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Finite Sums from Sequences Within Cells of a Partition of N

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The principal result of this paper establishes the validity of a conjecture by Graham and Rothschild. This states that, if the natural numbers are divided into two classes, then there is a sequence drawn from one of those classes such that all finite sums of distinct members of that sequence remain in the same class.

1. INTRODUCTION

Graham and Rothshild have asked [2] if, whenever $N = A_1 \cup A_2$, there must be some i in $\{1, 2\}$ and some sequence $\langle x_n \rangle_{n=1}^{\infty}$ such that $\sum_{n \in F} x_n \in A_i$ whenever F is a non-empty finite subset of N. This question was attributed to them as a conjecture by Erdös in [1]. The principal result of this paper establishes that this statement is true for any finite partition of N.

In an earlier paper [3] this author established the equivalence of this conjecture with the existence of an ultrafilter p on N such that

$$\{x \in N : A - x \in p\} \in p,$$

whenever $A \in p$, provided the continuum hypothesis holds. Thus the existence of this ultrafilter is obtained as a corollary. (The fact that this ultrafilter and the conjecture are related was suggested by F. Galvin.).

Section 2 consists of some technical lemmas. The main results are in Section 3.

2. SOME PRELIMINARY LEMMAS

The notation $F \subseteq_f A$ means that F is a non-empty finite subset of A.

2.1. DEFINITION. Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in N. $FS(\langle x_n \rangle_{n=1}^{\infty}) = \{\sum_{n \in F} x_n : F \subseteq_f N\}$.

We shall also write without confusion $FS(\langle x_n\rangle_{n=1}^{r})$ when $r \in N$. The following lemma is proved in [3, Lemma 2.3].

2.2. LEMMA. *If $\langle x_n\rangle_{n=1}^{\infty}$ is any sequence in N, then there exists a sequence $\langle y_n\rangle_{n=1}^{\infty}$ such that $FS(\langle y_n\rangle_{n=1}^{\infty}) \subseteq FS(\langle x_n\rangle_{n=1}^{\infty})$ and $2^s \mid y_{n+1}$ whenever $2^{s-1} \leqslant y_n$.*

The importance of this lemma lies in the fact that, if y_n and y_m are written in binary notation and $n \neq m$, then no carrying occurs in the addition of y_n and y_m. In particular, then, if $F \subseteq_f N$ and c is the largest element of F and $2^s \leqslant \sum_{n\in F} y_n$ then indeed $2^s \leqslant y_c$.

2.3. DEFINITION. Let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence in N such that $2^s \mid x_{n+1}$ whenever $2^{s-1} \leqslant x_n$. The natural map, τ, for $FS(\langle x_n\rangle_{n=1}^{\infty})$ is defined by the rule $\tau(\sum_{n\in F} x_n) = \sum_{n\in F} 2^{n-1}$.

Since every natural number has a unique binary expansion, and since $x_{n+1} > \sum_{i=1}^{n} x_i$ for every n, τ is easily seen to be one-to-one and onto N. When $A \subseteq N$ we shall use the notational convention that

$$\tau(A) = \{\tau(x) \colon x \in A \cap FS(\langle x_n\rangle_{n=1}^{\infty})\}.$$

A technique frequently used in this paper is to note that τ is almost an isomorphism. This statement is made precise in the following lemma.

2.4. LEMMA. *Let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence in N such that $2^s \mid x_{n+1}$ whenever $2^s \leqslant x_n$. Let $\{y_n : n < r\}$ be a subset of $FS(\langle x_n\rangle_{n=1}^{\infty})$ and, for each $n < r$, let $z_n = \tau(y_n)$ where τ is the natural map for $FS(\langle x_n\rangle_{n=1}^{\infty})$ and $r \in N \cup \{\infty\}$. Then the following two conditions are equivalent and each implies that $\sum_{n\in F} z_n = \tau(\sum_{n\in F} y_n)$ whenever $F \subseteq_f \{x \in N : 1 \leqslant x < r\}$:*

(1) *For each n less than $r - 1$, every element of F_n is smaller than every element of F_{n+1}, where $y_n = \sum_{t\in F_n} x_t$.*

(2) *For each n less than $r - 1$, $2^s \mid z_{n+1}$ whenever $2^{s-1} \leqslant z_n$.*

Proof. (1) implies (2). Let a be the largest element of F_n and let b be the smallest element of F_{n+1}. Then $z_n < 2^a$ and, since $a < b$, $2^a \mid 2^{t-1}$ for every t in F_{n+1}.

(2) implies (1). Let $a \in F_n$ and $b \in F_{n+1}$. Then $2^{a-1} \leqslant z_n$ so $2^a \mid z_{n+1}$. But z_{n+1} is the sum of distinct powers of 2 so 2^a divides each of them. In particular $2^a \mid 2^{b-1}$ and hence $a < b$.

Finally, note that whenever (1) is satisfied and $F \subseteq_f \{x \in N : 1 \leqslant x < r\}$ then $\{F_n : n \in F\}$ forms a pairwise disjoint family. Let $G = \bigcup_{n \in F} F_n$. Then

$$\sum_{n \in F} z_n = \sum_{n \in F} \Big(\sum_{t \in F_n} 2^{t-1} \Big) = \sum_{t \in G} 2^{t-1}$$

$$= \tau \Big(\sum_{t \in G} x_t \Big) = \tau \Big(\sum_{n \in F} \Big(\sum_{t \in F_n} x_t \Big) \Big) = \tau \Big(\sum_{n \in F} y_n \Big).$$

2.5. LEMMA. *Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in N such that $2^s \mid x_{n+1}$ whenever $2^{s-1} \leqslant x_n$. Let τ be the natural map for $FS(\langle x_n \rangle_{n=1}^{\infty})$ and let $\langle y_n \rangle_{n=1}^{\infty}$ be any sequence in N such that $FS(\langle y_n \rangle_{n=1}^{\infty}) \subseteq FS(\langle x_n \rangle_{n=1}^{\infty})$. Then there exists a sequence $\langle z_n \rangle_{n=1}^{\infty}$ such that $FS(\langle z_n \rangle_{n=1}^{\infty}) \subseteq FS(\langle y_n \rangle_{n=1}^{\infty})$ and $\tau(\sum_{n \in F} z_n) = \sum_{n \in F} \tau(z_n)$ whenever $F \subseteq_f N$.*

Proof. By Lemma 2.2 there exists a sequence $\langle z_n \rangle_{n=1}^{\infty}$ such that $FS(\langle z_n \rangle_{n=1}^{\infty}) \subseteq FS(\langle y_n \rangle_{n=1}^{\infty})$ and $2^s \mid z_{n+1}$ whenever $2^{s-1} \leqslant z_n$. In particular $FS(\langle z_n \rangle_{n=1}^{\infty}) \subseteq FS(\langle x_n \rangle_{n=1}^{\infty})$ so, by Lemma 2.4, it suffices to show that every element of F_n is less than every element of F_{n+1}, where $z_n = \sum_{t \in F_n} x_t$. Let a be the largest element of F_n and let b be the smallest element of F_{n+1}. Suppose that $b \leqslant a$ and let s be the largest integer such that $2^{s-1} \leqslant x_b$. Then $2^{s-1} \leqslant x_a \leqslant z_n$ so $2^s \mid z_{n+1}$. Also $2^s \mid x_t$ for every t in $F_{n+1} \backslash \{b\}$ since b is the smallest element of F_{n+1}. But $x_b = z_{n+1} - \sum \{x_t : t \in F_{n+1} \backslash \{b\}\}$ so $2^s \mid x_b$ and hence $2^s \leqslant x_b$, a contradiction.

2.6. LEMMA. *Let $k \in N$ and let $\{A(i, n) : i \in \{1, 2, ..., k\}$ and $n \in N\}$ be a collection of sets such that $A(i, n+1) \subseteq A(i, n)$ whenever $i \in \{1, 2, ..., k\}$ and $n \in N$. Then there exist a subset S of $\{1, 2, ..., k\}$, a sequence $\langle x_m \rangle_{m=1}^{\infty}$ in N, and an element M of N such that whenever $n \geqslant M$ and $\langle y_m \rangle_{m=1}^{\infty}$ is a sequence with $FS(\langle y_m \rangle_{m=1}^{\infty}) \subseteq FS(\langle x_m \rangle_{m=1}^{\infty})$ then $FS(\langle y_m \rangle_{m=1}^{\infty}) \cap A(i, n) \neq \varnothing$ if and only if $i \in S$.*

Proof. The proof is by induction on k. Let $k = 1$. If there are any n and any sequence $\langle x_m \rangle_{m=1}^{\infty}$ such that $FS(\langle x_m \rangle_{m=1}^{\infty}) \cap A(1, n) = \varnothing$, let $M = n$, $S = \varnothing$, and let $\langle x_m \rangle_{m=1}^{\infty}$ be as given. Otherwise let $M = 1$, $S = \{1\}$ and let $\langle x_m \rangle_{m=1}^{\infty}$ be any sequence whatever.

Now assume valid for $k - 1$ and let $\langle x_m' \rangle_{m=1}^{\infty}$, S', and M' be as given for $\{A(i, n) : i \in \{1, 2, ..., k-1\}$ and $n \in N\}$. If there are some $M'' \geqslant M'$ and $\langle y_m \rangle_{m=1}^{\infty}$ such that $FS(\langle y_m \rangle_{m=1}^{\infty}) \subseteq FS(\langle x_m' \rangle_{m=1}^{\infty})$ and

$$FS(\langle y_m \rangle_{m=1}^{\infty}) \cap A(k, M'') = \varnothing,$$

let $M = M''$, $S = S'$, and $\langle x_m \rangle_{m=1}^{\infty} = \langle y_m \rangle_{m=1}^{\infty}$. Otherwise let $M = M'$, $S = S' \cup \{k\}$, and $\langle x_m \rangle_{m=1}^{\infty} = \langle x_m' \rangle_{m=1}^{\infty}$.

2.7. DEFINITION. Let α be a finite partition of N, ($\alpha = \{A_i\}_{i=1}^{a}$). Let $n \in N$ and let $k < n$.

(a) $F_\alpha'(k, n) = \{x \in N : x \geqslant n$ *and there is some* i *in* $\{1, 2,\ldots, a\}$ such that $\{k, x, x + k\} \subseteq A_i\}$.

(b) $F_\alpha(k, n) = F_\alpha'(k, n) \backslash \bigcup_{j=1}^{k-1} F_\alpha'(j, n)$, if $k > 1$. $F_\alpha(1, n) = F_\alpha'(1, n)$.

(c) Let $i \in \{1, 2,\ldots, a\}$. $U_\alpha(i, n) = (A_i \cap \{x \in N : x \geqslant n\}) \backslash \bigcup_{k=1}^{n-1} F_\alpha(k, n)$.

If, for any n, $\bigcup_{k=1}^{n-1} F_\alpha(k, n) = \{x \in N : x \geqslant n\}$, the proof of the main theorem is quite easy. This is not, unfortunately, always the case. The result we now seek is that we can find a sequence $\langle x_m \rangle_{m=1}^{\infty}$ with

$$FS(\langle x_m \rangle_{m=1}^{\infty}) \subseteq \bigcup_{k=1}^{n-1} F_\alpha(k, n)$$

for some n. This result will be Lemma 2.10. This, together with the fact, guaranteed by Lemma 2.4, that the natural map τ for $FS(\langle x_m \rangle_{m=1}^{\infty})$ is "nearly" an isomorphism onto N will allow us to complete the proof. The following, exceedingly technical, lemma allows us to choose the desired sequence.

2.8. LEMMA. *Let* $\alpha = \{A_i\}_{i=1}^{a}$ *be a partition of* N. *Assume that for each* n *in* N *and sequence* $\langle y_m \rangle_{m=1}^{\infty}$ *in* N *one has* $FS(\langle y_m \rangle_{m=1}^{\infty}) \backslash \bigcup_{k=1}^{n-1} F_\alpha(k, n) \neq \varnothing$. *Then there exists* i *in* $\{1, 2,\ldots, a\}$ *such that for each* n *in* $N \cup \{0\}$ *there exist* x_n *and* M_n *in* N *and a sequence* $\langle x_{n,m} \rangle_{m=1}^{\infty}$ *in* N *such that for each* $p \geqslant M_n$ *there exists a set* $U(n, p)$ *satisfying*:

(1) *for each* m, *if* $2^{s-1} \leqslant x_{n,m}$ *then* $2^s \mid x_{n,m+1}$;

(2) *if* $p \geqslant M_n$ *and* $\langle y_m \rangle_{m=1}^{\infty}$ *is a sequence with*

$$FS(\langle y_m \rangle_{m=1}^{\infty}) \subseteq FS(\langle x_{n,m} \rangle_{m=1}^{\infty}),$$

then

$$FS(\langle y_m \rangle_{m=1}^{\infty}) \cap U(n, p) \neq \varnothing ;$$

(3) if $p \geqslant M_n$, then $U(n, p + 1) \subseteq U(n, p)$ *and* $U(n, p) \subseteq A_i$;

(4) *if* $n \geqslant 1$, *then* $M_n \geqslant M_{n-1}$ *and* $M_n > \sum_{j=1}^{n} x_j$;

(5) *if* $n \geqslant 1$ *and* $p \geqslant M_n$, *then* $U(n, p) \subseteq U(n - 1, p)$;

(6) *if* $n \geqslant 1$ *and* $p \geqslant M_n$ *and* $x \in U(n, p)$, *then* $x + x_n \in U(n - 1, M_{n-1})$

Proof. Let M, S, and $\langle w_m \rangle_{m=1}^{\infty}$ be as guaranteed by Lemma 2.6 for the family $\{U_\alpha(i, m) : i \in \{1, 2,\ldots, a\}$ and $m \in N\}$. By the hypothesis of the current lemma, $S \neq \varnothing$. (For, if $FS(\langle w_m \rangle_{m=1}^{\infty}) \cap U_\alpha(i, M) = \varnothing$ for each i in $\{1, 2,\ldots, a\}$, then $FS(\langle w_m \rangle_{m=1}^{\infty}) \subseteq \bigcup_{k=1}^{M-1} F_\alpha(k, M)$.) Let $i \in S$.

We now define x_n, M_n, $\langle x_{n,m}\rangle_{m=1}^{\infty}$, and, for each $p \geqslant M_n$, $U(n, p)$ inductively on n. Let $x_0 = 1$. (No requirements of the lemma affect x_0.) Let $M_0 = M$, let $\langle x_{0,m}\rangle_{m=1}^{\infty}$ be a sequence with $FS(\langle x_{0,m}\rangle_{m=1}^{\infty}) \subseteq FS(\langle w_m\rangle_{m=1}^{\infty})$ such that $2^s \mid x_{0,m+1}$ whenever $2^{s-1} \leqslant x_{0,m}$ (there is such a sequence by Lemma 2.2), and for each $p \geqslant M_0$ let $U(0, p) = U_\alpha(i, p)$. (Where M and $\langle w_m\rangle_{m=1}^{\infty}$ are as in the paragraph above.)

Conditions (4), (5), and (6) are satisfied vacuously, and $\langle x_{0,m}\rangle_{m=1}^{\infty}$ was chosen specifically to satisfy condition (1). Condition (2) is satisfied by Lemma 2.6, since $U(0, p) = U_\alpha(i, p)$, $i \in S$, and

$$FS(\langle x_{0,m}\rangle_{m=1}^{\infty}) \subseteq FS(\langle w_m\rangle_{m=1}^{\infty}).$$

Condition (3) is satisfied because $U(0, p) = U_\alpha(i, p)$.

We assume we have chosen x_k, M_k, $\langle x_{k,m}\rangle_{m=1}^{\infty}$, and, for each $p \geqslant M_k$, $U(k, p)$ satisfying each of the six conditions for every $k < n$. Let τ be the natural map for $FS(\langle x_{n-1,m}\rangle_{m=1}^{\infty})$ and let $p \geqslant M_{n-1}$. Consider $\tau(U(n-1, p))$. We claim that for each sequence $\langle z_m\rangle_{m=1}^{\infty}$ in N one has

$$FS(\langle z_m\rangle_{m=1}^{\infty}) \cap \tau U(n-1, p)) \neq \varnothing.$$

For, indeed, if there is a sequence with $FS(\langle z_m\rangle_{m=1}^{\infty}) \cap \tau(U(n-1, p)) = \varnothing$, then, by Lemma 2.2, we may suppose that $2^s \mid z_{m+1}$ whenever $2^{s-1} \leqslant z_m$. Consequently, if $y_m = \tau^{-1}(z_m)$ for each m, we have by Lemma 2.4 that $FS(\langle y_m\rangle_{m=1}^{\infty}) \cap U(n-1, p) = \varnothing$, an impossibility since

$$FS(\langle y_m\rangle_{m=1}^{\infty}) \subseteq FS(\langle x_{n-1,m}\rangle_{m=1}^{\infty})$$

(also by virtue of Lemma 2.4) and condition (2) holds at $n-1$. The claim is thus established.

Thus, in particular, there exists some b in N such that, for every x in N, $\{x+1, x+2,..., x+b\} \cap \tau(U(n-1, M_{n-1})) \neq \varnothing$. (For, if there were no bound on the gaps in $\tau(U(n-1, M_{n-1}))$, one could choose a sequence $\langle z_m\rangle_{m=1}^{\infty}$ inductively by picking z_m such that

$$\left\{z_m, z_m+1,..., z_m+\sum_{k=1}^{m-1} z_k\right\} \cap \tau(U(n-1), M_{n-1})) = \varnothing.$$

For this sequence we would have $FS(\langle z_m\rangle_{m=1}^{\infty}) \cap \tau(U(n-1, M_{n-1})) = \varnothing$.) Let M_n' be the larger of M_{n-1} and $\sum_{j=1}^{n-1} x_j + \tau^{-1}(b) + 1$ and let r be the largest integer such that $2^{r-1} \leqslant b$. For each j in $\{1, 2,..., b\}$ and for each $p \geqslant M_n'$ define $V(j, p) = \{x \in \tau(U(n-1, p)) : 2^r \mid x$ and

$$x + j \in \tau(U(n-1, M_{n-1}))\}.$$

Let $V(0, p) = \{x \in \tau(U(n-1, p)) : 2^r \nmid x\}$. Note that

$$\tau(U(n-1, p)) = \bigcup_{j=0}^{b} V(j, p).$$

Also since condition (3) holds at $n-1$ we have that $V(j, p) \supseteq V(j, p+1)$ whenever $p \geqslant M_n$

For $p < M_n{}'$ let $V(j, p) = V(j, M_n{}')$. Then by Lemma 2.6 there exist a subset S' of $\{0, 1, \ldots, b\}$, a sequence $\langle y_m \rangle_{m=1}^{\infty}$, and an element M_n'' of N such that, if $p \geqslant M_n''$ and $j \in S'$ and $\langle z_m \rangle_{m=1}^{\infty}$ is any sequence with

$$FS(\langle z_m \rangle_{m=1}^{\infty}) \subseteq FS(\langle y_m \rangle_{m=1}^{\infty}),$$

then $FS(\langle z_m \rangle_{m=1}^{\infty}) \cap V(j, p) \neq \varnothing$. We may assume, by Lemma 2.2, that $2^s \mid y_{m+1}$ whenever $2^{s-1} \leqslant y_m$. Note that $S' \neq \varnothing$. Otherwise, we would have that $FS(\langle y_m \rangle_{m=1}^{\infty}) \cap \tau(U(n-1, p) = \varnothing$ since

$$\tau(U(n-1, p)) = \bigcup_{j=0}^{b} V(j, p).$$

But that has already been established to be impossible. Note also that $0 \notin S'$ since, for all but finitely many terms of $\langle y_m \rangle_{m=1}^{\infty}$, $2^r \mid y_m$. Let $w \in S'$, let $x_n = \tau^{-1}(w)$, and let M_n be the larger of $M_n{}'$ and M_n''. For each m let $x_{n,m} = \tau^{-1}(y_m)$ and for each $p \geqslant M_n$ let $U(n, p) = \tau^{-1}(V(w, p))$.

To see that condition (1) is satisfied note that, since $x_{n,m} = \tau^{-1}(y_m)$ for each m, $x_{n,m} \in FS(\langle x_{n-1,t} \rangle_{t=1}^{\infty})$. For each m we have that $2^s \mid y_{m+1}$ whenever $2^{s-1} \leqslant y_m$ so by Lemma 2.4 and condition (1) applied to $FS(\langle x_{n-1,t} \rangle_{t=1}^{\infty})$ we have that each element of F_m is less than each element of F_{m+1}, where $x_{n,m} = \sum_{t \in F_m} x_{n-1,t}$. Thus, letting c be the largest element of F_m, we have that, if $2^{s-1} \leqslant x_{n,m}$, then $2^{s-1} \leqslant x_{n-1,c}$. Therefore, since condition (1) holds at $n-1$, $2^s \mid x_{n-1,t}$ for every t in F_{m+1}. That is, $2^s \mid x_{n,m+1}$.

To see that condition (2) holds let $\langle z_m \rangle_{m=1}^{\infty}$ be a sequence with

$$FS(\langle z_m \rangle_{m=1}^{\infty}) \subseteq FS(\langle x_{n,m} \rangle_{m=1}^{\infty})$$

and suppose that $FS(\langle z_m \rangle_{m=1}^{\infty}) \cap U(n, p) = \varnothing$. Then by Lemma 2.5 we may assume that, whenever $F \subseteq_f N$, $\tau(\sum_{m \in F} z_m) = \sum_{m \in F} \tau(z_m)$. Thus $FS(\langle \tau(z_m) \rangle_{m=1}^{\infty}) \cap V(w, p) = \varnothing$ while $FS(\langle \tau(z_m) \rangle_{m=1}^{\infty}) \subseteq FS(\langle y_m \rangle_{m=1}^{\infty})$, a contradiction. (The latter inclusion comes from the fact, a consequence of Lemma 2.4, that $FS(\langle \tau(x_{n,m}) \rangle_{m=1}^{\infty}) \subseteq FS(\langle y_m \rangle_{m=1}^{\infty})$.)

The verification of conditions (3), (4), and (5) is trivial. To see that condition (6) holds let $x \in U(n, p)$. Then $\tau(x) \in V(w, p)$ so $2^r \mid \tau(x)$ and $\tau(x) + w \in \tau(U(n-1, M_{n-1}))$. But $w \leqslant b$ so if $2^{s-1} \leqslant w$ then $s \leqslant r$ so $2^s \mid \tau(x)$. Thus, by Lemma 2.4, $x + \tau^{-1}(w) = x + x_n \in U(n-1, M_{n-1})$ as desired. The induction is complete.

2.9. LEMMA. *Let* $\alpha = \{A_j\}_{i=1}^{a}$ *be a partition of* N. *If, for each* n *in* N *and sequence* $\langle y_m \rangle_{m=1}^{\infty}$ *in* N *one has* $FS(\langle y_m \rangle_{m=1}^{\infty}) \backslash \bigcup_{k=1}^{n-1} F_\alpha(k, u) \neq \varnothing$, *then there are some* i *in* $\{1, 2,..., a\}$ *and some sequence* $\langle x_n \rangle_{n=1}^{\infty}$ *in* N *such that* $FS(\langle x_n \rangle_{n=1}^{\infty}) \cap A_i = \varnothing$.

Proof. Let i and $\langle x_n \rangle_{n=1}^{\infty}$ be as given by Lemma 2.8. Let $F \subseteq_f N$ and let t and r be, respectively, the smallest and largest elements of F. Let $x \in U(r, M_r)$. We show by induction on the number of elements in F that $x + \sum_{n \in F} x_n \in U(t-1, M_{t-1})$. In case F has one element we have by condition (6) of lemma 2.8 that $x + x_r \in U(r-1, M_{r-1}) = U(t-1, M_{t-1})$. Now assume F has more than one element and let $G = F \backslash \{t\}$. Let t' be the smallest element of G. By induction, $x + \sum_{n \in G} x_n \in U(t'-1, M_{t'-1})$. By condition (5) applied as often as needed $x + \sum_{n \in G} x_n \in U(t, M_{t'-1})$. (Of course, if $t = t' - 1$, condition (5) is not needed.) Then, by condition (6), $x + \sum_{n \in G} x_n + x_t \in U(t-1, M_{t-1})$. The induction is complete.

By condition (3) of lemma 2.8 we have that $x + \sum_{n \in F} x_n \in A_i$ and $x \in A_i$. But $x \in U(r, M_r)$ so by repeated application of condition (5) $x \in U(0, M_r) = U_\alpha(i, M_r)$. Thus $x \notin F_\alpha'(\sum_{n \in F} x_n, M_r)$. (By condition (4) $M_r > \sum_{n \in F} x_n$.) Thus it is not the case that

$$\left\{\sum_{n \in F} x_n, x, x + \sum_{n \in F} x_n\right\} \subseteq A_i.$$

That is, $\sum_{n \in F} x_n \notin A_i$ as desired.

2.10. LEMMA. *Let* $\alpha = \{A_i\}_{i=1}^{a}$ *be a partition of* N. *Then there exist* n *in* N *and a sequence* $\langle x_m \rangle_{m=1}^{\infty}$ *in* N *such that* $FS(\langle x_m \rangle_{m=1}^{\infty}) \subseteq \bigcup_{k=1}^{n-1} F_\alpha(k, n)$.

Proof. The proof is by induction on a, the number of elements of α. If $a = 1$ the result is trivial. Assume the lemma is valid for any partition with $a - 1$ elements.

Suppose the conclusion fails. Then by Lemma 2.9 we have some i in $\{1, 2,..., a\}$ and some sequence $\langle y_m \rangle_{m=1}^{\infty}$ such that $FS(\langle y_m \rangle_{m=1}^{\infty}) \cap A_i = \varnothing$. We may assume, by Lemma 2.2, that $2^s \mid y_{m+1}$ whenever $2^{s-1} \leqslant y_m$. Let τ be the natural map for $FS(\langle y_m \rangle_{m=1}^{\infty})$ and let $\beta = \{\tau(A_j) : j \in \{1, 2,..., a\}$ and $j \neq i\}$. Then, since $\tau(A_i) = \varnothing$, β is a partition of N with $a - 1$ elements. Consequently there exist r in N and $\langle z_m \rangle_{m=1}^{\infty}$ such that $FS(\langle z_m \rangle_{m=1}^{\infty}) \subseteq \bigcup_{k=1}^{r-1} F_\beta(k, r)$. We may assume, by Lemma 2.2, that $2^s \mid z_{m+1}$ whenever $2^{s-1} \leqslant z_m$. We also assume that $2^s \mid z_m$ whenever $2^{s-1} < r$.

Now, let $n = \tau^{-1}(r)$ and let $x_m = \tau^{-1}(z_m)$ for each m. We claim that $FS(\langle x_m \rangle_{m=1}^{\infty}) \subseteq \bigcup_{k=1}^{n-1} F_\alpha(k, n)$. To see this, let $F \subseteq_f N$. Then, by Lemma 2.4, $\tau(\sum_{m \in F} x_m) = \sum_{m \in F} z_m$. Thus $\tau(\sum_{m \in F} x_m) \in F_\beta(k, r)$ for some $k < r$. That is, $\{k, \tau(\sum_{m \in F} x_m), \tau(\sum_{m \in F} x_m) + k\} \subseteq \tau(A_j)$ for some j in $\{1, 2,..., a\}$. Thus, immediately $\tau^{-1}(k) \in A_j$ and $\sum_{m \in F} x_m \in A_j$. But $k < r$ so, if $2^{s-1} \leqslant k$, we have $2^{s-1} < r$ so that $2^s \mid z_m$ for each m. Thus $2^s \mid \sum_{m \in F} z_m$, so $2^s \mid \tau(\sum_{m \in F} x_m)$. Thus, by Lemma 2.4, $\sum_{m \in F} x_m + \tau^{-1}(k) \in A_j$. Noting finally that

$$\sum_{m \in F} x_m \geqslant \tau^{-1}(r) = n,$$

we have $\sum_{m \in F} x_m \in F_\alpha{}'(\tau^{-1}(k), n)$ as desried.

Lemma 2.12 is the only result needed to prove the main theorem. It uses in its proof the following lemma, which is a partial generalization of Corollary 4 of [2]. Graham and Rothschild attribute the result there to J. Folkman (in a personal communication), R. Rado [4], and J. Sanders [5].

2.11. Lemma. *For every partiton α of N with $\alpha = \{A_i\}_{i=1}^{a}$, there exists a function $f_\alpha : N \to N$ such that, for each r in N, there exist i in $\{1, 2,..., a\}$ and $\langle y_j \rangle_{j=1}^{r}$ satisfying*:

(1) $FS(\langle y_j \rangle_{j=1}^{r}) \subseteq A_i$;

(2) *if* $j \in \{1, 2,..., r-1\}$ *and* $2^{s-1} \leqslant y_j$, *then* $2^s \mid y_{j+1}$

(3) *if* $j \in \{1, 2,..., r\}$, *then* $y_j \leqslant f_\alpha(j)$.

Proof. For each α choose $p(\alpha)$ in N and a sequence $\langle x_{\alpha,m} \rangle_{m=1}^{\infty}$ such that $FS(\langle x_{\alpha,m} \rangle_{m=1}^{\infty}) \subseteq \bigcup_{k=1}^{p(\alpha)-1} F_\alpha(k, p(\alpha))$ and $2^s \mid x_{\alpha,m+1}$ whenever $2^{s-1} \leqslant x_{\alpha,m}$. We can assume in addition that, for each m, $2^s \mid x_{\alpha,m}$ whenever $2^{s-1} \leqslant p(\alpha)$. Let τ_α be the natural map for $FS(\langle x_{\alpha,m} \rangle_{m=1}^{\infty})$ and let $\beta(\alpha) = \{\tau_\alpha(F_\alpha(k, p(\alpha))) : k \in \{1, 2,..., p(\alpha) - 1\}\}$. Then $\beta(\alpha)$ is a partition of N.

We define $f_\alpha(n)$ inductively on n for every α at once. Let $f_\alpha(1) = p(\alpha) - 1$ and let $f_\alpha(n+1) = \tau_\alpha^{-1}(f_{\beta(\alpha)}(n))$.

Now, with f_α defined for every finite partition α of N we prove the lemma inductively on r. If $r = 1$, let $y_1 = 1$ and let i be the element of $\{1, 2,..., a\}$ that $1 \in A_i$. Condition (2) holds vacuously and (1) and (3) are trivial, since $p(\alpha) \geqslant 2$.

Let $r > 1$ and assume the lemma is valid for every partition α at $r - 1$. Let $\langle w_j \rangle_{j=1}^{r-1}$ and let k in $\{1, 2,..., p(\alpha) - 1\}$ be as guaranteed by the lemma for the partition $\beta(\alpha)$ at $r - 1$. Let i be that element of $\{1, 2,..., a\}$ such that $k \in A_i$. Let $y_1 = k$ and, for $j \in \{2,..., r\}$, let $y_j = \tau_\alpha^{-1}(w_{j-1})$.

To verify condition (1), first let $F \subseteq \{2,..., r\}$. Then, by Lemma 2.4 and condition (2) for $\beta(\alpha)$, $\tau_\alpha(\sum_{j \in F} y_j) = \sum_{j \in F} w_{j-1}$. But

$$\sum_{j \in F} w_{j-1} \in \tau_\alpha(F_\alpha(k, p(\alpha))) \quad \text{so} \quad \sum_{j \in F} y_j \in F_\alpha(k, p(\alpha)).$$

Thus $\sum_{j \in F} y_j \in A_i$ and $y_1 + \sum_{j \in F} y_j \in A_i$. Finally, since $y_1 \in A_i$ condition (1) is satisfied.

To see condition (2) note that, if $2^{s-1} \leqslant y_1$, then $2^{s-1} < p(\alpha)$ so $2^s \mid x_{\alpha,m}$ for every m. Consequently $2^s \mid y_2$. Now let $j \in \{2,..., r-1\}$. By Lemma 2.4 every element of F_j is less than every element of F_{j+1} where $y_j = \sum_{t \in F_j} x_{\alpha,t}$. Let c be the largest element of F_j. If $2^{s-1} \leqslant y_j$, then $2^{s-1} \leqslant x_{\alpha,c}$ so $2^s \mid x_{\alpha,t}$ for every t in F_{j+1} and consequently $2^s \mid y_{j+1}$.

Now consider condition (3). First $y_1 = k \leqslant p(\alpha) - 1$. Now let $j \in \{2,..., r\}$. Then $w_{j-1} \leqslant f_{\beta(\alpha)}(j-1)$ and τ_α is order preserving so $y_j \leqslant f_\alpha(j)$.

2.12. LEMMA. *For every partition α of N, with $\alpha = \{A_i\}_{i=1}^a$, there exist a function $f_\alpha : N \to N$ and an i in $\{1, 2,..., a\}$ such that, for every r in N, there exists $\langle y_j \rangle_{j=1}^r$ such that $FS(\langle y_j \rangle)_{j=1}^r \subseteq A_i$ and $y_j \leqslant f_\alpha(j)$ whenever $j \in \{1, 2,..., r\}$.*

Proof. Let f_α be as in Lemma 2.11. For each r in N let $i(r)$ be that element of $\{1, 2,..., a\}$ whose existence is guaranteed by Lemma 2.11. Let $i \in \{1, 2,..., a\}$ such that $i = i(r)$ for infinitely many r's.

Now let $r \in N$ and let $r' \in N$ such that $r' \geqslant r$ and $i = i(r')$. If $\langle y_j \rangle_{j=1}^{r'}$ is as guaranteed by Lemma 2.11, then $\langle y_j \rangle_{j=1}^r$ will work here.

3. THE MAIN RESULTS

The proof now rests only on the compactness of the product space $\{0, 1\}^N$. For an element s of $\{0, 1\}^N$ we define a sequence $\langle x_{s,m} \rangle_{m=1}^\infty$ in $N \cup \{0\}$ by agreeing that $x_{s,m} = k$ where k is the mth element of N such that $s_k = 1$. If s has fewer than m non-zero coordinates, we agree that $x_{s,m} = 0$.

3.1. THEOREM. *Let α be a finite partition of N with $\alpha = \{A_i\}_{i=1}^a$. There exist i in $\{1, 2,..., a\}$ and a sequence $\langle x_m \rangle_{m=1}^\infty$ such that $FS(\langle x_m \rangle_{m=1}^\infty) \subseteq A_i$.*

Proof. Let i and f_α be as guaranteed by Lemma 2.12. For each r and m in N let $A_{n,m} = \{s \in \{0, 1\}^N : \{x_{s,k} : k \in \{1, 2,..., n\}\} \subseteq \{1, 2,..., m\}$ and $FS(\langle x_{s,k} \rangle_{k=1}^n) \subseteq A_i\}$. Since whether or not $s \in A_{n,m}$ is determined by the first m coordinates of s, $A_{n,m}$ is closed. Now let $n \in N$ and let $\langle y_j \rangle_{j=1}^n$ be as guaranteed by Lemma 2.12. Let $s \in \{0, 1\}^N$ such that $s_{y_j} = 1$ for j in $\{1, 2,..., n\}$ and $s_k = 0$ otherwise, then $s \in \bigcap_{j=1}^n A_{j,f_\alpha(j)}$.

We thus have that $\{A_{n,m} : n \in N$ and $m = f_\alpha(n)\}$ is a family of closed sets in $\{0, 1\}^N$ with the finite intersection property. Consequently there exists s in $\bigcap_{n=1}^\infty A_{n,f_\alpha(n)}$. Let $x_m = x_{s,m}$ for every m. Let $F \subseteq_f N$ and let n be the largest element of F. Then $s \in A_{n,f_\alpha(n)}$ so $\sum_{m \in F} x_m \in A_i$. The proof is complete.

3.2. COROLLARY (Continuum Hypothesis). *There exists an ultrafilter p on N such that $\{x : A - x \in p\} \in p$ whenever $A \in p$. (Where*

$$A - x = \{y \in N : x + y \in A\}.)$$

Proof. This statement was shown in [3] to be equivalent, in the presence of the continuum hypothesis, to Theorem 3.1.

The author is grateful to R. Graham and B. Rothshild for pointing out that the following generalization of [2, Corollary 3] might also be obtained in this manner.

3.3. COROLLARY. *Let $\Pi = \{F : F \subseteq_f N\}$. If $\Pi = \bigcup_{i=1}^{a} \Gamma_i$, then there are a sequence $\langle F_n \rangle_{n=1}^{\infty}$* in Π *and an i in $\{1, 2,..., a\}$* such that $\bigcup_{n \in G} F_n \in \Gamma_i$ *whenever $G \subseteq_f N$.*

Proof. Define $\sigma : \Pi \to N$ by the rule $\sigma(F) = \sum_{n \in F} 2^{n-1}$. Then σ is one-to-one and onto. Let, for each i in $\{1, 2,..., a\}$, $A_i = \sigma(\Gamma_i)$. Then, by Theorem 3.1, there exist i in $\{1, 2,..., a\}$ and $\langle x_n \rangle_{n=1}^{\infty}$ such that

$$FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A_i .$$

By Lemma 2.2 we may suppose that $2^s \mid x_{n+1}$ whenever $2^{s-1} \leqslant x_n$.

Let $F_n = \sigma^{-1}(x_n)$, for each n in N. Then $\{F_n : n \in N\}$ form a pairwise disjoint collection. Thus, if $G \subseteq_f N$, we have that

$$\sigma\Big(\bigcup_{n \in G} F_n\Big) = \sum \Big\{2^{t-1}: t \in \bigcup_{n \in G} F_n\Big\} = \sum_{n \in G} \Big(\sum_{t \in F_n} 2^{t-1}\Big)$$
$$= \sum_{n \in F} \sigma(F_n) = \sum_{n \in G} x_n \in A_i .$$

Thus $\bigcup_{n \in G} F_n \in \sigma^{-1}(A_i) = \Gamma_i$ as desired.

The following very restricted partial generalizations of corollaries 1 and 2 of [2] are proved in a similar fashion, as was also noted by Graham and Rothschild.

3.4. COROLLARY. *Let A be an $\aleph_0$-dimensional affine space over the field of 2 elements. If $A = \bigcup_{i=1}^{n} B_i$, then there are an $\aleph_0$-dimensional affine subspace A' of A and an i in $\{1, 2,..., n\}$ such that $A' \subseteq B_i$.*

3.5. COROLLARY. *Let V be an $\aleph_0$-dimensional vector space over the field of 2 elements and let Π be the set of one-dimensional subspaces of V. If $\Pi = \bigcup_{i=1}^{n} \Gamma_i$, then there are an $\aleph_0$-dimensional subspace V' of V and an i in $\{1, 2,..., n\}$ such that every one-dimensional subspace of V' is an element of Γ_i.*

It should be remarked finally that Theorem 3.1 and Corollary 3.3 are not, strictly speaking, generalizations of Corollaries 4 and 3 of [2], respectively. For there is no bound given on x_i valid for all partitions with a given number of elements. Indeed, no such bound can be obtained, for one can let the first cell of a partition consist of arbitrarily long initial segments of N.

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