An asymptotic expansion for a Black–Scholes type model

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Received 1 February 2004; accepted 10 February 2004
Available online 18 May 2004

Abstract

We consider the Black–Scholes model where we add a perturbation term \( \sum \varepsilon_i \sigma_j \) to the model with diffusion coefficient \( \sigma_0(t) \). Then we derive an asymptotic expansion for the expected value of an European call option at time \( t \). This is done by applying methods of Malliavin calculus. Borel summability of the derived asymptotic expansion is proven.

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MSC: 60H07; 41A60; 42A24; 60H15

Keywords: Black–Scholes model; Asymptotic expansion; Malliavin calculus; Borel summability; Small diffusion; Laplace method

1. Introduction

One of the most popular option pricing models is the one developed by Black and Scholes [1] in 1973, which was formalized and extended in the same year by Merton. Within the Black–Scholes framework an investor can replicate an option’s return stream by continuously rebalancing a self-financing portfolio involving stocks and risk-free bonds. The value at time \( t \) of a replicating portfolio equals the arbitrage price of an option. Hence the main achievement of the Back–Scholes model is to derive closed-form expressions for both the option’s price and the replication strategy. The Black–Scholes partial differential equation is derived by constructing a risk-free portfolio containing an option and underlying stocks and considering the fact, that under the no-arbitrage condition, the return from such a portfolio equals the returns on risk-free bonds.

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In the Black–Scholes model it is assumed that the asset price $S(t)$ at time $t \in [0, \tau]$ with $\tau \in \mathbb{R}^+$ follows a diffusion process of the form

$$S(t) = S(0) + \int_0^t r \cdot S(v) \, dv + \int_0^t \sigma \cdot S(v) \, dB(v),$$

where $r > 0$ is the risk-free (constant) rate and $\sigma$ is the volatility constant. $B$ is the standard Brownian motion. From Itô’s Formula we know that the solution is given by

$$S(t) = S(0) \exp \left( \left( r - \frac{\sigma^2}{2} \right) t + \sigma \cdot B(t) \right).$$

The Black–Scholes model assumes a standard normalized distribution curve for future stock prices. The volatility $\sigma$ is derived from historical stock data and then replicated into the model. The more volatile a stock has been, the higher probability it has of being far from today’s price on expiration date. Thus to overcome this problem, the seller needs to receive more for the option, and the buyer needs to pay more for the possibility.

Extensions of the Black–Scholes model were studied for a long time. For example Merton [8] generalized the Black–Scholes formula to a deterministic time-dependent rather than constant volatility model already in the same year Black and Scholes published their paper. Later he introduced jump diffusion models for the price of the underlying assets (see [9]). However no ‘simple’ better model than the original Black–Scholes model has been universally accepted. A main problem that one recognizes in the real world is that the implied volatilities of market prices vary with strike price and time-to-maturity of the options contract whereas, in the Black–Scholes model, the volatility is assumed to be constant. In this context, one should also mention that the variation of implied volatility with strike price for options with the same time-to-maturity is often a $U$-shaped curve, which is generally called the smile curve, with minimum at or near the current asset price.

One tries to overcome the problem of discrepancy between variable implied volatility and constant volatility in the Black–Scholes model by introducing so called stochastic volatility models, i.e. the underlying asset price is modelled as a stochastic process driven by a random volatility Itô process. These models were first studied by Hull and White [2], Scott [13] and Wiggins [17]. A recent extension of these stochastic volatility models has been introduced by Sircar and Papanicolaou in [14]. The latter authors consider a volatility process fluctuating on a different time scale than the one of the price process. It is remarkable that in this framework pricing and hedging bands are not sensitive to how the volatility is modelled and the authors derive an explicit formula for the width of these bands in the special case of the Black–Scholes model, i.e. where the volatility does not depend on the underlying stock price. Our aim in this paper is to give an asymptotic expansion approach in the sense of small diffusion for Black–Scholes type models, based on the fact that volatilities for financial asset prices may vary over time, as noted above, but do not differ very much in comparison to the observed levels of asset prices. This method was introduced by Kunitomo and Takahashi [5,6]. They consider the valuation problem of interest rate based contingent claims. The main difficulties that occur here are that the payoff functions and the discount factors are usually non-linear functionals of bonds with different maturities and that the coupon-bearing bond prices are also complicated...
functions of the instantaneous forward rate processes. Thus it is in general not possible to obtain closed form solutions for the valuation problems of interest rate based contingent claims. To overcome these problems they developed a new asymptotic expansion approach by considering a family of instantaneous forward rate processes \( f^\varepsilon(s,t) \) which arise from a stochastic differential equation where the volatility function \( \sigma(f(s,t),s,t) \) in the general Heath, Jarrow and Morton forward rate setting is replaced by \( \varepsilon \sigma(f^\varepsilon(s,t),s,t) \) with a parameter \( \varepsilon \in (0,1) \). Under certain assumptions they derive an asymptotic expansion for the payoff \( V_t(T) \) of a contingent claim at the terminal period \( T \) given the information available at time \( t \).

In the present paper we give an asymptotic expansion approach in the above sense for the Black–Scholes model. That is we rigorously derive an expansion for the expected value of an European call option in the Black–Scholes framework where we substitute constant volatility by time and space dependent volatility because of the above mentioned failures of the Black–Scholes model. This is done in Section 4.2. Moreover we discuss the Borel summability of the derived asymptotic expansion for an explicit example.

2. Mathematical tools

2.1. Malliavin calculus

Let \((W^r, P)\) be the \(r\)-dimensional standard Wiener space, i.e. \(W^r\) is the space of all continuous mappings \(w : [0, 1] \ni t \mapsto w(t) \in \mathbb{R}^r\) such that \(w(0) = 0\) with the topology of uniform convergence, i.e.

\[
W = \{ w \in C([0, 1], \mathbb{R}^r); \ w(0) = 0 \}
\]

and \(P\) is the standard Wiener measure on \(W^r\). \textit{Wiener functionals} are \(P\)-measurable functions defined on the Wiener space \((W^r, P)\). Moreover, we consider a real separable Hilbert space \(E\) and a family of \(E\)-valued Wiener functionals \(\{F(\varepsilon, w); \ \varepsilon \in I\}\) for \(I = (0, 1]\), i.e. \(F(\varepsilon, \cdot); (W^r, P) \to E\) is \(P\)-measurable for every \(\varepsilon \in (0, 1]\) or generalized Wiener functionals (see below for definition). The \textit{Cameron–Martin subspace} \(H\) of \(W^r\), is the Hilbert subspace of \(W^r\) formed by all \(w \in W\) which are absolutely continuous in \(t\) with square integrable derivatives on \([0, 1]\) endowed with the Hilbert norm

\[
\|w\|_H^2 = \int_0^1 |\dot{w}(t)|^2 \, dt,
\]

where \(\dot{w}(t) = dw(t)/dt\).

A Wiener functional \(F : W \to \mathbb{R}\) is called a \textit{polynomial (smooth) functional} if there exist an integer \(n \in \mathbb{N}\), functions \(h_1, \ldots, h_n \in H\) and a real polynomial \(p(x_1, \ldots, x_n)\) (respectively a real tempered (in the sense of real tempered distributions) \(C^\infty\)-function \(f(x_1, \ldots, x_n)\)) of \(n\) variables such that

\[
F(w) = p([h_1](w), \ldots, [h_n](w)) \quad \text{(respectively } F(w) = f([h_1], \ldots, [h_n](w))\text{)},
\]
where $w$ is an element of the standard Wiener space $W$. The totality of polynomial functionals and smooth functionals are denoted by $\mathcal{P}$ and $\mathcal{S}$ respectively. An $E$-valued Wiener functional expressed in the form of a finite sum

$$F(w) = \sum F_i(w)e_i, \quad e_i \in E, \ w \in W, \text{ and } F_i \in \mathcal{P} (\mathcal{S})$$

is called an $E$-valued polynomial (respectively smooth) functional. The totality of all $E$-valued polynomial (smooth) functionals is denoted by $\mathcal{P}(E)$ (respectively $\mathcal{S}(E)$).

Now we can define the following spaces: $D^p_s(E)$ for $1 < p < \infty$ and $s \in \mathbb{R}$ is the completion of $\mathcal{P}(E)$ with respect to the norm $\| \cdot \|_{p,s} := \| (I - L)^{s/2} \cdot \|_p$ where $\| \cdot \|_p$ is the usual $L^p$-norm and the operator $L$ from $\mathcal{P}(E)$ into itself is the Ornstein–Uhlenbeck operator (see [3] for details).

$D^\infty_s(E)$ coincides with the completion of $\mathcal{S}(E)$ with respect to the norm $\| \cdot \|_{p,s}$. We define

$$\text{Then } D^\infty(E) \text{ is a complete countably normed space (Fréchet space) and } D^{-\infty}(E) \text{ is its dual. If } s < 0, \text{ the typical elements of } D^p_s(E) \text{ are not Wiener functionals and may be called generalized Wiener functionals.}$$

For $F \in \mathcal{P}(E)$ we can define its Fréchet derivative $DF(w) \in \mathcal{P}(H \otimes E)$ by

$$DF(w)[h] := \lim_{\varepsilon \downarrow 0} \frac{F(w - \varepsilon h) - F(w)}{\varepsilon}, \quad \text{for } h \in H, \ w \in W.$$ 

One can show as an application of Meyer’s theorem (see [10]) that the operator $D: \mathcal{P}(E) \to \mathcal{P}(H \otimes E)$ can be extended uniquely to an operator $D: D^{-\infty}(H \otimes E) \to D^{-\infty}(H \otimes E)$ so that its restriction $D: D^{p+1}_s(E) \to D^{p}_s(H \otimes E)$ is continuous for every $p \in (1, \infty)$ and $s \in \mathbb{R}$. 

**Definition 2.1.** A Wiener functional $F: W \to \mathbb{R}^d$ is said to be smooth in the sense of Malliavin if $F \in D^\infty(\mathbb{R}^d)$, i.e. $F = (F^1, \ldots, F^d)$ with $F^i \in D^\infty(\mathbb{R})$ for every $i = 1, \ldots, d$.

**Definition 2.2.** For $F(w) = (F^1(w), \ldots, F^d(w)) \in D^\infty(\mathbb{R}^d)$, $w \in W$, we define

$$\sigma^{i,j}(w) := \langle DF^i(w), DF^j(w) \rangle_H, \quad \text{for } i,j = 1, \ldots, d.$$ 

The matrix $\sigma(w) := (\sigma^{i,j}(w))$, denoted also by $\sigma_F(w)$, is called the Malliavin covariance of $F$. If

$$\left( \det \sigma_F(w) \right)^{-1} \in L_{-\infty},$$

then $F$ is called non-degenerate in the sense of Malliavin, where

$$L_{-\infty} := \bigcap_{1 < p < \infty} L_p.$$
Since we want to discuss an asymptotic expansion for the Black–Scholes model later on, we will have to define what we mean by that.

**Definition 2.3.** We say \( F(\varepsilon, w) = O(\varepsilon^k) \) in \( D^p(E) \), as \( \varepsilon \downarrow 0 \), if \( F(\varepsilon, w) \in D^p(E) \) for all \( \varepsilon \in (0, 1], w \in W \), and
\[
\limsup_{\varepsilon \downarrow 0} \left\| F(\varepsilon, w) \right\|_{s,p} / \varepsilon^k < \infty
\]
where \( k \) is some non-negative constant.

Let \( F(\varepsilon, w) \in D^\infty(E) \) \( (\tilde{D}^\infty(E), \tilde{D}^{-\infty}(E), D^{-\infty}(E)) \) for all \( \varepsilon \in I, w \in W \), and \( f_0, f_1, \ldots \in D^\infty(E) \) (respectively \( \tilde{D}^\infty(E), \tilde{D}^{-\infty}(E), D^{-\infty}(E) \)).

**Definition 2.4.** We say that \( F(\varepsilon, w) \), as above, has the asymptotic expansion \( F(\varepsilon, w) \sim f_0 + \varepsilon f_1 + \cdots \) in \( D^\infty(E) \) (respectively \( \tilde{D}^\infty(E), \tilde{D}^{-\infty}(E), D^{-\infty}(E) \)) as \( \varepsilon \downarrow 0 \) if, for every \( k = 1, 2, \ldots \),
\[
F(\varepsilon, w) - (f_0 + \varepsilon f_1 + \cdots + \varepsilon^{k-1} f_{k-1}) = O(\varepsilon^k)
\]
in \( D^\infty(E) \) (respectively \( \tilde{D}^\infty(E), \tilde{D}^{-\infty}(E), D^{-\infty}(E) \)) as \( \varepsilon \downarrow 0 \).

Let \( \sigma_\varepsilon(w) \) be the Malliavin covariance of \( F(\varepsilon, w) \in D^\infty(\mathbb{R}^d), \varepsilon \in I, w \in W \). Then this is said to be uniformly non-degenerate in the sense of Malliavin, if \( F(\varepsilon, w) \) is non-degenerate in the sense of Malliavin and furthermore
\[
\limsup_{\varepsilon \downarrow 0} \left\| (\det \sigma_\varepsilon)^{-1} \right\|_p < \infty \quad \text{for all } 1 < p < \infty.
\]

The following theorem gives a Taylor expansion formula for generalized Wiener functionals. The proof can be found for example in [16].

**Theorem 2.5.** Suppose that \( \{F(\varepsilon, w)\} \) is uniformly non-degenerate and \( F(\varepsilon, w) \) has the asymptotic expansion
\[
F(\varepsilon, w) \sim f_0 + \varepsilon f_1 + \cdots \quad \text{in } D^\infty(\mathbb{R}^d)
\]
as \( \varepsilon \downarrow 0 \). Then for every \( T \in S(\mathbb{R}^d)' \), \( T \circ F(\varepsilon, w) \) has the asymptotic expansion
\[
T \circ F(\varepsilon, w) \sim \phi_0 + \varepsilon \phi_1 + \cdots \quad \text{in } \tilde{D}^{-\infty}
\]
as \( \varepsilon \downarrow 0 \). Furthermore \( \phi_k \in \tilde{D}^{-\infty} \) is obtained by the asymptotic Taylor expansion in powers of \( \varepsilon \) of
\[
T \circ (f_0 + [\varepsilon f_1 + \varepsilon^2 f_2 + \cdots]) = \sum_{n \in \mathbb{Z}^d} \frac{1}{n!} (D^n T)(f_0)[\varepsilon f_1 + \varepsilon^2 f_2 + \cdots]^n
\]
where for \( n = (n_1, \ldots, n_d) \in (\mathbb{Z})^d \),
\[
n! = n_1! n_2! \cdots n_d!,
\]
\[
D^n = (\partial_1)^{n_1} \cdots (\partial_d)^{n_d}, \quad \text{and} \quad \partial_i = \frac{\partial}{\partial x_i}
\]
and
\[ a^n = a_1^{n_1} \ldots a_d^{n_d} \quad \text{for } a = (a_1, \ldots, a_d) \in \mathbb{R}^d. \]

Hence the \( \phi_k \)'s are given by
\[
\begin{align*}
\phi_0 &= T(f_0), \\
\phi_1 &= f_1 \cdot DT(f_0), \\
\phi_2 &= f_2 \cdot DT(f_0) + \frac{1}{2} \cdot f_1^2 \cdot D^2 T(f_0), \\
&\vdots
\end{align*}
\]

Here \( S(\mathbb{R}^d) \) is the Schwartz space of rapidly decreasing \( C^\infty \)-functions on \( \mathbb{R}^d \) and its dual \( S'(\mathbb{R}^d) \) is the Schwartz space of real tempered distributions on \( \mathbb{R}^d \). \( T \circ F \), denoted also by \( T(F) \), is called the composition of \( T \in S'(\mathbb{R}^d) \) and \( F \), or the lifting or pull-back of \( T \in S'(\mathbb{R}^d) \) under the Wiener map \( F: (W, P) \rightarrow \mathbb{R}^d \). See [3] or [16] for details.

**Proposition 2.6.** If
\[ \Phi(\varepsilon, w) \sim \phi_0 + \varepsilon \phi_1 + \cdots \text{ in } D^{-\infty} \text{ as } \varepsilon \downarrow 0, \]
then
\[ E[\Phi(\varepsilon, w)] = \langle \Phi(\varepsilon, w), 1 \rangle \sim E[\phi_0] + \varepsilon E[\phi_1] + \cdots \text{ as } \varepsilon \downarrow 0 \]
in the ordinary numerical sense (where \( \sim \) stands for equality with the asymptotic series on the right hand side).

**Remark 2.7.** Here \( E[\phi_n] \) denotes the generalized expectation of \( \phi_n \) which is defined as the coupling \( \langle \phi, 1 \rangle_E \) of the functional identically equal to 1 which is in the space \( D^\infty \) with the function \( \phi_n \) which is in \( D^{-\infty} \) since \( \Phi \) was supposed to be in \( D^{-\infty} \).

**Proof.** See [3], p. 221, for details.

Let us consider a family of vector fields \( \{V_\alpha; \alpha = 0, 1, \ldots, r\} \) on \( \mathbb{R}^d \) given in the form
\[ V_\alpha = \sum_{k=1}^{d} V^k_\alpha(x) \frac{\partial}{\partial x_k}. \quad \alpha = 0, 1, \ldots, r. \]

We assume that \( V^k_\alpha(x), \alpha = 0, 1, \ldots, r, k = 1, 2, \ldots, d, \) are \( C^\infty \) with bounded derivatives of all orders greater or equal to 1. On the \( r \)-dimensional Wiener space \( (W, P) \), we consider the following stochastic differential equation in the Stratonovich form:
\[
\begin{align*}
dX(t) &= \sum_{\alpha=1}^{r} V_\alpha(X(t)) \circ dw^\alpha(t) + V_0(X(t)) \, dt, \\
X(0) &= x \in \mathbb{R}^d.
\end{align*}
\]

This is equivalent to the Itô stochastic differential equation
\[ dX^k(t) = \sum_{\alpha=1}^{r} V^k_\alpha(X(t)) \, dw^\alpha(t) + \tilde{V}^k_0(X(t)) \, dt, \]

\[ X^k(0) = x^k, \quad k = 1, \ldots, d, \]

where

\[ \tilde{V}^k_0(x) = V^k_0 + \frac{1}{2} \sum_{\alpha=1}^{r} \sum_{j=1}^{d} (\partial_j V^k_\alpha)(x) V^j_\alpha(x), \quad \partial_j = \frac{\partial}{\partial x_j} \]

for \( x = (x^1, x^2, \ldots, x^d) \in \mathbb{R}^d \). It is known (see [4]) that there exists a unique solution \( X(t) = X(t,x,w) \) such that for almost all \( w(P) \),

(i) the mapping: \([0, \infty) \times \mathbb{R}^d \ni (t,x) \rightarrow X(t,x,w) \in \mathbb{R}^d \) is continuous,

(ii) for fixed \( t > 0 \), the mapping: \( \mathbb{R}^d \ni x \rightarrow X(t,x,w) \in \mathbb{R}^d \) is a diffeomorphism.

Following [3] we now return to the asymptotic expansion of solutions of stochastic differential equations, i.e. for a parameter \( \varepsilon \in (0,1] \) we consider the stochastic differential equation

\[ dX(t) = \varepsilon \sum_{\alpha=1}^{r} V^\alpha(X(t)) \circ dw^\alpha(t) + \varepsilon^2 V_0(X(t)) \, dt, \]

\[ X(0) = x \in \mathbb{R}^d. \]

The solution will be denoted by \( X^\varepsilon(t,x,w) \). The following important theorem is due to Watanabe [16].

**Theorem 2.8.** Suppose that all coefficients \( V^\alpha_\alpha, \alpha = 0, 1, \ldots, r, \, k = 1, 2, \ldots, d, \) are bounded. Let \( x \) be fixed in \( \mathbb{R}^d \). Then \( X^\varepsilon(1,x,w) \in D^\infty(\mathbb{R}^d) \) has the asymptotic expansion

\[ X^\varepsilon(1,x,w) \sim f_0 + \varepsilon f_1 + \cdots \quad \text{in } D^\infty(\mathbb{R}^d), \quad \varepsilon \downarrow 0 \]

and \( f_n, n = 0, 1, \ldots, \) are given by

\[ f_0 = x, \]

\[ f_1(w) = \sum_{\alpha=1}^{r} V^\alpha(x) w^\alpha(1), \]

\[ \vdots \]

\[ f_n(w) = \sum_{\alpha: ||\alpha|| = m} V^\alpha_0 V^\alpha_1 \cdots V^\alpha_2(V^\alpha_1)(x) S^\alpha(1, w), \]

where for \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \{0, 1, \ldots, r\}^m, \, m = 1, 2, \ldots, \)

\[ ||\alpha|| = m + \sharp\{v; \alpha_v = 0\}, \]

\[ S^\alpha(t,w) = \int_0^t \circ dw^{\alpha_1}(t_1) \int_0^{t_1} \circ dw^{\alpha_2}(t_2) \ldots \int_0^{t_{m-1}} \circ dw^{\alpha_m}(t_m) \]
and the vector field $V_\alpha$ is regarded as a differential operator.

We want to apply the above theorem (and the theory which leads to it) later on to our Black–Scholes type stochastic differential equation, which has a more general form. However for that application it suffices to study the case where the solution takes values in $\mathbb{R}$. Thus we consider for $0 < \varepsilon < 1$ the more general stochastic differential equation

$$dX_t = \sum_{i=0}^{n} \varepsilon^i \left( V_0^i(X_t) \, dt + V_1^i(X_t) \circ dw_t \right),$$

$$X_0 = x \in \mathbb{R}$$

(1)

for some $n \in \mathbb{N}$, where $V_0^i(x)$ and $V_1^i(x)$ for $i = 0, \ldots, n$ are real valued functions on $\mathbb{R}$. We suppose that $V_0^i(x)$ and $V_1^i(x)$ are $C^\infty$-functions with bounded derivatives of all orders.

Let $\hat{V}_0^i$ and $\hat{V}_1^i$ be the vector fields defined by

$$\hat{V}_0^i := V_0^i \frac{\partial}{\partial x},$$

$$\hat{V}_1^i := V_1^i \frac{\partial}{\partial x}$$

for $\alpha = 0; 1$. Let $I_k := \{0; 1\}^k$ and $J_k := \{0, 1, \ldots, n\}^k$ and for an element $j = (j_1, \ldots, j_k) \in J_k$ we set $\|j\| := \sum_{i=1}^{k} j_i$. Furthermore we define

$$Q^i(t) := \int_{0}^{t} \int_{0}^{t_1} \ldots \int_{0}^{t_{l-1}} dt_1 \ldots dt_{l+1} \circ dw_1 \circ \cdots \circ dw_l$$

where $i = (i_1, \ldots, i_k) \in I_k$ and $l := g[i_1; i_u = 1; u = 1, \ldots, k]$.

The solution of Eq. (1), supposed to exist (see below for sufficient conditions for this), will be denoted by $X^\varepsilon(t, x, w)$. Now a successive application of Itô’s formula yields

$$X^\varepsilon(t, x, w) - x$$

$$= \sum_{i=0}^{n} \varepsilon^i \left( \int_{0}^{t} V_0^i(X_s^\varepsilon) \, ds + \int_{0}^{t} V_1^i(X_s^\varepsilon) \circ dw_s \right) - x$$

$$= \sum_{i=0}^{n} \varepsilon^i \left[ V_1^i(x) w_t + \int_{0}^{t} \left( V_1^i(X_s) - V_1^i(x) \right) \circ dw_s + \int_{0}^{t} V_0^i(X_s^\varepsilon) \, ds \right]$$

$$= \sum_{i=0}^{n} \varepsilon^i \left[ V_1^i(x) w_t + \int_{0}^{t} \left( \int_{0}^{t} \sum_{j=0}^{n} \varepsilon^j \hat{V}_1^j(V_0^j(X_u^\varepsilon)) \circ dw_u \right) \circ dw_s \right. \right.$$

$$\left. \left. + \int_{0}^{t} \left( \int_{0}^{t} \sum_{j=0}^{n} \varepsilon^j \hat{V}_1^j(V_0^j(X_u^\varepsilon)) \, du \right) \circ dw_s \right) \right. \right. \right.$$

$$\left. \left. + \int_{0}^{t} \sum_{j=0}^{n} \varepsilon^j \hat{V}_1^j(V_0^j(X_s^\varepsilon)) \, ds \right) \right.$$

$$= \sum_{i=0}^{n} \varepsilon^i \left[ V_1^i(x) w_t + \left( V_0^i(x) t + \sum_{j=0}^{n} \varepsilon^j \hat{V}_1^j(V_0^j(x)) Q^{(1,1)} \right) \right]$$
Now we define real numbers $f_n$ by

$$f_0 = x + \sum_{k=1}^{\infty} \sum_{i \in I_k, j \in J_k, \|j\|=0} \hat{V}_{i_1}^{j_1} \circ \cdots \circ \hat{V}_{i_k}^{j_k-1} (V_{i_k}^{j_k})(x) \cdot Q^j(t),$$

$$f_n = \sum_{k=1}^{\infty} \sum_{i \in I_k, j \in J_k, \|j\|=n} \hat{V}_{i_1}^{j_1} \circ \cdots \circ \hat{V}_{i_k}^{j_k-1} (V_{i_k}^{j_k})(x) \cdot Q^j(t), \quad n \in \mathbb{N},$$

where the series are assumed to converge for all $x \in \mathbb{R}^d$ and $t \in [0, \infty)$.

From the observations above we immediately obtain the following result.

**Theorem 2.9.** Suppose that all coefficients $V_{i_j}^j$ for $i \in \{0; 1\}$ and $j \in \{0; 1; 2\}$ are bounded and that the sum

$$\sum_{k=1}^{\infty} \sum_{i \in I_k, j \in J_k, \|j\|=n} \hat{V}_{i_1}^{j_1} \circ \cdots \circ \hat{V}_{i_k}^{j_k-1} (V_{i_k}^{j_k})(x) \cdot Q^j(t)$$

converges for all $n \in \mathbb{N}$, and for all $x \in \mathbb{R}^d$, $t \in [0, \infty)$.

Then the solution $X(t, x, w)$ of Eq. (1) has the asymptotic expansion

$$X(t, x, w) \sim f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots$$

where the coefficients $f_i$ are defined as above.

This theorem will be of great importance in the next section where we consider the asymptotic expansion of a Black–Scholes type stochastic differential equation. Moreover we will need a condition under which the solution $X$ of a stochastic differential equation is in $D^\infty(\mathbb{R})$. Such a condition was derived in [12] for the case where both the drift and volatility term are globally Lipschitz and bounded. Let us thus consider a stochastic differential equation of the form

$$X(t) = x_0 + \sum_{j=1}^{d} \int_0^t A_j(s, X(s)) \, dw_j^s + \int_0^t B(s, X(s)) \, ds,$$

where $A_j, B : [0, T] \times \mathbb{R}^m \to \mathbb{R}^m$ for $1 \leq j \leq d$ are measurable functions satisfying the globally Lipschitz and boundedness conditions:

(a) $\sum_{j=1}^{m} |A_j(t, x) - A_j(t, y)| + |B(t, x) - B(t, y)| \leq K|x - y|$ for $x, y \in \mathbb{R}^d$, $t \in [0, T]$;
(b) \( t \mapsto A_j(t, 0) \) and \( t \mapsto B(t, 0) \) are bounded on \([0, T]\).

Then the following theorem holds.

**Theorem 2.10.** Let \( X \) be as above and assume that the coefficients \( A_i^j \) and \( B_i^j \) are infinitely differentiable functions in \( x \) with bounded derivatives of all orders greater than or equal to one and that the functions \( A_j^i(t, 0) \) and \( B_j^i(t, 0) \) are bounded. Then \( X_i(t) \) belongs to \( D^\infty \) for all \( t \in [0, T] \) and all \( i = 1, \ldots, m \).

For the proof see for example [12], p. 105.

This theorem is used later on to show the solution \( S_{\varepsilon}^t \) of the perturbed stochastic differential equation (Eq. (10)) is in the space \( D^\infty(\mathbb{R}) \) for every \( 0 < \varepsilon < 1 \). This is an important result since we want to apply Watanabe’s Theorem 2.5, which is only possible if the asymptotic expansion of \( S_{\varepsilon}^t \) is in \( D^\infty(\mathbb{R}) \) and uniformly non-degenerate in the sense of Malliavin.

### 2.2. Borel summability

An important tool for proving Borel summability is Watson’s Theorem (see e.g. [11]), which gives a sufficient condition for the function \( f(z) \) to equal the Borel sum of its asymptotic Taylor series.

**Theorem 2.11 (Watson’s Theorem).** Let \( f(z) \) be analytic in a sector \( |\arg z| < \pi/2 + \varepsilon, \| z \| < R \), for some \( 0 < \varepsilon < \pi/2 \), and let \( f(z) \) have there the asymptotic expansion

\[
 f(z) = \sum_{k=0}^{N-1} f_k z^k + R_N(z),
\]

with

\[
 |R_N(z)| \leq A \sigma^N N! |z|^N
\]

uniformly in \( N \) and in \( z \) in the sector where \( \sigma > 0 \) and \( A > 0 \) are constants. Under these assumptions, the following holds:

(a) \( B(t) = \sum f_n t^n / n! \) converges in the circle \( |t| < 1/\sigma \);
(b) \( B(t) \) has an analytic continuation to the sector \( |\arg t| < \varepsilon \); and
(c) the integral \( (1/z) \int_0^\infty e^{-t/z} B(t) dt \) is absolutely convergent for \( \text{Re} z^{-1} > R^{-1} \) and equals \( f(z) \).

An improvement of this theorem was introduced by [15]. Therein a necessary and sufficient characterization of a large class of Borel summable functions is given. This is a very helpful tool in cases where it is difficult to verify the analyticity and estimate (3) in a sector with \( \varepsilon > 0 \).

**Theorem 2.12 (Sokal).** Let \( f \) be analytic in \( C_R = \{ z : \text{Re} z^{-1} < R^{-1} \} \) and satisfy there estimates (2) and (3) uniformly in \( N \) and in \( z \in C_R \). Then \( B(t) = \sum f_n t^n / n! \) converges for
$|t| < 1/\sigma$ and has an analytic continuation to the striplike region $S_{\sigma} = \{ t : \text{dist}(t; \mathbb{R}_+) < 1/\sigma \}$ satisfying the bound

$$|B(t)| \leq K \exp(|t|/R)$$

uniformly in every $S_{\sigma'}$ with $\sigma' > \sigma$. Furthermore, $f$ can be represented by the absolutely convergent integral

$$f(z) = (1/z) \int_0^\infty e^{-1/z} B(t) \, dt,$$  \hspace{1cm} (5)

for any $z \in C_R$.

Conversely, if $B(t)$ is a function analytic in $S_{\sigma''}$ for $\sigma'' < \sigma$ and satisfying (4) in $S_{\sigma''}$, then the function $f(z)$ defined by (5) is analytic in $C_R$ and satisfies (2) and (3) uniformly in every $C_{R'}$ with $R' < R$.

An earlier improvement of Watson’s theorem was already proved 1918 by Nevanlinna in [11]. However, the above version fits our needs better.

3. The perturbation theory

3.1. The setting of the perturbation theory for the Black–Scholes model

We consider a continuous time economy with a trading interval $[0, \tau]$ where $\tau \in \mathbb{R}^+$. The market is assumed to be free of arbitrage and complete in the economic sense (see e.g. [7] for these concepts). Moreover we assume that there are no transaction costs or dividends and that short selling is permitted. Assets are supposed to be perfectly divisible. The risk-free rate $r$ is assumed to be constant and observable.

3.2. The idea of the perturbation theory

For the reason discussed in the introduction, i.e. the fact that the Black–Scholes model only partially takes into account the volatility changes in the markets, we want to add a correction term to the model, namely we consider a model where the volatility $\sigma$ is replaced by

$$\sigma^\varepsilon(S(t), t) = \sigma_0(t) \cdot S(t) + \sum_{k=1}^\infty \varepsilon^k \cdot \sigma_k(S(t), t)$$

for $0 < \varepsilon \leq 1$ (the series being assumed to be absolutely converging). Hence one adds a perturbation term $\varepsilon \sigma_1 + \varepsilon^2 \sigma_2 + \cdots$ to the model with diffusion coefficient $\sigma_0(t)$. Thus we consider a perturbation of the extended Black–Scholes model introduced by [8], i.e. where the volatility $\sigma_0$ is not constant anymore but depends on time.

We assume that the stock price $S^\varepsilon(t)$ for $0 \leq \varepsilon \leq 1$ follows a stochastic process

$$dS^\varepsilon_t = r \cdot S^\varepsilon_t \, dt + \sigma^\varepsilon(t, S^\varepsilon_t) \, dB_t,$$  \hspace{1cm} (6)
with \( \sigma^\varepsilon \) as above. If we suppose that \( S^\varepsilon \) is a European call option with strike price \( K \), the return at maturity is \( \max\{S^\varepsilon_T - K, 0\} \) where \( T \) is the maturity date. By applying certain techniques of Malliavin’s calculus we want to derive an asymptotic expansion for the expected value of a European call option \( E[(S^\varepsilon_T - K)^+] \).

4. Derivation of an asymptotic expansion for the expected value

We want to compute

\[
E\left[e^{-\int_0^t r \, dt} (S^\varepsilon_T - K)^+ \right] = e^{-rt} E\left[(S^\varepsilon_T - K)^+ \right]
\]

where \( K \) is the strike price and \( r \) is the constant positive interest rate. Since \( (S^\varepsilon_T - K)^+ \) is not differentiable, we first consider \( E[f(S^\varepsilon_T - K)] \) for some differentiable function \( f \). Then we will try to approximate \( (S^\varepsilon_T - K)^+ \) by a sequence of such functions.

We first discuss the case when the volatility \( \sigma^\varepsilon(t, x) \) is linear in the \( x \)-variable, i.e. when

\[
\sigma^\varepsilon(t, S^\varepsilon_t) := \sigma_0 \cdot S^\varepsilon_t + \varepsilon \sigma_1 \cdot S^\varepsilon_t,
\]

where \( \sigma_0 \) and \( \sigma_1 \) are positive constants, independent of \( t \) and \( \omega \). In this case we derive a generalized Black–Scholes differential equation from which we can compute the expected value of a European call option at time \( t \). Afterwards we will discuss the general case when the volatility function \( \sigma^\varepsilon \) is given as

\[
\sigma^\varepsilon(t, S^\varepsilon_t) := \sigma_0(t) \cdot S^\varepsilon_t + \sum_{i=1}^{\infty} \varepsilon^i \sigma_i(t, S^\varepsilon_t).
\]

This case is more complicated since we no longer have an explicit “Black–Scholes equation” that is easily solvable.

We distinguish these two cases because the linear case more relevant for large values of \( S_t \) and the general case is important for small values of \( S_t \). In fact, the general case can only be considered for values of \( S_t \) below a certain boundary since we want to apply the results from Malliavin calculus derived in Section 2.1. Therefore we have to include stopping times in our model. This will be explained in more detail in Section 4.2.

4.1. The linear case (constant drift and volatility)

First we start with \( \sigma^\varepsilon(S(t), t) = \sigma_0 \cdot S(t) + \varepsilon \cdot \sigma_1 S(t) \) where \( \sigma_0, \sigma_1 > 0 \) are constants, independent of \( t \) and \( \omega \), and \( 0 \leq \varepsilon \leq 1 \). Assuming this we obtain a Black–Scholes type model of the form

\[
S^\varepsilon(t) = S^\varepsilon(0) + \int_0^t r \cdot S^\varepsilon(v) \, dv + \int_0^t \sigma^\varepsilon(S^\varepsilon(v), v) \, dB(v)
\]

\[
= S^\varepsilon(0) + \int_0^t r \cdot S^\varepsilon(v) \, dv + \int_0^t \sigma_0 \cdot S^\varepsilon(v) \, dB(v) + \varepsilon \int_0^t \sigma_1 S^\varepsilon(v) \, dB(v).
\]
In this case we can compute the discounted value

\[ e^d = e^{\varepsilon t} \]

and

\[ S_\varepsilon = S_0 e^{\mu \varepsilon t} \exp\left(\int_0^t (r - \frac{(\sigma_0 + \varepsilon \sigma_1)^2}{2}) dt + (\sigma_0 + \varepsilon \sigma_1) B_t \right) \]

This is equivalent to

\[ \frac{\partial f}{\partial t} + r S_\varepsilon \frac{\partial f}{\partial S} + \frac{1}{2} \sigma_0^2 (S_\varepsilon)^2 \frac{\partial^2 f}{\partial S^2} = r f. \]

We know that the solution of the stochastic differential equation (7) is given by

\[ S_\varepsilon = S_0 \cdot \exp\left(\left( t - \frac{(\sigma_0 + \varepsilon \sigma_1)^2}{2}\right) + \int_0^t \frac{1}{\sqrt{2 \pi t}} e^{-\frac{y^2}{2}} dy \right) \]

In this case we can compute the discounted value \( e^{-rt} E[(S_\varepsilon^t - K)^+] \) as follows

\[ e^{-rt} E[(S_\varepsilon^t - K)^+] \]

\[ = e^{-rt} \int_{-\infty}^{\infty} S_0^t \cdot e^{\left( r - \frac{\mu \varepsilon t}{2}\right) t + \frac{1}{2} \sigma_0 (\sigma_0 + \varepsilon \sigma_1) x^2} e^{-\frac{1}{2} \sigma_0^2 \sigma_0 x^2} e^{-\frac{1}{2} \sigma_0^2 \sigma_0 x^2} dx \]

\[ = e^{-rt} \int_{-\infty}^{\infty} S_0^t \cdot e^{\left( r - \frac{\mu \varepsilon t}{2}\right) t + \frac{1}{2} \sigma_0 (\sigma_0 + \varepsilon \sigma_1) x^2} e^{-\frac{1}{2} \sigma_0^2 \sigma_0 x^2} e^{-\frac{1}{2} \sigma_0^2 \sigma_0 x^2} dx \]

\[ = e^{-rt} \int_{-\infty}^{\infty} S_0^t e^{\left( r - \frac{\mu \varepsilon t}{2}\right) t + \frac{1}{2} \sigma_0 (\sigma_0 + \varepsilon \sigma_1) x^2} e^{-\frac{1}{2} \sigma_0^2 \sigma_0 x^2} e^{-\frac{1}{2} \sigma_0^2 \sigma_0 x^2} dx \]

\[ = e^{-rt} \int_{-\infty}^{\infty} S_0^t e^{\left( r - \frac{\mu \varepsilon t}{2}\right) t + \frac{1}{2} \sigma_0 (\sigma_0 + \varepsilon \sigma_1) x^2} e^{-\frac{1}{2} \sigma_0^2 \sigma_0 x^2} e^{-\frac{1}{2} \sigma_0^2 \sigma_0 x^2} dx \]

\[ = e^{-rt} \int_{-\infty}^{\infty} S_0^t e^{\left( r - \frac{\mu \varepsilon t}{2}\right) t + \frac{1}{2} \sigma_0 (\sigma_0 + \varepsilon \sigma_1) x^2} e^{-\frac{1}{2} \sigma_0^2 \sigma_0 x^2} e^{-\frac{1}{2} \sigma_0^2 \sigma_0 x^2} dx \]

\[ = \left( X \cdot N(d_1) - K \cdot N(d_2) \right) \]

where

\[ d = \left( \frac{\ln S_0^t}{S_0^t} - \left( r - \frac{\mu \varepsilon t}{2}\right) t \right) \frac{1}{\sqrt{2 \pi t}} \]

and

\[ d_1 = \frac{\ln S_0^t}{S_0^t} + \left( r + \frac{\mu \varepsilon t}{2}\right) t \frac{1}{\sigma_0 + \varepsilon \sigma_1}, \quad d_2 = \frac{\ln S_0^t}{S_0^t} + \left( r + \frac{\mu \varepsilon t}{2}\right) t \frac{1}{\sigma_0 + \varepsilon \sigma_1}, \]

and \( N \) denotes the normal distribution.
4.2. Derivation of an asymptotic expansion for the expected value

Suppose that the volatility \( \sigma^\varepsilon(\omega, t, S^\varepsilon_t) \) is uniformly non-degenerate in the sense of section (2.1) and has the asymptotic expansion

\[
\sigma^\varepsilon(\omega, t, S^\varepsilon_t) \sim \sigma_0(\omega, t) \cdot S^\varepsilon_t + \varepsilon \sigma_1(\omega, t, S^\varepsilon_t) + \varepsilon^2 \sigma_2(\omega, t, S^\varepsilon_t) + \cdots
\]

in \( D^\infty(\mathbb{R}) \) as \( \varepsilon \downarrow 0 \).

Here we follow the notation of Section 2.1. Then the stochastic differential equation for the stock price has the asymptotic expansion

\[
dS^\varepsilon_t = r \cdot S^\varepsilon_t \, dt + \left( \sigma_0(t)S^\varepsilon_t + \varepsilon \sigma_1(t, S^\varepsilon_t) + \varepsilon^2 \sigma_2(t, S^\varepsilon_t) + \cdots \right) \, dB_t
\]

in \( D^\infty(\mathbb{R}) \) as \( \varepsilon \downarrow 0 \).

Changing this into a Stratonovich stochastic differential equation we obtain

\[
dS^\varepsilon_t = r \cdot S^\varepsilon_t \, dt + \sum_{i=0}^{\infty} \varepsilon^i \left( \sigma_i(t, S^\varepsilon_t) \circ dB_t - \frac{1}{2} \frac{\partial \sigma_i}{\partial x}(t, S_t) \, dt \right)
\]

\[
= r \cdot S^\varepsilon_t \, dt + \sum_{i=0}^{2} \varepsilon^i \left( \sigma_i(t, S^\varepsilon_t) \circ dB_t - \frac{1}{2} \frac{\partial \sigma_i}{\partial x}(t, S_t) \, dt \right) + O(\varepsilon^3)
\]

in \( D^\infty(\mathbb{R}) \),

where for simplicity of notation we denote \( \sigma_0(t, S^\varepsilon_t) := \sigma_0(t) \cdot S^\varepsilon_t \). We denote by \( \hat{V}_a^i; \ i = 0, 1, 2; \ a = 0, 1 \) the family of vector fields on \( \mathbb{R} \) given in the form

\[
\hat{V}_a^i = V_a^i(x, t) \frac{d}{dx}, \quad a = 0, 1; \ i = 0, 1, 2,
\]

such that

\[
dS^\varepsilon_t = r \cdot S^\varepsilon_t \, dt + \sum_{i=0}^{2} \varepsilon^i \left( \sigma_i(t, S^\varepsilon_t) \circ dB_t - \frac{1}{2} \frac{\partial \sigma_i}{\partial x}(t, S_t) \, dt \right) + O(\varepsilon^3)
\]

\[
= \sum_{i=0}^{2} \varepsilon^i \left( V_a^0(S^\varepsilon_t, t) \, dt + V_a^1(S^\varepsilon_t, t) \, dB_t \right) + O(\varepsilon^3).
\]
Hence the coefficients $V_i^{\alpha}, i = 0, 1, 2; \alpha = 0, 1$ are given by

\[
\begin{align*}
\hat{V}_0^0(t,x) &:= r \cdot x - \frac{1}{2} \sigma_0(t,x) \frac{\partial}{\partial x}, \\
\hat{V}_0^1(t,x) &:= \frac{1}{2} \left( \frac{\partial \sigma_1(t,x)}{\partial x} \right) \frac{\partial}{\partial x}, \\
\hat{V}_0^2(t,x) &:= \frac{1}{2} \left( \frac{\partial \sigma_2(t,x)}{\partial x} \right) \frac{\partial}{\partial x}, \\
\hat{V}_1^0(t,x) &:= \sigma_0(t,x) \frac{\partial}{\partial x}, \\
\hat{V}_1^1(t,x) &:= \sigma_1(t,x) \frac{\partial}{\partial x}, \\
\hat{V}_1^2(t,x) &:= \sigma_2(t,x) \frac{\partial}{\partial x}.
\end{align*}
\tag{9}
\]

Thus we consider the stochastic differential equation

\[
dS^\varepsilon_t = 2 \sum_{i=0}^2 \varepsilon_i \left( V_i^0(t,S^\varepsilon_t) dt + V_i^1(t,S^\varepsilon_t) \circ dB_t \right) + O(\varepsilon^3),
\tag{10}
\]

\[
S^\varepsilon_0 = S_0.
\]

Since we want to apply certain results from Section 2.1, $\sigma_i(t,x), i \geq 0,$ have to be bounded in the $x$-variable. Obviously this is not the case for $\sigma_0(t,x) := \sigma_0(t) \cdot x.$ As already mentioned above we therefore introduce a stopping time

\[
\tau(\omega):= \sup \{ t \in [0,T] \mid S^\varepsilon_u(\omega) < C \text{ for all } u \in [0,t] \},
\]

where $C > 0$ is a positive constant which defines a barrier between the linear and the general model in the sense that for $\tau(\omega) \geq T$ we apply the asymptotic expansion for the model with general $\sigma_i$ and for $\tau(\omega) < T$ we consider the model with linear volatility. Since all observations below are pathwise this distinction makes sense. We will now prove that $\tau$ indeed is a stopping time. Hence, we have to show that $[\tau \leq s]$ is $\mathcal{F}_s$-measurable for all $s \in [0,T]$, where $\{\mathcal{F}_s\}_{s \in [0,T]}$ is the filtration on the underlying probability space $(\Omega, \mathcal{F}, P)$ generated by the Brownian motion $\{B_s\}_{s \in [0,T]}.$ Indeed we have for $s \in [0,T]$

\[
[\tau \leq s] = \left\{ \sup_{t \in [0,T]} \left\{ S^\varepsilon_u(\omega) < C \text{ for all } u \in [0,t] \right\} \leq s \right\} = \left\{ \sup_{t \in [0,T]} \left\{ S^\varepsilon_u(\omega) < C \text{ for all } u \in [0,t] \right\} \leq s \right\} \in \mathcal{F}_s.
\]

Hence, $\tau$ is a stopping time. From now on we always assume that $\tau(\omega) \geq T.$ Then $\sigma_0(t,S^\varepsilon_t)$ is bounded in $S^\varepsilon_t$ and the $V_i^{j_h}$’s above are also bounded under certain assumptions which we will summarize in Assumption 4.1 below.

We define random variables $f_n$ by

\[
f_0 = S_0 + \sum_{k=1}^{\infty} \sum_{(i \in I_k, j \in J_k, ||j||=0)} \hat{V}_{i_1}^{j_1} \circ \cdots \circ \hat{V}_{i_{k-1}}^{j_{k-1}} (V_{i_k}^{j_k})(0,S_0) \cdot Q^i(t),
\]

\[
f_n = \sum_{k=1}^{\infty} \sum_{(i \in I_k, j \in J_k, ||j||=n)} \hat{V}_{i_1}^{j_1} \circ \cdots \circ \hat{V}_{i_{k-1}}^{j_{k-1}} (V_{i_k}^{j_k})(0,S_0) \cdot Q^i(t), \quad n \in \mathbb{N},
\tag{11}
\]
provided the series on the right hand side converge absolutely, where $Q^i(t)$ is defined as in Section 2.1 by

$$Q^i(t, \omega) := \int_0^t \int_0^t \ldots \int_0^t dt_l \ldots dt_{l+1} \circ dB_l(t) \circ \ldots \circ dB_1(t)$$

where $i = (i_1, \ldots, i_k) \in I_k$ and $l := [i_u, i_u = 1; u = 1, \ldots, k]$ and $I_k := \{0; 1\}^k$ and $J_k := \{0, 1, 2\}^k$ and for an element $j = (j_1, \ldots, j_k) \in J_k$ we set $\|j\| := \sum_{n=1}^k j_n$.

**Assumption 4.1.**

(1) Suppose that $\sigma_0(t, x), \sigma_1(t, x)$ and $\sigma_2(t, x)$ are infinitely differentiable functions in $x$ with bounded derivatives of all orders greater or equal one and that $\sigma_0, \sigma_1$ and $\sigma_2$ and their derivatives of all orders evaluated at $(0, S_0)$ are bounded by 1.

(2) We assume that the functions $\sigma_0(t, 0), \sigma_1(t, 0), \sigma_2(t, 0)$ are bounded.

(3) Assume that the functions $t \mapsto \sigma_i(t, 0), i = 0, 1, 2$, are bounded on $[0, T]$.

(4) We assume that the coefficients $\sigma_0(t,x), \sigma_1(t,x), \sigma_2(t,x)$ satisfy the global Lipschitz condition

$$|r \cdot x - r \cdot y| + |\sigma_0(t,x) + \varepsilon \sigma_1(t,x) + \varepsilon^2 \sigma_2(t,x) - \sigma_0(t,y)|$$

$$-\varepsilon \sigma_1(t,y) - \varepsilon^2 \sigma_2(t,y)| \leq K|x - y|$$

for any $x, y \in \mathbb{R}, t \in [0, T]$, where $K$ is a constant.

(5) Assume that the functions $\sigma_i(t, x)$ for $i = 0, 1, \ldots$ are non-negative.

Then all coefficients $f_n, n \in \mathbb{N}$, above converge since by assumption (1) in Assumption 4.1

$$\sum_{k=1}^{\infty} \sum_{(i \in I_k, j \in J_k, \|j\| = n)} \hat{V}_{i_{1}, j_1} \circ \ldots \circ \hat{V}_{i_{k-1}, j_{k-1}}(V_{i_k})(0, S_0) \cdot Q^i(t) < \infty$$

because the $V^i_j$’s are just combinations of the $\sigma_i$’s and there derivatives. Hence, by a straightforward extension of Theorem 2.9 in Section 2.1 to the case with time dependent coefficients, the solution $S^\varepsilon_t$ has an asymptotic expansion of the form

$$S^\varepsilon_t \sim f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \ldots$$

$$\sim S_0 + \sum_{k=0}^{\infty} \varepsilon^k \sum_{(i \in I_k, j \in J_k, \|j\| = k)} \hat{V}_{i_{1}, j_1} \circ \ldots \circ \hat{V}_{i_{k-1}, j_{k-1}}(V_{i_k})(0, S_0) \cdot Q^i(t).$$

(12)

To apply Theorem 2.9 in Section 2.1 we needed the assumption that all coefficients $V^i_j$ are bounded. This is were the considerations above concerning the stopping times enter. By assumption (1), (3) and (4) in Assumption 4.1 and Theorem 2.10 the solution $S^\varepsilon_t$ of Eq. (10) is in $D^\infty(\mathbb{R})$ for $0 < \varepsilon < 1$. Furthermore, one can show that $F(\varepsilon, x) := (S^\varepsilon_t - f_0)/\varepsilon$ is uniformly non-degenerate in the sense of Malliavin, since by a slight modification of Proposition 10.1 in [4] we get the following representation for the Malliavin covariance of $S^\varepsilon_t$. 
Proposition 4.2. Let $t > 0$ and $x = S(0) \in \mathbb{R}$ be fixed. Then $S^x_t = S^x_t(x, B)$ is smooth in the sense that

$$S^x_t \in D^\infty(\mathbb{R})$$

and the Malliavin covariance $\delta^x_t(B) := \langle DS^x_t(x, B), DS^x_t(x, B) \rangle_H$ is given by

$$\delta^x_t = \int_0^t \left( Y^x_t (Y^x_t)^{-1} \right)^2 \sum_{i,j=0}^\infty \varepsilon^{i+j} V^i_1(S^x_s) V^j_1(S^x_s) ds,$$

where the process $Y^x_t$ is defined by the following stochastic differential equation

$$dY^x_t = \sum_{i=0}^\infty \varepsilon^i \left( \frac{\partial}{\partial x} V^i_0(S^x_t) + \frac{\partial}{\partial x} V^i_1(S^x_t) \circ dB(t) \right),$$

$Y^0_0 = 1$.

(The series are convergent under the assumption (1) in Assumption 4.1.)

Proof. For simplicity of notation we will skip the parameter $\varepsilon$ in $S^\varepsilon_t$. For $S_t := S^\varepsilon_t(x, B)$ we define $S^n_t := S^\varepsilon_t(x, B)$ as

$$S^n_t := x + \int_0^t \left( \sum_{i=0}^\infty \varepsilon^i \sigma_i(S^\varepsilon_{\phi^n(s)}) \right) dB(s) + \int_0^t r S^n_{\phi^n(s)} ds$$

where $\phi^n(s) = \frac{k}{n}$ if $s \in [\frac{k}{n}, \frac{k+1}{n}]$ for $k = 0, 1, 2, \ldots$. We know that $S^n_t \in S$ and $\|S^n_t - S_t\|_p \to 0$ as $n \to \infty$ for every $1 < p < \infty$. It is easy to see that

$$D_h S^n_t := \langle DS^n_t, h \rangle_H$$

for $h \in H$ satisfies

$$D_h S^n_t = \int_0^t \left( \sum_{i=0}^\infty \varepsilon^i \frac{\partial}{\partial x} \sigma_i(S^\varepsilon_{\phi^n(s)}) D_h S^\varepsilon_{\phi^n(s)} \right) dB(s) + \int_0^t r D_h S^n_{\phi^n(s)} ds$$

and hence, if we denote

$$D_h S^n_t := \int_0^t \xi^n_t(v) h(v) dv$$

for $h = h(t) \in H$, then $\xi^n_t(v)$ for $t \geq v$ satisfies

$$\xi^n_t(v) = \int_{\phi^n(v) \leq t} \left( \sum_{i=0}^\infty \varepsilon^i \frac{\partial}{\partial x} \sigma_i(S^\varepsilon_{\phi^n(s)}(v)) \xi^n_{\phi^n(s)}(v) \right) dB(s)$$
\[ + \int \psi_n(\nu) \psi_n(\nu) \cdot \zeta_n(\nu) d\nu + \left( \sum_{i=0}^{\infty} \epsilon^i \sigma_i(S_n(\nu)) \right) \]

where \( \psi_n(\nu) = \frac{1}{k^n} \) if \( \nu \in \left( \frac{k-1}{2}, \frac{k}{2} \right) \) for \( k = 0, 1, 2, \ldots \) By applying Lemma 2.1 in [4], we obtain for every \( 1 < p < \infty \)

\[ E \left[ \sup_{s \in [0,t]} |S^n_s - S|_p \right] \rightarrow 0 \]

and

\[ \sup_{0 \leq \nu \leq t} E \left[ \sup_{s \in [\nu,t]} \left| \xi^n_{t-\nu} - \xi_{t} \right|_p \right] \rightarrow 0 \]

as \( n \rightarrow \infty \) where \( \xi_{t}(\nu) \) for \( t \geq \nu \geq 0 \) satisfies

\[ \xi_{t}(\nu) = \int \left( \sum_{i=0}^{\infty} \epsilon^i \frac{\partial}{\partial x} \sigma_i(S_{t}(\nu)) \right) \left( \sum_{i=0}^{\infty} \epsilon^i \sigma_i(S_{t}(\nu)) \right) dB(s) + \int r \cdot S_{t} \xi_{t}(\nu) d\nu + \sum_{i=0}^{\infty} \epsilon^i \sigma_i(S_{t}). \quad (14) \]

Note that \( \xi_{t}(\nu) \) is uniquely determined by (14) and is given by

\[ \xi_{t}(\nu) = Y_{t} Y_{\nu}^{-1} \sum_{i=0}^{\infty} \epsilon^i \sigma_i(S_{t}). \]

Indeed, it is immediately seen that \( \xi_{t}(\nu) \) given as above is the solution of (14) and the uniqueness is obvious. This proves

\[ S_t \in \bigcap_{1 < p < \infty} D_{p,1}(\mathbb{R}) \]

for all \( t \geq 0 \) and

\[ \langle DS_t, h \rangle = \int_{0}^{t} \xi_t(\nu) h(\nu) d\nu \]

for \( h \in H \). From this the representation (13) follows and \( S_t \in D_{\infty}(\mathbb{R}) \). \( \Box \)

Then a modification of Theorem 3.4 in [16] to our setting yields:

**Theorem 4.3.** The family \( F(\epsilon, B) = (S^\epsilon_t(x, B) - S(0)) \) is uniformly non-degenerate in the sense of Malliavin if and only if \( A(x) := \sum_{i,j=0}^{\infty} \epsilon^i \sigma_j(S_{t}(x)) \) is non-degenerate in the sense that \( A(x) > 0, \) where the \( V^k_j \)’s are given as functions of the \( \sigma_k \)’s as in (8) and (9). The series is convergent by Assumption 4.1.

**Proof.** Since \( F(\epsilon, B) \sim (f_0 - S(0)) + \epsilon f_1 + \epsilon^2 f_2 + \cdots \) where \( f_i \) are those in (11) and the Malliavin covariance \( \sum_{i,j=0}^{\infty} V^i_j(x) B_i \), which is the sum of the first summands of each \( f_n \), i.e. it is a summand of \( \sum_{n=0}^{\infty} f_n \), is given by \( A(x) := \sum_{i,j=0}^{\infty} \epsilon^i \sigma_j(S_{t}(x)) \), it
follows that $A(x)$ has to be non-degenerate in the sense that $A(x) > 0$ in order that $F(\varepsilon, B)$ be uniformly non-degenerate.

Conversely, suppose that

$$A(x) := \sum_{i,j=0}^{\infty} \varepsilon^{i+j}(V_{i}^{j}(x)V_{j}^{i}(x)) > 0.$$ 

Denoting by $\delta(\varepsilon)$ the Malliavin covariance of $F(\varepsilon, B)$, we have by Proposition 4.2

$$\delta(\varepsilon) = t \int_{0}^{t} (Y_{t}^{\varepsilon}(Y_{s}^{\varepsilon})^{-1})^{2} A(S_{t}^{\varepsilon}) ds.$$ 

Set

$$\tau = \inf\{s: (Y_{s}^{\varepsilon})^{-1} A(S_{s}^{\varepsilon})(Y_{s}^{\varepsilon})^{-1} \leq \frac{1}{2} A(x)\}.$$ 

Then by Lemma 10.5 in [4] we know that

$$P[\tau < 1/n] \leq c_{1} \exp(-c_{2} n^{c_{3}}), \quad n = 1, 2, \ldots,$$

where $c_{i}$ for $i = 1, 2, 3$ are positive constants independent of $\varepsilon$ and $n$. Note that

$$\sup_{\varepsilon \in (0,1)} \| (Y_{t}^{\varepsilon})^{-1} \|_{p} < \infty$$

for all $1 < p < \infty$, since $(Y_{t}^{\varepsilon})^{-1}$ is the solution of

$$Z_{t} = I - \int_{0}^{t} Z_{s} \left( \sum_{i=0}^{\infty} \varepsilon^{i} \frac{\partial}{\partial x} \sigma_{i}(Z_{s}) \right) \circ dB(s) - \int_{0}^{t} Z_{s} r ds.$$ 

Then

$$\delta(\varepsilon) \geq (Y_{t}^{\varepsilon})^{2} \int_{0}^{t} (Y_{s}^{\varepsilon})^{-1} A(S_{s}^{\varepsilon})(Y_{s}^{\varepsilon})^{-1} ds \geq \frac{1}{2} (Y_{t}^{\varepsilon})^{2} A(x)(t \wedge \tau)$$

and we can now easily conclude that

$$\sup_{\varepsilon \in (0,1)} \| (\delta(\varepsilon))^{-1} \|_{p} < \infty$$

for all $1 < p < \infty$, which completes the proof. \qed

The condition in the above theorem is fulfilled by assumption (5) in Assumption 4.1. In the following we assume that the initial value $S(0)$ equals 0. Then by the above theorem we know that $S_{t}^{\varepsilon}$ is uniformly non-degenerate in the sense of Malliavin and has the asymptotic expansion

$$S_{t}^{\varepsilon} \sim f_{0} + \varepsilon f_{1} + \varepsilon^{2} f_{2} + \cdots$$

$$\sim \sum_{\lambda=0}^{\infty} \varepsilon^{\lambda} \sum_{k=0}^{\infty} \sum_{(i \in I, j \in J, \|j\|=\lambda)} \hat{V}_{i_{1}}^{j_{1}} \circ \cdots \circ \hat{V}_{i_{k-1}}^{j_{k-1}} (V_{i_{k}}^{j_{k}})(0, S_{0}) \cdot Q^{i}(t).$$
Now let \( (T_n)_{n \in \mathbb{N}} \): \( T_n \in \mathcal{S}'(\mathbb{R}) \), be a sequence of functions such that
\[
T_n(S^c ε_t - K) \xrightarrow{n \to \infty} (S^c ε_t - K) +
\]
pointwise in \( \mathbb{R} \). If we set \( g(x) := (x - K)_+ \), we can for example choose \( T_n \) as the convolution of \( g \) with a sequence of Gaussian functions converging as \( n \to \infty \) to \( g \), i.e.
\[
T_n(x) := (g * \delta_n)(x) := \int_{\mathbb{R}} g(x - y) \delta_n(y) \, dy
\]
where \( \delta_n(y) := \frac{n}{\sqrt{2\pi}} e^{-\frac{y^2 n^2}{2}} \). Then all \( T_n \) are in \( C^\infty(\mathbb{R}) \) since \( \delta_n \in C^\infty(\mathbb{R}) \) and thus also in \( \mathcal{S}'(\mathbb{R}) \) and \( T_n(x) \) converges to \( g(x) \) for \( n \to \infty \) pointwise.

Hence from Watanabe’s Theorem 2.5, which conditions are fulfilled by the above observations, it follows that \( T_n(S^c ε_t) \in \tilde{\mathcal{D}}_{-\infty} \) has an asymptotic expansion
\[
T_n(S^c ε_t) \sim \Phi^*_n + \varepsilon \Phi^*_1 + \varepsilon^2 \Phi^*_2 + \cdots
\]
in \( \tilde{\mathcal{D}}_{-\infty} \) as \( \varepsilon \downarrow 0 \) (16) and \( \Phi^*_0, \Phi^*_1, \ldots \in \tilde{\mathcal{D}}_{-\infty} \) are determined by the Taylor expansion (see Theorem 2.5)
\[
T_n\left(f_0 + [\varepsilon f_1 + \varepsilon^2 f_2 + \cdots] \right) = \sum_k \frac{1}{k!} D^k T_n\left(f_0\right)[\varepsilon f_1 + \varepsilon^2 f_2 + \cdots]^k = \Phi_0 + \varepsilon \Phi_1 + \ldots.
\]
In particular, we have
\[
\Phi^*_0 \equiv T_n\left(f_0\right)
\]
\[
= \int_{\mathbb{R}} \frac{n}{\sqrt{2\pi}} e^{-\frac{x^2 n^2}{2}}
\]
\[
\times \left( y - K + \sum_{k=1}^{\infty} \sum_{i \in I_k, j \in J_k, |j|=1} \hat{V}^{j_1}_{i_1} \circ \cdots \circ \hat{V}^{j_{k-1}}_{i_{k-1}}(V^j_{i_k})(0,0) \cdot Q^j(t) \right) \, dy,
\]
\[
\Phi^*_1 \equiv f_1 \cdot \frac{\partial}{\partial x} T_n\left(f_0\right)
\]
\[
= \left( \sum_{k=1}^{\infty} \sum_{i \in I_k, j \in J_k, |j|=1} \hat{V}^{j_1}_{i_1} \circ \cdots \circ \hat{V}^{j_{k-1}}_{i_{k-1}}(V^j_{i_k})(0,0) \cdot Q^j(t) \right)
\]
\[
\times \left( \frac{\partial}{\partial x} \int_{\mathbb{R}} \frac{n}{\sqrt{2\pi}} e^{-\frac{x^2 n^2}{2}} \cdot (x - y)_+ \, dy \right)
\]
\[
\times \left( \sum_{k=1}^{\infty} \sum_{i \in I_k, j \in J_k, |j|=1} \hat{V}^{j_1}_{i_1} \circ \cdots \circ \hat{V}^{j_{k-1}}_{i_{k-1}}(V^j_{i_k})(0,0) \cdot Q^j(t) \right),
\]
\[
\Phi^*_2 \equiv f_2 \cdot \frac{\partial}{\partial x} T_n\left(f_0\right) + \frac{1}{2} f_1^2 \cdot \frac{\partial^2}{\partial x^2} T_n\left(f_0\right).
\]
Substituting for \( f_0, f_1, f_2 \) and \( T_n \) gives for any \( n \in \mathbb{N} \):
\[ \Phi_2^n = \left( \sum_{k=2}^{\infty} \sum_{i \in I_k, j \in J_k, i = j} \hat{\Phi}_{i_1}^{j_1} \cdots \hat{\Phi}_{i_{k-1}}^{j_{k-1}}(V_{i_k}^{j_k})(0,0) \cdot Q^i(t) \right)^2 \]
\[ \times \left( \frac{\partial}{\partial x} \int_{\mathbb{R}} \frac{n}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \cdot (x - y - K)_+ \, dy \right) \]
\[ \times \left( \sum_{k=1}^{\infty} \sum_{i \in I_1, j \in J_1, i = j} \hat{\Phi}_{i_1}^{j_1} \cdots \hat{\Phi}_{i_{k-1}}^{j_{k-1}}(V_{i_k}^{j_k})(0,0) \cdot Q^i(t) \right) \]
\[ \frac{1}{2} \left( \sum_{k=1}^{\infty} \sum_{i \in I_1, j \in J_1, i = j} \hat{\Phi}_{i_1}^{j_1} \cdots \hat{\Phi}_{i_{k-1}}^{j_{k-1}}(V_{i_k}^{j_k})(0,0) \cdot Q^i(t) \right) \]
\[ \times \left( \frac{\partial^2}{\partial x^2} \int_{\mathbb{R}} \frac{n}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \cdot (x - y - K)_+ \, dy \right) \]
\[ \times \left( \sum_{k=1}^{\infty} \sum_{i \in I_1, j \in J_1, i = j} \hat{\Phi}_{i_1}^{j_1} \cdots \hat{\Phi}_{i_{k-1}}^{j_{k-1}}(V_{i_k}^{j_k})(0,0) \cdot Q^i(t) \right). \]

The other coefficients can be computed analogously. An application of Proposition 2.6 now yields

\[ e^{-rt} E[T_n(S_t)] \sim e^{-rt} \left( E[\Phi_0^n] + \varepsilon E[\Phi_1^n] + \cdots \right) \tag{17} \]
as \( \varepsilon \downarrow 0 \), as an asymptotic series in the ordinary numerical sense. The existence and finiteness of the \( E[\phi^n] \) holds by Remark 2.7. Hence we have obtained an asymptotic expansion for the \( T_n \)-approximate return on an European call option \( T_n(S_t^\varepsilon) \) with strike price \( K \) when the volatility term is perturbed by a small term \( \varepsilon \).

An interesting question now is what happens when \( n \) tends to \( \infty \) in Eq. (17).

\[ |T_n(S_t^\varepsilon) - (S_t^\varepsilon - K)^+| = \left| \int_{\mathbb{R}} g(S_t^\varepsilon - y) \frac{n}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \, dy - (S_t^\varepsilon - K)^+ \right| \]
\[ \leq \left| \int_{-\infty}^{S_t^\varepsilon - K} (S_t^\varepsilon - y - K) \frac{n}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \, dy \right| \]
\[ \leq \left| \int_{-\infty}^{S_t^\varepsilon - K} |y| \frac{n}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \, dy \right| \leq \left| \frac{1}{n\sqrt{2\pi}} e^{-\frac{(S_t^\varepsilon - K)^2}{2}} \right| \]
\[ \leq \frac{1}{n\sqrt{2\pi}} \]
is finite for every \( n \in \mathbb{N} \) and vanishes as \( n \) tends to \( \infty \). Hence, \( |T_n(S_t^\varepsilon) - (S_t^\varepsilon - K)^+| \) is dominated by the function \( \frac{1}{\sqrt{2\pi}} \), and thus, we can apply the dominated convergence theorem to obtain for (15) that \( E[T_n(S_t^\varepsilon)] \to E[(S_t^\varepsilon - K)^+] \) for \( n \to \infty \). Moreover,
the leading coefficient on the right hand side $E[\Phi_0^n]$ converges, also by dominated convergence, to

$$E[\Phi_0^n] \xrightarrow{n \to \infty} E[(f_0 - K)^+]$$

since by definition $\Phi_0^n = T_n(f_0)$. The other coefficients $E[\phi_k^n]$ are all determined by a term $f_k \cdot \frac{\partial}{\partial x} T_n(f_0)$ and terms with derivatives of $T_n(f_0)$ of higher order than 1. Hence they converge, by dominated convergence, to

$$E[\Phi_k^n] \xrightarrow{n \to \infty} E[f_k 1_{(f_0 \geq K)}].$$

Since $S^\varepsilon_t$ has the asymptotic expansion

$$S^\varepsilon_t = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots + \varepsilon^n f_n + R_n(\varepsilon),$$

where $R_n(\varepsilon)$ is a rest term which vanishes for $n \to \infty$, one can easily see that

$$e^{-rt} E[(S^\varepsilon_t - K)^+] \sim E[\Phi_0^n] + \varepsilon E[\Phi_1^n] + \varepsilon^2 E[\Phi_2^n] + \cdots $$

the ‘downarrow’ indicates the limit for $n \to \infty$.

5. Borel-summability of the solution $S^\varepsilon_t$ in a special case

As in the former section we consider the general case where $\sigma^\varepsilon : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ has the asymptotic expansion

$$\sigma^\varepsilon(\omega, t, x) = \sum_{n=0}^{\infty} \sigma_n^\varepsilon(\omega, t, x) \cdot \varepsilon^n.$$
is of the order \( O(\varepsilon^3) \). From Section 4.2 we know that the solution \( S^\varepsilon_t \) of the perturbed Black–Scholes differential equation (6) has the asymptotic expansion (12)

\[
S^\varepsilon_t \sim f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots \\
\sim S_0 + \sum_{\lambda=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i \in I_k, j \in J_k, \|j\| = \lambda} \hat{V}^{j_i} \circ \cdots \circ \hat{V}^{j_{k-1}}(V^{j_k})(0, S_0) \cdot \mathcal{Q}^i(t).
\]

We suppose now that \( S^\varepsilon_t \) has an extension \( S_t(z) \) which is analytic in a sector \( C_R = \{ z : \Re z^{-1} < R^{-1} \} \) and has an asymptotic expansion

\[
S_t(z) = \sum_{n=0}^{N-1} f_n z^n + \tilde{R}_N(z)
\]

with a remainder satisfying the following bound

\[
|\tilde{R}_N(z)| \leq A \cdot \sigma^N N! \cdot \varepsilon^N
\]

uniformly in \( N \) and in \( z \in C_R \). Then, by Sokal’s Theorem 2.12, we obtain that

(a) \( b(t) = \sum f_n t^n / n! \) converges in the interval \( |t| < \frac{1}{\sigma} \).

(b) \( b(t) \) has an analytic continuation to the region \( S_\sigma = \{ t : \text{dist}(t, \mathbb{R}_+) < 1/\sigma \} \) satisfying the bound

\[
|b(t)| \leq K \exp(|t|/R),
\]

uniformly in every \( S_{\sigma'} \) with \( \sigma' > \sigma \).

(c) Furthermore, \( S_t \) can be represented by the absolutely convergent integral

\[
\frac{1}{z} \int_0^\infty e^{-\frac{t}{2}} b(t) \, dt
\]

for \( z \in C_R \).

Similarly we obtain that the expectation of the approximated payoff function \( E[T_n(S^\varepsilon_t)] \) is Borel-summable if \( E[T_n(S^\varepsilon_t)] \) has an extension which is analytic in a sector \( C_K = \{ z : \Re z^{-1} < K^{-1} \} \) and which asymptotic expansion has a remainder \( \tilde{R}_N(\varepsilon) \) that is bounded

\[
|\tilde{R}_N(z)| \leq C \delta^N N! z^N
\]

uniformly in \( N \) and in \( z \in C_K \).

The assumptions above, i.e. the condition that \( S_t \varepsilon \) and \( E[T_n(S^\varepsilon_t)] \) have analytic extensions in \( \varepsilon \) with uniformly bounded remainder, are rather technical and in most cases hard to verify. However we will give an example where Borel-summability of the solution can be shown.

**Example 5.1.** Suppose that the remainder \( R_3(\varepsilon) \) in Eq. (18) vanishes and that \( \sigma_0^\varepsilon \) and \( \sigma_1^\varepsilon \) are independent of \( x \), i.e.

\[
\sigma^\varepsilon(\omega, t, x) = \sigma_0^\varepsilon(\omega, t) + \sigma_1^\varepsilon(\omega, t) \cdot \varepsilon.
\]
This is obviously no longer the Black–Scholes setting since otherwise \( \sigma^0 \) would have to depend linearly on \( x \), i.e. on \( S_t \). In particular here the prices \( S_t \) can become negative, a fact which would require a separate discussion. We shall nevertheless comment on the asymptotics of this simple model since it allows a direct application of our methods. We have

\[
\frac{dS_t}{S_t} = r \cdot S_t dt + \left( \sigma^0(\omega, t) + \sigma^1(\omega, t) \cdot \varepsilon \right) dB_t.
\]

Then we obtain the following coefficients

\[
\hat{V}_0^0(t) = r \cdot x \frac{\partial}{\partial x},
\]

\[
\hat{V}_1^0(t) = \sigma^0(\omega, t) \frac{\partial}{\partial x},
\]

\[
\hat{V}_1^1(t) = \sigma^1(\omega, t) \frac{\partial}{\partial x}.
\]

We assume that \( \sigma^0, \sigma^1 \) satisfy the conditions in Assumption 4.1. Under these assumptions we can compute the \( f_n \)'s, which will consequently be convergent:

\[
f_0 = S_0 + \sum_{k=1}^{\infty} \sum_{(i \in I_k, j \in J_k, ||j||=0)} \hat{V}_{i_1}^{j_1} \circ \cdots \circ \hat{V}_{i_k}^{j_k-1}(V_{i_k}^{j_k})(0, S_0) \cdot Q^i(t)
\]

\[
= \sum_{k=1}^{\infty} \sum_{i \in I_k} \hat{V}_{i_1}^{i_1} \circ \cdots \circ \hat{V}_{i_k}^{i_k-1}(V_{i_k}^{i_k})(0, S_0) \cdot Q^i(t)
\]

\[
= \sum_{k=1}^{\infty} \left( \frac{t^{k-1}}{k!} S_0 + \sigma^0(0) t^{k-1} S_0 \cdot (Q_{(1,0,\ldots,0)}(t) + Q_{(0,1,0,\ldots,0)}(t) + \cdots + Q_{(0,\ldots,0,1)}(t)) \right) = S_0 + e^{rt} S_0 + \sum_{k=1}^{\infty} \sigma^0(0) t^{k-1} S_0 \cdot \left( \sum_{i=1}^{k} Q_{(i)}(t) \right),
\]

where \( Q_{(i)}(t) \) denoted \( Q_{(0,\ldots,0,1,0,\ldots,0)}(t) \) where the \( i \) th number is 1. Moreover we get

\[
f_1 = \sum_{k=1}^{\infty} \sum_{(i \in I_k, j \in J_k, ||j||=1)} \hat{V}_{i_1}^{j_1} \circ \cdots \circ \hat{V}_{i_k}^{j_k-1}(V_{i_k}^{j_k})(0, S_0) \cdot Q^i(t)
\]

\[
= \sum_{k=1}^{\infty} \sum_{i \in I_k} \hat{V}_{i_1}^{i_1} \circ \cdots \circ \hat{V}_{i_k}^{i_k-1}(V_{i_k}^{i_k})(0, S_0) \cdot Q^i(t)
\]

\[
= \sigma^1(0) \int_0^t \circ dB_s.
\]

All the other coefficients \( f_n, n \geq 2 \), vanish. Thus the expansion of the solution given by Eq. (12) can be simplified in the present case to the following expression

\[
S_t = S_0 \left( 1 + e^{rt} + \sum_{k=1}^{\infty} \sigma^0(0) t^{k-1} \cdot \sum_{i=1}^{k} Q_{(i)}(t) + \sigma^1(0) \int_0^t \circ dB_s \right).
\]
Acknowledgement

My advisor Professor Sergio Albeverio deserves my sincere gratitude for invaluable support, his continuous readiness for discussions, encouragement and for sharing with me his insights about financial mathematics and stochastic analysis.

References