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Torus and $\mathbb{Z}/p$ actions on manifolds

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Abstract

Let G be either a finite cyclic group of prime order or S^1. We show that if G acts on a manifold or, more generally, on a Poincaré duality space M, then each term of the Leray spectral sequence of the map $M \times_G EG \to BG$ satisfies a properly defined "Poincaré duality". As a consequence of this fact we obtain new results relating the cohomology groups of M and M^G. We apply our results to study group actions on 3-manifolds.

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1. Introduction

If G is either a finite cyclic group of prime order, $\mathbb{Z}/p$, or S^1 acting on a space M then the G-equivariant cohomology of M can be calculated from the Leray spectral sequence of the map $M \times_G EG \to BG$. If $G = S^1$ and M is a Poincaré duality space then the components of the second term of this spectral sequence (the cohomology groups of M) satisfy Poincaré duality. We show that if $M^{S^1} \neq \emptyset$ then each term of this spectral sequence satisfies a properly defined "Poincaré duality."

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Similarly, all terms of the corresponding Leray–Serre and Cartan spectral sequences satisfy Poincaré duality. These statements and similar statements for $\mathbb{Z}/p$-actions will be formulated precisely in Sections 3.1 and 3.3.

Using the notion of Poincaré duality for a spectral sequence we prove new results relating the cohomology of any Poincaré duality space with a torus or $\mathbb{Z}/p$ action, to the cohomology of the fixed point set of this action, see Theorems 1.1, 1.3, 1.4.

Since this work was motivated by a conjecture concerning group actions on 3-manifolds, we devote Sections 1.4–1.5 to discuss the consequences and the ramifications of the above results in 3-dimensional topology.

For the reader who is unfamiliar with group cohomology, equivariant cohomology, spectral sequences, or basic facts about group actions we suggest [1,3,7,9,15,23] as good sources of information on these subjects.

Throughout this paper we will consider paracompact spaces X of finite cohomological dimension (over $\mathbb{Z}$) only. If X is a manifold or CW-complex then the cohomological dimension of $X, cd\,X$, is equal to $\dim X$. For more information on $cd\,X$ see [6]. By H^* we will denote the sheaf cohomology groups with supports in closed sets. Recall that sheaf cohomology theory with constant coefficients agrees with Alexander–Spanier and Čech cohomology for paracompact spaces. Let $b^i(X)=\dim_{\mathbb{Q}} H^i(X;\mathbb{Q})$.

We say that a connected topological space X is a $PD_{\mathbb{K}}(n)$-space (Poincaré duality space of formal dimension n with respect to coefficients in a field $\mathbb{K}$) if $H^i(X;\mathbb{K})=0$ for $i>n$, $H^n(X;\mathbb{K})=\mathbb{K}$, and for all $0\leqslant i\leqslant n$ the cup product

$$H^i(X;\mathbb{K})\times H^{n-i}(X;\mathbb{K})\xrightarrow{\cup} H^n(X;\mathbb{K})\cong\mathbb{K}$$

is a non-degenerate bilinear form. We also assume that $\dim_{\mathbb{K}} H^*(X;\mathbb{K})<\infty$. $PD_{\mathbb{K}}(n)$-spaces will be usually denoted by letter M.

1.1. Torus actions

We say that a torus T action on a topological space X has finitely many connective orbit types (FMCOT) if the set $\{(T_x)^0\colon x\in X\}$ is finite. Here T_x denotes the stabilizer of $x, \{t\in T\colon tx=x\}$, and $(T_x)^0$ denotes the connected component of identity of T_x. Note that each S^1-action has FMCOT.

Theorem 1.1. *If a torus T action on a $PD_{\mathbb{Q}}(n)$-space M has FMCOT and*

$$\begin{aligned}&\text{either } n \text{ is even or}\\ &M^{\mathrm{T}}\neq\emptyset \text{ and } b^i(M)=0 \text{ for all even } i, 0<i\leqslant \tfrac{1}{2}(n-1)\end{aligned}\tag{1}$$

then

$$\sum_i b^i(M^{\mathrm{T}})\equiv\sum_i b^i(M)\bmod 4.$$

It will be seen in Sections 1.4 and 1.6 that condition (1) is necessary. The proof of Theorem 1.1 is given in Section 4.1.

The following well-known formulas provide additional information relating X to X^{T} for any torus action with FMCOT:

$$\chi(X^{\mathrm{T}}) = \chi(X), \tag{2}$$

$$\sum_{i=0}^{\infty} b^{k+2i}(X^{\mathrm{T}}) \leqslant \sum_{i=0}^{\infty} b^{k+2i}(X) \tag{3}$$

for all k, cf. [3, Theorems 3.1.13, 3.1.14]. Here, χ denotes the Euler characteristic.

1.2. $\mathbb{Z}/p$-actions

Let $\mathbb{F}_p$ denote the field of p elements. We are going to see that if $\mathbb{Z}/p$ acts on X then the numbers

$$t^i(X) = \dim_{\mathbb{F}_p} H^2(\mathbb{Z}/p, H^i(X;\mathbb{F}_p))$$

play similar role in relating X to $X^{\mathbb{Z}/p}$ as the Betti numbers in the study of torus actions. For example, if X is a finite dimensional $\mathbb{Z}/p$-CW complex or a finitistic space then (3) corresponds to

$$\sum_{i=0}^{\infty} t^{k+i}(X^{\mathbb{Z}/p}) \leqslant \sum_{i=0}^{\infty} t^{k+i}(X), \tag{4}$$

which holds for all k, see [3, Corollary 4.6.16]. [1]

Unlike for S^1-actions, the induced $\mathbb{Z}/p$-action on the cohomology groups of X may be non-trivial. For that reason, the results for $\mathbb{Z}/p$-actions analogous to (2) and Theorem 1.1 can be formulated and proved only if the action of $\mathbb{Z}/p$ on $H^*(X;\mathbb{F}_p)$ is *nice*.

Definition. An action of $\mathbb{Z}/p$ on an $\mathbb{F}_p$-vector space N is nice if N decomposes as $\mathbb{F}_p[\mathbb{Z}/p]$-module into $T \oplus F$, where T and F are trivial and free $\mathbb{F}_p[\mathbb{Z}/p]$-modules, respectively, i.e. $T = \bigoplus \mathbb{F}_p$, $F = \bigoplus \mathbb{F}_p[\mathbb{Z}/p]$. (In particular, trivial actions are nice.) We say that $\mathbb{Z}/p$ acts nicely on X if the induced $\mathbb{Z}/p$-action on $H^*(X;\mathbb{F}_p)$ is nice. Note that if $H^*(X;\mathbb{F}_p) = T^* \oplus F^*$ then

$$t^i(X) = \dim_{\mathbb{F}_p} T^i. \tag{5}$$

Proposition 1.2 (Proof in Section 4.2). *If* $p \neq 2$ *and* $\mathbb{Z}/p$ *acts nicely on a space* X *such that* $H^*(X;\mathbb{Z})$ *has no* p*-torsion then the following version of the Euler characteristic formula holds*:

$$\chi_t(X^{\mathbb{Z}/p}) = \chi_t(X),$$

where $\chi_t(X) = \sum_i (-1)^i t^i(X)$.

For completeness, we recall also the classical formula (see [7, Theorem III.4.3])

$$\chi(X) - \chi(X^{\mathbb{Z}/p}) = p(\chi(X/\mathbb{Z}/p) - \chi(X^{\mathbb{Z}/p})),$$

which holds if X is a finite dimensional $\mathbb{Z}/p$-CW complex or a finitistic space.

[1] In order to deduce (4) from [3, Corollary 4.6.16] we need to notice that if $\mathbb{Z}/p$ acts on an $\mathbb{F}_p$-vector space N then all Tate cohomology groups $\hat{H}^i(\mathbb{Z}/p, N)$ are equal to $H^2(\mathbb{Z}/p, N)$. This can be proved using Herbrand quotient or using the classification of $\mathbb{F}_p[\mathbb{Z}/p]$-modules given in the next section.

If $\mathbb{Z}/p$ acts on a $PD_{\mathbb{F}_p}(n)$-space M then $t^i(M) = t^{n-i}(M)$ by Corollary 2.2. Moreover, we have the following counterpart of Theorem 1.1 for $\mathbb{Z}/p$-actions.

Theorem 1.3 (Proof in Section 4.3). *Let $\mathbb{Z}/p$ act nicely on a $PD_{\mathbb{F}_p}(n)$-space M with no p-torsion in $H^*(M;\mathbb{Z})$. If $p \neq 2$, and*

$$\begin{aligned}&\text{either } n \text{ is even or}\\ &M^{\mathbb{Z}/p} \neq \emptyset \text{ and } t^l(M) = 0 \text{ for all even } 0 < l \leqslant \tfrac{1}{2}(n-1),\end{aligned} \tag{6}$$

then

$$\sum_i t^i(M^{\mathbb{Z}/p}) \equiv \sum_i t^i(M) \bmod 4.$$

The proof of the above theorem for even n is based on Proposition 1.2 and the standard properties of Poincaré duality spaces. For odd n, the proof uses the notion of Poincaré duality of spectral sequences (defined in Section 3) applied to the Leray spectral sequence associated with the $\mathbb{Z}/p$-action on M.

The next result shows the assumption about the lack of p-torsion in $H^*(M;\mathbb{Z})$ can be replaced by the following condition:

$$\begin{aligned}&\mathbb{Z}/p \text{ acts on a } PD_{\mathbb{F}_p}(n)\text{-space } M \text{ such that } d_r^{kl} = 0 \text{ for all odd}\\ &r > 1 \text{ and all } k \geqslant n \text{ in the Leray spectral sequence of the}\\ &\text{map } M \times_{\mathbb{Z}/p} E\mathbb{Z}/p \to B\mathbb{Z}/p \text{ with coefficients in } \mathbb{F}_p.\end{aligned} \tag{7}$$

Theorem 1.4 (Proof in Section 4.4). *Let $p \neq 2$ and let $\mathbb{Z}/p$ act nicely on a $PD_{\mathbb{F}_p}(n)$-space M in such a way that conditions (6) and (7) hold. If $M^{\mathbb{Z}/p} \neq 0$ then*

$$\sum_i t^i(M^{\mathbb{Z}/p}) \equiv \sum_i t^i(M) \bmod 4.$$

We will see in Proposition 3.12 that all nice $\mathbb{Z}/p$-actions on $PD_{\mathbb{F}_p}(n)$-spaces for $n \leqslant 3$ satisfy (7). Furthermore, one can show that if $p \neq 2$ and $\mathbb{Z}/p$ acts nicely on a $PD_{\mathbb{F}_p}(n)$-space M with $M^{\mathbb{Z}/p} \neq \emptyset$ then $d_r^{kl} = 0$ for all odd $r > 1$ and all odd $k \geqslant n$. Motivated by the above results, we conjectured in the previous version of this paper that condition (7) holds for all nice $\mathbb{Z}/p$-actions on $PD_{\mathbb{F}_p}(n)$-spaces with a non-empty fixed point set for $p \neq 2$. Recently Hanke showed that although this conjecture is not true in general, it does hold under the additional assumption that $H^*(M,\mathbb{Z})$ does not contain $\mathbb{Z}/p$ as a direct summand [13].

All other assumptions of Theorem 1.3 are necessary. Examples given in Sections 1.4 and 1.5 show that condition (6) cannot be dropped. We will also see that Theorem 1.3 fails if the $\mathbb{Z}/p$-action on M is not nice and $n > 2$. However, a much stronger statement holds for two-dimensional manifolds.

Theorem (Bryan [10]). *If $\mathbb{Z}/p$ acts on a connected surface F, $F^{\mathbb{Z}/p} \neq \emptyset$, then this action has $2 + \dim_{\mathbb{F}_p} H^1(\mathbb{Z}/p, H_1(F,\mathbb{F}_p))$ fixed points.*

1.3. S^1-actions on 3-manifolds

Since this paper was motivated by a conjecture concerning group actions on 3-manifolds, we devote this and the next subsection to present consequences and ramifications of our results to such actions.

By the slice theorem, if $G=S^1$ or $\mathbb{Z}/p$ acts smoothly on a closed, oriented, smooth manifold M with fixed points then M^G is a disjoint union of closed, orientable submanifolds of even codimension. Therefore, if $\dim M=3$ then M^G is a union of embedded circles.

By Theorem 1.1 and by (3) we have

Corollary 1.5. *If S^1 acts smoothly on a connected, closed, orientable 3-manifold M, $M^{S^1}\neq\emptyset$, then M^{S^1} is a union of s circles, where*:

- $s\leqslant 1+b_1(M)$ *and*
- $s\equiv 1+b_1(M) \bmod 2$.

This corollary can be also deduced from the classification of S^1-actions on 3-manifolds [16,19].

The statement of the corollary cannot be improved. Namely, given two integers s,b such that $0<s\leqslant 1+b$ and $s\equiv 1+b \bmod 2$, there is a 3-manifold M with an S^1-action such that $b_1(M)=b$ and M^{S^1} is a union of s circles. Such a manifold can be constructed as follows. Let $F_{g,s}$ denote a surface of genus $g=(b+1-s)/2$ with s boundary components, and let $M_0=F_{g,s}\times S^1$. The boundary of M_0 is a union of s tori and $b_1(M_0)=b_1(F_{g,s})+b_1(S^1)=b+1$. Choose s points $p_1,\dots,p_s\in\partial F_{g,s}$, each lying in a different component of $\partial F_{g,s}$. Now, attach s solid tori to M_0 along their boundaries, in such a way that the meridian of the ith solid torus is identified with $p_i\times S^1$. We denote the closed manifold obtained in this way by M. Note that after attaching the first solid torus the first Betti number decreases by 1, but after attaching the next tori, it stays unchanged. Therefore $b_1(M)=b$. Obviously, the S^1-action on M_0 extends on M and the fixed point set of the action is composed of the cores of the solid tori. Hence, M^{S^1} has exactly s components.

1.4. $\mathbb{Z}/p$-actions on 3-manifolds

If $\mathbb{Z}/p$ acts on a closed, connected, orientable 3-manifold M then (4) for $k=0$ implies that $M^{\mathbb{Z}/p}$ is a union of at most $1+t^1(M)$ circles. Since by Proposition 3.12 $\mathbb{Z}/p$-actions on 3-manifolds with fixed points always satisfy condition (7), the following result is a special case of Theorem 1.4.

Proposition 1.6. *Let $p\neq 2$. If $\mathbb{Z}/p$ acts nicely on a closed, connected, orientable 3-manifold M and if $M^{\mathbb{Z}/p}$ is composed of s circles, $s\neq 0$, then $s\equiv 1+t^1(M) \bmod 2$.*

The above proposition answers in affirmative a conjecture of Sokolov concerning p-periodic 3-manifolds, i.e. manifolds with a $\mathbb{Z}/p$-action with exactly one circle of fixed points. Sokolov conjectured the following statement.

Proposition 1.7. *If M is p-periodic then $H_1(M;\mathbb{F}_p)\neq\mathbb{F}_p$.*

Proof. If $H_1(M;\mathbb{F}_p)$ is a one-dimensional vector space then the $\mathbb{Z}/p$-action on $H_1(M;\mathbb{F}_p)$ is trivial, and therefore $t^1(M)=1$. Hence, $M^{\mathbb{Z}/p}=\emptyset$ or $S^1\cup S^1$. □

The statement of the above proposition makes an impression that it could be easily proved by elementary means of algebraic topology, Smith theory, or three-dimensional topology. We do not know any short proof of it, and we encourage the reader to try to find one by himself, in order to realize that this is not easy. After we proved the above proposition, Przytycki and Sokolov [17] found a different proof of it, which avoids using equivariant cohomology at the expense of an elaborate application of surgery theory.

Now we are about to show that Proposition 1.6 does not hold if we drop any of its assumptions. Observe first that any free $\mathbb{Z}/p$-action on S^3 is nice and $s=0$ and $t^1(S^3)=0$ for such action. Therefore, the assumption $s\neq 0$ in Proposition 1.6 is necessary.

If $p=2$ or if the action is not nice then the conclusion of Proposition 1.6 fails as well. To see this, we need to understand the possible $\mathbb{Z}/p$-actions on $H_1(M,\mathbb{F}_p)$. If $\mathbb{Z}/p$ acts on M then $H_1(M,\mathbb{F}_p)$ considered as a module over $R=\mathbb{F}_p[\mathbb{Z}/p]$ decomposes as a direct sum of indecomposable R-modules. We will see in Section 2 that each indecomposable R-module is isomorphic to

$$V_i=R/(t-1)^i=\mathbb{F}_p[t]/(t-1)^i$$

for a unique i between 1 and p. (Here, $R=\mathbb{F}_p[t]/(t^p-1)=\mathbb{F}_p[t]/(t-1)^p$.) A $\mathbb{Z}/p$-action on an $\mathbb{F}_p$-vector space N is nice if N decomposes as an R-module into a sum of V_1's and V_p's. By Corollary 2.2, a $\mathbb{Z}/p$-action on a 3-manifold M is nice if and only if the induced $\mathbb{Z}/p$-action on $H_1(M,\mathbb{F}_p)$ is nice.

A 3-manifold with a $\mathbb{Z}/p$-action which is not nice can be constructed as follows. Let S^3_1 and S^3_2 be two 3-spheres with some (not necessarily the same) $\mathbb{Z}/p$-actions. Choose 3-balls $B_1\subset S^3_1$, $B_2\subset S^3_2$, such that the orbit of B_i, $\bigcup_{g\in\mathbb{Z}/p} gB_i$, for $i=1,2$, is composed of p disjoint balls (on which $\mathbb{Z}/p$ acts freely). For all $g\in\mathbb{Z}/p$, remove the interiors of the balls gB_1, gB_2, from S^3_1 and S^3_2, respectively. Next, choose an arbitrary homeomorphism $\Psi:\partial B_1\to\partial B_2$ and identify $g\partial B_1$ with $g\partial B_2$, for any $g\in\mathbb{Z}/p$, via $g\Psi g^{-1}$. This construction gives a closed, orientable 3-manifold M_p, with the cyclic group, $\mathbb{Z}/p$, acting on it.

The proofs of the following remarks are easy and left to the reader.

Remarks. (i) $M_p\simeq \underbrace{(S^2\times S^1)\#\cdots\#(S^2\times S^1)}_{p-1}$.

(ii) $H_1(M_p,\mathbb{F}_p)\cong V_{p-1}$ for any $\mathbb{Z}/p$-action on M_p constructed as above. In particular, none of the $\mathbb{Z}/p$-actions on M is nice for $p\neq 2$.

(iii) On the other hand, since the only indecomposable $\mathbb{F}_2[\mathbb{Z}_2]$-modules are the trivial module, V_1, and the free module, V_2, all $\mathbb{Z}_2$-actions on vector spaces over $\mathbb{F}_2$ are nice. In particular, any $\mathbb{Z}_2$-action on M_2 is nice.

(iv) Since $\mathbb{Z}/p$ can act on S^3 with $(S^3)^{\mathbb{Z}/p}=S^1$ or $\emptyset$, there exist $\mathbb{Z}/p$-actions on M_p with $M_p^{\mathbb{Z}/p}=\emptyset, S^1$, and $S^1\cup S^1$.

(v) By (ii) and (iv) the statement of Proposition 1.6 fails for $\mathbb{Z}/p$-actions which are not nice.

(vi) By (ii) and (iii) the statement of Proposition 1.6 fails for $p=2$.

1.5. More examples

Theorems 1.1 and 1.3 may be useful for studying group actions on products of spheres. On the other hand, the analysis of examples of such actions shows that all the assumptions of Theorems 1.1 and 1.3 are necessary.

Example 1.8. Bredon in [7, VII, Section 10] constructs a circle action on $M = S^3 \times S^5 \times S^9$ with the fixed point set M^{S^1} being an S^7-bundle over $S^3 \times S^5$ with $b_i(M^{S^1}) = 1$ for $i = 0, 3, 5, 10, 12, 15$ and $b_i(M^{S^1}) = 0$ for all other i. This circle action does not satisfy the conclusion of Theorem 1.1. Therefore, condition (1) is necessary.

Example 1.9. There is a $\mathbb{Z}_3$-action on $M = S^n \times S^n$, for $n = 1, 3$ or 7, which is not nice and for which $M^{\mathbb{Z}_3} = (\text{point} + S^{n-1})$, see [7, VII, Section 9]. This shows that the restriction in Theorem 1.3 to nice actions is necessary. The action can be constructed as follows: Let R be the ring of complex numbers, quaternions, or Cayley numbers for $n = 1, 3, 7$, respectively. Let S denote the set of elements of norm 1 in $R, S \simeq S^n$, and let M be the space of all triples $(x, y, z) \in S \times S \times S$ such that $(xy)z = 1$. M is homeomorphic to $S^n \times S^n$ and since

$$(xy)z = 1 \quad \Leftrightarrow \quad (yz)x = 1 \quad \Leftrightarrow \quad (zx)y = 1$$

there is an action of $\mathbb{Z}_3$ on M by cyclic permutations. The fixed point set of this action is $\{x \in R \mid x^3 = 1\} \simeq \text{point} + S^{n-1}$.

The notation $X \sim_p Y$ in the next example means that X and Y are topological spaces with isomorphic cohomology rings with coefficients in $\mathbb{F}_p$.

Example 1.10. If $n \neq m$ and n, m are both even or both odd, or if the smaller of them is odd then any action of $\mathbb{Z}/p$ on $M = S^n \times S^m$ is nice and condition (6) is satisfied. In this situation Theorem 1.3 holds, and one can prove that $M^{\mathbb{Z}/p}$ is $\sim_p$-equivalent to one of the following spaces: $S^q \times S^r, S^q + S^r, P^3(2q), (\text{point} + P^2(2q))$; see [7, Theorem VII 9.1]. Here $P^n(2q)$ denotes a space whose cohomology ring with coefficients in $\mathbb{F}_p$ is $\mathbb{F}_p[x]/(x^{n+1})$, and $\deg x = 2q$.

However, there are known examples of $X \sim_p S^n \times S^m$ which do not satisfy the assumptions of the example above (i.e. $\min(n, m)$ is even and $\max(n, m)$ is odd) and which admit a $\mathbb{Z}/p$-action with $X^{\mathbb{Z}/p} \sim_p S^q$. Therefore (6) is a necessary condition for Theorem 1.3.

2. Classification of representations of $\mathbb{Z}/p$

In this section, we present a classification of all representations of $\mathbb{Z}/p$ over $\mathbb{F}_p$ and over the ring of integers localized at the prime ideal (p), $\mathbb{Z}_{(p)}$. This classification should help the reader to better understand the possible $\mathbb{Z}/p$-actions on the cohomology groups of X. We will classify indecomposable modules only, since all other modules are direct sums of these.

Note that $R = \mathbb{F}_p[\mathbb{Z}/p]$ is isomorphic to $\mathbb{F}_p[t]/(t^p - 1)$, and since $t^p - 1 = (t-1)^p \bmod p$, $R = \mathbb{F}_p[t]/(t-1)^p$. Therefore,

$$V_i = R/(t-1)^i = \mathbb{F}_p[t]/(t-1)^i$$

is an R-module for each $i = 1, \dots, p$.

Proposition 2.1. (i) *$V_1, \dots, V_p$ is the complete list of finitely generated indecomposable R-modules.*

(ii) *Each finitely generated R-module, N, decomposes as a finite sum*

$$N = V_{i_1} \oplus \cdots \oplus V_{i_k},$$

where $1 \leqslant i_1, \dots, i_k \leqslant p$ are unique up to a permutation.

Proof. (i) $\mathbb{F}_p[\mathbb{Z}/p] = \mathbb{F}_p[t]/(t-1)^p$ is a quotient of the ring of polynomials $\mathbb{F}_p[t]$, which is a principal ideal domain. Every indecomposable module over $\mathbb{F}_p[\mathbb{Z}/p]$ is also indecomposable over $\mathbb{F}_p[t]$ and, hence, cyclic. Such modules can be easily classified.

(ii) Note that $V_k/(t-1)^d = V_{\min(k,d)}$. Therefore, the number of components V_d in N is determined by difference in the dimensions of the vectors spaces $N/(t-1)^d$ and $N/(t-1)^{d+1}$. (Another way of proving the uniqueness of the decomposition of N is by applying the Krull–Schmidt Theorem, [12, 14.5].) □

Corollary 2.2. *If N and N' are $\mathbb{F}_p[\mathbb{Z}/p]$-modules and $\Psi : N \times N' \to \mathbb{F}_p$ is a non-degenerate bilinear $\mathbb{Z}/p$-equivariant form then N and N' are isomorphic as modules.*

Proof. N' is the dual module to N, i.e. N' is isomorphic to $Hom_{\mathbb{F}_p}(N, \mathbb{F}_p)$, where $g \in \mathbb{Z}/p$ sends $f : N \to \mathbb{F}_p$ to the homomorphism $x \to f(g^{-1}x)$. Since $(N_1 \oplus N_2)' = N_1' \oplus N_2'$, it is enough to assume that $N = V_k$. The module V_k' is generated by the homomorphism given by

$$f(t^i) = \begin{cases} 1 & \text{for } i = 0, \\ 0 & \text{for } 1 \leqslant i \leqslant k-1. \end{cases}$$

Therefore, V_k' is a cyclic R-module and hence $V_k' = V_i$ for certain i. Now $i = k$ since $\dim_{\mathbb{F}_p} V_k' = \dim_{\mathbb{F}_p} V_k$. □

Note that an $\mathbb{F}_p[\mathbb{Z}/p]$-module N is nice if it decomposes into a sum of V_1's and V_p's.

Later we will need the classification of $\mathbb{Z}/p$-modules over $\mathbb{Z}_{(p)}$. By modifying the proof of Theorem 74.3 in [12] one can show the following:

Proposition 2.3. *Every indecomposable $\mathbb{Z}/p$-module over $\mathbb{Z}_{(p)}$ which is free over $\mathbb{Z}_{(p)}$* is either

(i) *the trivial module, $\mathbb{Z}_{(p)}$, or*
(ii) *the free module, $\mathbb{Z}_{(p)}[\mathbb{Z}/p]$, or*
(iii) *the ring of cyclotomic integers, $\mathbb{Z}_{(p)}[\zeta_p] = \mathbb{Z}_{(p)}[x]/(1 + x + \cdots + x^{p-1})$.*

The action of the generator of $\mathbb{Z}/p$ on $\mathbb{Z}_{(p)}[\zeta_p]$ is given by the multiplication by ζ_p.

Note that $\mathbb{Z}_{(p)} \otimes \mathbb{F}_p = V_1$, $\mathbb{Z}_{(p)}[\mathbb{Z}/p] \otimes \mathbb{F}_p = V_p$, and $\mathbb{Z}_{(p)}[\zeta_p] \otimes \mathbb{F}_p = V_{p-1}$. Hence, if $H^*(M;\mathbb{Z})$ has no p-torsion then $H^*(M;\mathbb{F}_p)$ decomposes as a sum of V_1's, V_{p-1}'s, and V_p's.

The following lemma will be needed in Section 3.3.

Lemma 2.4. *If* $p \neq 2$ *and* $\mathbb{Z}/p$ *acts nicely on a space* X, *with no* p-*torsion in* $H^*(X;\mathbb{Z})$ *then*

$$H^k(\mathbb{Z}/p, H^*(X;\mathbb{Z})) = \begin{cases} H^2(\mathbb{Z}/p, H^*(X;\mathbb{F}_p)) & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd} \end{cases}$$

for $k > 0$.

Proof. By the universal coefficient theorem, $H^*(X;\mathbb{Z}_{(p)})$ is a free $\mathbb{Z}_{(p)}$-module and $H^*(X;\mathbb{F}_p) = H^*(X;\mathbb{Z}_{(p)}) \otimes \mathbb{F}_p$. Therefore, the sequence

$$0 \to H^*(X;\mathbb{Z}_{(p)}) \xrightarrow{\cdot p} H^*(X;\mathbb{Z}_{(p)}) \to H^*(X;\mathbb{F}_p) \to 0$$

is exact. By applying $H^*(\mathbb{Z}/p,\cdot)$ to that sequence we get

$$0 \to H^2(\mathbb{Z}/p, H^*(X;\mathbb{Z}_{(p)})) \to H^2(\mathbb{Z}/p, H^*(X;\mathbb{F}_p)) \to H^3(\mathbb{Z}/p, H^*(X;\mathbb{Z}_{(p)})) \to 0.$$

Now we use the classification of $\mathbb{Z}/p$ modules over $\mathbb{F}_p$ and $\mathbb{Z}_{(p)}$ described above. Since $\mathbb{Z}_{(p)}[\zeta_p] \otimes \mathbb{F}_p = V_{p-1}, H^*(X;\mathbb{Z}_{(p)})$ must be a direct sum of trivial and free $\mathbb{Z}/p$-modules. Therefore, $H^3(\mathbb{Z}/p, H^*(X;\mathbb{Z}_{(p)}))=0$ and, hence, $H^2(\mathbb{Z}/p, H^*(X;\mathbb{Z}_{(p)}))=H^2(\mathbb{Z}/p, H^*(X;\mathbb{F}_p))$. Since localization at (p) is an exact functor in the category of $\mathbb{Z}/p$-modules, $H^2(\mathbb{Z}/p, H^*(X;\mathbb{Z}))=H^2(\mathbb{Z}/p, H^*(X;\mathbb{Z}_{(p)}))$. Finally, the statement follows from the fact that $H^k(\mathbb{Z}/p,\cdot)$ is 2-periodic for $k > 0$. □

3. Poincaré duality on spectral sequences

Throughout this section we will make the following assumptions: Let $\mathbb{K}$ be a field and n be a positive integer. Let (E_*^{**}, d_*) be a spectral sequence whose each summand, E_r^{pq}, for $r \geqslant 2$, is a finite dimensional vector space over $\mathbb{K}$ and $E_2^{kl} = 0$ for $l < 0$ and $l > n$. Assume that E_r^{**} has a multiplicative structure for $r \geqslant 2$, i.e. there is a graded commutative product on each term, E_r^{**}, such that d_r^{**} is a derivation with respect to that product, and the product on E_{r+1}^{**} is induced from the product on E_r^{**}. Additionally, assume the following condition about the 0th row in E_*^{**}:

(ZR) $E_2^{*0} = E_\infty^{*0}$. Equivalently, the differentials $d_r^{*,r-1} : E_r^{*,r-1} \to E_r^{*+r,0}$ are 0 for all $r \geqslant 2$.

Proposition 3.1. *Let* $r \geqslant 2, k, k' \in \mathbb{Z}$ *and* $l, l' \geqslant 0$ *be such that* $E_r^{k+k',n} = E_r^{k+k',0} = \mathbb{K}, l + l' = n$, *and the following* $\mathbb{K}$-*bilinear maps are non-degenerate*

$$E_r^{kl} \times E_r^{k',l'} \xrightarrow{\cdot} E_r^{k+k',n} = \mathbb{K}, \tag{8}$$

$$E_r^{k+r,l-r+1} \times E_r^{k'-r,l'+r-1} \xrightarrow{\cdot} E_r^{k+k',n} = \mathbb{K}, \tag{9}$$

$$E_r^{k-r,l+r-1} \times E_r^{k'+r,l'-r+1} \xrightarrow{\cdot} E_r^{k+k',n} = \mathbb{K}. \tag{10}$$

(*A pairing* $V \times W \to \mathbb{K}$ *is non-degenerate if it induces an isomorphism* $V \to W^*$. *In particular, the pairing* $\{0\} \times \{0\} \xrightarrow{0} \mathbb{K}$ *is non-degenerate.*)

Assume additionally that (8) *and* (9) *are non-degenerate for* $l = n$, $l' = 0$.

Under the above assumptions, $E_{r+1}^{k+k',n} = \mathbb{K}$ *and the pairing*

$$E_{r+1}^{kl} \times E_{r+1}^{k',l'} \stackrel{\cdot}{\rightarrow} E_{r+1}^{k+k',n} = \mathbb{K} \tag{11}$$

is non-degenerate.

Proof. Consider the diagram

$$\begin{array}{ccccc} E_r^{k-r,l+r-1} & \xrightarrow{d_r^{k-r,l+r-1}} & E_r^{kl} & \xrightarrow{d_r^{kl}} & E_r^{k+r,l-r+1} \\ \downarrow & & \downarrow & & \downarrow \\ (E_r^{k'+r,l'-r+1})^* & \xrightarrow{(d_r^{k'l'})^*} & (E_r^{k'l'})^* & \xrightarrow{(d_r^{k'-r,l'+r-1})^*} & (E_r^{k'-r,l'+r-1})^* \end{array} \tag{12}$$

in which the vertical isomorphisms are induced by the non-degenerate pairings (8)–(10),

$$E_r^{kl} \ni \alpha \stackrel{\sim}{\rightarrow} (x \rightarrow \alpha \cdot x) \in (E_r^{k'l'})^*,$$

$$E_r^{k\pm r,l\mp r+1} \ni \beta \stackrel{\sim}{\rightarrow} (x \rightarrow \beta \cdot x) \in (E_r^{k'\mp r,l'\pm r-1})^*.$$

d_r^{kl} followed by the arrow pointing down on the right-hand side of the diagram sends $\alpha \in E_r^{kl}$ to the functional $x \rightarrow d_r^{kl}(\alpha) \cdot x$ in $(E_r^{k'-r,l'+r-1})^*$, and the arrow pointing down in the middle of the diagram followed by $(d_r^{k'-r,l'+r-1})^*$ sends α to $x \rightarrow \alpha \cdot d_r^{k'-r,l'+r-1}(x)$. Since $n + r - 1 > n$ and $\alpha \cdot x \in E_r^{k+k'-r,n+r-1}, \alpha \cdot x = 0$ and, hence, $d_r^{kl}(\alpha) \cdot x = \pm\alpha \cdot d_r^{k'-r,l'+r-1}(x)$. Therefore the right square in (12) is commutative up to sign. Similarly, we prove that the left square also commutes up to sign. Since the top and the bottom rows are isomorphic chain complexes, their cohomology is isomorphic, Hence, we have an isomorphism

$$E_{r+1}^{kl} \ni \alpha \rightarrow (x \rightarrow \alpha \cdot x) \in (E_{r+1}^{k'l'})^*. \tag{13}$$

By (ZR), $E_r^{k+k',0} = \mathbb{K}$ implies that $E_{r+1}^{k+k',0} = \mathbb{K}$. Now (13) for $l = n$ implies that $E_{r+1}^{k+k',n} = \mathbb{K}$. Finally, by (13), the pairing (11) is non-degenerate. □

Now, we are going to use Proposition 3.1 to prove the existence of "Poincaré duality" on spectral sequences. This duality will be useful for the study of group actions on Poincaré duality spaces.

Let (E_r^{**}, d_*) be a term of a spectral sequence with a multiplicative structure such that $E_r^{*l} = 0$ for all $l < 0$ and for all $l > n$ for a certain n. We will consider three types of Poincaré duality on (E_r^{**}, d_r):

We say that (E_r^{**}, d_r) satisfies **Poincaré duality**, denoted by $P_{\mathbb{K}}(n)$, if there exists N such that:

(i) $E_r^{k*} = 0$ for all odd $k > N$;
(ii) $E_r^{k0} = E_r^{kn} = \mathbb{K}$ for all even $k > N$;
(iii) $E_r^{kl} \times E_r^{k',l'} \stackrel{\cdot}{\rightarrow} E_r^{k+k',n} = \mathbb{K}$ is non-degenerate for all $l, l' \geqslant 0$ such that $l + l' = n$ and for all even $k, k' \geqslant N$.

We say that (E_r^{**}, d_r) satisfies **weak Poincaré duality**, denoted by $WP_{\mathbb{K}}(n)$, if there exists N such that:

(i) $E_r^{kn} = \mathbb{K}$ for all odd $k > N$, and
(ii) for all $k, k' > N$ of different parity and for all $0 \leqslant l \leqslant n$ and $r \geqslant 2$, the pairing $E_r^{kl} \times E_r^{k',n-l} \dot{\to} E_r^{k+k',n} = \mathbb{F}_p$ is non-degenerate.

Finally, (E_r^{**}, d_r) satisfies **strong Poincaré duality**, $SP_{\mathbb{K}}(n)$, if there exists N such that:

(i) $E_r^{kn} = \mathbb{K}$ for all $k > N$,
(ii) for all $k, k' > N$ such that at least one of them is even and for all $0 \leqslant l \leqslant n$ the pairing $E_r^{kl} \times E_r^{k',n-l} \dot{\to} E_r^{k+k',n} = \mathbb{K}$ is non-degenerate.

The following statement follows by induction from Proposition 3.1:

Proposition 3.2. *Let (E_*^{**}, d_*) have a multiplicative structure and satisfy (ZR).*

(i) *If (E_r^{**}, d_r) satisfies $P_{\mathbb{K}}(n)$ for $r = 2$ then it satisfies $P_{\mathbb{K}}(n)$ for all $r > 2$.*
(ii) *If (E_r^{**}, d_r) satisfies $WP_{\mathbb{K}}(n)$ for $r = 2$ then it satisfies $WP_{\mathbb{K}}(n)$ for all $r > 2$.*
(iii) *If (E_r^{**}, d_r) satisfies $SP_{\mathbb{K}}(n)$ and r is even then (E_{r+1}^{**}, d_{r+1}) satisfies $SP_{\mathbb{K}}(n)$ as well.*

Lemma 3.3. (i) *If $r \geqslant 2$ and (E_r^{**}, d_r) satisfies $P_{\mathbb{K}}(n)$ then there exists N, such that $E_r^{kl} \cong E_r^{k'l}$, rank $d_r^{kl} =$ rank $d_r^{k'l}$, and rank $d_r^{kl} =$ rank $d_r^{k',n-l+r-1}$ for all even $k, k' > N$ and all l.*

(ii) *If $r \geqslant 2$ and (E_r^{**}, d_r) satisfies $SP_{\mathbb{K}}(n)$ then there exists N, such that $E_r^{kl} \cong E_r^{k'l}$, for all $k, k' \geqslant N$. Additionally, if r is even then rank $d_r^{kl} =$ rank $d_r^{k'l}$, and rank $d_r^{kl} =$ rank $d_r^{k',n-l+r-1}$ for all $k, k' \geqslant n$ and $0 \leqslant l \leqslant n$.*

Proof. (i) $E_r^{kl} \cong (E_r^{k,n-l})^* \cong E_r^{k'l}$ implies the first claim. Since $d_r = 0$ for r odd, assume that r is even. Note that the vertical maps in (12) are isomorphisms for k, k', l, l' such that k, k' are even and sufficiently big and $l + l' = n$. Hence, *rank* $d_r^{kl} =$ *rank* $d_r^{k'-r,n-l+r-1}$ for any $0 \leqslant l \leqslant n$, and by substituting k' for $k' - r$ we get

$$\text{rank } d_r^{kl} = \text{rank } d_r^{k',n-l+r-1}.$$

By applying this identity twice, we get

$$\text{rank } d_r^{kl} = \text{rank } d_r^{k',n-l+r-1} = \text{rank } d_r^{k'l}.$$

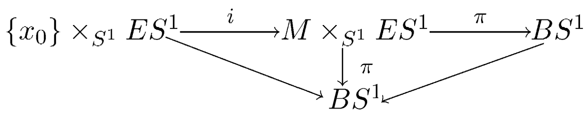
The proof of (ii) is analogous. □

3.1. Poincaré duality for spectral sequences for S^1-actions

Proposition 3.4 (PD for Leray spectral sequence). *If S^1 acts on a $PD_{\mathbb{Q}}(n)$-space M and $M^{S^1} \neq \emptyset$ then the Leray spectral sequence of the map $\pi: M \times_{S^1} ES^1 \to BS^1$ with coefficients in $\mathbb{Q}$ satisfies condition $P_{\mathbb{Q}}(n)$ for all $r \geqslant 2$.*

Proof. Since all cohomology groups are considered with supports in closed sets, by Bredon [6, Theorem IV.6.1] we have $E_2^{kl} = H^k(BS^1, \mathscr{H}^l(\pi; \mathbb{Q}))$, where $\mathscr{H}^l(\pi; \mathbb{Q})$ is the Leray sheaf of π.

Since BS^1 is simply connected, by the remark following [6, Theorem IV.8.2], $\mathscr{H}^l(\pi;\mathbb{Q})$ is the constant sheaf with stalks $H^l(X;\mathbb{Q})$. Hence, $E_2^{kl} \cong H^k(BS^1,\mathbb{Q}) \otimes H^l(M;\mathbb{Q})$. The Leray spectral sequence of $\pi, (E_*^{**}, d_*)$, has a multiplicative structure—see e.g. [6, IV.6.5] or [22, XII Section 3.2]. An argument similar to that used in the proof of Theorem III.15.11 in [5] shows that E_2^{**} and $H^*(BS^1;\mathbb{Q}) \otimes H^*(M;\mathbb{Q})$ are isomorphic as algebras. Since $H^*(BS^1) = \mathbb{Q}[t]$, where $deg\, t = 2, E_2^{**}$ satisfies condition $P_{\mathbb{Q}}(n)$. The statement for higher r follows from Proposition 3.2(i) once we show that (E_*^{**}, d_*) satisfies (ZR). To prove it, choose a fixed point $x_0 \in M$ of the action and consider the diagram



where i is the natural embedding, and the skew arrows represent the identity maps. Let $(\bar{E}_*^{**}, \bar{d}_*)$ denote the spectral sequence of the map $id : BS^1 \to BS^1$,

$$\bar{E}_r^{kl} = \begin{cases} \mathbb{Q} & \text{for } l = 0 \text{ and even } k \geqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

The horizontal maps of the diagram above induce morphisms of spectral sequences $\bar{E}_*^{**} \xrightarrow{\pi^*} E_*^{**} \xrightarrow{i^*} \bar{E}_*^{**}$. Since $i^*\pi^*$ is the identity on $\bar{E}_*^{**}, E_r^{k0} \neq 0$ for all $r \geqslant 2$ and any even $k \geqslant 0$. Hence (E_*^{**}, d_*) satisfies condition (ZR). □

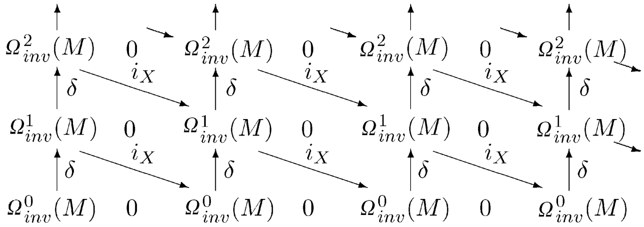
Similarly, by using [15, Theorem 5.2] and adopting the above argument we prove the following.

Proposition 3.5 (PD for Leray–Serre spectral sequence). *If S^1 acts on a $PD_{\mathbb{Q}}(n)$-space (with respect to singular cohomology) M, and $M^{S^1} \neq \emptyset$, then the Leray–Serre spectral sequence for singular cohomology of the fibration*

$$M \to M \times_{S^1} ES^1 \to BS^1$$

satisfies $P_{\mathbb{Q}}(n)$ for every $r \geqslant 2$.

A convenient way of calculating equivariant cohomology of a smooth closed manifold M with an S^1 action on it is by using the Cartan construction, [4, Section 4]: Let D^{**} be a bigraded $\mathbb{R}$-linear space whose $(2k, l)$th summand is the space of S^1-invariant, differential l-forms on M, $D^{2kl} = \Omega^l_{\text{inv}}(M) \subset \Omega^l(M)$ and $D^{2k+1,l} = 0$ for $k \geqslant 0$:



Let $X \in Vect(M)$ be the vector field on M induced by the S^1 action (the infinitesimal action), let $i_X : \Omega^k_{\text{inv}}(M) \to \Omega^{k-1}_{\text{inv}}(M)$ be the map $(i_X\omega)(\cdot,\dots,\cdot) = \omega(X,\cdot,\dots,\cdot)$, and let δ denote the exterior derivative of differential forms on M. Consider a differential $d : D^{**} \to D^{**}$,

$$d(\omega) = \begin{cases} \delta\omega - i_X\omega & \text{for } \omega \in D^{kl} = \Omega^l_{\text{inv}}(M) \text{ for even } k; \\ 0 & \text{for odd } k. \end{cases}$$

It is not difficult to see that the exterior product of forms induces a multiplicative structure on (D^{**}, d). By Theorem 4.13 and the following paragraphs in [4], the cohomology of the total complex of the above complex is isomorphic to $H^*_{S^1}(M; \mathbb{R})$. One can show that the vertical filtration of this double complex yields a spectral sequence which satisfies $P_{\mathbb{Q}}(n)$.

3.2. Spectral sequences for $\mathbb{Z}/p$-actions

Let $\mathbb{Z}/p$ act on a paracompact connected space X. Consider the standard fibration $X \to X_{\mathbb{Z}/p} \xrightarrow{\pi} B\mathbb{Z}/p$, where $X_{Z/p} = X \times_{\mathbb{Z}/p} B\mathbb{Z}/p$. There are three spectral sequences associated with the $\mathbb{Z}/p$-action on X involving cohomology of X with coefficients in a ring R:

(**Leray**) The Leray spectral sequence of π, compare [6, IV.6], [23, 5.8.6];

(**Serre**) The Leray–Serre spectral sequence of the fibration π (defined for singular cohomology theory) [15, Chapter 5]; [24, XIII.7].

(**Swan**) A spectral sequence defined as follows: Let (C^*, δ) be the cochain complex of X for sheaf (or Alexander–Spanier) cohomology with coefficients in R. There is a natural $\mathbb{Z}/p$-action on C^*. Let

$$\to P^2 \xrightarrow{\delta'} P^1 \xrightarrow{\delta'} P^0 \to R$$

be a projective resolution of the $R[\mathbb{Z}/p]$-module R with the trivial $\mathbb{Z}/p$-action. Then $D^{**} = Hom_{R[\mathbb{Z}/p]}(P^*, C^*)$ is a double complex with the differential $\delta_v = \delta : D^{kl} \to D^{k,l+1}$ and the differential $\delta_h : D^{kl} \to D^{k+1,l}$ dual to δ'. We consider the "first" spectral sequence associated with $(D^{**}, \delta_h, \delta_v)$ and for the purpose of this paper we will call it the Swan spectral sequence. (In [21] a similar construction based on complete projective resolutions is considered.) A version of Swan spectral sequence can be constructed for singular cochains of X and for cellular cochains if X is a CW-complex.

The total complex of $(D^{**}, \delta_h, \delta_v)$ is

$$D^s = \bigoplus_{k+l=s} D^{kl},$$

$$d(\alpha) = d_h(\alpha) + (-1)^k d_v(\alpha),$$

where $\alpha \in D^{kl}$. (Note that $d \circ d = 0$.) The "first" spectral sequence is the one induced by the vertical filtration of D^{**}.

Proposition 3.6. *If (E^{**}_*, d_*) is any of the three spectral sequences defined above for cohomology with (constant) coefficients in R then*:

(i) $E^{kl}_2 = H^k(\mathbb{Z}/p, H^l(X; R))$, *where $g \in \mathbb{Z}/p$ acts on $H^l(X;R)$ by the automorphism induced by $g^{-1} : X \to X$.*

(ii) *There is a multiplicative structure* $\cdot$ *on* (E_*^{**}, d_*) *such that* $\alpha\cdot\beta = (-1)^{k'l}\alpha\cup\beta$ *for* $\alpha\in E_2^{kl}$, $\beta\in E_2^{k'l'}$. (*Note that* $H^k(\mathbb{Z}/p, H^l(X;R))\times H^{k'}(\mathbb{Z}/p, H^{l'}(X;R))\xrightarrow{\cup} H^{k+k'}(\mathbb{Z}/p, H^{l+l'}(X;R))$ *is a well-defined cup-product on cohomology groups with non-constant coefficients.*)

(iii) *If* E_*^{**} *is the Leray or Leray–Serre spectral sequence then* E_*^{**} *converges to* $H^*(X_{\mathbb{Z}/p};R)$.

We do not know if the Swan spectral sequence converges to $H^*(X_{\mathbb{Z}/p};R)$. We do not know either under what conditions the above three spectral sequences are isomorphic.

Proof of Proposition 3.6. (**Leray**) (i) Since all cohomology groups are considered with supports in closed sets, by Bredon [6, Theorem IV.6.1] we have $E_2^{kl} = H^k(B\mathbb{Z}/p, \mathscr{H}^l(\pi;R))$, where $B\mathbb{Z}/p$ is locally contractible and $\mathscr{H}^l(\pi;R)$ is the Leray sheaf of π. By the remark following [6, Theorem IV.8.2], $\mathscr{H}^l(\pi;R)$ is locally constant on $B\mathbb{Z}/p$, and by careful retracing the relevant definitions, we see that Leray sheaf is given by the $\mathbb{Z}/p$-action on $H^*(X;R)$ described above. (ii) follows from [6, IV.6.5]. (iii) follows from [6, Theorem IV.6.1].

(**Leray–Serre**) The statement for Leray–Serre spectral sequence follows from [15, Theorem 5.2]. Compare also [24, XIII.8.10].

(**Swan**) We have $E_1^{kl} = Hom_{R[\mathbb{Z}/p]}(P_k, H^l(X;R))$, where $\mathbb{Z}/p$ acts on $H^l(X;R)$ as in Proposition 3.6(i). Therefore, $E_2^{kl} = H^k(\mathbb{Z}/p, H^l(X;R))$. If $\Delta: P^*\to P^*\otimes P^*$ is a diagonal approximation of (P^*,δ') then the cup product $D^{kl}\otimes D^{k'l'}\xrightarrow{\cup} D^{k+k',l+l'}$ is defined for any $\alpha\in D^{kl}$, $\beta\in D^{k'l'}$ by

$$P^{k+k'}\xrightarrow{\Delta_{kk'}} P^k\otimes P^{k'}\xrightarrow{\alpha\otimes\beta} C^l\otimes C^{l'}\xrightarrow{\cup} C^{l+l'}.$$

It has the following properties:

$$d_h(\alpha\cup\beta) = d_h(\alpha)\cup\beta + (-1)^k\alpha\cup d_h(\beta),$$

$$d_v(\alpha\cup\beta) = d_v(\alpha)\cup\beta + (-1)^l\alpha\cup d_v(\beta)$$

for $\alpha\in D^{kl}$, $\beta\in D^{k'l'}$. Let $\alpha\cdot\beta$ be a new product on D^{**} equal to $(-1)^{k'l}\alpha\cup\beta$, for α,β as above. The following lemma, whose proof is left to the reader, implies that $\cdot$ defines a multiplicative structure on E_*^{**}.

Proposition 3.7.

$$d(\alpha\cdot\beta) = d(\alpha)\cdot\beta + (-1)^{deg\,\alpha}\alpha\cdot d(\beta),$$

where $deg(\alpha) = k + l$.

This completes the proof of Proposition 3.6. □

Let (P_*,δ') be the standard resolution of R by free $R[\mathbb{Z}/p]$-modules: let $P_k = R[\mathbb{Z}/p]$ for all $k\geqslant 0$ and let $\delta': P_k\to P_{k-1}$ be

$$\delta'(\alpha) = \begin{cases} (t-1)\cdot\alpha & \text{for } k \text{ odd},\\ N\cdot\alpha & \text{for } k \text{ even},\end{cases}$$

where $t-1$, $N = 1+t+\cdots+t^{p-1}$ are elements of $R[\mathbb{Z}/p] = R[t]/(t^p-1)$. Let

$$\Delta: (P_*,\delta')\to(P_*,\delta')\otimes(P_*,\delta')$$

be the diagonal approximation whose (k,l)-component $\Delta_{kl}: P_{k+l} \to P_k \otimes P_l$ is given by

$$\Delta_{kl}(1) = \begin{cases} 1 \otimes 1 & \text{if } k \text{ even}, \\ 1 \otimes t & \text{if } k \text{ odd}, \ l \text{ even}, \\ \sum_{0 \leqslant i < j \leqslant p-1} t^i \otimes t^j & \text{if } k, l \text{ odd}, \end{cases}$$

cf. [9, Ex. V.1]. Therefore, after identifying $D^{kl} = Hom_{R[\mathbb{Z}/p]}(P_k, H^l(X;R))$ with $H^l(X;R)$ we have

$$\alpha \cup \beta = \begin{cases} \alpha \cup_X \beta & \text{if } k \text{ is even}, \\ \alpha \cup_X t\beta & \text{if } k \text{ is odd}, \ k' \text{ is even}, \\ \sum_{0 \leqslant i < j \leqslant p-1} t^i\alpha \cup_X t^j\beta & \text{if } k, k' \text{ are odd}, \end{cases} \tag{14}$$

where $\cup_X$ denotes the cup product on $H^*(X;R)$ and $\cup$ denotes the product on D^{**} defined before. (Recall that $\alpha \cdot \beta = (-1)^{k'l}\alpha \cup \beta$, for $\alpha \in D^{kl}$, $\beta \in D^{k'l'}$.)

Lemma 3.8. *If R is $\mathbb{Z}$ or $\mathbb{F}_p$ and the $\mathbb{Z}/p$-action on X has a fixed point then all three spectral sequences considered above satisfy condition (ZR) for $\mathbb{K} = \mathbb{F}_p$.*

Proof. Let E_*^{**} be the Leray or Leray–Serre or Swan spectral sequence associated with the $\mathbb{Z}/p$-action on X, let $x_0 \in X^{\mathbb{Z}/p}$ and let $\bar{E}_*^{**}$ be the corresponding spectral sequence associated with the trivial $\mathbb{Z}/p$-action on $\{x_0\}$. The $\mathbb{Z}/p$-equivariant maps: $\{x_0\} \hookrightarrow X$ and $X \to \{x_0\}$ induce maps

$$(\bar{E}_r^{**}, \bar{d}_r) \to (E_r^{**}, d_r) \to (\bar{E}_r^{**}, \bar{d}_r), \tag{15}$$

whose composition is the identity on $\bar{E}_r^{**}$ for $r \geqslant 1$. Since X is assumed connected and $R = \mathbb{Z}$ or $\mathbb{F}_p$, $E_2^{k0} = H^k(\mathbb{Z}/p, R)$ is either 0 or $\mathbb{F}_p$. Hence, if (ZR) is not satisfied then $E_2^{k0} = \mathbb{F}_p$ and $E_\infty^{k0} = 0$ for some k. This implies that $\bar{E}_2^{k0} = \mathbb{F}_p$, and since (15) is the identity map, $\bar{E}_\infty^{k0} = 0$. This leads to contradiction since $\bar{E}_2^{**}$ has only one non-zero row and $\bar{E}_\infty^{**} = \bar{E}_2^{**}$. □

3.3. Poincaré duality for spectral sequences for $\mathbb{Z}/p$-actions

Let $\mathbb{Z}/p$ act on a $PD_{\mathbb{F}_p}(n)$-space M with a fixed point and let (E_*^{**}, d_*) be either the Leray or Swan spectral sequence associated with that action with coefficients in $\mathbb{F}_p$.

Proposition 3.9. (E_r^{**}, d_r) *satisfies $WP_{\mathbb{F}_p}(n)$ for $r \geqslant 2$.*[2]

Proof. Let k, k' be of different parity, and let $0 \leqslant l \leqslant n$, $l' = n - l$. Since $\mathbb{Z}/p \subset \mathbb{Q}/\mathbb{Z}$, and

$$H^l(M;\mathbb{F}_p) = Hom(H^{l'}(M;\mathbb{F}_p), \mathbb{Z}/p) = Hom(H^{l'}(M;\mathbb{F}_p), \mathbb{Q}/\mathbb{Z})$$

[2] V. Puppe pointed to us that a similar result is hidden in the proof of the main theorem of [8].

as $\mathbb{F}_p[\mathbb{Z}/p]$-modules, the duality theorem for Tate cohomology, [9, Corollary VI.7.3], implies that

$$H^k(\mathbb{Z}/p, H^l(M;\mathbb{F}_p)) \times H^{k'}(\mathbb{Z}/p, H^{l'}(M;\mathbb{F}_p)) \to H^{k+k'}(\mathbb{Z}/p, H^n(M;\mathbb{F}_p)) = \mathbb{F}_p$$

is non-degenerate.[3] Therefore, by Proposition 3.6, E_2^{**} satisfies the weak Poincaré duality. Now the proposition follows from Lemma 3.8 and Proposition 3.2(ii). □

In order to say more about the multiplicative properties of (E_*^{**}, d_*) we need to assume that the $\mathbb{Z}/p$-action on M is nice, $H^*(M;\mathbb{F}_p) = T^* \oplus F^*$. Now $E_2^{kl} = H^k(\mathbb{Z}/p, H^l(M;\mathbb{F}_p)) = T^l$ for $k > 0$. Since the $\mathbb{Z}/p$ action on T^* is trivial, by (14) and Proposition 3.6 for $p \neq 2$ we have

$$\alpha \cdot \beta = \begin{cases} (-1)^{k'l} \alpha \cup \beta & \text{if } k \text{ or } k' \text{ is even,} \\ 0 & \text{if } k, k' \text{ are odd} \end{cases} \tag{16}$$

for $\alpha \in E_2^{kl}$, $\beta \in E_2^{k'l'}$. (Recall that $\alpha \cdot \beta = (-1)^{k'l} \alpha \cup \beta$. Furthermore, for k, k' odd we have $\alpha \cup \beta = \sum_{0 \leqslant i < j \leqslant p-1} \alpha \cup_X \beta = 0$ since $\binom{p}{2} \equiv 0 \bmod p$.)

Lemma 3.10. *If a $\mathbb{Z}/p$-action on M is nice and k or k' is even then the product $\cdot$ given by (16) is non-degenerate.*

Proof. Since for $k \not\equiv k' \bmod 2$ this follows from Proposition 3.9, we can assume that k, k' are even. Let $H^l(M;\mathbb{F}_p) = T^l \oplus F^l$, $H^{l'}(M;\mathbb{F}_p) = T^{l'} \oplus F^{l'}$, and let $\alpha_1, \ldots, \alpha_s$ be generators of the summands of $F^l = \mathbb{F}_p[\mathbb{Z}/p] \oplus \cdots \oplus \mathbb{F}_p[\mathbb{Z}/p]$. Let F_i^l be the $\mathbb{F}_p$-vector subspace of F^l generated by elements $(t-1)^i \alpha_j$ for $j = 1, \ldots, s$. Note that $F^l = F_0^l \oplus \cdots \oplus F_{p-1}^l$. Similarly we decompose $F^{l'}$ into $F_0^{l'} \oplus \cdots \oplus F_{p-1}^{l'}$. Since M is a $PD_{\mathbb{F}_p}(n)$-space, the matrix representing the product

$$(T^l \oplus F_{p-1}^l) \times (T^{l'} \oplus F_0^{l'} \oplus \cdots \oplus F_{p-1}^{l'}) \xrightarrow{\cup} \mathbb{F}_p$$

is of maximal rank, $\dim_{\mathbb{F}_p} T^l + s$. All columns of this matrix corresponding to spaces $F_i^{l'}$ for $i > 0$, are 0. Indeed, if $\beta \in T^l \oplus F_{p-1}^l = (H^l(M;\mathbb{F}_p))^{\mathbb{Z}/p}$ and $\beta' \in F_i^{l'}$ for $i > 0$ then there exists $\beta'' \in F_{i-1}^{l'}$ such that $\beta' = (t-1)\beta''$. Since

$$\beta \cup \beta'' = t(\beta \cup \beta'') = \beta \cup t\beta'',$$

we have $\beta \cup (t-1)\beta'' = \beta \cup \beta' = 0$. Therefore, the matrix of the cup product on

$$T^l \oplus F_{p-1}^l \times T^{l'} \oplus F_0^{l'}$$

is non-degenerate. By an argument similar to the above, $\beta \cup \beta' = 0$ for any $\beta \in F_{p-1}^l$ and $\beta' \in T^{l'}$. Hence, this matrix has a form

$$\begin{array}{cc} & \begin{array}{cc} T^{l'} & F_0^{l'} \end{array} \\ \begin{array}{c} T^l \\ F_{p-1}^l \end{array} & \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \end{array}.$$

Therefore, the matrix A associated with $T^l \times T^{l'} \xrightarrow{\cup} \mathbb{F}_p$ is non-degenerate. □

[3] Recall that k and k' have different parity.

The above lemma shows that E_2^{**} satisfies $SP_{\mathbb{F}_p}(n)$ for nice $\mathbb{Z}/p$-actions. If $p=2$ then it can be shown by induction on r and by using Proposition 3.1 that E_r^{**} satisfies $SP_{\mathbb{F}_p}(n)$ for all $r \geqslant 2$. However, we do not know if E_r^{**} satisfies $SP_{\mathbb{F}_p}(n)$ for $p \neq 2$ in general. This problem stems from the fact that the implication of Proposition 3.2(iii) does not hold for odd r. Therefore, for certain applications it is necessary to assume condition (7).

Lemma 3.11. *If condition (7) holds for a given $\mathbb{Z}/p$-action on M then (E_r^{**}, d_r) satisfies $SP_{\mathbb{F}_p}(n)$ for each $r \geqslant 2$.*

Proof. The statement follows from Proposition 3.2(iii) and Lemmas 3.8 and 3.10. □

Proposition 3.12. *Condition (7) holds for $n \leqslant 3$. Consequently, for any nice $\mathbb{Z}/p$-action on a $PD_{\mathbb{F}_p}(n)$-space for $n \leqslant 3$ each term of the induced Leray spectral sequence satisfies $SP_{\mathbb{F}_p}(n)$ for all r.*

Proof. For $n=1$ the statement is obvious. For $n=2$ the statement is a consequence of condition (ZR), cf. Lemma 3.8. Therefore, assume that $n=3$. Since $d_r=0$ for $r \geqslant 5$, it suffices to show that $d_3^{kl}=0$ for $k \geqslant n$. For $l=2$ it follows from Lemma 3.8. Hence, assume that $l=3$ and that $d_3^{kl}(\omega)=\alpha \neq 0$ for some $\omega \in E_3^{k3}=\mathbb{F}_p$. Since $\alpha \in E_3^{k+3,1}$, by the weak Poincaré duality there exists $\beta \in E_3^{k+2,2}$ such that $\alpha \cdot \beta \neq 0$. By Lemma 3.8, $d_3(\beta)=0$ and hence we get a contradiction:

$$0 = d_3(\omega\beta) = d_3(\omega) \cdot \beta + \omega \cdot d_3(\beta) = \alpha \cdot \beta \neq 0. \qquad \square$$

The next result concerns Poincaré duality for spectral sequences with integral coefficients.

Proposition 3.13. *If $\mathbb{Z}/p$ acts nicely on a $PD_{\mathbb{F}_p}(n)$-space M, with no p-torsion in $H^*(M;\mathbb{Z})$ and if $M^{\mathbb{Z}/p} \neq \emptyset$ then the Leray and the Swan spectral sequences for that action and for $R=\mathbb{Z}$ satisfy Poincaré duality, $P_{\mathbb{F}_p}(n)$, for all $r \geqslant 2$.*

Proof. The statement for $r=2$ follows from Lemmas 2.4 and 3.10. For $r>2$ the statement follows from Lemma 3.8 and Proposition 3.2(i). □

4. Proofs of the main results

Lemma 4.1. *Let (E_*^{**}, d_*) be a spectral sequence whose terms (E_r^{**}, d_r) for $r \geqslant 2$ are vector spaces over a field $\mathbb{K}$, char $\mathbb{K} \neq 2$, and satisfy either $P_{\mathbb{K}}(n)$ or $SP_{\mathbb{K}}(n)$. In the latter case we assume that $d_r=0$ for odd r. If*

(i) *n is even, or*
(ii) *$E_2^{kl}=0$ for all even l, $0<l\leqslant \frac{1}{2}(n-1)$, and for all sufficiently large k*

then

$$\sum_l \dim_{\mathbb{K}} E_\infty^{kl} \equiv \sum_l \dim_{\mathbb{K}} E_2^{kl} \bmod 4$$

for all sufficiently large k.

Proof. It is enough to prove that

$$\sum_l \dim E_{r+1}^{kl} \equiv \sum_l \dim E_r^{kl} \bmod 4$$

for all $r \geqslant 2$ and sufficiently large k. Since $E_r^{**} = E_{r+1}^{**}$ for r odd, assume that r is even. By Lemma 3.3, $rank\, d_r^{kl} = rank\, d_r^{k-r,l}$, for all sufficiently large k. Therefore,

$$\begin{aligned}\sum_l \dim E_{r+1}^{kl} &= \sum_l \dim Ker\, d_r^{kl} - \sum_l \dim Im\, d_r^{k-r,l+r-1}\\ &= \sum_l \dim E_r^{kl} - \sum_l rank\, d_r^{kl} - \sum_l rank\, d_r^{k-r,l+r-1}\\ &= \sum_l \dim E_r^{kl} - 2\sum_l rank\, d_r^{kl}.\end{aligned}$$

Therefore, we need to prove that

$$\sum_l rank\, d_r^{kl} \equiv 0 \bmod 2.$$

By Lemma 3.3,

$$\sum_l rank\, d_r^{kl} = 2 \cdot \sum_{l<n-l+r-1} rank\, d_r^{kl} + \begin{cases} rank\, d_r^{kl_0} & \text{if } l_0 = n - l_0 + r - 1,\\ 0 & \text{if there is no such } l_0.\end{cases}$$

For n even, there is no such l_0 and the proof is completed. Hence, assume that n is odd. If l_0 is odd then $l_0 - r + 1$ is even and $l_0 - r + 1 \leqslant \frac{1}{2}(n-1)$. Hence, $E_r^{k+r,l_0-r+1} = 0$ by the assumption of the lemma, and therefore $d_r^{kl_0} : E_r^{kl_0} \to E_r^{k+r,l_0-r+1}$ is 0. Therefore assume that l_0 is even. Since $d_r^{kl} = 0$ for odd k, assume also that k is even. Consider the bilinear form

$$\Psi : E_r^{kl_0} \times E_r^{kl_0} \to E_r^{2k+r,n} = \mathbb{K},$$

$\Psi(\alpha,\beta) = d_r(\alpha)\cdot\beta$. We have

$$d_r(\alpha)\cdot\beta + (-1)^{k+l_0}\alpha\cdot d_r(\beta) = d_r(\alpha\cdot\beta) = 0.$$

Since $deg(\alpha) = k + l_0$ is even, $\alpha\cdot d_r(\beta) = d_r(\beta)\cdot\alpha$ and

$$d_r(\alpha)\cdot\beta + d_r(\beta)\cdot\alpha = 0.$$

Therefore Ψ is skew-symmetric, and it has an even rank. But $rank\,\Psi = rank\, d_r^{kl_0}$, since $\alpha \in E_r^{kl_0}$, $d_r(\beta) \in E_r^{k+r,l_0-r+1}$ and the product $E_r^{kl_0} \times E_r^{k+r,l_0-r+1} \to E^{2k+r,n}$ is non-degenerate. □

4.1. Proof of Theorem 1.1

The following lemma shows that it is sufficient to prove Theorem 1.1 for circle actions.

Lemma 4.2. *If an action of a torus T on X has FMCOT then there exists $S^1 \subset T$ such that $X^{S^1} = X^{\mathrm{T}}$.*

Proof. The condition FMCOT implies that the set $\{(T_x)^0\colon\ x\in X\}$ is finite. Denote its elements different than T by $T_1,\dots,T_n$. Consider $S^1\subset T$ which does not lie inside T_i for any i. Then $S^1\cap T_i$ is finite for $1\leqslant i\leqslant n$. Since each T_i has only countably many finite extensions in T, the set $S^1\cap\left(\bigcup_{x\in X\setminus X^T} T_x\right)$ is at most countable. Therefore, there exists $t\in S^1$ such that the only points of X fixed by t are the elements of X^{T}. Hence $X^{S^1}=X^{\mathrm{T}}$. □

Assume now that $T=S^1$. The proof of Theorem 1.1 for n even follows immediately from (2), the lemma below, and the fact that M^{T} has finitely many components, each of which is a $PD_{\mathbb{Q}}(m)$-space, for m even (see [3, Theorem 5.2.1, Remark 5.2.4], cf. [11]).

Lemma 4.3. *If n is even and M is a $PD_{\mathbb{Q}}(n)$-space, then $\sum_i b^i(M)\equiv\chi(M) \bmod 4$.*

Proof. Since $b^i(M)=b^{n-i}(M)$, the difference between the left and the right-hand side of the above identity is

$$2\sum_{\text{odd }i} b^i(M)=4\sum_{\text{odd }i<n/2} b^i(M)+2\begin{cases} b^{n/2}(M) & \text{if } n/2 \text{ is odd},\\ 0 & \text{otherwise.}\end{cases}$$

This completes the proof for $n/2$ even. If $n/2$ is odd then the pairing

$$H^{n/2}(M;\mathbb{Q})\times H^{n/2}(M;\mathbb{Q})\xrightarrow{\cup}H^n(M;\mathbb{Q})=\mathbb{Q}$$

is non-degenerate and skew-symmetric. Hence $b^{n/2}(M)$ is even. □

Assume now that n is odd and that an action of S^1 on a $PD_{\mathbb{Q}}(n)$-space M satisfies all assumptions of Theorem 1.1.

By Proposition 3.10.9 and Corollary 3.10.12 in [3], $H^i_{S^1}(M,M^{S^1};\mathbb{Q})=0$ for $i>cd\,M$. Therefore, the long exact sequence of the equivariant cohomology groups for the pair (M,M^{S^1}) gives an isomorphism

$$H^s_{S^1}(M;\mathbb{Q})=H^s_{S^1}(M^{S^1};\mathbb{Q}) \tag{17}$$

for $s>cd\,M$.

$H^*_{S^1}(M^{S^1};\mathbb{Q})=H^*(M^{S^1};\mathbb{Q})\otimes H^*(BS^1)$ and $H^*(BS^1)=\mathbb{Q}[t]$, where $deg\,t=2$. Therefore,

$$\dim_{\mathbb{Q}} H^s_{S^1}(M^{S^1};\mathbb{Q})+\dim_{\mathbb{Q}} H^{s+1}_{S^1}(M^{S^1};\mathbb{Q})=\sum_i \dim_{\mathbb{Q}} H^i(M^{S^1};\mathbb{Q}). \tag{18}$$

The Leray spectral sequence (E^{**}_*,d_*) of the map $M\times_{S^1}ES^1\to BS^1$ with coefficients in $\mathbb{Q}$ converges to $H^*_{S^1}(M;\mathbb{Q})$. By Proposition 3.4, (E^{**}_r,d_r) satisfies condition $P_{\mathbb{Q}}(n)$ for all $r\geqslant 2$. Since by Lemma 3.3(i) the ranks of entries in E^{**}_∞ are 2-periodic, we have

$$\dim H^s_{S^1}(M;\mathbb{Q})+\dim H^{s+1}_{S^1}(M;\mathbb{Q})=\sum_{\substack{k+l=s\\ k\ \text{even}}}\dim E^{kl}_\infty+\sum_{\substack{k+l=s+1\\ k\ \text{even}}}\dim E^{kl}_\infty=\sum_l \dim E^{k_0 l}_\infty$$

for sufficiently large s and sufficiently large even k_0. By Lemma 4.1 the above expression is equal mod 4 to $\sum_l \dim E_2^{k_0 l} = \sum_l \dim H^l(M;\mathbb{Q})$. Hence,

$$\dim H^s_{S^1}(M;\mathbb{Q}) + \dim H^{s+1}_{S^1}(M;\mathbb{Q}) \equiv \sum_l \dim H^l(M;\mathbb{Q}) \bmod 4.$$

This equality together with (17) and (18) implies Theorem 1.1. □

4.2. Proof of Proposition 1.2

The following three lemmas will be needed in the proof of Proposition 1.2 and of Theorem 1.3:

Lemma 4.4. *If $\mathbb{Z}/p$ acts on a paracompact space X of finite cohomological dimension then the embedding $i : X^{\mathbb{Z}/p} \to X$ induces an isomorphism $i^* : H^s_{\mathbb{Z}/p}(X;A) \to H^s_{\mathbb{Z}/p}(X^{\mathbb{Z}/p};A)$ for $s > cd\,X$, where A is an arbitrary group of (constant) coefficients.*

Proof. By Proposition 3.10.9 in [3], $H^*_{\mathbb{Z}/p}(X, X^{\mathbb{Z}/p};A) \simeq H^*(X/(\mathbb{Z}/p), X^{\mathbb{Z}/p}/(\mathbb{Z}/p);A)$. Since by Quillen [18, Proposition A.11], $cd(X/(\mathbb{Z}/p)) \leqslant cd\,X$, we have $H^s(X/(\mathbb{Z}/p), X^{\mathbb{Z}/p}/(\mathbb{Z}/p);A) = 0$ for $s > cd\,X$. Now the proposition follows from the long exact sequence for the equivariant cohomology of the pair $(X, X^{\mathbb{Z}/p})$. □

Lemma 4.5 (Künneth formula for sheaf cohomology). *If R is a principal ideal domain, Y is a CW-complex and X is a paracompact space such that $H^l(X;R)$ is finitely generated R-module for each l, then there exists a split exact sequence*

$$0 \to \bigoplus_{k+l=s} H^k(X;R) \otimes H^l(Y;R) \to H^s(X \times Y;R)$$
$$\to \bigoplus_{k+l=s+1} Tor_R(H^k(X;R), H^l(Y;R)) \to 0.$$

Proof. By Hatcher [14, Proposition A.4] (cf. [20, Ex. Ch7 E5]) Y is a locally contractible space. By the remark following [6, Theorem IV.8.2], the Leray sheaf of the projection $\pi : X \times Y \to Y$ is the constant sheaf with the stalk $H^*(X;R)$. Therefore, the statement of proposition follows from [6, Ex. IV.18]. □

Lemma 4.6. *If $\mathbb{Z}/p$ acts trivially on X then*

$$H^s_{\mathbb{Z}/p}(X;\mathbb{Z}) \cong \bigoplus_{l \equiv s \bmod 2} H^l(X;\mathbb{F}_p)$$

for $s > cd\,X$.

Proof. By Lemma 4.5, $H^s_{\mathbb{Z}/p}(X;\mathbb{Z})$ is isomorphic to

$$\bigoplus_{k+l=s} H^k(B\mathbb{Z}/p, \mathbb{Z}) \otimes H^l(X;\mathbb{Z}) \oplus \bigoplus_{k+l=s+1} Tor_{\mathbb{Z}}(H^k(B\mathbb{Z}/p, \mathbb{Z}), H^l(X;\mathbb{Z})).$$

Since for $k > 0$ $H^k(B\mathbb{Z}/p, \mathbb{Z})$ is either $\mathbb{F}_p$ or 0 depending if k even or odd,

$$H^s_{\mathbb{Z}/p}(X;\mathbb{Z}) = \bigoplus_{l \equiv s \bmod 2} H^l(X;\mathbb{Z}) \otimes \mathbb{F}_p \oplus \bigoplus_{l \equiv s-1 \bmod 2} Tor_{\mathbb{Z}}(H^l(X;\mathbb{Z}), \mathbb{F}_p)$$

for $s > cd\,X$. But by the universal coefficient theorem for cohomology, the right side is isomorphic to $\bigoplus_{l \equiv s \bmod 2} H^l(X;\mathbb{F}_p)$. □

For the proof of Proposition 1.2 we will need the following version of the notion of Euler characteristic for double complexes: if D^{**} is a double complex of vector spaces over a field $\mathbb{K}$ then let

$$\chi(D^{**}) = \lim_{N\to\infty} \frac{1}{N} \sum_{\substack{0 \leqslant k \leqslant N \\ l \in \mathbb{Z}}} (-1)^{k+l} \dim_{\mathbb{K}} D^{kl}$$

if this limit exists.

Proposition 4.7. *If* (E^{**}_*, d_*) *is a spectral sequence such that* (a) E^{kl}_r *are vector spaces over a field* $\mathbb{K}$ *and* $\dim_{\mathbb{K}} E^{kl}_r \leqslant c$ *for all* k, l, *for a certain* c, (b) $E^{*l}_r = 0$ *for all* $l < 0$ *and* $l > n$ *for some* n, (c) $\chi(E^{**}_r)$ *exists, then*:

(i) E^{**}_{r+1} *satisfies conditions* (a), (b), (c) *as well, and*
(ii) $\chi(E^{**}_{r+1}) = \chi(E^{**}_r)$.

Proof. The only non-trivial statement of the proposition is that $\chi(E^{**}_{r+1})$ exists and it is equal to $\chi(E^{**}_r)$. Consider the cochain complex $(C^*_{kl,r}, d_r)$, where $C^i_{kl,r} = E_r^{k+ir,\,l-i(r-1)}$ for $i \in \mathbb{Z}$. Note that under the above assumptions the sums

$$\sum_{\substack{0 \leqslant k \leqslant N \\ l \in \mathbb{Z}}} (-1)^{k+l} \dim_{\mathbb{K}} E^{kl}_r$$

and

$$\sum_{\substack{0 \leqslant k \leqslant N \\ 0 \leqslant l \leqslant r-1}} (-1)^{k+l} \dim_{\mathbb{K}} \chi(C^*_{kl,r})$$

differ by a finite number of terms of the form $(-1)^{k+l} \dim_{\mathbb{K}} E^{kl}$, and that the number of such terms does not depend on N. Since $\dim_{\mathbb{K}} E^{kl}_r \leqslant c$, the difference between the above two sums is bounded uniformly in N and hence

$$\chi(E^{**}_r) = \lim_{N\to\infty} \frac{1}{N} \sum_{\substack{0 \leqslant k \leqslant N \\ 0 \leqslant l \leqslant r-1}} (-1)^{k+l} \dim_{\mathbb{K}} \chi(C^*_{kl,r}). \tag{19}$$

Since $\chi(C^*_{kl,r}) = \chi(C^*_{kl,r+1})$, the proof follows from (19) and the analogous equation for $r+1$. □

Let $p \neq 2$ and let $\mathbb{Z}/p$ act nicely on a space X with no p-torsion in $H^*(X;\mathbb{Z})$, and let (E^{**}_*, d_*) be the associated Leray spectral sequence with coefficients in $\mathbb{Z}$. Since $E^{kl}_2 = H^k(\mathbb{Z}/p, H^l(X;\mathbb{Z}))$, by

Lemma 2.4, $\chi(E_2^{**}) = \frac{1}{2}\chi_t(X)$. Therefore, by Proposition 4.7, $\chi(E_\infty^{**})$ exists and

$$\chi(E_\infty^{**}) = \frac{1}{2}\chi_t(X). \tag{20}$$

By an argument similar to that used in the proof above,

$$\chi(E_\infty^{**}) = \lim_{N\to\infty}\frac{1}{N}\sum_{0\leqslant s\leqslant N}(-1)^s\sum_k \dim_{\mathbb{F}_p} E_\infty^{k,s-k}.$$

By Lemmas 4.4 and 4.6,

$$\sum_k \dim_{\mathbb{F}_p} E_\infty^{k,s-k} \cong Gr\,H^s_{\mathbb{Z}/p}(X;\mathbb{Z}) \cong Gr\,H^s_{\mathbb{Z}/p}(X^{\mathbb{Z}/p};\mathbb{Z}) \cong \bigoplus_{l\equiv s \bmod 2} H^l(X^{\mathbb{Z}/p};\mathbb{F}_p)$$

for $s > cd\,X$. Hence,

$$\chi(E_\infty^{**}) = \frac{1}{2}\chi(H^*(X^{\mathbb{Z}/p};\mathbb{F}_p)).$$

Since $H^2(\mathbb{Z}/p, H^l(X^{\mathbb{Z}/p};\mathbb{F}_p)) = H^l(X^{\mathbb{Z}/p};\mathbb{F}_p)$,

$$\chi(E_\infty^{**}) = \frac{1}{2}\chi_t(X^{\mathbb{Z}/p}).$$

Now, by (20), the proof is completed.

4.3. Proof of Theorem 1.3

Let $\mathbb{Z}/p$ act on a $PD_{\mathbb{F}_p}(n)$ space M in such a way that the assumptions of Theorem 1.3 are satisfied. If n is even, than Theorem 1.3 can be given a proof analogous to that for S^1-actions. Indeed, by Theorem 5.2.1 and Remark 5.2.4 in [3] (cf. [11]), $M^{\mathbb{Z}/p}$ has finitely many components, each of which is a $PD_{\mathbb{F}_p}(m)$-space, for m even. Therefore, by Proposition 1.2 it is enough to prove that $\sum_i t^i(M) \equiv \chi_t(M) \bmod 4$. As in the proof of Lemma 4.3, it is sufficient to show that $t^{n/2}(M)$ is even if $n/2$ is odd. By Lemma 3.10, the cup product on $H^2(\mathbb{Z}/p, H^{n/2}(M;\mathbb{F}_p))$ is non-degenerate and, since it is skew-symmetric, $t^{n/2}(M) = \dim_{\mathbb{F}_p} H^2(\mathbb{Z}/p, H^{n/2}(M;\mathbb{F}_p))$ is even.

Assume now that n is odd and $M^{\mathbb{Z}/p} \neq \emptyset$. Consider the Leray spectral sequence, (E_*^{**}, d_*), with coefficients in $\mathbb{Z}$ associated with the $\mathbb{Z}/p$-action on M. Since (E_*^{**}, d_*) converges to $H^*_{\mathbb{Z}/p}(M;\mathbb{Z})$, there is a filtration of $H^*_{\mathbb{Z}/p}(M;\mathbb{Z})$ such that

$$Gr\,H^s_{\mathbb{Z}/p}(M;\mathbb{Z}) = \bigoplus_i F^i H^s_{\mathbb{Z}/p}(M;\mathbb{Z})/F^{i+1}H^s_{\mathbb{Z}/p}(M;\mathbb{Z}) = \bigoplus_{k+l=s} E_\infty^{kl}.$$

Since $H^l(M;\mathbb{Z})$ is finitely generated, $E_2^{kl} = H^k(\mathbb{Z}/p, H^l(M;\mathbb{Z}))$ is a finite dimensional vector space over $\mathbb{F}_p$ for $k > 0$.

Corollary 4.8. *If $s > cd\,X$ then $H^s_{\mathbb{Z}/p}(M;\mathbb{Z})$ is a finite p-group and $Gr\,H^s_{\mathbb{Z}/p}(M;\mathbb{Z})$ is a finite dimensional vector space over $\mathbb{F}_p$.*

By Proposition 3.13, (E_*^{**}, d_*) satisfies Poincaré duality $P_{\mathbb{F}_p}(n)$. Therefore, by Lemma 3.3(i), we have

$$\dim_{\mathbb{F}_p} Gr\,H^s_{\mathbb{Z}/p}(M;\mathbb{Z}) + \dim_{\mathbb{F}_p} Gr\,H^{s+1}_{\mathbb{Z}/p}(M;\mathbb{Z}) = \sum_l \dim_{\mathbb{F}_p} E_\infty^{kl}$$

for sufficiently large s and k, k even. By the assumptions of Theorem 1.3 and by Lemma 2.4, $E_2^{kl}=0$ for all even l, $0<l\leqslant\frac{1}{2}(n-1)$ and for $k>0$. Therefore, by Proposition 3.13, the assumptions of Lemma 4.1 are satisfied and the sum above equals to $\sum \dim_{\mathbb{F}_p} E_2^{kl} \bmod 4$. Therefore, by Lemma 2.4,

$$\dim_{\mathbb{F}_p} Gr\, H^s_{\mathbb{Z}/p}(M;\mathbb{Z}) + \dim_{\mathbb{F}_p} Gr\, H^{s+1}_{\mathbb{Z}/p}(M;\mathbb{Z}) \equiv \sum_l t^l(M) \bmod 4 \tag{21}$$

for sufficiently large s.

By Lemmas 4.4 and 4.6, the left-hand side of (21) is equal to $\sum_l \dim_{\mathbb{F}_p} H^l(M^{\mathbb{Z}/p};\mathbb{F}_p)$. Hence by (5),

$$\sum_l t^l(M^{\mathbb{Z}/p}) = \sum_l \dim_{\mathbb{F}_p} H^l(M^{\mathbb{Z}/p};\mathbb{F}_p) \equiv \sum_l t^l(M) \bmod 4. \quad \square$$

4.4. *Proof of Theorem 1.4*

Let $\mathbb{Z}/p$ act on M such that all assumptions of Theorem 1.4 are satisfied. The proof of Theorem 1.4 is analogous to the proof of Theorem 1.3, except that $\mathbb{F}_p$ is the ring of coefficients this time. Let (E_*^{**}, d_*) be the Leray spectral sequence with coefficients in $\mathbb{F}_p$ associated with the $\mathbb{Z}/p$-action on M. It converges to $H^s_{\mathbb{Z}/p}(M;\mathbb{F}_p)$. Since $H^s_{\mathbb{Z}/p}(M;\mathbb{F}_p)$ is a vector space over $\mathbb{F}_p$, $Gr\, H^s_{\mathbb{Z}/p}(M;\mathbb{F}_p) \cong H^s_{\mathbb{Z}/p}(M;\mathbb{F}_p)$ for any filtration of $H^s_{\mathbb{Z}/p}(M;\mathbb{F}_p)$, and hence

$$H^s_{\mathbb{Z}/p}(M;\mathbb{F}_p) \cong \sum_{k+l=s} E_\infty^{kl}.$$

By Lemmas 3.11 and 3.3(ii) the dimensions of E_r^{kl} over $\mathbb{F}_p$ do not depend on k for large k. Therefore, by Lemma 4.1

$$\begin{aligned}\dim_{\mathbb{F}_p} H^s_{\mathbb{Z}/p}(M;\mathbb{F}_p) &= \sum_l \dim_{\mathbb{F}_p} E_\infty^{k_0 l} \\ &\equiv \sum_l \dim_{\mathbb{F}_p} E_2^{k_0 l} = \sum_l t^l(M) \bmod 4\end{aligned} \tag{22}$$

for $s > cd\, M$ and large k_0.

On the other hand, by Lemma 4.5

$$H^s_{\mathbb{Z}/p}(M^{\mathbb{Z}/p};\mathbb{F}_p) \cong \bigoplus_{k+l=s} H^k(B\mathbb{Z}/p,\mathbb{F}_p)\otimes H^l(M^{\mathbb{Z}/p};\mathbb{F}_p) = \bigoplus_l H^l(M^{\mathbb{Z}/p};\mathbb{F}_p).$$

Therefore,

$$\dim_{\mathbb{F}_p} H^s_{\mathbb{Z}/p}(M^{\mathbb{Z}/p};\mathbb{F}_p) = \sum_l \dim_{\mathbb{F}_p} H^l(M^{\mathbb{Z}/p};\mathbb{F}_p) = \sum_l \dim_{\mathbb{F}_p} t^l(M^{\mathbb{Z}/p}). \tag{23}$$

By Lemma 4.4, the left-hand sides of (22) and (23) are equal, and hence the proof is completed. $\square$

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