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New approach to the incompressible Maxwell–Boussinesq approximation: Existence, uniqueness and shape sensitivity

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The Boussinesq approximation to the Fourier–Navier–Stokes (F–N–S) flows under the electromagnetic field is considered. Such a model is the so-called *Maxwell–Boussinesq approximation*. We propose a new approach to the problem. We prove the existence and uniqueness of weak solutions to the variational formulation of the model. Some further regularity in $W^{1,2+\delta}$, $\delta > 0$, is obtained for the weak solutions. The shape sensitivity analysis by the boundary variations technique is performed for the weak solutions. As a result, the existence of the strong material derivatives for the weak solutions of the problem is shown. The result can be used to establish the shape differentiability for a broad class of shape functionals for the models of Fourier–Navier–Stokes flows under the electromagnetic field.

1. Introduction

The magnetohydrodynamic flows have been studied in [1,18,19] for differential operators with constant coefficients. We would like also to mention the work of Duvaut and Lions [8], Sermange and

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Temam [25], Gerbeau and Le Bris [14,15] on the existence of solutions. However, the temperature dependent parameters seem to be more realistic in diverse situations. In what follows we deal with the existence and uniqueness of weak solutions to the thermoelectromagnetic flow (Fourier–Maxwell–Navier–Stokes) model with viscosity, as well as with electric and thermal conductivities and magnetic permeability dependent not only on the temperature but also on the space variables. The present model is provided for all magnetic fields $\mathbf{H} \in \mathbf{H}^1(\Omega)$ such that $\nabla \cdot \mathbf{H} = 0$ in Ω and $\mathbf{H} \cdot \mathbf{n} = 0$ on $\partial\Omega$. A different approach of the complete F–N–S system coupled with Maxwell equations can be found in [4], where the shape sensitivity is analyzed under the assumption that the velocity field of the perturbation of the domain is divergence free.

The shape derivatives of shape functionals governed by Navier–Stokes flow are investigated by Boisgérault and Zolésio [2] in the framework of the speed method. The expression for the Navier–Stokes equation transported to the fixed domain is given provided that the domain deformation for the purposes of shape sensitivity analysis is induced by a smooth vector field. Under regularity assumptions on the initial data and the forcing terms, and under a uniqueness assumption for the solution of a linearized version of the Navier–Stokes equation, the implicit function theorem is used to prove a regularity result for the solutions the Navier–Stokes equation transported to the fixed domain. The latter result is extended to the case of nonhomogeneous boundary conditions and shape-dependent forces in [3]. The shape sensitivity analysis of the Navier–Stokes system is investigated by Dziri, Moubachir and Zolésio [11] with respect to the dynamical deformations of the fluid domain boundary. The shape differentiability for the heat-conduction equation is considered in a work of Dziri and Zolésio [9]. The existence of shapes for the Navier–Stokes flow with heat convection is studied by Dziri and Zolésio [10]. The shape differentiability of solutions for compressible perturbations of Stokes problem is performed in [24] in the framework of local theory. The shape gradients are recovered by means of singular limits of the expressions obtained for the material derivatives, cf. [23].

In the present paper, the coefficients are assumed to be bounded and continuous, and there are no restrictions imposed on the derivatives of coefficients. The presented here model can also include the parameter dependent magnetic field if some boundedness and continuity properties of the model coefficients are still available. However, for the simplicity of the presentation we avoid such dependence.

The uniqueness of solutions is established under the smallness assumption on the data, related to the coefficients characterizing the viscous stress (the kinematic viscosity), the induced electric current (the electric conductivity) and the heat flux (the thermal conductivity). In the present setting some assumptions in the form of Lipschitz-type conditions on the parameters are also required. The shape analysis concerns the system with nonconstant coefficients as functions only on the temperature then the final elliptic system characterizing the strong material derivatives is provided if the additional C^1-assumption on the coefficients is taken into account.

The outline of this work is as follows. In Section 2 the thermoelectromagnetoflow is formulated as a Maxwell–Boussinesq approximation. In Section 3 we state the functional framework and the main existence, regularity and uniqueness results. Section 4 is devoted to some auxiliary existence results that will be needed in the proofs to the main theorems (Sections 5 and 6). In Section 7 we state some useful results in order to analyze the shape sensitivity. We deal with the shape sensitivity of the problem in Section 8. Applying the speed method introduced in the monograph by Sokolowski and Zolésio the problem of nonconservation of solenoidality appears. To overcome this difficulty we can use the test functions, which are not divergence free and to handle the estimate for pressure we apply the Bogovskii operator. We have to show that Bogovskii operator is independent of the shape perturbations parameter τ which governs small variations of the boundary $\partial\Omega$. Another possibility is to use the domain transformations built by the divergence free vector fields V of the speed method, or to apply the Piola transform to recover the solenoidal vector fields.

2. Thermoelectromagnetoflow model

Let Ω be an open bounded subset of $\mathbb{R}^3$ with the boundary $\partial\Omega$ of class $C^{1,1}$ which is split into two parts $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$, where Γ_D is an open nonempty subset of $\partial\Omega$ and $\Gamma_N = \partial\Omega \setminus \overline{\Gamma}_D$. The electromagnetic field is described by the Maxwell equations:

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0; \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0; \tag{2}$$

$$\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} = \nabla \times \mathbf{H}; \tag{3}$$

$$\nabla \cdot \mathbf{D} = \rho, \tag{4}$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic intensity fields, $\mathbf{D}$ and $\mathbf{B}$ are the electric and magnetic induction fields and $\mathbf{J}$ is the current density. The constitutive laws are

$$\mathbf{D} = \varepsilon \mathbf{E}, \qquad \mathbf{B} = \mu \mathbf{H},$$

where ε is the permittivity and μ is the magnetic permeability. The Ohm law reads

$$\mathbf{J} = \mathbf{J}_0 + \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}),$$

where $\mathbf{J}_0$ denotes a given applied current, σ is the electric conductivity and $\mathbf{u}$ is the fluid velocity vector.

It is known that:

- by (2) there is $\mathbf{A}$ such that $\mathbf{B} = \nabla \times \mathbf{A}$;
- introducing $\mathbf{A}$ in (1), we obtain $\nabla \times (\frac{\partial \mathbf{A}}{\partial t} + \mathbf{E}) = 0$;
- hence, there exists the potential ϕ, such that $\frac{\partial \mathbf{A}}{\partial t} + \mathbf{E} = -\nabla \phi$.

For the steady-state case, the latter equation is reduced to the well-known relation $\mathbf{E} = -\nabla \phi$.

The following coupled system of equations derived from the motion and energy equations models the steady-state motion of F–N–S fluids in Ω:

$$-\nabla \cdot \big(\nu(T) D\mathbf{u}\big) + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mu(T)\,\mathrm{rot}\,\mathbf{H} \times \mathbf{H} + \mathbf{f} - \mathbf{G}(T)T; \tag{5}$$

$$\mathrm{div}\,\mathbf{u} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = 0; \tag{6}$$

$$-\nabla \cdot \big(k(T)\nabla T\big) + \mathbf{u} \cdot \nabla T = f; \tag{7}$$

where T is the temperature, $D\mathbf{u} = (D_{ij}) = (\partial_i u_j + \partial_j u_i)/2$ $(i,\ j = 1, 2, 3)$ is the symmetrized gradient of the velocity, ν is the viscosity, p denotes the pressure, k is the thermal conductivity, $\mathbf{f}$ and f denote the external forces and heat sources, respectively. The buoyancy force as in the Boussinesq approximation is described by $\mathbf{G}(T) = \beta(T)(0, 0, g)^\top$, where β denotes the coefficient of thermal dilatation and g is the constant of gravity. The mass density is assumed to be constant, we set $\rho = 1$. The existence of two body forces in the fluid, the Lorentz force $\mathbf{J} \times \mathbf{B} = (\nabla \times \mathbf{H}) \times (\mu \mathbf{H})$ and the buoyancy force, results from the presence of the magnetic field.

The thermoelectromagnetoflow problem under study has the following boundary conditions

$$\mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega, \tag{8}$$

$$T = 0 \quad \text{on } \Gamma_D, \qquad k(T)\frac{\partial T}{\partial \mathbf{n}} + \alpha T = h \quad \text{on } \Gamma_N, \tag{9}$$

$$\mathbf{H} \cdot \mathbf{n} = 0, \qquad \mathbf{E} \times \mathbf{n} = \mathbf{0} \quad \text{on } \partial\Omega, \tag{10}$$

where α represents the convective heat transfer coefficient. To simplify the presentation it is assumed that $\mathbf{g}=\mathbf{0}$. Indeed, under a convenient choice of a lift the general case is recovered.

For the steady-state, taking the divergence of Eq. (3) leads to

$$\nabla\cdot\big(\sigma(T)\mathbf{E}\big)=-\nabla\cdot\big(\mathbf{J}_0+\sigma(T)\mu(T)\mathbf{u}\times\mathbf{H}\big).$$

Let us consider the electric field $\mathbf{E}=-\nabla\phi$ and the mixed boundary value problem for the potential ϕ of the form

$$-\nabla\cdot\big(\sigma(T)\nabla\phi\big)=-\nabla\cdot\big(\mathbf{J}_0+\sigma(T)\mu(T)\mathbf{u}\times\mathbf{H}\big)\quad\text{in }\Omega,\tag{11}$$

$$\phi=0\quad\text{on }\Gamma_D\quad\text{and}\quad\frac{\partial\phi}{\partial\mathbf{n}}=0\quad\text{on }\Gamma_N.\tag{12}$$

3. Assumptions and main existence results

We need some assumptions on the model, which are listed below.

Let us assume that

(H1) $\nu,\mu,\sigma,k:\Omega\times\mathbb{R}\to\mathbb{R}$ are Caratheodory functions such that

$$\exists\nu^\#,\nu_\#>0:\quad \nu_\#\leqslant\nu(\cdot,\xi)\leqslant\nu^\#,\quad\text{a.e. in }\Omega,\ \forall\xi\in\mathbb{R};\tag{13}$$

$$\exists\mu^\#,\mu_\#>0:\quad \mu_\#\leqslant\mu(\cdot,\xi)\leqslant\mu^\#,\quad\text{a.e. in }\Omega,\ \forall\xi\in\mathbb{R};\tag{14}$$

$$\exists\sigma^\#,\sigma_\#>0:\quad \sigma_\#\leqslant\sigma(\cdot,\xi)\leqslant\sigma^\#,\quad\text{a.e. in }\Omega,\ \forall\xi\in\mathbb{R};\tag{15}$$

$$\exists k^\#,k_\#>0:\quad k_\#\leqslant k(\cdot,\xi)\leqslant k^\#,\quad\text{a.e. in }\Omega,\ \forall\xi\in\mathbb{R};\tag{16}$$

(H2) $\mathbf{G}=(0,0,\beta_T g)$ where β_T is a real, continuous, and bounded function and g is a constant, we denote $G^\#=g\beta^\#$, where $\beta^\#$ denotes the upper bound for the function β_T;

(H3) $\alpha\in L^2_+(\Gamma_N)=\{\alpha\in L^2(\Gamma_N):\ \alpha\geqslant 0\}$;

(H4) And

$$\mathbf{f}\in\mathbf{L}^2(\Omega),\qquad \mathbf{J}_0\in\mathbf{L}^2(\Omega),\qquad f\in L^2(\Omega)\quad\text{and}\quad h\in L^2(\Gamma_N).\tag{17}$$

(H5) In the variable domain setting (see Section 7), the function β_T is given by the restriction to Ω_τ of a given H^1-function defined in $\mathbb{R}^3$.

(H6) In the variable domain setting, the elements

$$\mathbf{f}^\tau\in\mathbf{L}^2(\Omega_\tau),\qquad \mathbf{J}_0^\tau\in\mathbf{L}^2(\Omega_\tau),\qquad f^\tau\in L^2(\Omega_\tau)\quad\text{and}\quad h^\tau\in L^2\big(\Gamma_N^\tau\big),$$

stand for the data in boundary value problems in Ω_τ, are simply given by restrictions to Ω_τ of some functions

$$\mathbf{f}\in\mathbf{H}^1\big(\mathbb{R}^3\big),\qquad \mathbf{J}_0\in\mathbf{H}^1\big(\mathbb{R}^3\big),\qquad f\in H^1\big(\mathbb{R}^3\big)\quad\text{and}\quad h\in H^1\big(\mathbb{R}^3\big),\tag{18}$$

defined in all space. In this way the shape derivatives of all the data vanish, except for h, and the material derivatives are just given by the scalar products of the gradients of the data with respect to spatial variables with the velocity vector field, e.g., $\dot f=\nabla f\cdot V$, provided that all data are given in the Sobolev spaces $H^1(\mathbb{R}^3)$.

In the framework of function spaces of the Lebesgue and Sobolev type, the norms are denoted by the symbols $\|\cdot\|, \|\cdot\|_1, \|\cdot\|_{\Gamma_N}$ in spaces $L^2(\Omega), H^1(\Omega), L^2(\Gamma_N)$, respectively, and there scalar and vector function spaces are not distinguished in our notations. Providing that the meaning remains clear, the canonical norm in $L^p(\Omega)$ for $p \neq 1, 2$ is denoted by $\|\cdot\|_p$. We introduce the Hilbert spaces

$$\mathbf{V} = \{\mathbf{v} \in \mathbf{H}^1_0(\Omega)\colon \operatorname{div}\mathbf{v} = 0 \text{ in } \Omega\},$$
$$Z = \{\xi \in H^1(\Omega)\colon \xi = 0 \text{ on } \Gamma_D\},$$

equipped with their standard scalar products. We recall that the norm $\|\cdot\|_Z$ is equivalent to the usual seminorm $\|\nabla\cdot\|$ and also to the norm $\|\cdot\|_1$ on space $H^1(\Omega)$.

We state the main results of the paper considering a free magnetic field.

Theorem 3.1. *Under the above assumptions* (H1)–(H4), *the problem* (5)–(9), (11)–(12) *has a weak solution in the following sense*:

For every magnetic field $\mathbf{H} \in \mathbf{H}^1(\Omega)$ *such that* $\nabla \cdot \mathbf{H} = 0$ *in* Ω *and* $\mathbf{H} \cdot \mathbf{n} = 0$ *on* $\partial\Omega$, *there is the triplet* $(\mathbf{u}, \phi, T) \in \mathbf{V} \times Z^2$ *which satisfies the following integral identities*

$$\int_\Omega \nu(T) D\mathbf{u} : D\mathbf{v}\,dx + \int_\Omega (\mathbf{v} \otimes \mathbf{u}) : \nabla\mathbf{u}\,dx = \int_\Omega \big(\mu(T)(\nabla \times \mathbf{H}) \times \mathbf{H} + \mathbf{f} - \mathbf{G}(T)T\big) \cdot \mathbf{v}\,dx, \quad \forall \mathbf{v} \in \mathbf{V}; \tag{19}$$

$$\int_\Omega \sigma(T)\nabla\phi \cdot \nabla\psi\,dx = \int_\Omega \big(\mathbf{J}_0 - \sigma(T)\mu(T)\mathbf{H} \times \mathbf{u}\big) \cdot \nabla\psi\,dx, \quad \forall \psi \in Z; \tag{20}$$

$$\int_\Omega k(T)\nabla T \cdot \nabla\eta\,dx + \int_\Omega \mathbf{u} \cdot \nabla T\,\eta\,dx + \int_{\Gamma_N} \alpha T\eta\,ds = \int_\Omega f\eta\,dx + \int_{\Gamma_N} h\eta\,ds, \quad \forall \eta \in Z. \tag{21}$$

Theorem 3.2. *Assume that* $\mathbf{f} \in \mathbf{L}^{2+\delta_1}(\Omega)$ *and* $\mathbf{H} \in \mathbf{W}^{1,2+\delta_1}(\Omega)$, *where* $\delta_1 > 2/5$, *then the weak solution obtained by Theorem* 3.1 *enjoys the additional regularity, actually* $(\mathbf{u}, \phi, T) \in \mathbf{W}^{1,2+\delta}(\Omega) \times W^{1,2+\epsilon}(\Omega) \times W^{1,2+\varepsilon}(\Omega)$ *for some* $\delta, \epsilon, \varepsilon > 0$. *Furthermore, under the following Lipschitz-type continuity assumptions on the temperature dependent function parameters of the model*

$$\exists \bar{\nu} > 0: \quad |\nu(T^2) - \nu(T^1)| \leqslant \bar{\nu}|T^2 - T^1|^{3\delta/(2+\delta)}, \tag{22}$$

$$\exists \bar{\mu} > 0: \quad |\mu(T^2) - \mu(T^1)| \leqslant \bar{\mu}|T^2 - T^1|, \tag{23}$$

$$\exists \bar{\beta} > 0: \quad |\beta(T^2) - \beta(T^1)| \leqslant \bar{\beta}|T^2 - T^1|, \qquad \bar{G} = g\bar{\beta}, \tag{24}$$

$$\exists \bar{\sigma} > 0: \quad |\sigma(T^2) - \sigma(T^1)| \leqslant \bar{\sigma}|T^2 - T^1|^{3\epsilon/(2+\epsilon)}, \tag{25}$$

$$\exists \bar{k} > 0: \quad |k(T^2) - k(T^1)| \leqslant \bar{k}|T^2 - T^1|^{3\varepsilon/(2+\varepsilon)}, \quad \forall T^2, T^1 \in \mathbb{R}, \tag{26}$$

the weak solution is unique for small data.

The existence of the pressure p in the space of distributions follows from the well-known results by using the divergence-free test functions $\mathbf{v} \in \mathbf{C}_0^\infty(\Omega)$ in (19). Moreover, the pressure is unique up to a constant.

Finally, we recall the Tychonoff extension to weak topologies of the Schauder fixed point theorem [7, pp. 453–456 and 470].

Theorem 3.3. *Let K be a nonempty weakly sequentially compact convex subset of a locally convex linear topological vector space V. Let $\mathcal{L}: K \to K$ be a weakly sequentially continuous operator. Then $\mathcal{L}$ has at least one fixed point.*

4. Auxiliary results

In this section it is assumed that there are given elements $\mathbf{w}, \xi$ with the following properties. The vector function $\mathbf{w} \in \mathbf{L}^4(\Omega)$ is the solenoidal function, i.e., $\operatorname{div} \mathbf{w} = 0$ in the sense of the distributions, and $\mathbf{w} \cdot \mathbf{n} = 0$ on $\partial\Omega$, furthermore $\xi \in L^1(\Omega)$.

In order to apply the fixed point argument (cf. Theorem 3.3), let us introduce the nonlinear mapping

$$\mathcal{L}: (\mathbf{w}, \xi) \mapsto (\mathbf{u}, T), \tag{27}$$

where T and $\mathbf{u}$ will be solutions to the variational problems (28) and (30), respectively. Note that the potential ϕ can be determined provided that the pair $(\mathbf{u}, T)$ is given. The existence of solutions for the following elliptic boundary value problems is the straightforward application of the classical existence theory, hence their proofs are omitted here.

Proposition 4.1. *Assume that conditions* (16), (H3) *and* (17) *are fulfilled. Then there exists a unique $T \in Z$ such that*

$$\int_\Omega k(\xi)\nabla T \cdot \nabla\eta\,dx + \int_\Omega \mathbf{w} \cdot \nabla T\eta\,dx + \int_{\Gamma_N} \alpha T\eta\,ds = \int_\Omega f\eta\,dx + \int_{\Gamma_N} h\eta\,ds, \quad \forall \eta \in Z. \tag{28}$$

Moreover, the energy estimate holds

$$k_\# \|T\|_1 \leqslant \|f\| + \|h\|_{\Gamma_N}. \tag{29}$$

Proposition 4.2. *Assume that conditions* (13), (H2) *and* (17) *are fulfilled. Then there exists a unique $\mathbf{u} \in \mathbf{V}$ such that*

$$\int_\Omega \nu(\xi) D\mathbf{u} : D\mathbf{v}\,dx + \int_\Omega (\mathbf{v} \otimes \mathbf{w}) : \nabla\mathbf{u}\,dx = \int_\Omega \big(\mu(\xi)(\nabla \times \mathbf{H}) \times \mathbf{H} + \mathbf{f} - \mathbf{G}(T)T\big) \cdot \mathbf{v}\,dx, \quad \forall \mathbf{v} \in \mathbf{V}. \tag{30}$$

Moreover, the energy estimate holds

$$\nu_\# \|\mathbf{u}\|_1 \leqslant \mu^\# \|\nabla \times \mathbf{H}\| \|\mathbf{H}\|_{L^3} + \|\mathbf{f}\| + G^\# \|T\|_{L^{6/5}}. \tag{31}$$

Proposition 4.3. *Assume that conditions* (14)–(15) *and* (17) *are fulfilled. Then there exists a unique* $\phi \in Z$ *satisfying* (20)*. Moreover, the energy estimate holds*

$$\sigma_\# \|\phi\|_1 \leqslant \|\mathbf{J}_0\| + \mu^\# \sigma^\# \|\mathbf{H} \times \mathbf{u}\|. \tag{32}$$

We point out that in order to prove Theorem 3.1 by the fixed point argument, it suffices to establish the continuity result for the mapping $\mathcal{L}$.

Lemma 4.4 *(Weak continuity of $\mathcal{L}$). Let* $\{(\mathbf{w}_m, \xi_m)\}$ *be a sequence such that* $\mathbf{w}_m \to \mathbf{w}$ *in* $\mathbf{L}^4(\Omega)$ *and* $\xi_m \to \xi$ *in* $L^1(\Omega)$*. Let* $\{(\mathbf{u}_m, T_m)\}$ *be the corresponding sequence of weak solutions given by Propositions* 4.1 *and* 4.2*. Then there exists the weak solution* $(\mathbf{u}, T)$ *for the limit functions* $(\mathbf{w}, \xi)$ *such that*

$$\mathbf{u}_m \rightharpoonup \mathbf{u} \quad \text{in } \mathbf{V}; \qquad T_m \rightharpoonup T \quad \text{in } Z.$$

Proof. Take $\mathbf{w}_m \to \mathbf{w}$ in $\mathbf{L}^4(\Omega)$ and $\xi_m \to \xi$ in $L^1(\Omega)$ and denote by $(\mathbf{u}_m, T_m)$ the corresponding solutions to the integral identities (30) and (28), for $m \in \mathbb{N}$. From the estimates (31) and (29), the sequence $(\mathbf{u}_m, T_m)$ is bounded in $\mathbf{V} \times Z$, which implies the existence of a limit, $(\mathbf{u}, T) \in \mathbf{V} \times Z$ such that the weak convergence holds, possibly for a subsequence, still denoted by $(\mathbf{u}_m, T_m)$. In order to show that the limit $(\mathbf{u}, T)$ is a solution of the required problem (30) and (28), we pass to the limit as $m \to +\infty$ in the integral identities (30) and (28), simply replacing $\mathbf{w}, \xi, \mathbf{u}$ and T by the sequences $\mathbf{w}_m, \xi_m, \mathbf{u}_m$ and T_m, respectively. Indeed, the passage to the limit can be justified due to the continuity properties of the Nemytskii operators in the coefficients combined with the standard arguments. □

5. Proof of Theorem 3.1

Let $\mathcal{L}$ be the mapping defined by (27). In view of Propositions 4.1 and 4.2, the associated operator $\mathcal{L}$ is well defined. Lemma 4.4 yields its weak continuity, since we have the compact embeddings

$$\mathbf{V} \hookrightarrow\hookrightarrow \{\mathbf{w} \in \mathbf{L}^4(\Omega)\colon\ \nabla \cdot \mathbf{w} = 0 \text{ in } \Omega,\ \mathbf{w} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\},$$
$$Z \hookrightarrow\hookrightarrow L^1(\Omega).$$

Finally, $\mathcal{L}$ maps the ball $K = \{(\mathbf{w}, \xi) \in \mathbf{V} \times Z\colon\ \|\mathbf{w}\|_1 + \|\xi\|_1 \leqslant R\}$ into itself, and taking into account (29) and (31) we set

$$R := \frac{1}{\nu_\#}\big(\mu^\# \|\nabla \times \mathbf{H}\| \|\mathbf{H}\|_{L^3} + \|\mathbf{f}\|\big) + \left(\frac{G^\#}{\nu_\#} + 1\right)\frac{1}{k_\#}\big(\|f\| + \|h\|_{\Gamma_N}\big).$$

Therefore, Theorem 3.3 can be applied to the mapping $\mathcal{L}$ and in this way the existence of a weak solution $(\mathbf{u}, T)$ is established. Then, the complete solution is obtained in view of Proposition 4.3.

6. Proof of Theorem 3.2

For weak solutions, the property $(\mathbf{u}, \phi, T) \in \mathbf{W}^{1,2+\delta}(\Omega) \times W^{1,2+\epsilon}(\Omega) \times W^{1,2+\varepsilon}(\Omega)$, for some δ, ϵ, $\varepsilon > 0$, is a consequence of the following regularity results.

Proposition 6.1. *Assume that* $\mathbf{f} \in \mathbf{L}^{2+\delta_1}(\Omega)$ *and* $\mathbf{H} \in \mathbf{W}^{1,2+\delta_1}(\Omega)$*, for some* $\delta_1 > 2/5$*, then there exists a constant* $\delta > 0$ *such that the weak solution* $\mathbf{u} \in \mathbf{V}$ *of* (19) *belongs to* $\mathbf{W}^{1,2+\delta}(\Omega)$*, i.e.*

$$\|\nabla \mathbf{u}\|_{2+\delta} \leqslant K_1,$$

with a constant $K_1 > 0$ *only dependent on the data.*

Proof. For every $x_0 \in \overline{\Omega}$, $0 < r < R$ small enough, $\Omega(x_0, R) := \Omega \cap B(x_0, R)$, $\theta \in]0, 1[$ and some positive constants B_1, B_2, independent of $\mathbf{u}$ and T, we have the following reverse estimate (cf. [5, Lemma 3.2])

$$\left(\int_{\Omega(x_0,r)} |\nabla \mathbf{u}|^2\,dx\right)^{1/2} \leqslant \theta\left(\int_{\Omega(x_0,R)} |\nabla \mathbf{u}|^2\,dx\right)^{1/2} + \frac{B_1}{R-r}\left(\int_{\Omega(x_0,R)} |\nabla \mathbf{u}|^{6/5}\,dx\right)^{5/6} + \frac{B_2}{R-r}\left(\int_{\Omega(x_0,R)} \left(|\mathbf{u}\otimes\mathbf{u}|^2 + |\mathbf{F}|^2 + 1\right)dx\right)^{1/2}.$$

By the Sobolev embedding $\mathbf{W}^{1,2+\delta_1}(\Omega) \hookrightarrow \mathbf{L}^{3(2+\delta_1)/(1-\delta_1)}(\Omega)$ it follows that $\operatorname{rot}\mathbf{H} \times \mathbf{H} \in \mathbf{L}^{3(2+\delta_1)/(4-\delta_1)}(\Omega)$. In view of $3(2+\delta_1)/(4-\delta_1) = 2 + \varkappa$, where $\varkappa > 0$ since $\delta_1 > 2/5$, we get $\mathbf{F} = \mathbf{f} + \mu(T)\operatorname{rot}\mathbf{H} \times \mathbf{H} - \mathbf{G}(T)T \in \mathbf{L}^{2+\varkappa}(\Omega)$. Since $\mathbf{u}\otimes\mathbf{u} \in \mathbf{L}^3(\Omega)$ then the Gehring inequality [13] guarantees the higher integrability $\mathbf{u} \in \mathbf{W}^{1,2+\delta}(\Omega)$ for some $0 < \delta < \varkappa < 1$. □

For the regularity of the potential ϕ and the temperature T it will be sufficient to state the following simplified version of the general result on the higher regularity for weak solutions to the mixed boundary value problems (cf. [16, Theorem 1]).

Theorem 6.2. *Let Ω be a bounded domain of class C^1 and $u \in Z$ be a solution to the second-order elliptic differential equation $Au = F \in Z'$, and satisfy natural boundary conditions on $\partial\Omega \setminus \overline{\Gamma}_D$. Set $A : Z \to Z'$ the operator such that $Au = -\nabla \cdot (a(\cdot, u)\nabla u)$ where $a : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function such that there exist constants $a^\# > a_\# > 0$*

$$a_\# \leqslant a(\cdot, \xi) \leqslant a^\#, \quad \text{a.e. in } \Omega,\ \forall \xi \in \mathbb{R}. \tag{33}$$

Then A maps $Z_p = \{v \in W^{1,p}(\Omega):\ v = 0 \text{ on } \Gamma_D\}$ onto Z'_p for some $p > 2$.

Thus we can prove the following two results.

Proposition 6.3. *If $\mathbf{J}_0 \in \mathbf{L}^2(\Omega)$ then there exists a constant $\epsilon > 0$ such that the weak solution $\phi \in Z$ of* (20) *belongs to $W^{1,2+\epsilon}(\Omega)$, i.e.*

$$\|\nabla\phi\|_{2+\epsilon} \leqslant K_2,$$

with a constant $K_2 > 0$ only dependent on the data.

Proof. In order to apply Theorem 6.2, let $u = \phi$ be a weak solution verifying (20) and denote the operator A by

$$\langle A\phi, \psi\rangle = \int_\Omega \sigma(T)\nabla\phi \cdot \nabla\psi\,dx.$$

The boundedness property (33) is fulfilled, considering that the assumption on σ (15) holds under $T \in L^1(\Omega)$. Since

$$F = -\nabla \cdot \big(\mathbf{J}_0 - \sigma(T)\mu(T)\mathbf{H} \times \mathbf{u}\big) \in Z' \hookrightarrow Z'_p, \quad \forall p \geqslant 2,$$

then Theorem 6.2 guarantees that $\phi \in W^{1,2+\epsilon}(\Omega)$ for some $\epsilon > 0$. □

Proposition 6.4. *If $f \in L^2(\Omega)$ and $h \in L^2(\Gamma_N)$ then there exists a constant $\varepsilon > 0$ such that the weak solution $T \in Z$ of (21) belongs to $W^{1,2+\varepsilon}(\Omega)$, i.e.*

$$\|\nabla T\|_{2+\varepsilon} \leqslant K_3,$$

with a constant $K_3 > 0$ only dependent on the data.

Proof. In order to apply Theorem 6.2, let $u = T$ be a weak solution verifying (21) and denote the operator A by

$$\langle AT, \eta\rangle = \int_\Omega k(T)\nabla T \cdot \nabla\eta\, dx.$$

The boundedness property (33) is fulfilled, considering that the assumption on k (16) holds. Since $\mathbf{u} \in \mathbf{V}$, $T \in H^1(\Omega)$ and (H3) holds, we have $f - \mathbf{u} \cdot \nabla T, h - \alpha T \in Z' \hookrightarrow Z'_p$, for all $p \geqslant 2$, then Theorem 6.2 guarantees that $T \in W^{1,2+\varepsilon}(\Omega)$ for some $\varepsilon > 0$. □

To show the uniqueness let us assume that $(\mathbf{u}^1, \phi^1, T^1)$ and $(\mathbf{u}^2, \phi^2, T^2)$ are two weak solutions to problem (19)–(21). Thus, the differences $\bar{\mathbf{u}} = \mathbf{u}^1 - \mathbf{u}^2$, $\bar{\phi} = \phi^1 - \phi^2$ and $\bar{T} = T^1 - T^2$ satisfy

$$\begin{aligned}\int_\Omega \nu(T^1)|D\bar{\mathbf{u}}|^2\, dx &= \int_\Omega \big(\nu(T^2) - \nu(T^1)\big)D\mathbf{u}^2 : D\bar{\mathbf{u}}\, dx - \int_\Omega (\bar{\mathbf{u}} \otimes \bar{\mathbf{u}}) : \nabla\mathbf{u}^2\, dx \\ &\quad + \int_\Omega \big(\mu(T^1) - \mu(T^2)\big)(\nabla \times \mathbf{H}) \times \mathbf{H} \cdot \bar{\mathbf{u}}\, dx - \int_\Omega \big(\mathbf{G}(T^1)T^1 - \mathbf{G}(T^2)T^2\big) \cdot \bar{\mathbf{u}}\, dx;\end{aligned}$$

$$\begin{aligned}\int_\Omega \sigma(T^1)|\nabla\bar{\phi}|^2\, dx &= \int_\Omega \big(\sigma(T^2) - \sigma(T^1)\big)\nabla\phi^2 \cdot \nabla\bar{\phi}\, dx \\ &\quad + \int_\Omega \big(\mu(T^2)\sigma(T^2)\mathbf{H} \times \mathbf{u}^2 - \mu(T^1)\sigma(T^1)\mathbf{H} \times \mathbf{u}^1\big) \cdot \nabla\bar{\phi}\, dx;\end{aligned}$$

$$\begin{aligned}\int_\Omega k(T^1)|\nabla\bar{T}|^2\, dx + \int_{\Gamma_N} \alpha|\bar{T}|^2\, ds &= \int_\Omega \big(k(T^2) - k(T^1)\big)\nabla T^2 \cdot \nabla\bar{T}\, dx \\ &\quad - \int_\Omega \bar{\mathbf{u}} \cdot \nabla T^2\bar{T}\, dx.\end{aligned}$$

Using the assumptions (H3), (13)–(16), and applying Hölder and Young inequalities leads to

$$\begin{aligned}\frac{\nu_\#}{2}\|D\bar{\mathbf{u}}\|^2 &\leqslant \frac{1}{\nu_\#}\big\|\big(\nu(T^2) - \nu(T^1)\big)D\mathbf{u}^2\big\|^2 + \big\||\bar{\mathbf{u}}|^2\big\|\,\|\nabla\mathbf{u}^2\| \\ &\quad + \frac{C_1}{\nu_\#}\big(\big\|\big(\mu(T^1) - \mu(T^2)\big)(\nabla \times \mathbf{H}) \times \mathbf{H}\big\|_{6/5} + \big\|\mathbf{G}(T^1)T^1 - \mathbf{G}(T^2)T^2\big\|_{6/5}\big)^2,\end{aligned}$$

$$\frac{\sigma_\#}{2}\|\nabla\bar\phi\|^2 \leqslant \frac{1}{\sigma_\#}\left\|\left(\sigma\left(T^2\right)-\sigma\left(T^1\right)\right)\nabla\phi^2\right\|^2 + \frac{C_1}{\sigma_\#}\left\|\mu\left(T^2\right)\sigma\left(T^2\right)\mathbf{H}\times\mathbf{u}^2-\mu\left(T^1\right)\sigma\left(T^1\right)\mathbf{H}\times\mathbf{u}^1\right\|^2,$$

$$\frac{k_\#}{2}\|\nabla\bar T\|^2 \leqslant \frac{1}{k_\#}\left\|\left(k\left(T^2\right)-k\left(T^1\right)\right)\nabla T^2\right\|^2 + \frac{C_1}{k_\#}\left\|\bar{\mathbf{u}}\cdot\nabla T^2\right\|_{6/5}^2,$$

where C_1 is the Sobolev constant of the embedding $H^1(\Omega)\hookrightarrow L^6(\Omega)$. Furthermore, using this time the Lipschitz continuity assumptions (22)–(26), and applying Hölder and Young inequalities result in

$$\frac{\nu_\#}{2}\|D\bar{\mathbf{u}}\|^2 \leqslant \frac{\bar\nu}{\nu_\#}\|\bar T\|_6^{6\delta/(2+\delta)}\left\|D\mathbf{u}^2\right\|_{2+\delta}^2 + C_2^2\|D\bar{\mathbf{u}}\|^2\left\|\nabla\mathbf{u}^2\right\| + \frac{C_1}{\nu_\#}\left(\bar\mu\|\bar T\|_6\left\|(\nabla\times\mathbf{H})\times\mathbf{H}\right\|_{3/2} + G^\#\|\bar T\|_{6/5} + \bar G\|\bar T\|_6\left\|T^2\right\|_{3/2}\right)^2; \tag{34}$$

$$\frac{\sigma_\#}{2}\|\nabla\bar\phi\|^2 \leqslant \frac{\bar\sigma}{\sigma_\#}\|\bar T\|_6^{6\epsilon/(2+\epsilon)}\left\|\nabla\phi^2\right\|_{2+\epsilon}^2 + \frac{C_1}{\sigma_\#}\Big(\mu^\#\sigma^\#\|\mathbf{H}\|_4\|\bar{\mathbf{u}}\|_4 + \mu^\#\bar\sigma\|\bar T\|_6^{3\epsilon/(2+\epsilon)}\|\mathbf{H}\|_{2(2+\epsilon)/(4-\epsilon)}\left\|\mathbf{u}^2\right\|_6 + \bar\mu\sigma^\#\|\bar T\|_6\|\mathbf{H}\|_6\left\|\mathbf{u}^2\right\|_6\Big)^2; \tag{35}$$

$$\frac{k_\#}{2}\|\nabla\bar T\|^2 \leqslant \frac{\bar k}{k_\#}\|\bar T\|_6^{6\varepsilon/(2+\varepsilon)}\left\|\nabla T^2\right\|_{2+\varepsilon}^2 + \frac{C_1}{k_\#}\|\bar{\mathbf{u}}\|_6^2\left\|\nabla T^2\right\|_{3/2}^2, \tag{36}$$

where C_2 is the Sobolev constant of the embedding $H^1(\Omega)\hookrightarrow L^4(\Omega)$. In the sequel C stands for different Sobolev constants. Let K_1, K_2 and K_3 be the upper bounds derived in Propositions 6.1, 6.3 and 6.4, respectively, and K_4 and K_5 be the upper bounds in estimates (29) and (31), namely,

$$K_5 = \frac{1}{\nu_\#}\left(\mu^\#\|\nabla\times\mathbf{H}\|\|\mathbf{H}\|_1 + \|\mathbf{f}\| + G^\# K_4\right), \qquad K_4 = \frac{1}{k_\#}\left(\|f\| + \|h\|_{\Gamma_N}\right).$$

Now, summing up (34)–(36) we get

$$\left(\frac{\nu_\#}{2} - C^2K_5 - \frac{C\mu^\#\sigma^\#}{\sigma_\#}\|\mathbf{H}\|_1^2 - \frac{C}{k_\#}K_4^2\right)\|D\bar{\mathbf{u}}\|^2 + \frac{\sigma_\#}{2}\|\nabla\bar\phi\|^2 + \left(\frac{k_\#}{2} - \frac{C\bar\nu}{\nu_\#}K_1^2 - \frac{C}{\nu_\#}\left(\bar\mu\left\|(\nabla\times\mathbf{H})\times\mathbf{H}\right\|_{3/2} + G^\# + \bar G K_4\right)^2 - \frac{C}{\sigma_\#}\left(\bar\sigma K_2^2 + \left(\mu^\#\bar\sigma + \bar\mu\sigma^\#\right)\|\mathbf{H}\|_1^2K_5^2\right) - \frac{C\bar k}{k_\#}K_3^2\right)\|\nabla\bar T\|^2 \leqslant 0,$$

and the uniqueness holds for small data. Note that the uniqueness of the potential $\phi\in Z$ is obtained without any smallness assumption on the data.

7. Shape sensitivity preliminaries

We apply the framework established in [26] to the shape sensitivity analysis. A slightly different approach to the stationary compressible flows is proposed in [22,24] (see also [6,20,21]). The main outcome from the approach is the direct evaluation of the shape gradients by means of the appropriate singular limits of volume integrals. It seems that the approach is the only possibility to derive the shape derivatives of solutions and of the functionals for the nonlinear boundary value problems in fluid and gas flows.

To investigate the sensitivity of solutions to the Boussinesq approximation (5)–(9) under the free magnetic field problem (11)–(12) with respect to perturbations of the shape we use the speed method developed, among others, by Sokolowski and Zolésio, all details can be found in [26]. We derive the strong material derivatives for weak solutions of the model under study. Such shape sensitivity results can be used, in particular, to obtain the form of the shape gradients for a broad class of shape functionals governed by the Maxwell–Navier–Stokes model investigated in the paper.

The speed method is briefly described with our applications in mind. First, a family of mappings $\mathcal{T}_\tau\colon \mathbb{R}^3 \to \mathbb{R}^3$ associated with a given velocity field $V(\tau,x)$ is constructed. We define the family of perturbations of a given initial configuration Ω by $\Omega_\tau = \mathcal{T}_\tau(\Omega)$, each specific family parametrized by τ is defined in the direction of a given vector field V. The evolution of geometrical domains, if the vector field V is chosen, is governed by the real parameter τ, and the so-called variable domain $\Omega_\tau = \mathcal{T}_\tau(\Omega)$ depends on *two parameters*, a vector field V and the real variable τ, therefore, the variable τ has the meaning of the time in our setting. The field V is compactly supported with respect to the spatial variable x, i.e.,

$$V \in C\big(-\tau_1, \tau_1; \mathcal{D}^2\big(\Omega; \mathbb{R}^3\big)\big), \qquad \operatorname{supp} V \subset \Omega,$$

for some positive constant τ_1. The mapping is given by the system of differential equations

$$\frac{d}{d\tau}x(\tau) = V\big(\tau, x(\tau)\big), \qquad x(0) = X, \tag{37}$$

with the solution denoted by $x(\tau) = x(\tau, X)$, $\tau \in (-\tau_1, \tau_1)$, $X \in \mathbb{R}^3$. Then the variable domains are defined by the images of the mapping, and denoted by $\Omega_\tau = \{x \in \mathbb{R}^3 \mid x = x(\tau, X),\ X \in \Omega\}$.

In order to measure how the weak solutions depends on the geometrical domain, it is convenient to define our model in the variable domain setting. However, the direct shape sensitivity analysis in the variable domain setting, which results in the so-called shape derivatives, is usually difficult to justify and requires some additional regularity of the weak solutions to the boundary value problem under study. Therefore, the shape differentiability is shown in the fixed domain setting, the domain perturbations are transformed into the variable coefficients of the integral variational formulations of the equations in question. In our setting all equations defined in variable domain Ω_τ can be transported to the reference domain which is also called the fixed domain Ω, using the inverse transformation $\mathcal{T}_\tau^{-1}: \Omega_\tau \to \Omega$.

7.1. Properties of the mapping $\mathcal{T}_\tau$

We consider the general case of the domain transformations $\mathcal{T}_\tau$, for $\tau \in \mathbb{R}$. Let D be a domain in $\mathbb{R}^N$ with the boundary ∂D piecewise C^k for a given integer $k \geqslant 0$. Let $\mathcal{T}_\tau$ be a one-to-one mapping from $\overline{D}$ onto $\mathcal{T}_\tau(\overline{D})$ such that

(A1) $\mathcal{T}_\tau$ and $\mathcal{T}_\tau^{-1}$ belong to $\mathbf{C}^k(\overline{D}) = C^k(\overline{D}; \mathbb{R}^N)$ and $\mathbf{C}^k(\mathcal{T}_\tau(\overline{D}))$, respectively;
(A2) $\tau \mapsto \mathcal{T}_\tau(X)$, $\tau \mapsto \mathcal{T}_\tau^{-1}(x) \in C(-\tau_1, \tau_1)$, $\forall X \in \overline{D}$, $x \in \mathcal{T}_\tau(\overline{D})$.

Thus $(\tau, X) \mapsto \mathcal{T}_\tau(X) \in C(-\tau_1, \tau_1; \mathbf{C}^k(\overline{D}))$. For any $X \in \overline{D}$, the point $x(\tau) = \mathcal{T}_\tau(X)$ moves along the trajectory $x(\cdot)$ with the velocity

$$\frac{d}{d\tau}x(\tau) = \frac{\partial}{\partial\tau}\mathcal{T}_\tau(X). \tag{38}$$

It is obvious that $V(\tau, x)$ takes the form

$$V(\tau, x) = \left(\frac{\partial}{\partial\tau}\mathcal{T}_\tau\right) \circ \mathcal{T}_\tau^{-1}(x). \tag{39}$$

Theorem 7.1 below fully characterizes the families of domains transformations and by the Nagumo Theorem the transformations leave invariant, by construction, the *hold-all-domain* D. Such a construction serves for the investigations of shape differentiability of solutions to boundary value problems as well as of the shape functionals. In particular, by the constructions, all admissible domains for a specific problem of shape optimization are included in the sufficiently large hold-all-domain D.

By (38) and (39), the vector field $V(\tau)$ defined as $V(\tau)(x) = V(\tau, x)$ satisfies the relation

$$V \in C\big(-\tau_1, \tau_1; C^k\big(\overline{D}; \mathbb{R}^N\big)\big). \tag{40}$$

If V is a vector field such that (40) holds, then the transformation $\mathcal{T}_\tau$, depending on V, and such that conditions (38), (39) are satisfied, is defined by (37).

Theorem 7.1. *Let D be a bounded domain in $\mathbb{R}^N$ with the piecewise smooth boundary ∂D, and $V \in C(-\tau_1, \tau_1; C^k(\overline{D}; \mathbb{R}^N))$ be a given vector field which satisfies*

$$V(\tau, x) \cdot n(x) = 0 \quad \textit{for a.e. } x \in \partial D, \tag{41}$$

and we set

$$V(\tau, x) = 0 \quad \textit{if } n = n(x) \textit{ is not defined as a singular point } x \in \partial D. \tag{42}$$

Then there exists an interval $I \subset (-\tau_1, \tau_1)$, $0 \in I$, and the one-to-one transformation $T_\tau(V): \mathbb{R}^N \to \mathbb{R}^N$ such that $T_\tau(V)$ maps $\overline{D}$ onto $T_\tau(V)(\overline{D})$. Furthermore $T_\tau(V)$ satisfies the conditions (38), (39). *In particular the vector field V can be written in the form*

$$V = \partial_\tau T_\tau(V) \circ T_\tau(V)^{-1}. \tag{43}$$

On the other hand, if $T_\tau = T_\tau(V)$ is a transformation of $\overline{D}$, T_τ satisfies (38), (39) *and $\widetilde{V}$ is defined by the formula*

$$\widetilde{V} = \partial_\tau T_\tau \circ T_\tau^{-1}, \tag{44}$$

then (41) *and* (42) *hold for $\widetilde{V}$. Furthermore, $\widetilde{V} \in C(-\tau_1, \tau_1; C^k(\overline{D}; \mathbb{R}^N))$ and the transformation $\mathcal{T}_\tau(X) = x(\tau, X)$ is defined as the local solution to the system of ordinary differential equations* (37), *that is, $\mathcal{T}_\tau = T_\tau(V)$.*

Remark 7.2. There exists an interval $I = (-\delta, \delta)$, $0 < \delta \leqslant \tau_1$ and a one-to-one transformation $T_\tau(V)$ for each $\tau \in I$ which satisfies all properties of Theorem 7.1.

For the strong differentiability of composed mapping, we refer the reader to [26]. We recall here some results required in our framework of boundary variations technique for the shape sensitivity analysis. If D is the hold-all-domain, it means that by the Nagumo Theorem the boundary ∂D is invariant under the flow of admissible vector fields which satisfy conditions (41), (42). Given function defined on D, or on $\mathbb{R}^3$, we can obtain the explicit form of the material and shape derivatives of the function in the direction of an admissible vector field V.

Proposition 7.3. *Let $f \in H^1(D)$, $V \in C(-\tau_1, \tau_1; \mathcal{D}^k(\mathbb{R}^N_+; \mathbb{R}^N))$, $k \geqslant 1$ is a given vector field such that conditions* (41), (42) *are satisfied, then the mapping $\tau \mapsto f \circ \mathcal{T}_\tau$ is strongly differentiable in the space $L^2(D)$, and its derivative is $\nabla f \cdot V$.*

For the proof see [26]. We recall here, that the above result is in fact equivalent to the shape differentiability of the integral shape functional $J(\Omega_\tau) = \int_{\Omega_\tau} f(x)\,dx$, where $\Omega_\tau \subset \overline{D}$ for $\tau \in (-\delta, \delta)$. It is clear, that the results established in our paper lead to the shape differentiability for a class of shape functionals, however this subject is not developed here, we restrict ourselves to the strong shape differentiability of solutions to the boundary value problems.

Very important is also the property of a weak differentiability of $\mathcal{T}_\tau$ with respect to τ. We address the question, if for a given domain $D \subset \mathbb{R}^N$ and a given function f in L^2 the composed mapping $\tau \mapsto f \circ \mathcal{T}_\tau$ is strongly differentiable in $H^{-1}(D)$. The answer is negative, for a counterexample see [26], however the differentiability in the weak topology of the space $H^{-1}(D)$ can be shown.

Proposition 7.4. *Let $f \in L^2(D)$, $V \in C(-\tau_1, \tau_1; \mathcal{D}^k(\mathbb{R}^N_+; \mathbb{R}^N))$ be given, $k \geqslant 1$, then the mapping $\tau \mapsto f \circ \mathcal{T}_\tau$ is weakly differentiable in the space $H^{-1}(D)$.*

For the proof see [26].

Proposition 7.5. *Let $f \in L^2(\mathbb{R}^N)$, $V \in C(-\tau_1, \tau_1; \mathcal{D}^k(\mathbb{R}^N; \mathbb{R}^N))$ be given, then the mapping $\tau \to f \circ \mathcal{T}_\tau$ is strongly differentiable in the space $H^{-2}(\mathbb{R}^N)$.*

For the proof see [26].

7.2. Bogovskii operator

We need some elementary properties of the operator which defines, in an appropriate way, the inverse of the *div* differential operator, the so-called Bogovskii operator. Consider an auxiliary problem:

Given

$$g \in L^q(\Omega_\tau), \quad \int_{\Omega_\tau} g\,dx_\tau = 0, \quad 1 < q < \infty, \tag{45}$$

find a vector field $\mathbf{v} = \mathcal{B}_\tau[g]$ such that

$$\mathbf{v} \in \mathbf{W}_0^{1,q}(\Omega_\tau), \qquad \operatorname{div} \mathbf{v} = g \quad \text{a.a. in } \Omega_\tau \quad \text{and} \quad |v|_{1,q} \leqslant c\|g\|_q, \tag{46}$$

where $c = c(N, q, \Omega)$.

We recall the following results and definition, for more details see Chapters II and III.3 in Galdi [12].

Definition 7.6. The domain Ω is called *star-shaped* domain if there exist a point $\bar{x} \in \Omega$ and a continuous positive function h on the unit sphere such that

$$\Omega = \left\{ x \in \mathbb{R}^N \colon |x - \bar{x}| < h\left(\frac{x - \bar{x}}{|x - \bar{x}|} \right) \right\}.$$

Proposition 7.7. *Let Ω be locally Lipschitzian. Then there exist m locally Lipschitzian domains $G_1, \ldots, G_m$ such that*

(i) $\partial\Omega \subset \bigcup_{i=1}^m G_i$;
(ii) *the domains $\Omega_i = \Omega \cap G_i$, $i = 1, \ldots, m$ are* (*locally Lipschitzian and*) *star-shaped with respect to every point of a ball B_i with $\overline{B}_i \subset \Omega_i$.*

Proposition 7.8. *Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, be such that*

$$\Omega = \bigcup_{k=1}^{M} \Omega_k, \quad M \geqslant 1, \tag{47}$$

where each Ω_k is star-shaped domain with respect to some open ball B_k with $\overline{B}_k \subset \Omega_k$, and let $g \in L^q(\Omega)$ satisfy (45)*. Then, there exist M functions g_k such that for all $k = 1, \ldots, M$:*

(i) $g_k \in L^q(\Omega)$;
(ii) $\operatorname{supp}(g_k) \subset \overline{\Omega}_k$;
(iii) $\int_{\Omega_k} g_k = 0$;
(iv) $\|g_k\|_q \leqslant C\|g\|_q$, *with*

$$C = \left(1 + \frac{|\Omega_k|}{|\Omega_k \cap D_k|}\right) \prod_{i=1}^{k-1} \left(1 + |F_i|^{1/q-1} |D_i - \Omega_i|^{1-1/q}\right), \tag{48}$$

and where $D_i = \bigcup_{s=i+1}^{M} \Omega_s$ and $F_i = |\Omega_i \cap D_i|$, $i = 1, \ldots, M-1$.

Theorem 7.9. *Let Ω be a bounded domain of $\mathbb{R}^N$, $N \geqslant 2$, such that* (47) *holds where each Ω_k is star-shaped with respect to some open ball B_k with $\overline{B}_k \subset \Omega_k$. For instance Ω satisfies the cone condition. Then, given $f \in L^q(\Omega)$ verifying $\int_\Omega f = 0$, there exists at least one solution v to* (46)*. Furthermore, the constant c entering in inequality* (46) *admits the following estimate*:

$$c \leqslant c_0 C \left(\frac{\delta(\Omega)}{R_0}\right)^N \left(1 + \frac{\delta(\Omega)}{R_0}\right), \tag{49}$$

where R_0 is the smallest radius of the balls B_k, $c_0 = c_0(N, q)$, C is given as (48)*, and $\delta(\Omega)$ is the diameter of Ω,*

$$\delta(\Omega) = \sup_{x,y \in \Omega} |x - y|.$$

Finally, if f is of compact support in Ω so is v.

Proposition 7.10. *For each $\tau > 0$ small enough there is a solution operator $\mathcal{B}_\tau$ associated to problem* (45)–(46) *such that*

$$\left\|\mathcal{B}_\tau[g]\right\|_{\mathbf{W}_0^{1,q}(\Omega_\tau)} \leqslant c(N, q, \tau)\|g\|_{L^q(\Omega_\tau)}. \tag{50}$$

Moreover, the norm of $\mathcal{B}_\tau$ is independent of τ.

Proof. It is known that a domain $\Omega \subset \mathbb{R}^N$ is $C^{k,1}$ if the boundary $\partial\Omega$ can be expressed as a graph of a $C^{k,1}$ function

$$a : S \cap \mathbb{R}^{N-1} \times \{0\} \to \mathbb{R}$$

with an appropriate open set $S \subset \mathbb{R}^N$ and $k \in \mathbb{N}$. We assume that $\Omega = \bigcup_{i=1}^{M} \Omega_i$ and we know that on every Ω_i Bogovskii operator is valid. Now we transform from Ω to Ω_τ and we know that this transformation satisfies that $(\tau, X) \mapsto \mathcal{T}_\tau(X) \in C(-\tau_1, \tau_1; \mathbf{C}^k(\overline{D}))$. It is known that if we introduce the transformation of $\mathbb{R}^N$ into itself then constant c in inequality (46) does not change in case of homothetic transformation or rotation (see Galdi [12, Chapter III, Lemma 3.3.]). In our case it can be seen that in our more general situation we can choose supremum over τ in $\delta(\Omega_\tau)$ to get independent constant c on τ in the estimate for Bogovskii operator. □

8. Shape sensitivity analysis

We consider the model (5)–(9), (11)–(12), under the assumption that all coefficients ν, μ, k and σ are continuous functions only on the temperature. We perform the shape sensitivity analysis of the considered model (cf. Theorem 3.1).

8.1. Perturbed problem in variable domain

Definition 8.1. The following system of equations defined in Ω_τ is called a perturbed problem to the model (5)–(9), (11)–(12),

$$\begin{aligned} &-\nabla\cdot\big(\nu\big(T^\tau\big)D\mathbf{u}^\tau\big)+\big(\mathbf{u}^\tau\cdot\nabla\big)\mathbf{u}^\tau+\nabla p^\tau \\ &\quad=\mu\big(T^\tau\big)\operatorname{rot}\mathbf{H}^\tau\times\mathbf{H}^\tau+\mathbf{f}^\tau-\mathbf{G}\big(T^\tau\big)T^\tau;\qquad \operatorname{div}\mathbf{u}^\tau=0; \end{aligned} \tag{51}$$

$$-\nabla\cdot\big(\sigma\big(T^\tau\big)\nabla\phi^\tau\big)=-\nabla\cdot\big(\mathbf{J}_0^\tau+\sigma\big(T^\tau\big)\mu\big(T^\tau\big)\mathbf{u}^\tau\times\mathbf{H}^\tau\big); \tag{52}$$

$$-\nabla\cdot\big(k\big(T^\tau\big)\nabla T^\tau\big)+\mathbf{u}^\tau\cdot\nabla T^\tau=f^\tau, \tag{53}$$

along with the boundary conditions:

$$\mathbf{u}^\tau=\mathbf{0}\quad\text{on }\partial\Omega_\tau; \tag{54}$$

$$T^\tau=\phi^\tau=0\quad\text{on }\Gamma_D^\tau; \tag{55}$$

$$k\big(T^\tau\big)\frac{\partial T^\tau}{\partial\mathbf{n}^\tau}+\alpha^\tau T^\tau=h^\tau\quad\text{and}\quad\frac{\partial\phi^\tau}{\partial\mathbf{n}^\tau}=0\quad\text{on }\Gamma_N^\tau. \tag{56}$$

We introduce the Hilbert space

$$Z^\tau=\big\{\xi\in H^1(\Omega_\tau)\colon \xi=0\text{ on }\Gamma_D^\tau\big\}$$

equipped with its standard inner product.

Theorem 8.2. *Under the assumptions* (H1)–(H6), *the problem* (51)–(56) *has a weak solution in the following sense*:

For every magnetic field $\mathbf{H}^\tau\in\mathbf{H}^1(\Omega_\tau)$ such that $\nabla\cdot\mathbf{H}^\tau=0$ in Ω_τ and $\mathbf{H}^\tau\cdot\mathbf{n}^\tau=0$ on $\partial\Omega_\tau$, there exists $(\mathbf{u}^\tau,\phi^\tau,T^\tau,p^\tau)\in\mathbf{H}_0^1(\Omega_\tau)\times(Z^\tau)^2\times L^2(\Omega_\tau)$ satisfying, for all $\mathbf{v}^\tau\in\mathbf{H}_0^1(\Omega_\tau)$ and $\psi^\tau,\eta^\tau\in Z^\tau$,

$$\begin{aligned} &\int_{\Omega_\tau}\nu\big(T^\tau\big)D\mathbf{u}^\tau:D\mathbf{v}^\tau\,dx_\tau+\int_{\Omega_\tau}\big(\mathbf{v}^\tau\otimes\mathbf{u}^\tau\big):\nabla\mathbf{u}^\tau\,dx_\tau \\ &\quad=\int_{\Omega_\tau}p^\tau\nabla\cdot\mathbf{v}^\tau\,dx_\tau+\int_{\Omega_\tau}\big(\mu\big(T^\tau\big)\big(\nabla\times\mathbf{H}^\tau\big)\times\mathbf{H}^\tau+\mathbf{f}^\tau-\mathbf{G}\big(T^\tau\big)T^\tau\big)\cdot\mathbf{v}^\tau\,dx_\tau, \end{aligned} \tag{57}$$

$$\int_{\Omega_\tau}\sigma\big(T^\tau\big)\nabla\phi^\tau\cdot\nabla\psi^\tau\,dx_\tau=\int_{\Omega_\tau}\big(\mathbf{J}_0^\tau-\sigma\big(T^\tau\big)\mu\big(T^\tau\big)\mathbf{H}^\tau\times\mathbf{u}^\tau\big)\cdot\nabla\psi^\tau\,dx_\tau, \tag{58}$$

$$\int_{\Omega_\tau} k(T^\tau)\nabla T^\tau \cdot \nabla \eta^\tau \, dx_\tau + \int_{\Omega_\tau} \mathbf{u}^\tau \cdot \nabla T^\tau \eta^\tau \, dx_\tau + \int_{\Gamma_N^\tau} \alpha^\tau T^\tau \eta^\tau \, ds_\tau$$
$$= \int_{\Omega_\tau} f^\tau \eta^\tau \, dx_\tau + \int_{\Gamma_N^\tau} h^\tau \eta^\tau \, ds_\tau. \tag{59}$$

Proof. Only difference is in term with pressure. Indeed we can argue as in the proof of Theorem 3.1 and then apply the De Rham Theorem to obtain the existence of the pressure. To estimate the pressure we use as a test function

$$\mathbf{v}^\tau = \mathcal{B}_\tau\left[p^\tau - \frac{1}{|\Omega_\tau|}\int_{\Omega_\tau} p^\tau \, dx_\tau\right]$$

and we get

$$\|p^\tau\|_2 \leqslant c. \qquad \square$$

Theorem 8.3. *Assume $\mathbf{f}^\tau \in \mathbf{L}^{2+\delta_1}(\Omega_\tau)$ and $\mathbf{H}^\tau \in \mathbf{W}^{1,2+\delta_1}(\Omega_\tau)$, for some $\delta_1 > 2/5$, then the solution in accordance to Theorem* 8.2 *is for some δ, ϵ, $\varepsilon > 0$ such that $(\mathbf{u}^\tau, \phi^\tau, T^\tau) \in \mathbf{W}^{1,2+\delta}(\Omega_\tau) \times W^{1,2+\epsilon}(\Omega_\tau) \times W^{1,2+\varepsilon}(\Omega_\tau)$ and it is unique under small data on* (24)*.*

Proof. See the proof of Theorem 3.2. $\square$

8.2. Transported problem

The transported solution to the fixed domain is denoted by $\mathbf{u}_\tau = \mathbf{u}^\tau \circ \mathcal{T}_\tau$, $\mathbf{H}_\tau = \mathbf{H}^\tau \circ \mathcal{T}_\tau$, $\phi_\tau = \phi^\tau \circ \mathcal{T}_\tau$, $T_\tau = T^\tau \circ \mathcal{T}_\tau$ with data $\mathbf{f}_\tau = \mathbf{f}^\tau \circ \mathcal{T}_\tau$, $\mathbf{G}_\tau = \mathbf{G}^\tau \circ \mathcal{T}_\tau$, $\mathbf{J}_{0\tau} = \mathbf{J}_0^\tau \circ \mathcal{T}_\tau$, $\alpha_\tau = \alpha^\tau \circ \mathcal{T}_\tau$, $f_\tau = f^\tau \circ \mathcal{T}_\tau$ and $h_\tau = h^\tau \circ \mathcal{T}_\tau$.

Remark 8.4. The shape analysis for the Maxwell equations coupled with heat equation and an evolution equation for the volume fraction of the high temperature phase in steel was investigated by Hömberg and Sokolowski [17]. Here we deal with the more general situation: incompressible heat conductive fluid with Maxwell equations.

We recall the following important results.

Proposition 8.5. *Denote by $J\mathcal{T}_\tau$ the Jacobian of $\mathcal{T}_\tau$ and for any matrix B the transposed matrix is denoted by ${}^*B$. Then we have*

(i) $(\operatorname{grad} w) \circ \mathcal{T}_\tau = ({}^*J\mathcal{T}_\tau^{-1}\nabla)(w \circ \mathcal{T}_\tau)$ *for all* $w \in H^1(\Omega)$;
(ii) $(\operatorname{div} \mathbf{w}) \circ \mathcal{T}_\tau = \zeta(\tau)^{-1}(\zeta(\tau)J\mathcal{T}_\tau^{-1}\nabla) \cdot (\mathbf{w} \circ \mathcal{T}_\tau)$ *for all* $\mathbf{w} \in \mathbf{H}^1(\Omega)$;
(iii) $(\operatorname{curl} \mathbf{w}) \circ \mathcal{T}_\tau = ({}^*J\mathcal{T}_\tau^{-1}\nabla) \times (\mathbf{w} \circ \mathcal{T}_\tau)$ *for all* $\mathbf{w} \in \mathbf{H}^1(\Omega)$.

Remark 8.6. From Proposition 8.5, it follows that functions which are divergence free on Ω_τ generally loose this property when they are transported to the fixed domain. For more details see [17, Remark 6.2].

Proposition 8.7. *(See [26, Proposition 2.47].) For any $f \in L^1(\Gamma_\tau)$,*

$$\int_{\Gamma_\tau} f\, ds_\tau = \int_{\Gamma} f \circ \mathcal{T}_\tau \left\| M(\mathcal{T}_\tau) \cdot \mathbf{n} \right\|_{\mathbb{R}^3} ds,$$

where $M(\mathcal{T}_\tau) = \det(J\mathcal{T}_\tau)^* J\mathcal{T}_\tau^{-1}$ is the cofactor matrix of the Jacobian matrix $J\mathcal{T}_\tau$.

We introduce the following notations

$$\zeta(\tau) = \det(J\mathcal{T}_\tau),$$

$$\varrho(\tau) = {}^* J\mathcal{T}_\tau^{-1},$$

$$A(\tau) = \zeta(\tau)^* \varrho(\tau)\varrho(\tau),$$

$$B(\tau) = \zeta(\tau)\varrho(\tau),$$

$$\omega(\tau) = \left\| M(J\mathcal{T}_\tau) \cdot \mathbf{n} \right\|_{\mathbb{R}^3},$$

and we define the following forms

(F1) $\alpha_0(\tau, \mathbf{u}, T, \mathbf{v}) = \int_\Omega \nu(T)\zeta(\tau)(\varrho(\tau)D\mathbf{u}) : (\varrho(\tau)D\mathbf{v})\, dx = \int_\Omega \nu(T)A(\tau) : (D\mathbf{u}D\mathbf{v})\, dx$,
(F2) $\alpha_1(\tau, \mathbf{u}, \mathbf{v}) = \int_\Omega \zeta(\tau)(\varrho(\tau)\nabla\mathbf{u}) : (\mathbf{v} \otimes \mathbf{u})\, dx = \int_\Omega B(\tau)\nabla\mathbf{u} : (\mathbf{v} \otimes \mathbf{u})\, dx$,
(F3) $\alpha_2(\tau, \mathbf{H}, T, \mathbf{v}) = \int_\Omega \mu(T)\zeta(\tau)(((\varrho(\tau)\nabla) \times \mathbf{H}) \times \mathbf{H}) \cdot \mathbf{v}\, dx$,
(F4) $\alpha_3(\tau, \mathbf{f}, T, \mathbf{v}) = \int_\Omega \zeta(\tau)(\mathbf{f} - \mathbf{G}(T)T) \cdot \mathbf{v}\, dx$,
(F5) $\alpha_4(\tau, p, \mathbf{v}) = \int_\Omega \zeta(\tau)p(\varrho(\tau)\nabla) \cdot \mathbf{v}\, dx = \int_\Omega pB(\tau) : \nabla\mathbf{v}\, dx$,
(F6) $\beta_1(\tau, T, \phi, \psi) = \int_\Omega \sigma(T)(\varrho(\tau)\nabla\phi) \cdot (\varrho(\tau)\nabla\psi)\, dx = \int_\Omega \sigma(T)A(\tau) : (\nabla\phi \otimes \nabla\psi)\, dx$,
(F7) $\beta_2(\tau, \mathbf{J}_0, T, \mathbf{H}, \mathbf{u}, \psi) = \int_\Omega B(\tau) : ((\mathbf{J}_0 - \sigma(T)\mu(T)\mathbf{H} \times \mathbf{u}) \otimes \nabla\psi)\, dx$,
(F8) $\gamma_1(\tau, T, \eta) = \int_\Omega k(T)A(\tau) : (\nabla T \otimes \nabla\eta)\, dx$,
(F9) $\gamma_2(\tau, \mathbf{u}, T, \eta) = \int_\Omega \zeta(\tau)\mathbf{u} \cdot (\varrho(\tau)\nabla)T\eta\, dx = \int_\Omega B(\tau) : (\mathbf{u} \otimes \nabla T)\eta\, dx$,
(F10) $\gamma_3(\tau, \alpha, T, \eta) = \int_{\Gamma_N} \alpha T\eta\omega(\tau)\, ds$,
(F11) $\gamma_4(\tau, f, \eta) = \int_\Omega f\eta\zeta(\tau)\, dx$,
(F12) $\gamma_5(\tau, h, \eta) = \int_{\Gamma_N} h\eta\omega(\tau)\, ds$.

In view of Propositions 8.5 and 8.7, the problem (57)–(59) in variable domains can be transported to the fixed domain, and rewritten in the weak form as the following system of integral identities.

Definition 8.8. The following system of integral identities is called the transported problem to the fixed domain

$$\alpha_0(\tau, \mathbf{u}_\tau, T_\tau, \mathbf{v}) + \alpha_1(\tau, \mathbf{u}_\tau, \mathbf{v}) = \alpha_2(\tau, \mathbf{H}_\tau, T_\tau, \mathbf{v}) + \alpha_3(\tau, \mathbf{f}_\tau, T_\tau, \mathbf{v}) + \alpha_4(\tau, p_\tau, \mathbf{v}),$$

$$\beta_1(\tau, T_\tau, \phi_\tau, \psi) = \beta_2(\tau, \mathbf{J}_{0\tau}, T_\tau, \mathbf{H}_\tau, \mathbf{u}_\tau, \psi),$$

$$\gamma_1(\tau, T_\tau, \eta) + \gamma_2(\tau, \mathbf{u}_\tau, T_\tau, \eta) + \gamma_3(\tau, \alpha_\tau, T_\tau, \eta) = \gamma_4(\tau, f_\tau, \eta) + \gamma_5(\tau, h_\tau, \eta),$$

for all $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$, $\psi, \eta \in Z$. Here

$$\mathbf{H}^\tau \cdot \mathbf{n}^\tau = 0,$$
$$\left(\operatorname{div}\mathbf{H}^\tau\right) \circ \mathcal{T}_\tau = \frac{1}{\zeta(\tau)} \operatorname{div}\left(\zeta(\tau) J\mathcal{T}_\tau^{-1} \mathbf{H}_\tau\right),$$

and

$$\phi_\tau = T_\tau = 0 \quad \text{on } \Gamma_D \quad \text{and} \quad \mathbf{u}_\tau = \mathbf{0} \quad \text{on } \partial\Omega.$$

Remark 8.9. First, we prove the formula for the transport of the divergence operator (to be sure). We have in the variable domain

$$\int_{\Omega_\tau} \operatorname{div}\mathbf{v}\varphi\, dx_\tau = -\int_{\Omega_\tau} \mathbf{v} \cdot \nabla\varphi\, dx_\tau$$

then, the transport to the fixed domain leads to, denote $\psi = \varphi \circ \mathcal{T}_\tau$,

$$\begin{aligned}\int_\Omega \left[(\operatorname{div}\mathbf{v}) \circ \mathcal{T}_\tau\right]\psi\zeta(\tau)\, dx &= -\int_\Omega [\mathbf{v} \circ \mathcal{T}_\tau] \cdot \left[(\nabla\varphi) \circ \mathcal{T}_\tau\right]\zeta(\tau)\, dx \\ &= -\int_\Omega [\mathbf{v} \circ \mathcal{T}_\tau] \cdot \left({}^* J\mathcal{T}_\tau^{-1} \nabla\psi\right)\zeta(\tau)\, dx\end{aligned}$$

which confirms the above formula.

For a given divergence free vector field $\mathbf{v}^\tau$ defined in the variable domain Ω_τ it follows that the transported divergence is given by the expression

$$\left(\operatorname{div}\mathbf{v}^\tau\right) \circ \mathcal{T}_\tau = \frac{1}{\zeta(\tau)} \operatorname{div}\left(\zeta(\tau) J\mathcal{T}_\tau^{-1} \mathbf{v}_\tau\right)$$

which shows the following equivalence for $\mathbf{v}_\tau = \mathbf{v}^\tau \circ \mathcal{T}_\tau$

$$\operatorname{div}\mathbf{v}^\tau = 0 \quad \text{in } \Omega \quad \text{if and only if} \quad \operatorname{div}\left(\zeta(\tau) J\mathcal{T}_\tau^{-1} \mathbf{v}_\tau\right) = 0 \quad \text{in } \Omega_\tau.$$

Therefore, if we introduce the new function $\mathfrak{v}_\tau = \zeta(\tau) J\mathcal{T}_\tau^{-1} \mathbf{v}_\tau$ defined in the fixed domain, then it follows that

$$\operatorname{div}\mathfrak{v}_\tau = \operatorname{div}\left(\zeta(\tau) J\mathcal{T}_\tau^{-1} \mathbf{v}_\tau\right) = \zeta(\tau)\left(\operatorname{div}\mathbf{v}^\tau\right) \circ \mathcal{T}_\tau = 0$$

for the divergence free field $\mathbf{v}^\tau$. In this way we have $\mathbf{v}_\tau = \zeta^{-1}(\tau) J\mathcal{T}_\tau \mathfrak{v}_\tau$ and $\mathbf{v}^\tau = \zeta^{-1}(\tau) J\mathcal{T}_\tau \mathfrak{v}^\tau$. Therefore, to keep divergence free the unknown velocity field we should replace in the transported problem $\mathbf{u}_\tau$ by $\mathbf{u}_\tau = \zeta^{-1}(\tau) J\mathcal{T}_\tau \mathfrak{u}_\tau$, and we have $\operatorname{div}\mathfrak{u}_\tau = 0$ since $\operatorname{div}\mathbf{u}^\tau = 0$.

Lemma 8.10. *Under our assumptions on the domain transformations the functions $\tau \mapsto \zeta(\tau)$, $A(\tau)$, $B(\tau)$ are differentiable at $\tau = 0$. Furthermore, for $|\tau| \leqslant \tau_1$ and τ_1 small enough, the following Taylor expansions are obtained*

(i) $\zeta(\tau) = 1 + \tau\zeta'(0) + o(\tau)$,
(ii) $\varrho(\tau) = I + \tau\varrho'(0) + O(\tau)$,
(iii) $A(\tau) = I + \tau A'(0) + O(\tau)$,
(iv) $B(\tau) = I + \tau B'(0) + O(\tau)$,

where the derivatives at $\tau = 0$ are given by

(v) $\zeta'(0) = \operatorname{div} V(0)$,
(vi) $\varrho'(0) = -{}^*JV(0)$,
(vii) $A'(0) = \operatorname{div} V(0)I - 2D(V(0))$,
(viii) $B'(0) = \operatorname{div} V(0)I - {}^*JV(0)$.

As in Section 2, $D(V(0))$ denotes the symmetrized part of $JV(0)$, i.e. $D(V(0)) = \frac{1}{2}(JV(0) + {}^*JV(0))$.

For the proof see Sokolowski and Zolésio [26, Section 2.13].

Theorem 8.11. *Suppose that the assumptions (H1)–(H6) are fulfilled. For every magnetic field $\mathbf{H}_\tau \in \mathbf{H}^1(\Omega)$ such that $\nabla \cdot \mathbf{H}_\tau = 0$ in Ω and $\mathbf{H}_\tau \cdot \mathbf{n}_\tau = 0$ on $\partial\Omega$, there exists a weak solution $(\widetilde{\mathbf{u}}, \widetilde{\phi}, \widetilde{T}, \widetilde{p}) \in \mathbf{H}_0^1(\Omega) \times Z^2 \times L^2(\Omega)$ to the transported problem in accordance to Definition 8.8.*

Proof. The existence of a solution is consequence of Theorem 3.1. Only difference is in term with pressure. To estimate the pressure we use as a test function

$$\mathbf{v}_\tau = \mathcal{B}\left[p_\tau - \frac{1}{|\Omega|}\int_\Omega p_\tau\,dx\right]$$

and we get

$$\|p_\tau\|_2 \leqslant c. \qquad \square$$

Theorem 8.12. *Assume $\mathbf{f}_\tau \in \mathbf{L}^{2+\delta_1}(\Omega)$ and $\mathbf{H}_\tau \in \mathbf{W}^{1,2+\delta_1}(\Omega)$, for some $\delta_1 > 2/5$, then the solution in accordance to Theorem 8.11 belongs to $\mathbf{W}^{1,2+\delta}(\Omega) \times W^{1,2+\epsilon}(\Omega) \times W^{1,2+\varepsilon}(\Omega)$ for some $\delta, \epsilon, \varepsilon > 0$ and the solution is unique under small data on (24), that is,*

$$(\widetilde{\mathbf{u}}, \widetilde{\phi}, \widetilde{T}, \widetilde{p}) = (\mathbf{u}_\tau, \phi_\tau, T_\tau, p_\tau).$$

Proof. See the proof of Theorem 3.2. $\square$

Consequence of Lemma 8.10 is the following corollary.

Corollary 8.13. *Let $|\tau| \leqslant \tau_1$ and τ_1 be small enough, then there exist realvalued functions g_i satisfying $g_i(\tau) = o(\tau)$, $i = 0, \ldots, 11$ and forms $\widetilde{\alpha}_i(\tau, \ldots)$, $i = 0, 1, 2, 3, 4$ and $\widetilde{\beta}(\tau, \ldots)$, $i = 1, 2$, and $\widetilde{\gamma}(\tau, \ldots)$, $i = 1, \ldots, 5$, such that the following statements are valid.*

(B1) *For all $\mathbf{u}, \mathbf{v} \in \mathbf{H}_0^1(\Omega)$ and $T \in L^1(\Omega)$,*

$$\alpha_0(\tau, \mathbf{u}, T, \mathbf{v}) = \alpha_0\big(0, \mathbf{u}, T, \mathbf{v}\big) + \tau\alpha_{0,\tau}\big(0, \mathbf{u}, T, \mathbf{v}\big) + \widetilde{\alpha_0}\big(\tau, \mathbf{u}, T, \mathbf{v}\big),$$

$$\alpha_{0,\tau}\big(0, \mathbf{u}, T, \mathbf{v}\big) = \int_\Omega \nu(T)A'(0) : (D\mathbf{u}D\mathbf{v})\,dx,$$

$$\widetilde{\alpha_0}\big(\tau, \mathbf{u}, T, \mathbf{v}\big) \leqslant g_0(\tau)\nu^{\#}\|\mathbf{u}\|_1\|\mathbf{v}\|_1.$$

(B2) *For all* $\mathbf{u}, \mathbf{v} \in \mathbf{H}^1_0(\Omega)$,

$$\alpha_1\big(\tau, \mathbf{u}, \mathbf{v}\big) = \alpha_1\big(0, \mathbf{u}, \mathbf{v}\big) + \tau\alpha_{1,\tau}\big(0, \mathbf{u}, \mathbf{v}\big) + \widetilde{\alpha_1}\big(\tau, \mathbf{u}, \mathbf{v}\big),$$

$$\alpha_{1,\tau}\big(0, \mathbf{u}, \mathbf{v}\big) = \int_\Omega B'(0)\nabla\mathbf{u} : (\mathbf{v} \otimes \mathbf{u})\, dx,$$

$$\widetilde{\alpha_1}\big(\tau, \mathbf{u}, \mathbf{v}\big) \leqslant g_1(\tau)\|\mathbf{u}\|_1^2\|\mathbf{v}\|_1.$$

(B3) *For all* $\mathbf{H} \in \mathbf{H}^1(\Omega)$, $T \in L^1(\Omega)$ *and* $\mathbf{v} \in \mathbf{H}^1_0(\Omega)$,

$$\alpha_2\big(\tau, \mathbf{H}, T, \mathbf{v}\big) = \alpha_2\big(0, \mathbf{H}, T, \mathbf{v}\big) + \tau\alpha_{2,\tau}\big(0, \mathbf{H}, T, \mathbf{v}\big) + \widetilde{\alpha_2}\big(\tau, \mathbf{H}, T, \mathbf{v}\big),$$

$$\alpha_{2,\tau}\big(0, \mathbf{H}, T, \mathbf{v}\big) = \int_\Omega \mu(T)\big(\zeta'(0)(\nabla \times \mathbf{H}) \times \mathbf{H} + \big(\big(\varrho'(0)\nabla\big) \times \mathbf{H}\big) \times \mathbf{H}\big) \cdot \mathbf{v}\, dx,$$

$$\widetilde{\alpha_2}\big(\tau, \mathbf{H}, T, \mathbf{v}\big) \leqslant g_2(\tau)\mu^\#\|\nabla \times \mathbf{H}\|\|\mathbf{H}\|_1\|\mathbf{v}\|_1.$$

(B4) *For all* $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $T \in Z$ *and* $\mathbf{v} \in \mathbf{H}^1_0(\Omega)$,

$$\alpha_3(\tau, \mathbf{f}, T, \mathbf{v}) = \alpha_3(0, \mathbf{f}, T, \mathbf{v}) + \tau\alpha_{3,\tau}(0, \mathbf{f}, T, \mathbf{v}) + \widetilde{\alpha_3}(\tau, \mathbf{f}, T, \mathbf{v}),$$

$$\alpha_{3,\tau}(0, \mathbf{f}, T, \mathbf{v}) = \int_\Omega \zeta'(0)\big(\mathbf{f} - \mathbf{G}(T)T\big) \cdot \mathbf{v}\, dx,$$

$$\widetilde{\alpha_3}(\tau, \mathbf{f}, T, \mathbf{v}) \leqslant g_3(\tau)\big(\|\mathbf{f}\| + G^\#\|T\|\big)\|\mathbf{v}\|.$$

(B5) *For all* $p \in L^2(\Omega)$ *and* $\mathbf{v} \in \mathbf{H}^1_0(\Omega)$,

$$\alpha_4(\tau, p, \mathbf{v}) = \alpha_4(0, p, \mathbf{v}) + \tau\alpha_{4,\tau}(0, p, \mathbf{v}) + \widetilde{\alpha_4}(\tau, p, \mathbf{v}),$$

$$\alpha_{4,\tau}(0, p, \mathbf{v}) = \int_\Omega pB'(0) : \nabla\mathbf{v}\, dx,$$

$$\widetilde{\alpha_4}(\tau, p, \mathbf{v}) \leqslant g_4(\tau)\|p\|\|\mathbf{v}\|_1.$$

(B6) *For all* $T \in L^1(\Omega)$ *and* $\phi, \psi \in Z$,

$$\beta_1(\tau, T, \phi, \psi) = \beta_1(0, T, \phi, \psi) + \tau\beta_{1,\tau}(0, T, \phi, \psi) + \widetilde{\beta_1}(\tau, T, \phi, \psi),$$

$$\beta_{1,\tau}(0, T, \phi, \psi) = \int_\Omega \sigma(T)A'(0) : (\nabla\phi \otimes \nabla\psi)\, dx,$$

$$\widetilde{\beta_1}(\tau, T, \phi, \psi) \leqslant g_5(\tau)\sigma^\#\|\phi\|_1\|\psi\|_1.$$

(B7) *For all* $\mathbf{J}_0 \in \mathbf{L}^2(\Omega)$, $T \in L^1(\Omega)$, $\mathbf{H} \in \mathbf{H}^1(\Omega)$, $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ *and* $\psi \in Z$,

$$\beta_2(\tau, \mathbf{J}_0, T, \mathbf{H}, \mathbf{u}, \psi) = \beta_2(0, \mathbf{J}_0, T, \mathbf{H}, \mathbf{u}, \psi) + \tau \beta_{2,\tau}(0, \mathbf{J}_0, T, \mathbf{H}, \mathbf{u}, \psi) + \widetilde{\beta_2}(\tau, \mathbf{J}_0, T, \mathbf{H}, \mathbf{u}, \psi),$$

$$\beta_{2,\tau}(0, \mathbf{J}_0, T, \mathbf{H}, \mathbf{u}, \psi) = \int_\Omega B'(0) : \big(\big(\mathbf{J}_0 - \sigma(T)\mu(T)\mathbf{H} \times \mathbf{u}\big) \otimes \nabla\psi\big)\, dx,$$

$$\widetilde{\beta_2}(\tau, \mathbf{J}_0, T, \mathbf{H}, \mathbf{u}, \psi) \leqslant g_6(\tau)\big(\|\mathbf{J}_0\| + \sigma^\# \mu^\# \|\mathbf{H}\|_1 \|\mathbf{u}\|_1\big)\|\psi\|_1.$$

(B8) *For all* $T, \eta \in Z$,

$$\gamma_1(\tau, T, \eta) = \gamma_1(0, T, \eta) + \tau \gamma_{1,\tau}(0, T, \eta) + \widetilde{\gamma_1}(\tau, T, \eta),$$

$$\gamma_{1,\tau}(0, T, \eta) = \int_\Omega k(T) A'(0) : (\nabla T \otimes \nabla \eta)\, dx,$$

$$\widetilde{\gamma_1}(\tau, T, \eta) \leqslant g_7(\tau) k^\# \|T\|_1 \|\eta\|_1.$$

(B9) *For all* $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ *and* $T, \eta \in Z$,

$$\gamma_2(\tau, \mathbf{u}, T, \eta) = \gamma_2(0, \mathbf{u}, T, \eta) + \tau \gamma_{2,\tau}(0, \mathbf{u}, T, \eta) + \widetilde{\gamma_2}(\tau, \mathbf{u}, T, \eta),$$

$$\gamma_{2,\tau}(0, \mathbf{u}, T, \eta) = \int_\Omega B'(0) : (\mathbf{u} \otimes \nabla T)\eta\, dx,$$

$$\widetilde{\gamma_2}(\tau, \mathbf{u}, T, \eta) \leqslant g_8(\tau)\|\mathbf{u}\|_1 \|T\|_1 \|\eta\|_1.$$

(B10) *For all* $\alpha \in L^2(\Gamma_N)$ *and* $T, \eta \in Z$,

$$\gamma_3(\tau, \alpha, T, \eta) = \gamma_3(0, \alpha, T, \eta) + \tau \gamma_{3,\tau}(0, \alpha, T, \eta) + \widetilde{\gamma_3}(\tau, \alpha, T, \eta),$$

$$\gamma_{3,\tau}(0, \alpha, T, \eta) = \int_{\Gamma_N} \alpha T \eta \omega'(0)\, ds,$$

$$\widetilde{\gamma_3}(\tau, \alpha, T, \eta) \leqslant g_9(\tau)\|\alpha\|_{\Gamma_N} \|T\|_1 \|\eta\|_1.$$

(B11) *For all* $f \in L^2(\Omega)$ *and* $\eta \in Z$,

$$\gamma_4(\tau, f, \eta) = \gamma_4(0, f, \eta) + \tau \gamma_{4,\tau}(0, f, \eta) + \widetilde{\gamma_4}(\tau, f, \eta),$$

$$\gamma_{4,\tau}(0, f, \eta) = \int_\Omega f \eta \zeta'(0)\, dx,$$

$$\widetilde{\gamma_4}(\tau, f, \eta) \leqslant g_{10}(\tau)\|f\|\|\eta\|.$$

(B12) *For all* $h \in L^2(\Gamma_N)$ *and* $\eta \in Z$,

$$\gamma_5(\tau, h, \eta) = \gamma_5(0, h, \eta) + \tau \gamma_{5,\tau}(0, h, \eta) + \widetilde{\gamma_5}(\tau, h, \eta),$$

$$\gamma_{5,\tau}(0, h, \eta) = \int_{\Gamma_N} h \eta \omega'(0)\, ds,$$

$$\widetilde{\gamma_5}(\tau, h, \eta) \leqslant g_{11}(\tau)\|h\|_{\Gamma_N} \|\eta\|_1.$$

Applying Taylor polynomials of degree one (cf. Lemma 8.10) we can prove the stability result.

Proposition 8.14. *Under the assumptions of Theorem* 8.12*, if* $(\mathbf{u}_\tau, \phi_\tau, T_\tau)$ *is a transported solution which converges to* $(\mathbf{u}, \phi, T)$ *with* $\tau \to 0$*, and in addition the following assumptions on the continuity of the data are fulfilled*:

(M1) $\|\mathbf{H}_\tau - \mathbf{H}\|_1 \leqslant C|\tau|$,
(M2) $\|\mathbf{f}_\tau - \mathbf{f}\| \leqslant C|\tau|$,
(M3) $\|\mathbf{G}(T_\tau)T_\tau - \mathbf{G}(T)T\|_{6/5} \leqslant C|\tau|$,
(M4) $\|\mathbf{J}_{0\tau} - \mathbf{J}_0\| \leqslant C|\tau|$,
(M5) $\|\alpha_\tau - \alpha\|_{\Gamma_N} \leqslant C|\tau|$,
(M6) $\|h_\tau - h\|_{\Gamma_N} \leqslant C|\tau|$,
(M7) $\|f_\tau - f\| \leqslant C|\tau|$,

then we have

$$\|\mathbf{u}_\tau - \mathbf{u}\|_1 \leqslant C|\tau|; \tag{60}$$

$$\|\phi_\tau - \phi\|_1 \leqslant C|\tau|; \tag{61}$$

$$\|T_\tau - T\|_1 \leqslant C|\tau|; \tag{62}$$

$$\|p_\tau - p\| \leqslant C|\tau|, \tag{63}$$

where C stands for a generic constant.

Proof. For τ small enough and $\xi_i \in [0, \tau]$, $i = 0, \dots, 4$, we can write

$$\zeta(\tau) = 1 + \tau\zeta'(\xi_0),$$

$$\varrho(\tau) = I + \tau\varrho'(\xi_1),$$

$$A(\tau) = I + \tau A'(\xi_2),$$

$$B(\tau) = I + \tau B'(\xi_3),$$

$$\omega(\tau) = 1 + \tau\omega'(\xi_4),$$

where $\zeta(\tau) \geqslant c_{\tau_1} > 0$ for $|\tau| \leqslant \tau_1$ and A, B, ϱ are positive definite for $|\tau| \leqslant \tau_1$.

Observing that α_4 is linear with respect to the second argument, we write

$$\alpha_4(\tau, p_\tau, \mathbf{v}) - \alpha_4(0, p, \mathbf{v}) = \alpha_4(0, p_\tau - p, \mathbf{v}) + \tau \int_\Omega p_\tau B'(\xi_3) : \nabla\mathbf{v}\, dx. \tag{64}$$

Observe that α_i $(i = 0, 1, 2, 3)$ is no more linear with respect to its arguments, thus we write

$$\begin{aligned} &\alpha_0(\tau, \mathbf{u}_\tau, T_\tau, \mathbf{v}) - \alpha_0(0, \mathbf{u}, T, \mathbf{v}) \\ &\quad = \alpha_0(0, \mathbf{u}_\tau, T_\tau, \mathbf{v}) - \alpha_0(0, \mathbf{u}, T, \mathbf{v}) + \tau \int_\Omega \nu(T_\tau) A'(\xi_2) : (D\mathbf{u}_\tau D\mathbf{v})\, dx; \end{aligned} \tag{65}$$

$$\alpha_1(\tau,\mathbf{u}_\tau,\mathbf{v})-\alpha_1(0,\mathbf{u},\mathbf{v})$$
$$=\alpha_1(0,\mathbf{u}_\tau,\mathbf{v})-\alpha_1(0,\mathbf{u},\mathbf{v})+\tau\int_\Omega B'(\xi_3)\nabla\mathbf{u}_\tau:(\mathbf{v}\otimes\mathbf{u}_\tau)\,dx; \tag{66}$$

$$\alpha_2(\tau,\mathbf{H}_\tau,T_\tau,\mathbf{v})-\alpha_2(0,\mathbf{H},T,\mathbf{v})$$
$$=\alpha_2(0,\mathbf{H}_\tau,T_\tau,\mathbf{v})-\alpha_2(0,\mathbf{H},T,\mathbf{v})$$
$$+\tau\int_\Omega \mu(T_\tau)\big(\zeta'(\xi_0)(\nabla\times\mathbf{H}_\tau)\times\mathbf{H}_\tau+\big(\big(\varrho'(\xi_1)\nabla\big)\times\mathbf{H}_\tau\big)\times\mathbf{H}_\tau\big)\cdot\mathbf{v}\,dx; \tag{67}$$

$$\alpha_3(\tau,\mathbf{f}_\tau,T_\tau,\mathbf{v})-\alpha_3(0,\mathbf{f},T,\mathbf{v})$$
$$=\alpha_3(0,\mathbf{f}_\tau,T_\tau,\mathbf{v})-\alpha_3(0,\mathbf{f},T,\mathbf{v})+\tau\int_\Omega \zeta'(\xi_0)\big(\mathbf{f}_\tau-\mathbf{G}(T_\tau)T_\tau\big)\cdot\mathbf{v}\,dx. \tag{68}$$

Considering that $\mathbf{u}$ is the particular case $(\tau=0)$ to the perturbed $\mathbf{u}_\tau$ it follows that

$$\text{RHS of (65)}+\text{RHS of (66)}=\text{RHS of (67)}+\text{RHS of (68)}+\text{RHS of (64)}.$$

Next, we write

(N1) $\beta_1(\tau,T_\tau,\phi_\tau,\psi)-\beta_1(0,T,\phi,\psi)=\beta_1(0,T_\tau,\phi_\tau,\psi)-\beta_1(0,T,\phi,\psi)+\tau\int_\Omega\sigma(T_\tau)A'(\xi_2):(\nabla\phi_\tau\otimes\nabla\psi)\,dx$;
(N2) $\beta_2(\tau,\mathbf{J}_{0\tau},T_\tau,\mathbf{H}_\tau,\mathbf{u}_\tau,\psi)-\beta_2(0,\mathbf{J}_0,T,\mathbf{H},\mathbf{u},\psi)=\beta_2(0,\mathbf{J}_{0\tau},T_\tau,\mathbf{H}_\tau,\mathbf{u}_\tau,\psi)-\beta_2(0,\mathbf{J}_0,T,\mathbf{H},\mathbf{u},\psi)+\tau\int_\Omega B'(\xi_3):((\mathbf{J}_{0\tau}-\sigma(T_\tau)\mu(T_\tau)\mathbf{H}_\tau\times\mathbf{u}_\tau)\otimes\nabla\psi)\,dx$;
(N3) $\gamma_1(\tau,T_\tau,\eta)-\gamma_1(0,T,\eta)=\gamma_1(0,T_\tau,\eta)-\gamma_1(0,T,\eta)+\tau\int_\Omega k(T_\tau)A'(\xi_2):(\nabla T_\tau\otimes\nabla\eta)\,dx$;
(N4) $\gamma_2(\tau,\mathbf{u}_\tau,T_\tau,\eta)-\gamma_2(0,\mathbf{u},T,\eta)=\gamma_2(0,\mathbf{u}_\tau,T_\tau,\eta)-\gamma_2(0,\mathbf{u},T,\eta)+\tau\int_\Omega B'(\xi_3):(\mathbf{u}_\tau\otimes\nabla T_\tau)\eta\,dx$;
(N5) $\gamma_3(\tau,\alpha_\tau,T_\tau,\eta)-\gamma_3(0,\alpha,T,\eta)=\gamma_3(0,\alpha_\tau,T_\tau,\eta)-\gamma_3(0,\alpha,T,\eta)+\tau\int_{\Gamma_N}\alpha_\tau T_\tau\eta\omega'(\xi_4)\,ds$;
(N6) $\gamma_4(\tau,f_\tau,\eta)-\gamma_4(0,f,\eta)=\gamma_4(0,f_\tau-f,\eta)+\tau\int_\Omega f_\tau\eta\zeta'(\xi_0)\,dx$;
(N7) $\gamma_5(\tau,h_\tau,\eta)-\gamma_5(0,h,\eta)=\gamma_5(0,h_\tau-h,\eta)+\tau\int_{\Gamma_N}h_\tau\eta\omega'(\xi_4)\,ds$.

Considering that ϕ and T are the particular case $(\tau=0)$ to the perturbed ϕ_τ, and T_τ, respectively, we can argue as in the proof of Theorem 3.2, setting $\mathbf{v}=\mathbf{u}_\tau-\mathbf{u}$, $\psi=\phi_\tau-\phi$ and $\eta=T_\tau-T$ to get the following estimate

$$\begin{aligned}
&\left(\nu_\#-C^2K_5-\frac{C\mu^\#\sigma^\#}{\sigma_\#}\|\mathbf{H}\|_1^2-\frac{C}{k_\#}K_4^2\right)\|\mathbf{u}_\tau-\mathbf{u}\|_1^2+\sigma_\#\|\phi_\tau-\phi\|_1^2\\
&\quad+\left(k_\#-\frac{C\bar{\nu}}{\nu_\#}K_1^2-\frac{C}{\nu_\#}\big(\bar{\mu}\big\|(\nabla\times\mathbf{H})\times\mathbf{H}\big\|_{3/2}+G^\#+\bar{G}K_4\big)^2\right.\\
&\quad\left.-\frac{C}{\sigma_\#}\big(\bar{\sigma}K_2^2+\big(\mu^\#\bar{\sigma}+\bar{\mu}\sigma^\#\big)\|\mathbf{H}\|_1^2K_5^2\big)-\frac{C\bar{k}}{k_\#}K_3^2\right)\|T_\tau-T\|_1^2\\
&\leqslant\frac{C}{\nu_\#}\big(\mu^\#\big\|(\nabla\times\mathbf{H}_\tau)\times\mathbf{H}_\tau-(\nabla\times\mathbf{H})\times\mathbf{H}\big\|_{6/5}+\|\mathbf{f}_\tau-\mathbf{f}\|+\|p_\tau-p\|+|\tau|\big)^2\\
&\quad+\frac{C}{\sigma_\#}\big(\|\mathbf{J}_{0\tau}-\mathbf{J}_0\|+\mu^\#\sigma^\#\|\mathbf{H}_\tau-\mathbf{H}\|_1+|\tau|\big)^2\\
&\quad+\frac{C}{k_\#}\big(\|\alpha_\tau-\alpha\|_{\Gamma_N}+\|f_\tau-f\|+\|h_\tau-h\|_{\Gamma_N}+|\tau|\big)^2.
\end{aligned}$$

Under the smallness condition on the data, we argue as in Theorem 8.11 and from assumptions (M1)–(M7), and we get (60)–(63). □

8.3. Material derivative

Definition 8.15. The following limit in the function space norm $\mathcal{H}$

$$\dot{f} = \lim_{\tau \to 0} \frac{f(\tau) - f(0)}{\tau}$$

is called the strong material derivative $\dot{f}$ of f in $\mathcal{H}$.

Definition 8.16. The shape derivative u' of $u(\tau)$ in the direction of the vector field V is defined by the formula

$$u' = \dot{u} - \nabla u \cdot V$$

provided that there exists the material derivative $\dot{u}$.

We recall that $A(0) = B(0) = \varrho(0) = I$, $\zeta(0) = \omega(0) = 1$, $\dot{\varrho} = \varrho'(0)$, $\dot{A} = A'(0)$, $\dot{B} = B'(0)$, $\dot{\zeta} = \zeta'(0)$ and $\dot{\omega} = \omega'(0)$, and we state the following result on the existence of material derivatives.

Theorem 8.17. *Under the assumptions $\dot{\mathbf{f}} \in \mathbf{L}^2(\Omega)$, $\dot{\mathbf{j}} \in \mathbf{L}^2(\Omega)$, $\dot{f} \in L^2(\Omega)$, $\dot{\alpha}, \dot{h} \in L^2(\Gamma_N)$, for every magnetic field $\dot{\mathbf{H}} \in \mathbf{H}^1(\Omega)$ such that $\nabla \cdot \dot{\mathbf{H}} = 0$ in Ω, $\dot{\mathbf{H}} \cdot \dot{\mathbf{n}} = 0$ on $\partial\Omega$ and $\nu, \mu, \sigma, k \in C^1(\Omega)$ the triple $(\dot{\mathbf{u}}, \dot{\phi}, \dot{T}) \in \mathbf{V} \times Z^2$ and it satisfies*:

Momentum equations:

$$\begin{aligned}
&\int_\Omega \nu(T)(\dot{A} D\mathbf{u} + D\dot{\mathbf{u}}) : D\mathbf{v}\, dx + \int_\Omega \nu'(T)\dot{T} D\mathbf{u} : D\mathbf{v}\, dx \\
&\quad + \int_\Omega (\dot{B}\nabla\mathbf{u} + \nabla\dot{\mathbf{u}}) : (\mathbf{v} \otimes \mathbf{u})\, dx + \int_\Omega \nabla\mathbf{u} : (\mathbf{v} \otimes \dot{\mathbf{u}})\, dx \\
&\quad + \int_\Omega \big\{(\dot{\zeta} p + \dot{p})\nabla \cdot \mathbf{v} + p(\dot{\varrho}\nabla) \cdot \mathbf{v}\big\}\, dx \\
&= \int_\Omega \mu(T)\dot{\zeta}\big((\nabla \times \mathbf{H}) \times \mathbf{H}\big) \cdot \mathbf{v}\, dx + \int_\Omega \mu(T)\Big(\big((\dot{\varrho}\nabla) \times \mathbf{H}\big) \times \mathbf{H}\Big) \cdot \mathbf{v}\, dx \\
&\quad + \int_\Omega \mu(T)\big((\nabla \times \dot{\mathbf{H}}) \times \mathbf{H} + (\nabla \times \mathbf{H}) \times \dot{\mathbf{H}}\big) \cdot \mathbf{v}\, dx \\
&\quad + \int_\Omega \big(\mu'(T)\dot{T}(\nabla \times \mathbf{H}) \times \mathbf{H}\big) \cdot \mathbf{v}\, dx \\
&\quad + \int_\Omega \big(\dot{\mathbf{f}} - \dot{\mathbf{G}}(T)T - \mathbf{G}(T)\dot{T} + \big(\mathbf{f} - \mathbf{G}(T)T\big)\dot{\zeta}\big) \cdot \mathbf{v}\, dx, \quad \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega);
\end{aligned} \tag{69}$$

Equation for the electric field:

$$\begin{aligned}
&\int_\Omega \sigma(T)(\dot{A}\nabla\phi + \nabla\dot{\phi})\cdot\nabla\psi\,dx + \int_\Omega \sigma'(T)\dot{T}\nabla\phi\cdot\nabla\psi\,dx \\
&\quad = \int_\Omega \dot{B} : \big(\mathbf{J}_0 - \sigma(T)\mu(T)\mathbf{H}\times\mathbf{u}\big)\otimes\nabla\psi\,dx \\
&\qquad + \int_\Omega \big(\dot{\mathbf{J}}_0 - \sigma(T)\mu(T)\dot{\mathbf{H}}\times\mathbf{u} - \sigma(T)\mu(T)\mathbf{H}\times\dot{\mathbf{u}}\big)\cdot\nabla\psi\,dx \\
&\qquad + \int_\Omega \big(\big(\sigma'(T)\dot{T}\mu(T) + \sigma(T)\mu'(T)\dot{T}\big)\mathbf{H}\times\mathbf{u}\big)\cdot\nabla\psi\,dx, \quad \forall\psi\in Z;
\end{aligned} \tag{70}$$

Energy equation:

$$\begin{aligned}
&\int_\Omega k(T)(\dot{A}\nabla T + \nabla\dot{T})\cdot\nabla\eta\,dx + \int_\Omega k'(T)\dot{T}\nabla T\cdot\nabla\eta\,dx \\
&\quad + \int_\Omega (\dot{B} : \mathbf{u}\otimes\nabla T + \dot{\mathbf{u}}\cdot\nabla T + \mathbf{u}\cdot\nabla\dot{T})\eta\,dx \\
&\quad + \int_{\Gamma_N} \alpha(\dot{T} + T\dot{\omega})\eta\,ds + \int_{\Gamma_N} \dot{\alpha}T\eta\,ds \\
&= \int_\Omega (\dot{f} + f\dot{\zeta})\eta\,dx + \int_{\Gamma_N} (\dot{h} + h\dot{\omega})\eta\,ds, \quad \forall\eta\in Z;
\end{aligned} \tag{71}$$

and the following estimate

$$\begin{aligned}
\|\dot{T}\|_1 + \|\dot{\mathbf{u}}\|_1 + \|\dot{\phi}\|_1 \leqslant C\Big(&\big(1 + \|\mathbf{u}\|_1 + \|\dot{\alpha}\|_{\Gamma_N} + \|\alpha\|_{\Gamma_N}\big)\|T\|_1 \\
&+ \big(1 + \|\mathbf{u}\|_1\big)\|\mathbf{u}\|_1 + \|\phi\|_1 + \|\dot{f}\| + \|f\| + \|\dot{h}\|_{\Gamma_N} + \|h\|_{\Gamma_N} \\
&+ \big(\|\dot{\mathbf{H}}\|_1 + \|\mathbf{H}\|_1\big)\|\mathbf{H}\|_1 + \|\dot{\mathbf{f}}\| + \|\mathbf{f}\| + \big(\dot{G}^{\#} + G^{\#}\big)\|T\|_1 \\
&+ \|\dot{\mathbf{H}}\times\mathbf{u}\| + \|\mathbf{H}\times\dot{\mathbf{u}}\| + \|\mathbf{H}\times\mathbf{u}\| + \|\dot{\mathbf{J}}_0\| + \|\mathbf{J}_0\|\Big),
\end{aligned} \tag{72}$$

where C denotes a positive constant depending on the upper and lower bounds of ν, μ, σ, k and its derivatives.

Proof. The estimate (72) we get as before, namely from (69), (70) and (71) using the uniqueness argument of Theorem 3.2. Now, it remains to prove that they are material derivative.

The transported solution given by Theorem 8.12 verifies, for all $\mathbf{v}\in\mathbf{H}^1_0(\Omega)$ and $\psi,\eta\in Z$,

$$\begin{aligned}
&\int_\Omega \nu(T_\tau)A(\tau) : (D\mathbf{u}_\tau D\mathbf{v}) + \int_\Omega B(\tau)\nabla\mathbf{u}_\tau : (\mathbf{v}\otimes\mathbf{u}_\tau) \\
&\quad = \int_\Omega p_\tau B(\tau) : D\mathbf{v} + \zeta(\tau)\big(\mu(T_\tau)\big(\big(\varrho(\tau)\nabla\big)\times\mathbf{H}_\tau\big)\times\mathbf{H}_\tau + \mathbf{f}_\tau - \mathbf{G}(T_\tau)T_\tau\big)\cdot\mathbf{v}\,dx;
\end{aligned} \tag{73}$$

$$\int_\Omega \sigma(T_\tau)A(\tau):(\nabla\phi_\tau\otimes\nabla\psi)\,dx$$
$$=\int_\Omega B(\tau):\big(\big(\mathbf{J}_{0\tau}-\sigma(T_\tau)\mu(T_\tau)\mathbf{H}_\tau\times\mathbf{u}_\tau\big)\otimes\nabla\psi\big)\,dx; \tag{74}$$

$$\int_\Omega k(T_\tau)A(\tau):(\nabla T_\tau\otimes\nabla\eta)\,dx+\int_\Omega B(\tau):(\mathbf{u}_\tau\otimes\nabla T_\tau)\eta\,dx+\int_{\Gamma_N}\alpha_\tau T_\tau\eta\omega(\tau)\,ds$$
$$=\int_\Omega f_\tau\eta\zeta(\tau)dx+\int_{\Gamma_N}h_\tau\eta\omega(\tau)\,ds. \tag{75}$$

The solution given by Theorem 3.2 verifies (20)–(21), and applying the De Rham Theorem, also

$$\int_\Omega \nu(T)D\mathbf{u}:D\mathbf{v}\,dx+\int_\Omega D\mathbf{u}:(\mathbf{u}\otimes\mathbf{v})\,dx$$
$$=\int_\Omega p\nabla\cdot\mathbf{v}\,dx+\int_\Omega\big(\mu(T)(\nabla\times\mathbf{H})\times\mathbf{H}+\mathbf{f}-\mathbf{G}(T)T\big)\cdot\mathbf{v}\,dx,\quad\forall\mathbf{v}\in\mathbf{H}_0^1(\Omega). \tag{76}$$

Let

$$\mathbf{z}_\tau=\frac{1}{\tau}(\mathbf{u}_\tau-\mathbf{u});\qquad \Upsilon_\tau=\frac{1}{\tau}(T_\tau-T);\qquad \Phi_\tau=\frac{1}{\tau}(\phi_\tau-\phi);\qquad \pi_\tau=\frac{1}{\tau}(p_\tau-p).$$

From Proposition 8.14 we find the following results (as $\tau\to 0$):

(i) $\mathbf{z}_\tau\to\dot{\mathbf{u}}$ weakly in $\mathbf{H}_0^1(\Omega)$, $\mathbf{u}_\tau\to\mathbf{u}$ strongly in $\mathbf{H}_0^1(\Omega)$;
(ii) $\Upsilon_\tau\to\dot T$ weakly in Z, $T_\tau\to T$ strongly in Z;
(iii) $\Phi_\tau\to\dot\phi$ weakly in Z, $\phi_\tau\to\phi$ strongly in Z;
(iv) $\pi_\tau\to\dot p$ weakly in $L^2(\Omega)$, $p_\tau\to p$ strongly in $L^2(\Omega)$.

Applying the Sobolev–Kondrachov compact embedding to (i)–(iii), the sequences $\mathbf{z}_\tau$, Υ_τ and Φ_τ are strongly convergent in $L^q(\Omega)$ for any $1\leqslant q<6$ and, in particular, it follows $\mathbf{u}_\tau\to\mathbf{u}$, $T_\tau\to T$ and $\phi_\tau\to\phi$ strongly in $\mathbf{L}^6(\Omega)$, $L^6(\Omega)$ and $L^6(\Omega)$, respectively. Thus, considering the relation

$$\frac{1}{\tau}\big(\nu(T_\tau)-\nu(T)\big)=\nu'(T_\nu)\Upsilon_\tau,\quad\text{for some } T_\nu \text{ between } T_\tau \text{ and } T,$$

applying (22) and the continuity of ν' we obtain

(v) $\frac{\nu(T_\tau)-\nu(T)}{\tau}\to\nu'(T)\dot T$ weakly in $L^{2(2+\delta)/\delta}(\Omega)$ and strongly in $L^{q(2+\delta)/(3\delta)}(\Omega)$.

Recalling that $D\mathbf{u}\in\mathbf{L}^{2+\delta}(\Omega)$ it results from (v) that $D\mathbf{u}(\nu(T_\tau)-\nu(T))/\tau\to\nu'(T)\dot T D\mathbf{u}$ weakly in $\mathbf{L}^2(\Omega)$.

Analogously we have

(vi) $\frac{\mu(T_\tau)-\mu(T)}{\tau}\to\mu'(T)\dot T$ weakly in $L^6(\Omega)$ and strongly in $L^q(\Omega)$;
(vii) $\frac{\sigma(T_\tau)-\sigma(T)}{\tau}\to\sigma'(T)\dot T$ weakly in $L^{2(2+\epsilon)/\epsilon}(\Omega)$ and strongly in $L^{q(2+\epsilon)/(3\epsilon)}(\Omega)$;
(viii) $\frac{k(T_\tau)-k(T)}{\tau}\to k'(T)\dot T$ weakly in $L^{2(2+\varepsilon)/\varepsilon}(\Omega)$ and strongly in $L^{q(2+\varepsilon)/(3\varepsilon)}(\Omega)$.

Subtracting (73), (74) and (75) by (76), (20) and (21), respectively, dividing by $\tau \neq 0$ we get:
Momentum equations:

$$\begin{aligned}
&\int_\Omega \nu(T_\tau)\left(\frac{1}{\tau}\big(A(\tau)-I\big):D\mathbf{u}D\mathbf{v}+A(\tau):D\mathbf{z}_\tau D\mathbf{v}\right)dx\\
&\quad+\int_\Omega \frac{1}{\tau}\big(\nu(T_\tau)-\nu(T)\big)D\mathbf{u}:D\mathbf{v}\,dx\\
&\quad+\int_\Omega B(\tau)\nabla\mathbf{z}_\tau:\mathbf{v}\otimes\mathbf{u}_\tau\,dx+\int_\Omega B(\tau)D\mathbf{u}:\mathbf{v}\otimes\mathbf{z}_\tau\,dx\\
&\quad+\int_\Omega \frac{1}{\tau}\big(B(\tau)-I\big)\nabla\mathbf{u}:(\mathbf{v}\otimes\mathbf{u})\,dx\\
&\quad+\int_\Omega \left(p_\tau B(\tau)+p\left(\frac{1}{\tau}\big(B(\tau)-I\big)\right)\right):D\mathbf{v}\,dx\\
&=\int_\Omega \mu(T_\tau)\zeta(\tau)\left(\big(\varrho(\tau)\nabla\big)\times\frac{\mathbf{H}_\tau-\mathbf{H}}{\tau}\right)\times\mathbf{H}_\tau\cdot\mathbf{v}\,dx\\
&\quad+\int_\Omega \mu(T_\tau)\zeta(\tau)\big(\big(\varrho(\tau)\nabla\big)\times\mathbf{H}_\tau\big)\times\frac{1}{\tau}(\mathbf{H}_\tau-\mathbf{H})\cdot\mathbf{v}\,dx\\
&\quad+\int_\Omega \mu(T_\tau)\zeta(\tau)\left(\left(\left(\frac{1}{\tau}\big(\varrho(\tau)-I\big)\nabla\right)\times\mathbf{H}_\tau\right)\times\mathbf{H}_\tau\right)\cdot\mathbf{v}\,dx\\
&\quad+\int_\Omega \frac{1}{\tau}\big(\mu(T_\tau)-\mu(T)\big)\zeta(\tau)\big(\big(\big(\rho(\tau)\nabla\big)\times\mathbf{H}\big)\times\mathbf{H}\big)\cdot\mathbf{v}\,dx\\
&\quad+\int_\Omega \left(\frac{1}{\tau}\big(\zeta(\tau)-I\big)\right)\big(\mu(T)\big(\big(\varrho(\tau)\nabla\big)\times\mathbf{H}\big)\times\mathbf{H}+\mathbf{f}-\mathbf{G}(T)T\big)\cdot\mathbf{v}\,dx\\
&\quad+\int_\Omega \zeta(\tau)\left(\frac{1}{\tau}(\mathbf{f}_\tau-\mathbf{f})-\left(\frac{1}{\tau}\big(\mathbf{G}(T_\tau)-\mathbf{G}(T)\big)\right)T_\tau+\frac{1}{\tau}(T_\tau-T)\mathbf{G}(T)\right)\cdot\mathbf{v}\,dx,\\
&\forall\mathbf{v}\in\mathbf{H}_0^1(\Omega);
\end{aligned}$$

Electric equation:

$$\begin{aligned}
&\int_\Omega \sigma(T_\tau)A(\tau):(\nabla\Phi_\tau\otimes\nabla\psi)\,dx+\int_\Omega \sigma(T_\tau)\left(\frac{1}{\tau}\big(A(\tau)-I\big)\right):(\nabla\phi\otimes\nabla\psi)\,dx\\
&\quad+\int_\Omega \frac{1}{\tau}\big(\sigma(T_\tau)-\sigma(T)\big)\nabla\phi\cdot\nabla\psi\,dx\\
&=\int_\Omega B(\tau):\left(\frac{1}{\tau}(\mathbf{J}_{0\tau}-\mathbf{J}_0)-\sigma(T_\tau)\mu(T_\tau)\left(\frac{\mathbf{H}_\tau-\mathbf{H}}{\tau}\times\mathbf{u}+\mathbf{H}_\tau\times\mathbf{z}_\tau\right)\right)\otimes\nabla\psi\,dx
\end{aligned}$$

$$+\int_{\Omega} B(\tau) : \left(\frac{1}{\tau}\big(\big(\sigma(T_\tau)-\sigma(T)\big)\mu(T)+\sigma(T)\big(\mu(T_\tau)-\mu(T)\big)\big)\mathbf{H}\times\mathbf{u}\right)\otimes\nabla\psi\,dx$$

$$+\int_{\Omega}\left(\frac{1}{\tau}\big(B(\tau)-I\big)\right) : \big(\big(\mathbf{J}_{0\tau}-\sigma(T)\mu(T)\mathbf{H}_\tau\times\mathbf{u}_\tau\big)\otimes\nabla\psi\big)\,dx,\quad \forall\psi\in Z;$$

Energy equation:

$$\int_{\Omega} k(T_\tau)A(\tau) : (\nabla\Upsilon_\tau\otimes\nabla\eta)dx+\int_{\Omega} k(T_\tau)\left\{\frac{1}{\tau}\big(A(\tau)-I\big)\right\} : (\nabla T\otimes\nabla\eta)\,dx$$

$$+\int_{\Omega}\frac{1}{\tau}\big(k(T_\tau)-k(T)\big)\nabla T\cdot\nabla\eta\,dx$$

$$+\int_{\Omega}\left(\left\{\frac{1}{\tau}\big(B(\tau)-I\big)\right\}\mathbf{u}\cdot\nabla T+B(\tau)\mathbf{z}_\tau\cdot\nabla T_\tau+B(\tau)\mathbf{u}\cdot\nabla\Upsilon_\tau\right)\eta\,dx$$

$$+\int_{\Gamma_N}\alpha_\tau\Upsilon_\tau\eta\omega(\tau)\,ds+\int_{\Gamma_N}\alpha_\tau T\eta\left(\frac{1}{\tau}\big(\omega(\tau)-I\big)\right)ds+\int_{\Gamma_N}\left(\frac{1}{\tau}(\alpha_\tau-\alpha)\right)T\eta\,ds$$

$$=\int_{\Omega}\left(\frac{1}{\tau}(f_\tau-f)\right)\eta\zeta(\tau)\,dx+\int_{\Omega} f_\tau\eta\left(\frac{1}{\tau}\big(\zeta(\tau)-I\big)\right)dx$$

$$+\int_{\Gamma_N}\left(\frac{1}{\tau}(h_\tau-h)\right)\eta\omega(\tau)\,ds+\int_{\Gamma_N} h_\tau\eta\left(\frac{1}{\tau}\big(\omega(\tau)-I\big)\right)ds,\quad\forall\eta\in Z.$$

Therefore passing to the limit in the momentum, electric and energy equations as τ tends to zero, applying the convergences (i)–(viii), Lemma 8.10 and Corollary 8.13 we conclude (69), (70) and (71). □

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