mathfrak{M}}$ semi-simple algebra in $\mathfrak{K}$ is a finite direct sum of simple algebras from $\mathfrak{M}$[8, Theorem 46, page 121].

Suppose now that $\phi_n \in \mathfrak{M} \rightarrow \phi_m \in \mathfrak{M}$ for all m such that $1 \leqslant m \leqslant n$. Let R be $\bigcup_{\mathfrak{M}}$ semi-simple and I a nonzero onesided ideal of R with I in $\bigcup_{\mathfrak{M}}$. We know that $R = \sum_1^k \oplus R_i$, where the $R_i \in \mathfrak{M}$, $i = 1,..., k$. Let $h_i : R \to R_i$ be the natural homomorphism from R onto R_i. Then there must exist a j such that $h_j : I \to I' \neq 0$, else $I \subseteq \bigcap_i$ kernels $h_i = 0$. Then I' is a nonzero onesided ideal of $R_j = \phi_n$. Let W be the classical nilpotent radical in $\mathfrak{K}$. It is known that W is a strong radical. Now ϕ_n is W semi-simple and therefore I' cannot be a W-ring. Then $I'/W(I')$ is nonzero and is a finite direct sum of matrix algebras each of degree $\leqslant n$. Thus I' and therefore I, can be mapped onto some ϕ_m with $1 \leqslant m \leqslant n$. By our condition this ϕ_m is in $\mathfrak{M}$ and therefore I can be mapped onto a ring in $\mathfrak{M}$. However this contradicts the assumption that I is in $\bigcup_{\mathfrak{M}}$. Therefore $\bigcup_{\mathfrak{M}}$ is a strong radical.

Conversely suppose that $\bigcup_{\mathfrak{M}}$ is a strong radical property. Take ϕ_n in $\mathfrak{M}$ and let m be any integer such that $1 \leqslant m \leqslant n$. Define

I_m .≡. all matrices $A = (\alpha_{ij})$ in ϕ_n such that $\alpha_{ik} = 0$ for all i and all k such that $m < k \leqslant n$.

Then I_m is a left ideal of ϕ_n. Define

P_m .≡. all matrices $A = (\alpha_{ij})$ in I_m such that $\alpha_{ij} = 0$ for any $i < m$.

Then P_m is a two sided ideal of I_m. Pictorially

$$I_m = \left\{ \underset{m}{\left(\begin{array}{c|c} * & 0 \\ \hline * & 0 \end{array} \right)} m \right\}; \qquad P_m = \left\{ \underset{m}{\left(\begin{array}{c|c} 0 & 0 \\ \hline * & 0 \end{array} \right)} m \right\}.$$

Define a mapping h from I_m to ϕ_m which sends

$$A = \left(\begin{array}{c|c} * & 0 \\ \hline * & 0 \end{array} \right) \text{ in } I_m \text{ to } (*),$$

the top left hand corner of A. Thus h: $A = (\alpha_{ij})$ in $I_m \to (\alpha_{ij})$ in ϕ_m, where i and $j = 1, 2,..., m$.

This map h is onto ϕ_m and it clearly preserves addition. To see that it also preserves multiplication, take $A = (\alpha_{ij})$ and $B = (\beta_{ij})$ in I_m. Then $AB = (\gamma_{ij})$ where $\gamma_{ij} = \sum_{k=1}^m \alpha_{ik}\beta_{kj}$, since $\alpha_{ik} = 0$ for $k > m$. Therefore $h(AB) = h(A) \cdot h(B)$.

The kernel of h is precisely P_m. Thus $I_m/P_m \cong \phi_m$. Now $P_m{}^2 = 0$ and thus P_m is in $\bigcup_{\mathfrak{M}}$. If ϕ_m is not an $\mathfrak{M}$ ring then since it is simple, it must be in $\bigcup_{\mathfrak{M}}$. Since $\bigcup_{\mathfrak{M}}$ is a radical property, P_m and I_m/P_m in $\bigcup_{\mathfrak{M}}$ implies I_m in $\bigcup_{\mathfrak{M}}$. This is impossible since the $\bigcup_{\mathfrak{M}}$ semi-simple ring ϕ_n cannot contain a nonzero left $\bigcup_{\mathfrak{M}}$ ideal. Therefore ϕ_m is in $\mathfrak{M}$, and the theorem is finished.

REFERENCES

1. S. AMITSOR, A general theory of radicals II, *Amer. J. Math.* **76** (1954), 100–125.
2. S. AMITSUR, A general theory of radicals III, *Amer. J. Math.* **76** (1954), 126–136.
3. S. AMITSUR, Algebras over infinite fields, *Proc. Amer. Math. Soc.* **7** (1956), 35–48.
4. ANDERSON, DIVINSKY, AND SULINSKI, Hereditary radicals in associative and alternative rings, *Can. J. Math.* **17** (1965), 594–603.
5. SULINSKI, ANDERSON, AND DIVINSKY, Lower radical properties for associative and alternative rings, *J. Lond. Math. Soc.* **41** (1966), 417–424.
6. ARMENDARIZ AND LEAVITT, The hereditary property in the lower radical construction, *Can. J. Math.* **20** (1968), 474–476.
7. S. E. DICKSON, A note on hypernilpotent radical properties for associative rings, *Can. J. Math.* **19** (1967), 447–448.
8. N. DIVINSKY, "Rings and Radicals," University of Toronto Press, 1965.
9. A. HEINICKE, A note on lower radical constructions for associative rings, *Can. Math. Bull.* **11** (1968), 23–30.
10. N. JACOBSON, "Structure of Rings," A.M.S. Colloquium Publications, Providence, R. I., 1964.
11. Y. M. RUBIKIN, Radical theory for nonassociative rings, *Math. Isledovani* **3** (1968), 86–99.
12. J. F. WATTERS, Lower radicals in associative rings, *Can. J. Math.* **21** (1969), 466–476.