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## On Calculating with $B$ -Splines

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### INTRODUCTION

In computational dealings with splines, the question of representation is of primary importance. For splines of fixed order on a fixed partition, this is a question of *choice of basis*, since such splines form a linear space. Only three kinds of bases for spline spaces have actually been given serious attention; those consisting of truncated power functions, of cardinal splines, and of  $B$ -splines. Truncated power bases are known to be open to severe ill-conditioning, while cardinal splines are difficult to calculate. By contrast, bases consisting of  $B$ -splines are well-conditioned, at least for orders  $\leq 20$ . Such bases are also local in the sense that at every point only a fixed number (equal to the order) of  $B$ -splines is nonzero.  $B$ -splines are also evaluated quite easily, using their definition as a divided difference of the truncated power function. Unfortunately, such calculations are ill-conditioned, particularly for partitions of widely varying interval lengths, as is indicated by the fact that special measures have to be taken in case of coincident knots.

In this note, a different way of evaluating  $B$ -splines is discussed which is very well conditioned yet efficient, and which needs no special adjustments in case of coincident knots. It is also shown that the condition of the  $B$ -spline basis increases exponentially with the order.

#### 1. DEFINITIONS AND BASIC PROPERTIES OF (NORMALIZED) $B$ -SPLINES

$B$ -splines were first introduced by Schoenberg in [5, 2]. A nice compendium of many of their algebraic properties can be found in [3]. These functions are

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also known as hump functions, patch functions or hill functions. In this section, we list a few facts about  $B$ -splines for later reference.

For simplicity, we deal with splines on a bi-infinite partition

$$\pi = \{t_i\}_{i=-\infty}^{\infty}; \quad t_i \leq t_{i+1}, \quad \text{all } i, \quad (1)$$

of the (open) subinterval  $I = (\lim_{i \rightarrow -\infty} t_i, \lim_{i \rightarrow \infty} t_i)$  of the real line. Because of the localness of the  $B$ -splines, it is then a simple matter to specialize to the case of a finite partition of a finite interval (see, e.g., [3 or 1]). With  $k$  a positive integer, let

$$g_k(s; t) = (s - t)_+^{k-1} = \begin{cases} (s - t)^{k-1}, & s \geq t \\ 0, & s < t. \end{cases} \quad (2)$$

Then, the  $B$ -spline  $M_{i,k}(t)$  is given as the  $k$ -th divided difference of  $g_k(s; t)$  in  $s$  on  $t_i, \dots, t_{i+k}$  for fixed  $t$ , i.e.,

$$M_{i,k}(t) = g_k(t_i, \dots, t_{i+k}; t), \quad (3)$$

while the *normalized  $B$ -spline*  $N_{i,k}(t)$  is

$$\begin{aligned} N_{i,k}(t) &= (t_{i+k} - t_i) M_{i,k}(t) \\ &= g_k(t_{i+1}, \dots, t_{i+k}; t) - g_k(t_i, \dots, t_{i+k-1}; t). \end{aligned} \quad (4)$$

If  $k > 1$  and if  $\pi$  is a  $k$ -extended partition [1], i.e., if at most  $k - 1$  consecutive  $t_j$ 's coincide, then both  $M_{i,k}(t)$  and  $N_{i,k}(t)$ , as given by (3) and (4), respectively, are well-defined continuous functions. Otherwise, (3) and (4) make, in general, sense only for  $t \neq t_j$ , all  $j$ , because of the jump discontinuity of

$$(\partial/\partial s)^{k-1} g_k(s; t)$$

at  $s = t$ . Whenever this situation arises, we assume the definitions (3) and (4) to be augmented by the (admittedly arbitrary) demand that  $N_{i,k}(t)$  and  $M_{i,k}(t)$  be *right-continuous* everywhere. For instance, we let

$$M_{i,1}(t) = \begin{cases} (t_{i+1} - t_i)^{-1}, & t_i \leq t < t_{i+1} \\ 0, & \text{otherwise,} \end{cases} \quad (5)$$

hence

$$N_{i,1}(t) = \begin{cases} 1, & t_i \leq t < t_{i+1} \\ 0, & \text{otherwise.} \end{cases} \quad (6)$$

Note that these definitions imply

$$M_{i,1}(t) \equiv N_{i,1}(t) \equiv 0, \quad \text{whenever } t_i = t_{i+1}. \quad (7)$$

Most of the known properties of  $B$ -splines can be derived from the simple identity

$$M_{i,k}(t) = \frac{t - t_i}{t_{i+k} - t_i} M_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} M_{i+1,k-1}(t), \quad (8)$$

which we now prove<sup>1</sup>. For the proof, recall Leibniz' formula

$$h(s_0, \dots, s_k) = \sum_{r=0}^k f(s_0, \dots, s_r) g(s_r, \dots, s_k) \quad (9)$$

for the  $k$ -th divided difference of the function

$$h(s) = f(s) g(s)$$

in terms of the divided differences of  $f(s)$  and  $g(s)$ . Apply (9) to the function

$$h(s) = g_k(s; t) = g_{k-1}(s; t)(s - t)$$

to get

$$g_k(t_i, \dots, t_{i+k}; t) = g_{k-1}(t_i, \dots, t_{i+k-1}; t) \cdot 1 + g_{k-1}(t_i, \dots, t_{i+k}; t) \cdot (t_{i+k} - t),$$

since all divided differences of  $(s - t)$  of order 2 or higher vanish. Hence, with (3),

$$M_{i,k}(t) = M_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_i} (M_{i+1,k-1}(t) - M_{i,k-1}(t)),$$

which is (8), slightly rewritten.

In terms of the  $N_{i,k}$ , (8) reads

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t). \quad (10)$$

The identity (8) states that, for any  $t$ ,  $M_{i,k}(t)$  is an average (or, a linear cross mean, as Aitken would call it) of the numbers  $M_{i,k-1}(t)$  and  $M_{i+1,k-1}(t)$ . Further, for  $t_i < t < t_{i+k}$ ,  $M_{i,k}(t)$  is a strictly convex combination of these two numbers. Since  $M_{i,1}(t)$  is positive for  $t_i \leq t < t_{i+1}$  and zero otherwise, it therefore follows at once from (8) (by induction on  $k$ ) that, for  $k > 1$ ,  $M_{i,k}(t)$  is positive for  $t_i < t < t_{i+k}$  and zero otherwise. The normalized  $B$ -spline  $N_{i,k}(t)$  satisfies, of course, the same condition.

It is this feature which makes the  $N_{i,k}(t)$  so attractive for calculations. In order to evaluate the function

$$F(t) = \sum_i A_i N_{i,k}(t) \quad (11)$$

<sup>1</sup> This identity was also found by Lois Mansfield.

at a point  $\hat{t} \in [t_j, t_{j+1})$ , it is merely necessary to calculate the  $k$  numbers

$$N_{i,k}(\hat{t}), \quad i = j - k + 1, \dots, j;$$

$F(\hat{t})$  is then given by

$$F(\hat{t}) = \sum_{i=j-k+1}^j A_i N_{i,k}(\hat{t}).$$

Differentiation of  $F(t)$  is equally simple. One has

$$\begin{aligned} N_{i,k}^{(1)}(t) &= (d/dt)[g_k(t_{i+1}, \dots, t_{i+k}; t) - g_k(t_i, \dots, t_{i+k-1}; t)] \\ &= -(k-1)[M_{i+1,k-1}(t) - M_{i,k-1}(t)]. \end{aligned}$$

Hence,

$$\begin{aligned} F^{(1)}(t) &= (k-1) \sum_i A_i [M_{i,k-1}(t) - M_{i+1,k-1}(t)] \\ &= (k-1) \sum_i A_i^{(1)} N_{i,k-1}(t), \end{aligned} \tag{12}$$

where

$$A_i^{(1)} = (A_i - A_{i-1}) / (t_{i+k-1} - t_i). \tag{13}$$

More generally, with

$$\begin{aligned} A_i^{(0)} &= A_i, \\ A_i^{(j)} &= (A_i^{(j-1)} - A_{i-1}^{(j-1)}) / (t_{i+k-j} - t_i), \quad j > 0, \end{aligned} \tag{14}$$

one has

$$F^{(j)}(t) = (k-1) \cdots (k-j) \sum_i A_i^{(j)} N_{i,k-j}(t). \tag{15}$$

If  $\pi$  is uniform,

$$t_i = t_0 + ih, \quad \text{all } i,$$

then (15) reduces to

$$F^{(j)}(t) = h^{-j} \sum_i (\nabla^j A_i) N_{i,k-j}(t)$$

which is familiar from [5]. We return to the evaluation of the spline function  $F(t)$  and its derivatives in Section 2.

Using the identity (8), it is possible to rewrite  $F(t)$  in terms of normalized  $B$ -splines of lower order, with certain polynomial coefficients. One has

$$\begin{aligned} F(t) &= \sum_i A_i N_{i,k}(t) \\ &= \sum_i A_i \{ (t - t_i) M_{i,k-1}(t) + (t_{i+k} - t) M_{i+1,k-1}(t) \} \\ &= \sum_i \{ A_i(t - t_i) + A_{i-1}(t_{i+k-1} - t) \} M_{i,k-1}(t) \\ &= \sum_i A_i^{[1]}(t) N_{i,k-1}(t), \end{aligned}$$

where

$$A_i^{[1]}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} A_i + \frac{t_{i+k-1} - t}{t_{i+k-1} - t_i} A_{i-1}.$$

More generally, setting

$$A_i^{[j]}(t) = \begin{cases} A_i, & j = 0 \\ \frac{t - t_i}{t_{i+k-j} - t_i} A_i^{[j-1]}(t) + \frac{t_{i+k-j} - t}{t_{i+k-j} - t_i} A_{i-1}^{[j-1]}(t), & j > 0, \end{cases} \quad (16)$$

one gets

$$F(t) = \sum_i A_i^{[j]}(t) N_{i,k-j}(t). \quad (17)$$

Since  $N_{i,1}(t) = 1$  for  $t_i \leq t < t_{i+1}$  and is zero otherwise, it follows that

$$F(t) = A_i^{[k-1]}(t), \quad t_i \leq t < t_{i+1}. \quad (18)$$

Hence, if  $t \in [t_i, t_{i+1})$ , then  $F(t)$  can also be found, from  $A_{i-k+1}, \dots, A_i$ , by forming repeatedly certain convex combinations according to (16).

Finally, we mention the important identity

$$(s - t)^{k-1} = \sum_i \varphi_{i,k}(s) N_{i,k}(t), \quad \text{with} \quad \varphi_{i,k}(s) = \prod_{r=1}^{k-1} (s - t_{i+r}), \quad \text{all } i, \quad (19)$$

which was first proved by Marsden [4], and which simplifies many dealings with splines. Its proof is straightforward : Setting

$$A_i^{[0]}(t) = A_i = \varphi_{i,k}(s), \quad \text{all } i,$$

one gets from (16) that

$$\begin{aligned} A_i^{[1]}(t) &= \{(t - t_i) \varphi_{i,k}(s) + (t_{i+k-1} - t) \varphi_{i-1,k}(s)\} / (t_{i+k-1} - t_i) \\ &= \varphi_{i,k-1}(s) \{(t - t_i)(s - t_{i+k-1}) + (t_{i+k-1} - t)(s - t_i)\} / (t_{i+k-1} - t_i) \\ &= \varphi_{i,k-1}(s)(s - t); \end{aligned}$$

hence

$$\sum_i \varphi_{i,k}(s) N_{i,k}(t) = (s - t) \sum_i \varphi_{i,k-1}(s) N_{i,k-1}(t).$$

Since

$$\sum_i \varphi_{i,1}(s) N_{i,1}(t) \equiv \sum_i N_{i,1}(t) \equiv 1,$$

induction on  $k$  now proves (19).

By expanding both sides of (19) in powers of  $s$  and comparing coefficients of like powers, it follows, e.g., that

$$\sum_i N_{i,k}(t) \equiv 1, \quad (20)$$

reaffirming the conclusion from (16) and (18) that, for  $t \in [t_i, t_{i+1})$ , the number  $F(t)$  is a *convex* combination of the numbers  $A_{i-k+1}, \dots, A_i$ .

It also follows from (19) that

$$(s - t)^{k-1} = \sum_i \psi_{i,k}(t) N_{i,k}(s), \quad \psi_{i,k}(t) = (t_{i+1} - t) \cdots (t_{i+k-1} - t); \quad (21)$$

hence

$$(s - t_j)_+^{k-1} = \sum_i \psi_{i,k}^+(t_j) N_{i,k}(s), \quad \psi_{i,k}^+(t) = (t_{i+1} - t)_+ \cdots (t_{i+k-1} - t)_+. \quad (22)$$

## 2. EVALUATION OF $B$ -SPLINES

The material in the preceding section suggests (at least) two stable, yet efficient ways to evaluate the function

$$F(t) = \sum_i A_i N_{i,k}(t)$$

at any particular  $t$ , which we now discuss. The resulting algorithms can, of

course, also be used to evaluate the single  $B$ -spline  $N_{i,k}(t)$ , merely by specializing to the situation

$$A_j = \delta_{ij}, \quad \text{all } j.$$

The more obvious of the two algorithms is based on (16)-(18). Having found  $i$  such that  $t_i \leq t < t_{i+1}$ , one generates all the entries in the following table, using (16):

$$\begin{array}{cccc} A_{i-k+1}^{[0]}(t) & & & \\ A_{i-k+2}^{[0]}(t) & A_{i-k+2}^{[1]} & & \\ \vdots & \vdots & \ddots & \\ A_{i-1}^{[0]}(t) & A_{i-1}^{[1]}(t) & \cdots & A_{i-1}^{[k-2]}(t) \\ A_i^{[0]}(t) & A_i^{[1]}(t) & \cdots & A_i^{[k-2]}(t) \quad A_i^{[k-1]}(t) \end{array}$$

The right-most entry is then the desired number  $F(t)$ .

Set

$$\begin{aligned} A(r, s) &= A_{i-k+r}^{[s-1]}(t), \quad r = s, \dots, k; \quad s = 1, \dots, k, \\ \left. \begin{aligned} DP(r) &= t_{i+r} - t, \\ DM(r) &= t - t_{i-k+r}, \end{aligned} \right\} r = 1, \dots, k, \end{aligned} \quad (23)$$

to simplify notation. Then

$$\begin{aligned} A(r, 1) &= A_{i-k+r}, \quad r = 1, \dots, k, \\ A(r, s+1) &= (DM(r) * A(r, s) + DP(r-s) * A(r-1, s)) / (DM(r) + DP(r-s)) \\ &\quad r = s+1, \dots, k; \quad s = 1, \dots, k-1. \end{aligned} \quad (24)$$

Note that

$$\begin{aligned} DM(r) + DP(r-s) &= t - t_{i-k+r} + t_{i+r-s} - t \\ &= t_{i+r-s} - t_{i+r-k} \geq t_{i+1} - t_i > 0, \end{aligned}$$

so that coincident points in the partition  $\pi$  cause no additional difficulty if, as we assume,  $i$  is chosen so that  $t_i \leq t < t_{i+1}$ .

The calculation of the  $A(r, s)$  can either be carried out column by column, i.e., with

$$r = s, \dots, k; \quad s = 2, \dots, k$$

or row by row, i.e., with

$$s = 2, \dots, r; \quad r = 2, \dots, k$$

or downward diagonal by downward diagonal, i.e., with

$$s = 2, \dots, j, \quad r = s + k - j; \quad j = 2, \dots, k.$$

Each way requires only one one-dimensional array with  $k$  entries for the actual storage of the successively calculated numbers  $A(r, s)$ . In the first way, one would precalculate the array  $DP$ , in the last, one would precalculate the array  $DM$ , while the second would require initial calculation of both the  $DP$  and the  $DM$  array.

If the value of  $F(t)$  and of some of its derivatives are required at the same time, then it is probably better to use an algorithm which generates at the same time all the numbers  $N_{i,j}(t)$  which are not zero for the given  $t$ . With the assumption that

$$t_i \leq t < t_{i+1},$$

this amounts to generating all the entries of the following triangular table:

$$\begin{array}{cccccc} N_{i,1}(t) & N_{i-1,2}(t) & \cdots & N_{i-k+2,k-1}(t) & N_{i-k+1,k}(t) & \\ & N_{i,2}(t) & \cdots & N_{i-k+3,k-1}(t) & N_{i-k+2,k}(t) & \\ & & \ddots & \vdots & \vdots & \\ & & & N_{i,k-1}(t) & N_{i-1,k}(t) & \\ & & & & N_{i,k}(t) & \end{array}$$

The  $(k - j)$ th column of this table contains the numbers needed for the evaluation of  $F^{(j)}(t)$  using (15),  $j = 0, \dots, k - 1$ . For this reason, we describe here only how to generate this table column by column.

Set

$$\begin{aligned} N(r, s) &= N_{i+r-s,s}(t), \\ \left. \begin{aligned} DP(r) &= t_{i+r} - t, \\ DM(r) &= t - t_{i+1-r}, \end{aligned} \right\} r = 1, \dots, k, \end{aligned} \tag{25}$$

to simplify notation. The needed table entries are then

$$N(r, s), \quad r = 1, \dots, s; \quad s = 1, \dots, k,$$



while

$$N(r, s) = 0, \quad \text{for } r > s \quad \text{or } r < 1. \quad (26)$$

With the abbreviations (25), we get from (10) that

$$\begin{aligned} N(r, s + 1) = & DM(s + 1 - r + 1) \frac{N(r - 1, s)}{DP(r - 1) + DM(s + 1 - r + 1)} \\ & + DP(r) \frac{N(r, s)}{DP(r) + DM(s + 1 - r)}. \end{aligned} \quad (27)$$

We emphasize that, once again, this formula is unaffected by the possible presence of coincident  $t_j$ 's, since

$$DP(r) + DM(s + 1 - r) = t_{i+r} - t_{i+r-s} \geq t_{i+1} - t_i > 0$$

for all values of  $r$  and  $s$  of interest.

Equations (25) and (27) lead to the following algorithm for the generation of the  $N(r, s)$ :

```

Set  $N(1, 1) = 1$ ;
for  $s = 1, \dots, k - 1$ , do:
  :   set  $DP(s) = t_{i+s} - t$ ,  $DM(s) = t - t_{i+1-s}$ ,
  :   set  $N(1, s + 1) = 0$ ;
  :   for  $r = 1, \dots, s$ , do:
  :     :   set  $M = N(r, s)/(DP(r) + DM(s + 1 - r))$ ,
  :     :   set  $N(r, s + 1) = N(r, s + 1) + DP(r) * M$ ,
  :     :   set  $N(r + 1, s + 1) = DM(s + 1 - r) * M$ .

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This algorithm can, of course, be modified so as to use only a one-dimensional array of  $k$  entries for the storage of the  $N(r, s)$ , by overwriting successive columns.

### 3. CONDITION OF THE $B$ -SPLINE BASIS

A limited number of numerical experiments have shown both algorithms presented in the preceding section to be extremely stable when used for the evaluation of  $F(t)$ . This is not surprising, since both algorithms arrive at  $F(t)$  by repeatedly forming convex combinations. Even for  $k = 80$ , the absolute error in the computed value for  $F(t)$  was only about the size of roundoff in the coefficients  $A_i$ .

These numerical experiments showed, incidentally, the unpleasant but important fact that the condition number of the normalized  $B$ -spline basis increases exponentially with the order  $k$ . This can be confirmed, in the case of a uniform partition, by the following calculations.

The condition number  $\kappa(k, \pi)$  of the normalized  $B$ -spline basis  $\{N_{i,k}\}_i$  for the partition  $\pi = \{t_i\}$  is defined by

$$\kappa(k, \pi) = \sup_{\|A\|_\infty=1} \left\| \sum_i A_i N_{i,k} \right\|_\infty / \inf_{\|A\|_\infty=1} \left\| \sum_i A_i N_{i,k} \right\|_\infty, \quad (28)$$

where

$$\|A\|_\infty = \sup_i |A_i|$$

and

$$\|f\|_\infty = \sup_{t \in I} |f(t)|,$$

$I$  being the interval for which  $\pi$  is a partition. It is assumed, of course, that  $\{N_{i,k}\}$  is linearly independent, i.e., that no  $t_i$  agrees with more than  $k - 1$  other  $t_j$ 's. By (20),

$$\left\| \sum_i A_i N_{i,k} \right\|_\infty \leq \|A\|_\infty$$

with equality when  $A_i = 1$ , all  $i$ . Hence

$$\kappa(k, \pi) = 1 / \inf_{\|A\|_\infty=1} \left\| \sum_i A_i N_{i,k} \right\|_\infty. \quad (29)$$

It was proved in [1] that, for each  $k$ ,

$$D_k = \sup_\pi \kappa(k, \pi) < \infty$$

Preliminary calculations based on the argument in [1] give upper bounds for  $D_k$  which increase about as fast as  $k!$ , as  $k$  increases. These bounds are probably not sharp. But it can be shown that  $D_k$  must increase at least exponentially with  $k$ .

**THEOREM.** *If the partition  $\pi$  is uniform,*

$$t_j = t_0 + jh, \quad \text{all } j,$$

then

$$\kappa(k, \pi) = 1/\varphi_k(\pi), \quad (30)$$

where

$$\varphi_k(u) = \sum_j \psi_k(u + 2\pi j),$$

$$\psi_k(u) = \left( \frac{\sin(u/2)}{u/2} \right)^k.$$

Hence,

$$\kappa(k, \pi) \geq (\pi/2)^{k-2}, \quad \text{for } k > 1,$$

$$\lim_{k \rightarrow \infty} \kappa(k, \pi) / (\pi/2)^k = 2. \quad (31)$$

*Proof.* We show how to obtain (30) from Schoenberg's recent paper [6] on Cardinal Interpolation. First, we note that  $\kappa(k, \pi)$  is invariant under a linear change of the independent variable; hence we may restrict attention to the particular uniform partition

$$\pi = \{j\}.$$

If  $A = (A_i)$  is any sequence, then

$$\left\| \sum_i A_i N_{i,k} \right\|_{\infty} \geq \|C_A\|_{\infty}, \quad (32)$$

with

$$C_A(i) = \sum_j A_j N_{j,k}(i + k/2), \quad \text{all } i.$$

Since we are dealing with the particular partition  $\pi = \{j\}$ , we have

$$N_{j,k}(t) = N_{0,k}(t - j) = M_k(t - j - k/2),$$

where, in the notation of [6 or 5],

$$M_k(t) = \frac{1}{(k-1)!} \delta_{t+}^{k-1}.$$

Therefore,

$$C_A(i) = \sum_j A_j M_k(i - j), \quad \text{all } i.$$

In this form, the linear sequence-to-sequence transformation

$$A \rightarrow C_A$$

has been studied in detail in [6], where it is proved (see, in particular Section 6 of [6]) that

$$\|C_A\|_\infty \geq \varphi_k(\pi) \|A\|_\infty, \quad \text{with equality if } A_i = -A_{i+1}, \quad \text{all } i. \quad (33)$$

Combining (32) and (33), we get that

$$\inf_{\|A\|_\infty=1} \left\| \sum_j A_j N_{j,k} \right\|_\infty \geq \varphi_k(\pi). \quad (34)$$

On the other hand, if

$$A_i = -A_{i+1}, \quad \text{all } i,$$

then it easily follows (e.g., from Lemma 12 of [6]) that

$$\left\| \sum_j A_j N_{j,k} \right\|_\infty = \|C_A\|_\infty,$$

hence, by (33), then

$$\left\| \sum_j A_j N_{j,k} \right\|_\infty = \varphi_k(\pi) \|A\|_\infty.$$

Combining this with (34) and (29) gives (30);

Q.E.D.

We note that the inequality (31) alone can be derived directly by calculation of  $\sum_j (-1)^j N_{j,k}(k/2)$ . Further, the theorem implies that, for a uniform partition,

$$\varkappa(k, \pi) \approx 10^{k/5}. \quad (35)$$

This implies that, on a typical 7 decimal digit machine and with  $k = 40$ , the calculated value of  $F(t) = \sum_j A_j N_{j,k}(t)$  at some point  $t$  may well be inaccurate in the first nonzero digit. Since, on the other hand, the normalized  $B$ -spline basis is, at present, the only suitable basis for dealing with splines in computations, this seems to limit the use of splines in solving functional equations on a computer to splines of relatively low order, say of order  $k < 20$ , unless one is willing to pay the price of multiple-precision arithmetic.

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*Added in proof:* The following additional references are of interest.

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