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Densest lattice packings of 3-polytopes

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Abstract

Based on Minkowski's work on critical lattices of 3-dimensional convex bodies we present an efficient algorithm for computing the density of a densest lattice packing of an arbitrary 3-polytope. As an application we calculate densest lattice packings of all regular and Archimedean polytopes. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Throughout this paper, $\mathbb{R}^d$ denotes the d-dimensional Euclidean space with origin 0, Euclidean norm $\|\cdot\|$, inner product $\langle\cdot,\cdot\rangle$ and unit sphere S^{d-1}. $\mathcal{K}^d$ denotes the set of all convex bodies $K \subset \mathbb{R}^d$ with nonempty interior $\mathrm{int}(K)$ and $\mathcal{K}_0^d$ denotes the subset of $\mathcal{K}^d$ consisting of all bodies which are centrally symmetric with respect to the origin. For a set $M \subset \mathbb{R}^d$ we denote by $\mathrm{vol}(M)$ its volume with respect to its affine hull $\mathrm{aff}(M)$. Furthermore, $\mathrm{conv}(M)$, $\mathrm{lin}(M)$ denotes the convex hull, linear hull of M, respectively. The boundary of $K \in \mathcal{K}^d$ is denoted by $\mathrm{bd}(K)$.

By a lattice $\Lambda \subset \mathbb{R}^d$ with basis $B = \{b^1, \ldots, b^d\}$, where $b^1, \ldots, b^d \in \mathbb{R}^d$ are linearly independent, we understand the set

$$\Lambda = \{z_1 b^1 + \cdots + z_d b^d \colon z_1, \ldots, z_d \in \mathbb{Z}\} = B\mathbb{Z}^d.$$

The determinant $\det \Lambda$ of Λ is the volume of the parallelepiped spanned by $b^1, \ldots, b^d$, i.e., $\det \Lambda = |\det B|$. A lattice Λ is called a packing lattice for $K \in \mathcal{K}^d$ if $x + K$ and $y + K$ do not overlap for

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Table 1

Body	Author
Regular octahedron	Minkowski [20], 1904
Truncated cubes; for $0 < \lambda \leqslant 3$:	Whitworth [29], 1948
$\{x \in \mathbb{R}^3\colon\ \lvert x_i\rvert \leqslant 1,\ \lvert x_1+x_2+x_3\rvert \leqslant \lambda\}$	
$\{x \in \mathbb{R}^3\colon\ (x_1^2+x_2^2)^{1/2} + \lvert x_3\rvert \leqslant 1\}$	Whitworth [30], 1951
Frustrum of a sphere; for $0 < \lambda \leqslant 1$:	Chalk [5], 1950
$\{x \in \mathbb{R}^3\colon\ x_1^2+x_2^2+x_3^2 \leqslant 1,\ \lvert x_3\rvert \leqslant \lambda\}$	
Tetrahedron and cubeoctahedron	Hoylman [16], 1970

$x, y \in \Lambda$, $x \neq y$. There always exists a packing lattice $\Lambda^*(K)$ with minimal determinant. Such a lattice is called a densest packing lattice and the quantity

$$\delta^*(K) = \frac{\operatorname{vol}(K)}{\det \Lambda^*(K)}$$

is called the lattice packing density of K or the density of a densest lattice packing of K. There is a large amount of literature on packings. Some books, which give a good description of the background of our work, are [10,14,23,33] as well as the Diploma thesis [15]. For a more general survey on the theory of packings we refer to [11] and the references within.

The lattice packing problem for a general body in $\mathcal{K}^d$ is very hard. In fact, for $d \geqslant 4$ the only exact results are on space fillers (cf. [18,27]) for which $\delta^*(K) = 1$ and on the unit ball B^d where $\delta^*(B^d)$ is known for $d \leqslant 8$ (see, e.g., [7,34]). In contrast for a fixed body $K \in \mathcal{K}^2$ there are several techniques to solve the problem (see, e.g., [14, p. 241]) and there exists also an algorithm, due to Mount and Silverman [22], that determines $\delta^*(P)$ for a centrally symmetric n-gon in time $\mathrm{O}(n)$.

However, already in 3-space the situation is rather more complicated. Apart from the 3-dimensional space-fillers (for a classification see [14, p. 164]) and from cylinders based on a convex disk, in which case it can be shown that the problem is equivalent to the determination of the lattice packing density of the convex disk (cf. [23, p. 13]), densest lattice packings are only known for the bodies in $\mathbb{R}^3$, which are listed in Table 1.

It is worth to mention that the family of frustrums of a sphere includes the 3-ball as a limiting case, for which the packing density was determined already by Gauss [12].

All the computations of $\delta^*(K)$ for the bodies in Table 1 may be regarded as an application of a general method developed by Minkowski [20] which characterizes densest packing lattices of a 3-dimensional convex body by certain properties (see also [14, p. 340]). However, this method was considered as rather impractical and in 1964 Rogers [23] wrote "*Despite considerable theoretical advances in the Geometry of Numbers since Minkowski's time, the problem of determining the value of $\delta^*(K)$ for a given convex* 3-*dimensional body K remains a formidable task*". Indeed the only change in the list of known densest lattice packings since the publication of Rogers' book is the addition of the tetrahedron and cubeoctahedron. And even in 1990 Gruber mentioned the determination of $\delta^*(K)$ of a 3-dimensional convex body as one of the important open problems in Geometry of Numbers [13].

Here we use Minkowski's work as a starting point for the construction of a practicable algorithm to compute the packing density of an arbitrary polytope in $\mathbb{R}^3$. We proceed as follows. In Section 2 we adapt Minkowski's work to our purposes. Moreover we discuss in this section the principal applicability of his work in higher dimensions.

Having adapted Minkowski's method we still have to solve two problems to obtain a practicable algorithm. First we have to determine the local minima of a polynomial of degree three in three variables. Somewhat surprisingly there appears to be no general purpose algorithm of numerical analysis which suits our needs. Thus we develop an ad hoc method for our problem. This will be done in Section 4. Before doing this we deal in Section 3 with a more geometric problem. It turns out that by the methods of Section 2 we have essentially to look at every choice of 7 facets of the polytope. This leaves us with more or less $\binom{n}{7}$ cases for a polytope with n facets and in every step we have to solve a nonlinear minimization problem. This limits the applicability of the algorithm to polytopes with few facets. Thus we develop in Section 3 some methods to reduce the number of cases. While we do not give an exact worst case analysis it should be possible to reduce the number of cases to possibly as few as $\mathrm{O}(n^{3/2})$ in typical cases by showing that optimal packings must lead to feasible points of some related simple linear optimization problems. This will be done in Section 3. It should however be mentioned that in our program we do not fully exploit this reduction as an implementation would become rather complex. As an application of our work we present in Section 5 optimal lattice packings for all regular and Archimedean polytopes.

2. Necessary conditions for optimal lattices

In this section we state without proof Minkowski results. In fact, we have summarized the relevant facts in Theorem 2.1. We have included some more results, as we discuss at the end of the section, whether the algorithm could be extended to higher dimensions. Further we have changed and modernized the notation. We remark that a good part of Minkowski theory is exposed in [14,33], though in both books Theorem 2.1 is stated in a slightly weaker form.

For $K_i \in \mathcal{K}^d$, $\lambda_i \in \mathbb{R}$, $i = 1, 2$, we denote by $\lambda_1 K_1 + \lambda_2 K_2$ the set $\{\lambda_1 x^1 + \lambda_2 x^2\colon\ x^1 \in K_1,\ x^2 \in K_2\}$. Minkowski observed that

$$\Lambda \text{ is a packing lattice of } K \quad \Longleftrightarrow \quad \Lambda \text{ is a packing lattice of } \tfrac{1}{2}(K - K). \tag{2.1}$$

Since the difference body $\frac{1}{2}(K - K)$ belongs to the class $\mathcal{K}_0^d$, in the following we assume that all bodies are centrally symmetric.

A lattice Λ is called admissible for $K \in \mathcal{K}_0^d$, if $\operatorname{int} K \cap \Lambda = \{0\}$. It is well known and easy to see that Λ is admissible if and only if 2Λ is a packing lattice. The value

$$\Delta(K) = \min\{\det \Lambda\colon\ \Lambda \text{ admissible for } K\}$$

is called the critical determinant of K and an admissible lattice Λ satisfying $\Delta(K) = \det \Lambda$ is called a critical lattice. Thus $\delta^*(K) = \operatorname{vol}(K)/(2^d \Delta(K))$ and the problems of constructing a densest packing lattice and a critical lattice are equivalent.

While we naturally do not know a basis of a critical lattice beforehand, we shall see that we have a great amount of information on the behaviour of certain lattice points with fixed coordinates with respect to such a basis. For a basis $B = \{b^1, \ldots, b^d\}$ of $\mathbb{R}^d$ and a point $x \in \mathbb{R}^d$ we denote by $x_B = (x_1, \ldots, x_d)_B^{\mathrm{T}}$ the vector given by $x_B = \sum_{i=1}^n x_i b^i$.

The construction of critical lattices is based on the connection of lattice crosspolytopes and primitive vectors. As usual, a set of lattice vectors $b^1, \ldots, b^k$ of a lattice $\Lambda \subset \mathbb{R}^d$ is called *primitive* iff this set can be extended to a basis of the lattice Λ. Let $\Lambda \subset \mathbb{R}^d$ be a lattice and let $b^1, \ldots, b^k \in \Lambda$ be linearly independent. The crosspolytope $C = \mathrm{conv}\{\pm b^1, \ldots, \pm b^k\}$ is called a *lattice crosspolytope* iff $\Lambda \cap \mathrm{int}(C) = \{0\}$. If we even have $\Lambda \cap \mathrm{bd}(C) = \{\pm b^1, \ldots, \pm b^k\}$ then it is called a *free lattice crosspolytope*. Clearly, the convex hull of every primitive set $b^1, \ldots, b^k$ forms a free lattice crosspolytope and every free lattice crosspolytope is a lattice crosspolytope, while the converse is not true. A complete characterization of (free) lattice crosspolytopes of dimension up to three is given in Lemma 2.1.

Lemma 2.1 (Minkowski, 1904). *Let Λ be a lattice in $\mathbb{R}^3$.*
- (i) *For $k = 1$ the vertices of any lattice crosspolytope are primitive.*
- (ii) *For $k = 2$ the vertices of any free lattice crosspolytope are primitive, while for the non-free lattice crosspolytopes C there exists a basis B of the lattice such that $C = \mathrm{conv}\{(1,0,0)_B, (1,2,0)_B\}$.*
- (iii) *For $k = 3$ there are two types of free lattice crosspolytopes. One has primitive vertices* (crosspolytope of the first type)*, the other has the vertices $(1,0,0)_B, (0,1,0)_B, (1,1,2)_B$ for a basis B* (crosspolytope of the second type). *For every non-free lattice crosspolytope C there exists a basis B such that $C = \mathrm{conv}\{(1,0,0)_B, (0,1,0)_B, p\}$, where p is element of the set* $\{(0,1,2)_B, (1,1,2)_B, (1,2,2)_B, (1,1,3)_B, (1,2,3)_B, (2,2,3)_B, (1,2,4)_B, (2,3,4)_B\}$.

Using free lattice crosspolytopes we can identify critical lattices for every $K \in \mathcal{K}_0^3$. To this end we use the following abbreviation: For a basis $B = \{b^1, b^2, b^3\}$ of $\mathbb{R}^3$ let

$$\begin{aligned}
\mathcal{U}_B^1 &= \{(1,0,0)_B, (0,1,0)_B, (0,0,1)_B, (0,1,-1)_B, (-1,0,1)_B, (1,-1,0)_B\},\\
\mathcal{U}_B^2 &= \{(1,0,0)_B, (0,1,0)_B, (0,0,1)_B, (0,1,1)_B, (1,0,1)_B, (1,1,0)_B\},\\
\mathcal{U}_B^3 &= \{(1,0,0)_B, (0,1,0)_B, (0,0,1)_B, (0,1,1)_B, (1,0,1)_B, (1,1,0)_B, (1,1,1)_B\}.
\end{aligned} \tag{2.2}$$

Lemma 2.2 (Minkowski, 1904). *Let $K \in \mathcal{K}_0^3$. Then there exists a critical lattice Λ with basis B such that one of the following cases holds*:
(1) $\mathcal{U}_B^1 \subset \mathrm{bd}(K)$ *and K contains no lattice crosspolytope of the second type,*
(2) $\mathcal{U}_B^2 \subset \mathrm{bd}(K)$ *and* $(1,1,1)_B \notin K$,
(3) $\mathcal{U}_B^3 \subset \mathrm{bd}(K)$.

While the distinction between cases (2) and (3) may look artificial, we shall see that it leads to rather different cases in the actual computation of a critical lattice. For any lattice with basis B which satisfies a condition of the previous lemma we can easily check its admissibility:

Lemma 2.3 (Minkowski, 1904). *Let $K \in \mathcal{K}_0^3$ and Λ be a lattice with basis B. Then Λ is admissible, if one of the following conditions is satisfied*:
(1) $\mathcal{U}_B^1 \subset \mathrm{bd}(K)$ and $(-1,1,1)_B, (1,-1,1)_B, (1,1,-1)_B \notin \mathrm{int}\, K$,
(2) $\mathcal{U}_B^2 \subset \mathrm{bd}(K)$ and $(1,1,1)_B \notin \mathrm{int}\, K$.

Proof. The second statement follows immediately from Minkowski's characterizations of lattice crosspolytopes (cf. [33, Lemma 4.9, 2*]), whereas item (1) is contained in a more implicitly way in his paper. However, from the work of Minkowski we know (cf. [33, Lemma 4.9, 1^*]) that in the first case, i.e., $\mathcal{U}_B^1 \subset \mathrm{bd}(K)$, the lattice Λ is admissible if

$$\text{(a1)}\ (0, \pm 1, \pm 1)_B \notin K, \qquad \text{(a2)}\ (1, \pm 1, \pm 1)_B \notin K, \qquad \text{(a3)}\ (2, \pm 1, \pm 1)_B \notin K, \tag{2.3}$$

as well as all points arising from permutations of the coordinates of the above points are not contained in K. Therefore, in order to prove (1) we have to show that the points $(0,1,1)_B$, $(1,1,1)_B$, $(2,\pm 1,\pm 1)_B$ and the corresponding permutations are not contained in K. To this end let $f:\mathbb{R}^3 \to \mathbb{R}_{\geqslant 0}$ be the distance function of K, i.e., $f(x) = \min\{\lambda\colon\ \lambda \in \mathbb{R}_{\geqslant 0} \text{ and } x \in \lambda K\}$. Then $K = \{x \in \mathbb{R}^3\colon\ f(x) \leqslant 1\}$ and since $K \in \mathcal{K}_0^3$ the function f describes a norm on $\mathbb{R}^3$. Since $\mathcal{U}_B^1 \subset \mathrm{bd}(K)$ we find

$$\begin{aligned}
f\big((0,1,1)_B\big) &\geqslant f\big((0,0,2)_B\big) - f\big((0,-1,1)_B\big) = 1,\\
f\big((1,1,1)_B\big) &\geqslant f\big((3,0,0)_B\big) - f\big((1,-1,0)_B\big) - f\big((1,0,-1)_B\big) = 1,\\
f\big((2,1,-1)_B\big) &\geqslant f\big((2,0,0)_B\big) - f\big((0,-1,1)_B\big) = 1,\\
f\big((2,-1,1)_B\big) &\geqslant f\big((2,0,0)_B\big) - f\big((0,1,-1)_B\big) = 1,\\
f\big((2,-1,-1)_B\big) &\geqslant f\big((2,-2,0)_B\big) - f\big((0,-1,1)_B\big) = 1.
\end{aligned}$$

Obviously, the same inequalities hold for the points given by all permutations of the coordinates of the points of the left hand side and thus

$$f\big((2,1,1)_B\big) \geqslant f\big((2,2,0)_B\big) - f\big((0,1,-1)_B\big) \geqslant 1. \qquad \square$$

We call the sets $\mathcal{U}_B^j$, $j = 1,2,3$ (cf. (2.2)) *test sets* of the first, second or third kind, respectively.

Now suppose that $B = \{b^1, b^2, b^3\}$ is a basis of a critical lattice Λ of K and let $\mathcal{U}_B = \{u_B^1, \ldots, u_B^k\}$, $k = 6$ or $k = 7$, be one the three test sets such that $\mathcal{U}_B \subset \mathrm{bd}(K)$ (cf. Lemma 2.2). Let H_i be *any* supporting hyperplane of K containing u_B^i, $i = 1,\ldots,k$, and let

$$\mathcal{S}_{H_1,\ldots,H_k} = \big\{W \in \mathbb{R}^{3\times 3}\colon\ u_W^i \in H_i,\ 1 \leqslant i \leqslant k\big\}. \tag{2.4}$$

On this space we consider the function

$$f_{H_1,\ldots,H_k} : \mathcal{S}_{H_1,\ldots,H_k} \to \mathbb{R} \quad \text{given by} \quad f_{H_1,\ldots,H_k}(W) = \big|\det(W)\big|. \tag{2.5}$$

If we assume for a moment that $\mathcal{U}_B = \Lambda \cap \mathrm{bd}(K)$ then

$$B \text{ is a local minimum of } f_{H_1,\ldots,H_k}(W), \quad W \in \mathcal{S}_{H_1,\ldots,H_k}. \tag{2.6}$$

Otherwise there exists a $W \in \mathcal{S}_{H_1,\ldots,H_k}$ in a sufficiently small neighborhood of B such that $|\det(W)| < \det \Lambda$ and $K \cap (W\mathbb{Z}^d)\backslash\{0\} \subset \{u_W^1, \ldots, u_W^k\}$. However, this set of vectors is contained in the supporting hyperplanes $H_1,\ldots,H_k$ for any element $W \in \mathcal{S}(H_1,\ldots,H_k)$. Thus the lattice $W\mathbb{Z}^d$ is admissible and hence Λ cannot be critical.

In general the situation is much more complicated as $\mathcal{U}_B \cap \mathrm{bd}(K)$ may just be a proper subset of $\Lambda \cap \mathrm{bd}(K)$. But by a close examination of all possible cases Minkowski found the following theorem.

Theorem 2.1 (Minkowski, 1904). *Let $K \in \mathcal{K}_0^3$. Then there exists a critical lattice Λ of K with basis B such that* $\mathrm{bd}(K) \cap \Lambda$ *contains a test set $\mathcal{U}_B^j = \{u_B^1, \ldots, u_B^k\}$ for a $j \in \{1,2,3\}$, such that for any choice of supporting hyperplanes H_i of K containing u_B^i, $1 \leqslant i \leqslant k$, one of the following* 4 *cases holds*:

I. $j = 1$ *and* (2.6) *holds,*

II. $j = 2$ *and* (2.6) *holds,*

III. $j = 3$ *and* (2.6) *holds,*

IV. $j = 3$ *and there are scalars* $\lambda_1, \lambda_2, \lambda_6 > 0$, $\lambda_3 \in \mathbb{R}$, *such that for the outer normal vectors* v^1, v^2, v^3, v^6 *of the hyperplanes* H_1, H_2, H_3, H_6, *respectively, holds*: (a) $\lambda_1 v^1 + \lambda_2 v^2 + \lambda_3 v^3 = \lambda_6 v^6$, *and* (b) *the hyperplane* $\widetilde{H}_6$ *with outer normal vector* $\lambda_1 v^1 + \lambda_2 v^2$ *containing* u_B^6 *is a supporting hyperplane of* K *and* (2.6) *holds with* H_6 *replaced by* $\widetilde{H}_6$.

At this point we should remark that in case I Minkowski gave a somewhat stronger condition, but our form appears to be more suitable for automatic computation.

Of course, given an arbitrary convex body K we do not know how to exploit Theorem 2.1, but if we consider only polytopes then for the supporting hyperplanes H_i in Theorem 2.1 we may always choose the supporting hyperplanes of the facets of the polytope. As a polytope has only finitely many facets we obtain the following frame of an algorithm for the computation of a critical lattice of a polytope.

Algorithm 2.1. Let $P \in \mathcal{K}_0^3$.

- For each of the cases I–IV of Theorem 2.1 do
  - For every choice of k facets with supporting planes $H_1, \ldots, H_k$ of P ($k = 6$ in the first two cases and $k = 7$ in the latter ones) do
    - S1. Determine $\mathcal{S}_{H_1,\ldots,H_k}$.
    - S2. Find the local minima $\mathcal{M}_{H_1,\ldots,H_k}$ of $f_{H_1,\ldots,H_k}$ (cf. (2.6)).
    - S3. For each $M \in \mathcal{M}_{H_1,\ldots,H_k}$ check whether $M \cdot \mathbb{Z}^d$ is an admissible lattice, i.e., M satisfies the criterion (1) of Lemma 2.3 in the first case and criterion (2) in the remaining cases.
- Among all calculated admissible lattices find one with minimal determinant. The corresponding lattice is a critical lattice of P.

It turns out that at this point we are left with two problems. First there appears to be no general purpose algorithm to find all local minima of a function like $f_{H_1,\ldots,H_k}$ as we have to do in step S2. Moreover, a priori we cannot assume that the local minima of this function are isolated points. In general, they may form a manifold and we have the problem to parameterize such a manifold in order to carry out step S3. In Section 4 we shall show how one can overcome these problems. Another problem is just the number of steps of the algorithm. In a straightforward implementation, we have to consider every choice of 6 and 7 facets of the polytope and hence we have about $\binom{n}{7}$ steps. Of course, this limits the algorithm to polytopes having only few facets. Hence in order to get an efficient algorithm we have to reduce the number of steps. This can be done very effectively by some considerations given in Section 3.

We close the section with some remarks on the extension of the algorithm to higher dimensions. While Minkowski settles his work in 3-space, the ideas principally work in higher dimensions as well. In fact there has been an enumeration of lattice crosspolytopes in dimension 4 [2–4,17,21,31] and dimension 5 [28]. Beside this classification we have to determine the number of different test sets. For the cardinality of a test set we have the general natural lower bound of $d(d+1)/2$ given in [26]. There is no obvious upper bound for their cardinality, but the results in dimension two and three suggest the upper bound of one half of the maximal number of lattice points contained in the boundary of a lattice point free strictly convex set. Due to a result of Minkowski this number is bounded by $2^d - 1$. Finally, we have to take into account the additional lattice points (not contained in the test sets) lying in the boundary, which are responsible for the split of test sets of the third kind into two separate cases in dimension 3. Thus even in dimension 4 there should be a large number of cases which have to be considered separately and there has been no attempt to give an enumeration of these cases.

Further in each case we have to consider all possible choices of facets and even though we could apply the results of the next section it appears to be computationally difficult to choose at least 10 out of n facets. Moreover, even in dimension 4 we have to solve a minimization problem for polynomials of degree 4 with up to six variables. Again it appears to be not an easy task to find all local minima reliably and quickly. Thus without introduction of new ideas we have the impression that the algorithm is practically restricted to 3-space.

3. Necessary conditions for test sets

In the following let $P \in \mathcal{K}_0^3$ be a centrally symmetric polytope with n facets F_i and let $H_i = \mathrm{aff}(F_i)$, $1 \leqslant i \leqslant n$. By H_i^+ we denote the halfspace bounded by H_i containing P and let $H_i^- = \mathbb{R}^3 \backslash H_i^+ \cup H_i$. We always assume that we have a *lattice description* of the polytope, i.e., we know the face lattice of P specified by its Hasse diagram and the vertices and edges of P (cf. [25]). In particular, for each facet F_i we have a list $\mathcal{N}(F_i)$ of its edge-neighbors, i.e.,

$$\mathcal{N}(F_i) = \{F_j\colon \dim(F_i \cap F_j) = 1\}.$$

We remark that such a lattice description can be computed from the description $P = \bigcap_{i=1}^n H_i^+$ in time $\mathrm{O}(n \log n)$ [6]. Regarding the combinatorics of polytopes we refer to the books [19,32].

As pointed out in the last section one crucial point of Algorithm 2.1 is the number of choices of facets (or hyperplanes H_i) which have to be considered for each case of the algorithm. In this section we show how one can reduce this number. However, since it turns out that the most time consuming step of Algorithm 2.1 is step S2, we are also looking for ways to reduce the number of executions of step S2 as well. An exact worst case analysis of the complexity of the resulting algorithm appears to be not completely straightforward. But we show that for some rather natural classes of polytopes with n facets we can eliminate "most" possible choices of facets in time $\mathrm{O}(n^2)$ and we have to carry out only $\mathrm{O}(n^{3/2})$ times the steps S1–S3.

Of course, using the central symmetry of the polytope and the arbitrariness of the order of the basis of a lattice we can reduce the number of possible choices of hyperplanes H_i in the first two cases to $\frac{1}{12}\binom{n}{6}$ and to $2 \cdot \frac{1}{12}\binom{n}{7}$ choices in the remaining cases. This does not really help and so one could try to make further use of the symmetries of a given polytope, as Minkowski did in his study of the octahedron, where he managed to reduce the number of cases to 1 (!). Thus he did not need to carry out one step S2. However, for polytopes with little symmetry this would not be of great help and therefore we use a different approach.

With respect to step S3 of Algorithm 2.1 we are only interested in a selection of hyperplanes H_i, $1 \leqslant i \leqslant k$ (say), such that

$$\mathcal{S}_{H_1,\dots,H_k} \cap \{W \in \mathbb{R}^{3\times 3}\colon \mathcal{U}_W \subset \mathrm{bd}(P)\} \neq \emptyset, \tag{3.1}$$

where $\mathcal{U}_W$ is a test set of the first, second or third kind corresponding to the case we are studying. If $\mathcal{U}_W = \{u_W^1, \dots, u_W^k\}$ then (3.1) just says that the point u_W^i should not only lie in the hyperplane H_i, but in the facet F_i. Hence (3.1) can be reformulated as the condition that

$$\widetilde{\mathcal{S}}_{H_1,\dots,H_K} = \{W \in \mathbb{R}^{3\times 3}\colon\ u_W^i \in H_i \text{ and } u_W^i \in H_{i_j}^+ \text{ for all } F_{i_j} \in \mathcal{N}(F^i),\ 1 \leqslant i \leqslant k\} \neq \emptyset. \tag{3.2}$$

To check whether $\widetilde{\mathcal{S}}_{H_1,\dots,H_K}$ is empty is just an instance of a feasibility problem of Linear Programming and may be easily solved by any LP-solver (cf. e.g. [24]).

Since the vectors in a test set are linearly dependent the set $\widetilde{\mathcal{S}}_{H_1,\dots,H_K}$ will be empty for "most choices" of hyperplanes H_i. For instance: Let $\sigma = -1$ if we are dealing with case 1 and let $\sigma = 1$ otherwise. Then the vectors of a test set satisfy the relations

$$u_W^1 + \sigma u_W^2 = u_W^6, \qquad u_W^2 + \sigma u_W^3 = u_W^4, \qquad u_W^1 + \sigma u_W^3 = u_W^5. \tag{3.3}$$

If we have fixed a facet F_1 with hyperplane $H_1 = \{x \in \mathbb{R}^3\colon\ (a^1)^{\mathrm{T}} x = b_1\}$, say, for the vector u_W^1, then by the first relation of (3.3) we get $\sigma (a^1)^{\mathrm{T}} u_W^2 \leqslant 0$. Otherwise the sum of these two vectors would be separated by H_1 from the polytope, but the sum u_W^6 has to lie in a facet. This trivial observation already reduces the possible choices for H_2 to almost $n/2$ and obviously we can apply the same argumentation to the other hyperplanes. However, as we shall see at the end of the section, a detailed analysis of the linear dependencies will give a much better reduction for certain classes of polytopes.

Of course, the determination of the emptiness of (3.2) would reduce the number of executions of step S2, but we still have to consider all possible choices. Hence we have the problem to find a "fast" way to exclude "almost" all choices of facets (hyperplanes H_i) with $\widetilde{\mathcal{S}}_{H_1,\dots,H_K} = \emptyset$. To this end we make use of (3.3). Let

$$\mathcal{G} = \{(F_i, F_j, F_k)\colon\ (F_i + \sigma F_j) \cap F_k \neq \emptyset\}. \tag{3.4}$$

Obviously,

$$\widetilde{\mathcal{S}}_{H_{l_1},\dots,H_{l_k}} \neq \emptyset \quad\Longrightarrow\quad (F_{l_1}, F_{l_2}, F_{l_6}), (F_{l_2}, F_{l_3}, F_{l_4}), (F_{l_1}, F_{l_3}, F_{l_5}) \in \mathcal{G},$$

and to test whether a tuple (F_i, F_j, F_k) belongs to $\mathcal{G}$ is just a feasibility problem of Linear Programming, namely:

$$\begin{aligned}(F_i, F_j, F_k) \in \mathcal{G} \quad\Longleftrightarrow\quad & \{(w^1, w^2) \in \mathbb{R}^{3\times 2}\colon\ w^1 \in H_i,\ w^1 \in H_{i_l}^+ \text{ for } F_{i_l} \in \mathcal{N}(F_i),\\ & w^2 \in H_j,\ w^2 \in H_{j_l}^+ \text{ for } F_{j_l} \in \mathcal{N}(F_j),\\ & w^1 + \sigma w^2 \in H_k,\ w^1 + \sigma w^2 \in H_{k_l}^+ \text{ for } F_{k_l} \in \mathcal{N}(F_k)\} \neq \emptyset.\end{aligned} \tag{3.5}$$

The construction of the set $\mathcal{G}$ by enumeration clearly involves the consideration of $\mathrm{O}(n^3)$ possibilities. While in the numerical examples given in the last section the most time consuming part of the algorithm was the solution of (3.2), the series of examples at the end of the section indicates that for large polytopes the determination of $\mathcal{G}$ could be the hardest part. Therefore we show now how the geometry of the polytope can be used to do this effectively. To this end let

$$\mathcal{G}(F_i) = \{F_j\colon\ (F_i + \sigma F_j) \cap \mathrm{bd}(P) \neq \emptyset\}, \quad 1 \leqslant i \leqslant n.$$

With this notation we have the following lemma.

Lemma 3.1. *Let F_i be a facet of P and let $F_i^{\cup} = \bigcup_{F_j \in \mathcal{G}(F_i)} F_j$. The set $F_i^{\cup}$ is edge-connected, i.e., any two points $x, y \in F_i^{\cup}$ can be connected by a continuous path contained in $F_i^{\cup}$ without crossing a vertex of P.*

Proof. For a fixed point $v \in F_i$ let $\mathcal{I}(v) = \mathrm{bd}(P) \cap \sigma(\mathrm{bd}(P) - v) = \mathrm{bd}(P) \cap (\mathrm{bd}(P) - \sigma v)$. Then we have $\mathcal{I}(v) = \{y \in P \cap (P - \sigma v)\colon\ y + \mu v, \mu \in \mathbb{R}$, is a supporting line of $P \cap (P - \sigma v)\}$ and thus $\mathcal{I}(v)$ is the shadow boundary of $P \cap (P - \sigma v)$ in direction v. Hence two points of $\mathcal{I}(v)$ can be connected by a continuous path. Now let

$$\mathcal{I}(v)^{\cup} = \bigcup_{\{F_j\colon\ F_j \cap \mathcal{I}(v) \neq \emptyset\}} F_j.$$

Then, by construction, the set $\mathcal{I}(v)^{\cup}$ is edge-connected and so it is the union $\bigcup_{v \in F_i} \mathcal{I}(v)^{\cup} = F_i^{\cup}$. □

Once we have found one element of the set $\mathcal{G}(F_i)$, the lemma says that we can determine the other elements by recursively checking the neighbors of the elements which were already found and since

$$(F_i, F_j, F_k) \in \mathcal{G} \quad \Longrightarrow \quad F_j \in \mathcal{G}(F_i), \tag{3.6}$$

we may use Lemma 3.1 in order to construct the set $\mathcal{G}$. In order to present an algorithm for computing $\mathcal{G}$ as well as the sets $\mathcal{G}(F_i)$ we need one more notation: For $l, u \in \mathbb{R}^3$, $l \leqslant u$, let $B(l, u) = \{x \in \mathbb{R}^3\colon\ l \leqslant x \leqslant u\}$ be the box parallel to the coordinate axes with lower vertex l and upper vertex u and for a facet F_i let $B(l^i, u^i)$ be the minimal box containing F_i, which can be computed from the coordinates of the vertices of F_i in time $\mathrm{O}(n)$. A necessary condition for $(F_i + \sigma F_j) \cap F_k \neq \emptyset$ is given by

$$\bigl(B\bigl(l^i, u^i\bigr) + \sigma B\bigl(l^j, u^j\bigr)\bigr) \cap B\bigl(l^k, u^k\bigr) = \begin{cases} B(l^i + l^j, u^i + u^j) \cap B(l^k, u^k) \neq \emptyset, & \sigma = 1, \\ B(l^i - u^j, u^i - l^j) \cap B(l^k, u^k) \neq \emptyset, & \sigma = -1, \end{cases} \tag{3.7}$$

which can easily be checked since

$$B(l, u) \cap B\bigl(\tilde{l}, \tilde{u}\bigr) \neq \emptyset \quad \Longleftrightarrow \quad l \leqslant \tilde{u} \text{ and } \tilde{l} \leqslant u. \tag{3.8}$$

Therefore, if we want to find for a given pair F_i, F_j all facets F_k with $(F_i, F_j, F_k) \in \mathcal{G}$ we just have to consider the facets corresponding to boxes $B(l^k, u^k)$ satisfying (3.7). For polytopes with many facets "most" facets will be "far away" from $F_i + F_j$ and thus (3.7) won't be fulfilled for "most" facets.

The next lemma says how we can find one facet lying in a set $\mathcal{G}(F_i)$ with the help of these boxes.

Lemma 3.2. *Let $\nu(P) = \max\{\#\mathcal{N}(F_i)\colon\ 1 \leqslant i \leqslant n\}$, let $\beta(P)$ be the maximal number of boxes $B(l^k, u^k)$ intersecting a box of the form $B(l^i + u^i) + \sigma B(l^j + u^j)$, $1 \leqslant i, j \leqslant n$, and let $\eta(P)$ be the maximal number of facets of P having a nonempty intersection with a fixed hyperplane containing the origin. Then a facet F_j lying in a set $\mathcal{G}(F_i)$ can be found in time $\mathrm{O}(n + \eta(P)\log^2(n) + \eta(P)\beta(P)\nu(P))$.*

Proof. Let $v \in F_i$ and let H be any hyperplane containing v and the origin. Then there exists a facet $F_j \in \mathcal{G}(F_i)$ having a nonempty intersection with $H \cap P$. Using the edges of P we can easily determine all facets $\{F_{j_1}, \ldots, F_{j_l}\}$, $j_l \leqslant \eta(P)$, with this property. Since the polytope has $\mathrm{O}(n)$ edges this can be carried out in time $\mathrm{O}(n)$. For each j_i the boxes $B(l^k, u^k)$, $k \leqslant \beta(P)$, intersecting $(B(l^i, u^i) + \sigma B(l^{j_i}, u^{j_i}))$ can be determined in time $\mathrm{O}(\log^2(n) + \beta(P))$ by well-known methods from computational geometry about range searching (cf. [1]). For each possible choice F_i, F_{j_i}, F_k we use (3.5) to verify whether $(F_i, F_{j_i}, F_k) \in \mathcal{G}$ and thus $F_{j_i} \in \mathcal{G}(F_i)$. Now each (3.5) is a feasibility problem of Linear Programming with $\mathrm{O}(\nu(P))$ constraints in dimension 6 and this can be solved with $\mathrm{O}(\nu(P))$ arithmetic operations (cf. [24, p. 199]). □

Using the last lemma we have the following algorithm for computing the set $\mathcal{G}$ and the sets $\mathcal{G}(F_i)$, $1 \leqslant i \leqslant n$.

Algorithm 3.1 (Computing $\mathcal{G}(F_i)$ and $\mathcal{G}$).
Input: A polytope $P \in \mathcal{K}_0^3$ given by the supporting hyperplanes H_i, $1 \leqslant i \leqslant n$, of its facets F_i, as well as a lattice description of P.
Output: $\mathcal{G}$ and $\mathcal{G}(F_i)$, $1 \leqslant i \leqslant n$.
- Let $U = \emptyset$ and determine the boxes $B(l^i, u^i)$, $1 \leqslant i \leqslant n$.
- For each facet F_i do
  - G1. Find a facet $F_j \in \mathcal{G}(F_i)$. Let $U_i = \emptyset$ and $N = \{F_j\}$.
  - G2. While $N \neq \emptyset$ do
    - G3. For $F_{j_1} \in N$ determine all facets $F_{k_1}, \ldots, F_{k_p}$ such that $(B(l^i, u^i) + \sigma B(l^{j_1}, u^{j_1})) \cap B(l^{k_l}, u^{k_l}) \neq \emptyset$ (cf. (3.7), (3.8)).
    - G4. Use (3.5) to determine all $(F_i, F_{j_1}, F_{k_l}) \in \mathcal{G}$, $k_l \in \{k_1, \ldots, k_p\}$, and add them to the set U.
    - G5. Let $U_i = U_i \cup \{F_{j_1}\}$ and $N = (N \cup \mathcal{N}(F_{j_1})) \backslash U_i$, if there exists a $(F_i, F_{j_1}, F_{k_l}) \in \mathcal{G}$, otherwise let $N = N \backslash F_{j_1}$.

Lemma 3.3. *Let $\gamma(P) = \max\{\#\mathcal{G}(F_i)\colon\ 1 \leqslant i \leqslant n\}$, and let $\nu(P)$, $\eta(P)$, $\beta(P)$ as in Lemma* 3.2. *Then Algorithm* 3.1 *determines $\mathcal{G}$ and $\mathcal{G}(F_i)$, $1 \leqslant i \leqslant n$, in time*

$$\mathrm{O}\big(n^2 + n\log^2(n)\big(\eta(P) + \gamma(P)\nu(P)\beta(P)\big) + n\beta(P)\nu(P)\big(\eta(P) + \gamma(P)\nu(P)\big)\big).$$

Proof. By Lemma 3.1 the algorithm finds all $(F_i, F_j, F_k) \in \mathcal{G}$ and at the end we have $U = \mathcal{G}$. Furthermore, at the end of each loop G2 the set U_i coincides with $\mathcal{G}(F_i)$.

To find a first facet $F_j \in \mathcal{G}(F_i)$ we need by Lemma 3.2 at most $\mathrm{O}(n + \eta(P)\log^2(n) + \eta(P)\beta(P)\nu(P))$ operations. For the estimation of the steps G3 and G4 we can proceed as in the proof of Lemma 3.2. The boxes in step G3 can be found in time $\mathrm{O}(\log^2(n) + \beta(P))$ and in step G4 we have to solve at most $\beta(P)$ feasibility problems (cf. (3.5)) with at most $\nu(P)$ constraints in dimension 6 which can be done with $\mathrm{O}(\nu(P))$ arithmetic operations. Finally, we observe that for each facet F_i the loop G2 is executed at most $\gamma(P)\nu(P)$ times. □

Remarks.
- (i) The bound on the running time given in the last lemma is useless for a worst-case analysis, because there exist polytopes such that each of the numbers $\eta(P)$, $\beta(P)$, $\gamma(P)$, $\nu(P)$ is of order $\mathrm{O}(n)$. In this case Lemma 3.3 would give an $\mathrm{O}(n^5)$-algorithm and of course, one can determine the sets $\mathcal{G}$, $\mathcal{G}(F_i)$ by a trivial $\mathrm{O}(n^4)$-algorithm. Nevertheless we shall see in Theorem 3.1 that this lemma gives an $\mathrm{O}(n^2)$ bound for a rather natural class of polytopes.
- (ii) As we have $F_j \in \mathcal{G}(F_i) \Leftrightarrow F_i \in \mathcal{G}(F_j)$ we may use the facets belonging to a set $\mathcal{G}(F_i)$ as starting facets in step G1. It is not hard to see that we can determine all starting facets except the first one in this way. However, for simplification we do not exploit this fact.

Altogether the previous observations lead to the following refinement of Algorithm 2.1.

Algorithm 3.2.
Input: A polytope $P \in \mathcal{K}_0^3$ given by the supporting hyperplanes H_i, $1 \leqslant i \leqslant n$, of its facets F_i, as well as a lattice description of P.
Output: A densest packing lattice.

- For each of the cases I–IV of Theorem 2.1 do
  - I1. Compute the sets $\mathcal{G}(F_i)$ and $\mathcal{G}$ with Algorithm 3.1.
    - – For three facets $F_{l_1}, F_{l_2}, F_{l_3}$ satisfying $F_{l_2} \in \mathcal{G}(F_{l_1})$ and $F_{l_3} \in \mathcal{G}(F_{l_1}) \cap \mathcal{G}(F_{l_2})$ do
      - – For every choice of facets F_{l_i}, $4 \leqslant i \leqslant k$, with $(F_{l_1}, F_{l_2}, F_{l_6}), (F_{l_2}, F_{l_3}, F_{l_4}), (F_{l_1}, F_{l_3}, F_{l_5}) \in \mathcal{G}$ do
        - S0. If $\widetilde{\mathcal{S}}_{H_{l_1},\ldots,H_{l_k}} \neq \emptyset$ (cf. (3.2)) do
          - S1. Determine $\mathcal{S}_{H_{l_1},\ldots,H_{l_k}}$.
          - S2. Find the local minima $\mathcal{M}_{H_{l_1},\ldots,H_{l_k}}$ of $f_{H_{l_1},\ldots,H_{l_k}}$.
          - S3. For each $M \in \mathcal{M}_{H_{l_1},\ldots,H_{l_k}}$ check whether $M \cdot \mathbb{Z}^d$ is an admissible lattice, i.e., M satisfies the criterion (1) of Lemma 2.3 in the first case and criterion (2) in the remaining cases.
- Among all calculated admissible lattices find one with minimal determinant. The corresponding lattice is a critical lattice of P.

It seems to be a nontrivial problem to give a "nontrivial" worst case analysis of the algorithm for an arbitrary polytope. In the following we want to demonstrate, by a rather natural series of polytopes, the improvement of Algorithm 3.2 compared with the brute force method (see Algorithm 2.1) which involves the examination of $\Omega(n^7)$ steps S0–S3.

For a facet F of a polytope P we denote by $R(F)$ its circumradius and by $r(F)$ its inradius with respect to its affine hull. For a real $c > 0$ we say that the facets of P are of c-uniform shape, if

$$\min\{r(F):\ F \text{ facet of } P\} \geqslant c \cdot \max\{R(F):\ F \text{ facet of } P\}.$$

Theorem 3.1. *Let $\{P_m\}_{m\in\mathbb{N}}$ be a series of polytopes $P_m \in \mathcal{K}_0^3$ such that*
(i) *all P_m have facets of c-uniform shape for some fixed c, and*
(ii) *$\{P^m\}$ converges to the* 3*-dimensional unit ball with respect to the Hausdorff metric.*
Let f_m be the number of facets of the polytope P_m. Then the number of all choices of facets which will be examined by Algorithm 3.2 *in the steps* S0–S3 *is of size* $\mathrm{O}(f_m^{3/2})$ *and these possible choices of facets can be determined in time* $\mathrm{O}(f_m^2)$.

Proof. We project $\mathrm{bd}(P_m)$ by the radial projection ρ onto the unit sphere S^2 and we carry out all calculations on S^2. It can easily be checked that our asymptotical estimates remain correct. The facets of P_m will be denoted by F_i^m, $1 \leqslant i \leqslant f_m$, and in the following we shall denote by c_i certain positive constants. First we observe that the uniformity of our sequence implies that for the spherical diameter $d(\rho(F_i^m))$ and the spherical area $A(\rho(F_i^m))$ of the facets holds

$$\frac{c_1}{\sqrt{f_m}} \leqslant d\big(\rho\big(F_i^m\big)\big) \leqslant \frac{c_2}{\sqrt{f_m}}, \qquad \frac{c_3}{f_m} \leqslant A\big(\rho\big(F_i^m\big)\big) \leqslant \frac{c_4}{f_m}. \tag{3.9}$$

Next we note that for a point $x \in S^2$ we have

$$\big\{y \in S^2:\ y + x \in S^2\big\} = \big\{y \in S^2:\ \langle x, y\rangle = -1/2\big\}. \tag{3.10}$$

Now let F_i^m, F_j^m be two facets of P_m such that $(F_i^m + \sigma F_j^m) \cap \mathrm{bd}(P_m) \neq \emptyset$ and let $x \in F_i^m$. We find by (3.9), (3.10) that

$$F_j^m \subset \left\{ y \in \mathrm{bd}(P_m)\colon -\frac{\sigma}{2} - \frac{c_5}{\sqrt{f_m}} \leqslant \langle \rho(y), \rho(x) \rangle \leqslant -\frac{\sigma}{2} + \frac{c_5}{\sqrt{f_m}} \right\}. \tag{3.11}$$

As the spherical area of the set on the right is bounded by $c_6/\sqrt{f_m}$ we find by (3.9) that for fixed F_i^m there are at most $\mathrm{O}(\sqrt{f_m})$ faces F_j^m such that $(F_i^m + \sigma F_j^m) \cap \mathrm{bd}(P_m) \neq \emptyset$. For two faces F_i^m and F_j^m we have $R(F_i^m + \sigma F_j^m) \leqslant R(F_i^m) + R(F_j^m) \leqslant c_7/\sqrt{f_m}$. Proceeding as before we see that there can be at most $\mathrm{O}(1)$ faces of P_m which intersect $(F_i^m + \sigma F_j^m)$. Altogether we have found that

$$\#\mathcal{G} = \mathrm{O}\big(f_m^{3/2}\big) \quad \text{and} \quad \#\mathcal{G}\big(F_i^m\big) = \mathrm{O}\big(\sqrt{f_m}\,\big). \tag{3.12}$$

Next we ask for the efficiency of Algorithm 3.1 to determine these sets. Let $B(l^{m,i}, u^{m,i})$ be a minimal box containing F_i^m. By the above estimate for the circumradius of two facets F_i^m, F_j^m we have $|l_k^{m,i} + l_k^{m,j} - u_k^{m,i} - u_k^{m,j}| = \mathrm{O}(1/\sqrt{f_m})$ for each coordinate k. Again using the area bound (3.9) we see that there are at most $\mathrm{O}(1)$ facets F_s^m with $u_1^{m,s} \geqslant l_1^{m,i} + l_1^{m,j}$ (or $u_1^{m,s} \geqslant l_1^{m,i} - u_1^{m,j}$) and $l_1^{m,s} \leqslant u_1^{m,i} + u_1^{m,j}$ (or $l_1^{m,s} \leqslant u_1^{m,i} - l_1^{m,j}$) (cf. (3.8)). Hence the number $\beta(P)$ of Lemma 3.3 is of order $\mathrm{O}(1)$. Next we note that on account of our assumptions the maximal number $\nu(P)$ of neighbors of a given facet is constant, if f_m is large enough. Moreover, by (3.11) we also see that the number $\eta(P)$ is of order $\sqrt{f_m}$ and therefore Algorithm 3.1 determines the set $\mathcal{G}$ as well as the sets $\mathcal{G}(F_i^m)$ in time $\mathrm{O}(f_m^2)$ (cf. Lemma 3.3).

Since we have already proven that $\#\mathcal{G}(F_i^m) = \mathrm{O}(\sqrt{f_m}\,)$ and that for two given facets $F_{l_1}^m, F_{l_2}^m$ there are only $\mathrm{O}(1)$ many facets $F_{l_k}^m$ with $(F_{l_1}^m, F_{l_2}^m, F_{l_k}^m) \in \mathcal{G}$, it remains to show that $\#(\mathcal{G}(F_{l_1}^m) \cap \mathcal{G}(F_{l_2}^m)) = \mathrm{O}(1)$ for $F_{l_2}^m \in \mathcal{G}(F_{l_1}^m)$ (cf. Algorithm 3.2). Let $x^i \in \mathcal{G}(F_{l_i}^m)$. Then we have (cf. (3.11))

$$-\frac{\sigma}{2} - \frac{c_5}{\sqrt{f_m}} \leqslant \langle \rho(x^1), \rho(x^2) \rangle \leqslant -\frac{\sigma}{2} + \frac{c_5}{\sqrt{f_m}}, \quad \text{and}$$

$$\mathcal{G}\big(F_{l_1}^m\big) \cap \mathcal{G}\big(F_{l_2}^m\big) \subset \left\{ y \in \mathrm{bd}(P_m)\colon -\frac{\sigma}{2} - \frac{c_5}{\sqrt{f_m}} \leqslant \langle \rho(y), \rho(x^i) \rangle \leqslant -\frac{\sigma}{2} + \frac{c_5}{\sqrt{f_m}},\ i = 1, 2 \right\}.$$

The radial projection of the latter set is a spherical parallelogram with "edge length" $\mathrm{O}(1/\sqrt{f_m}\,)$. Hence its spherical area is $\mathrm{O}(1/f_m)$ and together with (3.9) this shows $\#(\mathcal{G}(F_{l_1}^m) \cap \mathcal{G}(F_{l_2}^m)) = \mathrm{O}(1)$. □

Remark. Obviously, in the cases II–IV the sets $\mathcal{G}(F_i)$, $\mathcal{G}$ coincide. Furthermore, since $(F_i - F_j) \cap F_k = (F_i + (-F_j)) \cap F_k$ and $-F_j$ is a facet of P it suffices to determine $\mathcal{G}(F_i)$, $\mathcal{G}$ only for one case.

4. Determination of the local minima

In this section we concentrate on the steps S1–S3 of Algorithm 3.2. To this end let $P \in \mathcal{K}_0^3$ be a centrally symmetric polytope with n facets given by the inequalities

$$P = \big\{ x \in \mathbb{R}^3\colon\ (a^i)^{\mathrm{T}} x \leqslant b_i,\ 1 \leqslant i \leqslant n \big\}, \tag{4.1}$$

where $a^i \in \mathbb{R}^3$ and $b_i \in \mathbb{R}_{>0}$. Let $H_i = \{x \in \mathbb{R}^3\colon\ (a^i)^{\mathrm{T}} x = b_i\}$ be the supporting hyperplane of the facet F_i. Since the cases I–IV of the algorithm can be treated " more or less" in the same way, in the following we shall focus only on case I and at the end of the section we shall discuss the necessary changes and adoptions for the other cases.

For six hyperplanes H_{i_j}, $1 \leqslant j \leqslant 6$, the set $\mathcal{S}_{H_{i_1},\dots,H_{i_6}}$ (see step S1 and (2.4)) is given by all $W = (w^1, w^2, w^3) \in \mathbb{R}^{3\times 3}$ satisfying

$$\begin{pmatrix} (a^{i_1})^{\mathrm{T}} & 0 & 0 \\ 0 & (a^{i_2})^{\mathrm{T}} & 0 \\ 0 & 0 & (a^{i_3})^{\mathrm{T}} \\ 0 & (a^{i_4})^{\mathrm{T}} & -(a^{i_4})^{\mathrm{T}} \\ -(a^{i_5})^{\mathrm{T}} & 0 & (a^{i_5})^{\mathrm{T}} \\ (a^{i_6})^{\mathrm{T}} & -(a^{i_6})^{\mathrm{T}} & 0 \end{pmatrix} \begin{pmatrix} w^1 \\ w^2 \\ w^3 \end{pmatrix} = \begin{pmatrix} b_{i_1} \\ b_{i_2} \\ b_{i_3} \\ b_{i_4} \\ b_{i_5} \\ b_{i_6} \end{pmatrix}. \tag{4.2}$$

We denote the matrix on the left $A_{i_1,\dots,i_6} \in \mathbb{R}^{6\times 9}$ and with suitable matrices $C_{i_1,\dots,i_6}$, $M^j_{i_1,\dots,i_6}$, $1 \leqslant j \leqslant 9 - \operatorname{rank}(A_{i_1,\dots,i_6})$ we may write

$$\mathcal{S}_{H_{i_1},\dots,H_{i_6}} = \left\{ W \in \mathbb{R}^{3\times 3}\colon\ W = C_{i_1,\dots,i_6} + \sum_{j=1}^{9-\operatorname{rank}(A_{i_1,\dots,i_6})} \lambda_j \cdot M^j_{i_1,\dots,i_6},\ \lambda_j \in \mathbb{R} \right\}. \tag{4.3}$$

We remark that the matrices $C_{i_1,\dots,i_6}$, $M^j_{i_1,\dots,i_6}$ can easily be determined by any program for solving systems of linear equations.

Lemma 4.1. *If* $\operatorname{rank}(A_{i_1,\dots,i_6}) < 6$ *then the function* $f_{H_{i_1},\dots,H_{i_6}}(W)$ (*cf.* (2.5)) *has no local minimum* $W \in \mathcal{S}_{H_{i_1},\dots,H_{i_6}}$ *with* $f_{H_{i_1},\dots,H_{i_6}}(W) > 0$.

Proof. Since the vectors a^{j_i} correspond to facets defining hyperplanes the vectors $(a^{i_1}, 0, 0)^{\mathrm{T}}$, $(0, a^{i_2}, 0)^{\mathrm{T}}$, $(a^{i_6}, -a^{i_6}, 0)^{\mathrm{T}} \in \mathbb{R}^9$ are linearly independent. Otherwise we can assume that $a^{i_6} = a^{i_1} = \pm a^{i_2}$ and we get $(a^{i_6})^{\mathrm{T}}(w^1 - w^2) = b_{i_1} \mp b_{i_2} \neq b_{i_1} = b_{i_6}$. Hence the vectors $\{a^{i_3}, a^{i_4}, a^{i_5}\}$ are linearly dependent, because otherwise $\operatorname{rank}(A_{i_1,\dots,i_6}) = 6$. In the same way we find that $\{a^{i_1}, a^{i_5}, a^{i_6}\}$ and $\{a^{i_2}, a^{i_4}, a^{i_6}\}$ are linearly dependent. Thus we can find three nontrivial vectors $v^1, v^2, v^3 \in \mathbb{R}^3\backslash\{0\}$ such that

$$v^1 \in \operatorname{lin}\{a^{i_3}, a^{i_4}, a^{i_5}\}^{\perp}, \qquad v^2 \in \operatorname{lin}\{a^{i_1}, a^{i_5}, a^{i_6}\}^{\perp}, \qquad v^3 \in \operatorname{lin}\{a^{i_2}, a^{i_4}, a^{i_6}\}^{\perp},$$

where $\operatorname{lin} U^{\perp}$ denotes the orthogonal complement of $\operatorname{lin} U$. Now let $W = (w^1, w^2, w^3) \in \mathcal{S}_{H_{i_1},\dots,H_{i_6}}$ with $\det(W) \neq 0$, and let us assume that W is a local minimum. By the choice of v^i we have $(w^1 + \lambda v^1, w^2 + \mu v^2, w^3 + \nu v^3) \in \mathcal{S}_{H_{i_1},\dots,H_{i_6}}$ for all $\lambda, \mu, \nu \in \mathbb{R}$ and therefore let

$$\begin{aligned} g(\lambda, \mu, \nu) &= \det\bigl(w^1 + \lambda v^1, w^2 + \mu v^2, w^3 + \nu v^3\bigr) \\ &= \det(W) + \lambda \det\bigl(v^1, w^2, w^3\bigr) + \mu \det\bigl(w^1, v^2, w^3\bigr) + \nu \det\bigl(w^1, w^2, v^3\bigr) \\ &\quad + \lambda\mu \det\bigl(v^1, v^2, w^3\bigr) + \mu\nu \det\bigl(w^1, v^2, v^3\bigr) + \lambda\nu \det\bigl(v^1, w^2, v^3\bigr) \\ &\quad + \lambda\mu\nu \det\bigl(v^1, v^2, v^3\bigr). \end{aligned}$$

It is easy to see that this function has a local extremum at $(0, 0, 0)$ if and only if it is constant, i.e., $g(\lambda, \mu, \nu) = \det(W)$. In particular we have $\det(v^1, w^2, w^3) = \det(w^1, v^2, w^3) = \det(v^1, v^2, w^3) = 0$. Since w^1, w^2, w^3 are linearly independent this implies that v^1 or v^2 belongs to $\operatorname{lin}\{w^3\}$, and in the same way we find that v^2 or v^3 lies in $\operatorname{lin}\{w^1\}$ and v^1 or v^3 belongs to $\operatorname{lin}\{w^2\}$. However, since $v^i \in \mathbb{R}^3\backslash\{0\}$ this yields the contradiction $|\det(v^1, v^2, v^3)| = |\det(W)|$. $\square$

Since we are only interested in local minima W of the functions $f_{H_{i_1},\dots,H_{i_6}}$ with $f_{H_{i_1},\dots,H_{i_6}}(W) > 0$, in the following we assume that $\operatorname{rank}(A_{i_1,\dots,i_6}) = 6$. Instead of searching for the local minima of

$f_{H_{i_1},\dots,H_{i_6}}(W) = |\det(W)|$ it is more practical to look for all local extrema of the function $\det(W)$, which can be parameterized by $\mathfrak{p}_{i_1,\dots,i_6} : \mathbb{R}^3 \to \mathbb{R}$ given by (see (4.3))

$$\mathfrak{p}_{i_1,\dots,i_6}(x, y, z) = \det\bigl(C_{i_1,\dots,i_6} + x \cdot M^1_{i_1,\dots,i_6} + y \cdot M^2_{i_1,\dots,i_6} + z \cdot M^3_{i_1,\dots,i_6}\bigr). \tag{4.4}$$

$\mathfrak{p}_{i_1,\dots,i_6}$ is a polynomial in 3 variables (x, y, z), where each monomial has total degree 3 at most. In what follows we are mainly interested in general properties of such a polynomial, and therefore we shall write $\mathfrak{p}$ instead of $\mathfrak{p}_{i_1,\dots,i_6}$. So $\mathfrak{p}$ can be written as

$$\mathfrak{p}(x, y, z) = \sum_{0\leqslant i,j,k\leqslant 3,\ i+j+k\leqslant 3} \alpha_{i,j,k} \cdot x^i y^j z^k,$$

for some scalars $\alpha_{i,j,k} \in \mathbb{R}$. The canonical first step in order to find the local extrema is to calculate the set $\mathcal{V}(\nabla\mathfrak{p})$ where the gradient $\nabla\mathfrak{p}$ vanishes, i.e.,

$$\mathcal{V}(\nabla\mathfrak{p}) = \bigl\{(x, y, z) \in \mathbb{R}^3\colon\ \nabla\mathfrak{p}(x, y, z) = 0\bigr\}.$$

Thus we are interested in the common roots of the partial derivatives

$$\begin{aligned}
\mathfrak{p}_1 &= \frac{\partial\mathfrak{p}}{\partial x} = \chi_1 x^2 + \mathfrak{l}_1(y, z) \cdot x + \mathfrak{q}_1(y, z),\\
\mathfrak{p}_2 &= \frac{\partial\mathfrak{p}}{\partial y} = \chi_2 x^2 + \mathfrak{l}_2(y, z) \cdot x + \mathfrak{q}_2(y, z),\\
\mathfrak{p}_3 &= \frac{\partial\mathfrak{p}}{\partial z} = \chi_3 x^2 + \mathfrak{l}_3(y, z) \cdot x + \mathfrak{q}_3(y, z),
\end{aligned} \tag{4.5}$$

where $\chi_i \in \mathbb{R}$, $\mathfrak{l}_i = \chi_{i,2}y + \chi_{i,3}z + \chi_{i,0}$, $\chi_{i,j} \in \mathbb{R}$, and $\mathfrak{q}_i = \chi_{i,2,0}y^2 + \chi_{i,1,1}yz + \chi_{i,0,2}z^2 + \chi_{i,1,0}y + \chi_{i,0,1}z + \chi_{i,0,0}$, $\chi_{i,j,k} \in \mathbb{R}$, $1 \leqslant i \leqslant 3$. As $\mathfrak{p}$ is a polynomial of total degree of at most 3, $\mathcal{V}(\nabla\mathfrak{p})$ has the following nice property.

Lemma 4.2. *Let $m \in \mathcal{V}(\nabla\mathfrak{p})$ be a local extremum of the function $\mathfrak{p}$ and let $\mathcal{C} \subset \mathcal{V}(\nabla\mathfrak{p})$ be a (path-) connected component containing m. Then* $\operatorname{aff}(\mathcal{C}) \subset \mathcal{V}(\nabla\mathfrak{p})$.

Proof. Let $n \in \mathcal{C}$, $m \neq n$. Since there exists a path from m to n in $\mathcal{C}$ and since the gradient vanishes on $\mathcal{C}$ we have $\mathfrak{p}(m) = \mathfrak{p}(n)$. For $t \in \mathbb{R}$ let $s(t) = m + t \cdot (n - m)$. The function $\mathfrak{p}(s(t))$ is an univariate polynomial in t of degree at most three with $\mathfrak{p}(s(0)) = \mathfrak{p}(s(1))$ and the derivative vanishes at 0 and 1. Thus $\mathfrak{p}(s(t)) = \mathfrak{p}(m)$ for all $t \in \mathbb{R}$. Since m is assumed to be a local extremum there exists a $\bar{t} \in (0, 1)$ such that $s(\bar{t})$ is a local extremum, too. Hence we have found three points on the line $s(t)$ where the gradient of $\mathfrak{p}$ vanishes. Since all partial derivatives are polynomials of total degree at most two we have shown that $\nabla\mathfrak{p}(s(t)) = 0$ for all $t \in \mathbb{R}$. □

The last lemma tells us that in order to locate all possible local extrema of $\mathfrak{p}$ it suffices to find all *isolated affine subspaces* of $\nabla\mathfrak{p} = 0$, i.e.,

$$\begin{aligned}\mathcal{V}_{\mathrm{aff}}(\nabla\mathfrak{p}) = \bigl\{\mathcal{C} \subset \mathcal{V}(\nabla\mathfrak{p})\colon\ &\mathcal{C} = \operatorname{aff}(\mathcal{C}) \text{ and there exists no connected}\\ &\text{component } \mathcal{U} \subset \mathcal{V}(\nabla\mathfrak{p}) \text{ with } \mathcal{C} \subsetneq \mathcal{U}\bigr\}.\end{aligned} \tag{4.6}$$

To our surprise we have not found any efficient algorithm for the determination of the set $\mathcal{V}$ or $\mathcal{V}_{\mathrm{aff}}$ or, in general, for the determination of the local extrema of the polynomial $\mathfrak{p}$. Therefore we have developed an ad hoc method for our purposes which is based on resultants of polynomials and Lemma 4.2. For a detailed treatment of resultants we refer to [8,9]. Here we just collect some of their basic properties.

To this end we denote for a polynomial $\mathfrak{f} \in \mathbb{R}[x_1, \ldots, x_d]$ by $\deg(\mathfrak{f})$ its total degree and by $\deg(\mathfrak{f}, x_i)$ we denote the degree of the variable x_i if we consider $\mathfrak{f}$ as a polynomial over $\mathbb{R}[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d]$. For a polynomial or a system of polynomials $\mathcal{I} \subset \mathbb{R}[x_1, \ldots, x_d]$ we denote by $\mathcal{V}(\mathcal{I}) = \{x \in \mathbb{R}^n\colon \mathfrak{f}(x) = 0$ for all $\mathfrak{f} \in \mathcal{I}\}$. Furthermore, let $\mathcal{V}_{\mathrm{aff}}(\mathcal{I})$ be the set of all isolated affine subspaces of $\mathcal{V}(\mathcal{I})$ in the sense of (4.6) and let

$$\mathcal{V}^j_{\mathrm{aff}}(\mathcal{I}) = \big\{\mathcal{C} \in \mathcal{V}_{\mathrm{aff}}(\mathcal{I})\colon \dim(\mathcal{C}) = j\big\}, \quad j = 0, \ldots, d.$$

The elements of $\mathcal{V}^0_{\mathrm{aff}}$ will be called *isolated roots*. Since in general is seems to be a hard problem to determine exactly the set $\mathcal{V}_{\mathrm{aff}}(\mathcal{I})$ we only look for an approximation of this set. This means we want to determine a set

$$\widetilde{\mathcal{V}}_{\mathrm{aff}}(\mathcal{I}) \textit{ consisting of finitely many affine subspace } \mathcal{C} \subset \mathcal{V}(\mathcal{I}) \textit{ and } \mathcal{V}_{\mathrm{aff}}(\mathcal{I}) \subset \widetilde{\mathcal{V}}_{\mathrm{aff}}(\mathcal{I}). \tag{4.7}$$

Now let $\mathfrak{f}^j \in \mathbb{R}[x_1, \ldots, x_d]$, $j = 1, 2$, be two polynomials and with respect to the coefficient ring $\mathbb{R}[x_2, \ldots, x_d]$ we write

$$f^j(x_1, \ldots, x_d) = \sum_{i=0}^{m_j} f_{i,j}\, x_1^i,$$

with $f_{i,j} \in \mathbb{R}[x_2, \ldots, x_d]$, $m_j = \deg(\mathfrak{f}^j, x_1)$ and let $m_1, m_2 \geqslant 1$. The *resultant* of f and g with respect to x_1 will be denoted by $\operatorname{res}(\mathfrak{f}^1, \mathfrak{f}^2, x_1) \in \mathbb{R}[x_2, \ldots, x_d]$ and it is given by the determinant of the Sylvester matrix of $\mathfrak{f}^1$ and $\mathfrak{f}^2$ with respect to x_1, i.e.,

$$\operatorname{res}\big(\mathfrak{f}^1, \mathfrak{f}^2, x_1\big) = \det \begin{pmatrix} f_{m_1,1} & & & f_{m_2,2} & & \\ f_{m_1-1,1} & \ddots & & f_{m_2-1,2} & \ddots & \\ \vdots & & f_{m_1,1} & \vdots & & f_{m_2,2} \\ \vdots & & f_{m_1-1,1} & \vdots & & f_{m_2-1,2} \\ f_{0,1} & & \vdots & & & \vdots \\ & & \vdots & f_{0,2} & & \vdots \\ & & f_{0,1} & & & f_{0,2} \end{pmatrix}, \tag{4.8}$$

where the columns corresponding to $\mathfrak{f}^1$, $\mathfrak{f}^2$ are repeated m_2-times, m_1-times, respectively. The definition can be extended to the case $m_1 + m_2 \geqslant 1$ by setting $\operatorname{res}(\mathfrak{f}^1, \mathfrak{f}^2, x_1) = \mathfrak{f}^j$ if $m_j = 0$.

Lemma 4.3 [8, p. 150].

(i) *Let* $(u_1, u_2, \ldots, u_d) \in \mathbb{R}^d$ *be a common root of* $\mathfrak{f}^1$ *and* $\mathfrak{f}^2$. *Then* $\operatorname{res}(\mathfrak{f}^1, \mathfrak{f}^2, x_1)(u_2, \ldots, u_d) = 0$.

(ii) *Let* $u = (u_2, \ldots, u_d) \in \mathbb{R}^{d-1}$ *such that* $\operatorname{res}(\mathfrak{f}^1, \mathfrak{f}^2, x_1)(u) = 0$ *but* $f_{m_1,1}(u) \neq 0$ *or* $f_{m_2,2}(u) \neq 0$. *Then there exists a* $u_1 \in \mathbb{C}$ *such that* $\mathfrak{f}^1(u_1, u) = \mathfrak{f}^2(u_1, u) = 0$.

(iii) $\operatorname{res}(\mathfrak{f}^1, \mathfrak{f}^2, x_1) = 0$ *if and only if* $\mathfrak{f}^1$ *and* $\mathfrak{f}^2$ *have a common factor* $g \in \mathbb{R}[x_1, \ldots, x_d]$ *with* $\deg(\mathfrak{g}, x_1) \geqslant 1$.

In general we would like to use resultants in the following way. Let $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{p}_3$ be the partial derivatives of the polynomial $\mathfrak{p}$ (see (4.5)). First we compute the two resultants $\operatorname{res}_{1,2} = \operatorname{res}(\mathfrak{p}_1, \mathfrak{p}_2, x) \in$

$\mathbb{R}[y,z]$, $\mathrm{res}_{2,3} = \mathrm{res}(\mathfrak{p}_2, \mathfrak{p}_3, x) \in \mathbb{R}[y,z]$ and then the resultant $\mathrm{res}_{1,2,3} = \mathrm{res}\big(\mathrm{res}_{1,2}, \mathrm{res}_{2,3}, y\big) \in \mathbb{R}[z]$. Next we determine the real roots of $\mathrm{res}_{1,2,3}$, and for each root $\overline{z}$ we determine the common roots of $\mathrm{res}_{1,2}(y,\overline{z}) = 0$ and $\mathrm{res}_{2,3}(y,\overline{z}) = 0$. So we get a couple of common roots of $\mathrm{res}_{1,2}$ and $\mathrm{res}_{2,3}$. Again we put each pair of those common roots into $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$ and solve the three polynomials with respect to x.

Now, as we shall see, if $\mathrm{res}_{1,2,3} \neq 0$ (and thus all resultants are nontrivial), then we can compute $\mathcal{V}(\nabla\mathfrak{p})$ by this method. However, in general some of the resultants may vanish and hence we have to find common factors of the polynomials (cf. Lemma 4.3(iii)). According to Lemma 4.2 we are mainly interested in factors corresponding to affine subspaces. Therefore we call a polynomial $\mathfrak{l} \in \mathbb{R}[x_1,\dots,x_d]$ *linear* if $\deg(\mathfrak{l}) = 1$, i.e.,

$$\mathfrak{l} = l_0 + \sum_{i=1}^{d} l_i \cdot x_i, \quad l_i \in \mathbb{R}.$$

Remark 4.1. Let $\mathfrak{f} = \sum_{i=0}^{m} f_i\, x_1^i$, $\mathfrak{l} = l_0 + \sum_{i=1}^{d} l_i \cdot x_i \in R[x_1,\dots,x_d]$ with $f_i \in \mathbb{R}[x_2,\dots,x_d]$, $l_i \in \mathbb{R}$ and $f_m, l_1 \neq 0$. Then $\mathfrak{l}$ is a factor of $\mathfrak{f}$ if and only if $\mathfrak{f}\big((l_1 x_1 - \mathfrak{l})/l_1, x_2,\dots,x_d\big) = 0$.

Proof. Obviously, if $\mathfrak{l}$ is factor of $\mathfrak{f}$ then the statement holds. Without loss of generality let $l_1 = 1$ and $\overline{\mathfrak{l}} = \mathfrak{l} - x_1$ and let $\mathfrak{q} = \sum_{i=0}^{m_1} q_i\, x_1^i$ be the polynomial whose coefficients are recursively defined by

$$q_{m-1} = f_m \quad \text{and} \quad q_i = f_{i+1} - q_{i+1}\overline{\mathfrak{l}}, \quad i = m-2,\dots,0. \tag{4.9}$$

Multiplication of $\mathfrak{l}$ and $\mathfrak{q}$ yields $\mathfrak{l}\cdot\mathfrak{q} = \mathfrak{f} - f_0 + \overline{\mathfrak{l}}\cdot q_0$. Since $f_0, \overline{\mathfrak{l}}\cdot q_0 \in \mathbb{R}[x_2,\dots,x_d]$, $\mathfrak{l}(-\overline{\mathfrak{l}}, x_2,\dots,x_d) = \mathfrak{f}(-\overline{\mathfrak{l}}, x_2,\dots,x_d) = 0$ we must have $-f_0 + \overline{\mathfrak{l}}\cdot q_0 = 0$. $\square$

Hence the $(d-1)$-dimensional affine subspaces, where a polynomial vanishes are given by the linear factors, and visa versa. Therefore for a set $\mathcal{I}$ of polynomials in $\mathbb{R}[x_1,\dots,x_d]$ let

$$\mathcal{W}^{d-1}(\mathcal{I}) = \big\{\mathcal{C} \subset \mathcal{V}(\mathcal{I})\colon \mathcal{C} = \mathrm{aff}(\mathcal{C}) \text{ and } \dim(\mathcal{C}) = d-1\big\}.$$

On account of Remark 4.1 we can easily determine all linear factors and thus $\mathcal{W}^{d-1}(\mathfrak{f})$ of a given polynomial by the following procedure.

Algorithm 4.1 (Determining a linear factor of a polynomial $\mathfrak{f}$).

- For each variable x_j do (without loss of generality let $j = 1$)
  - F1. Write $\mathfrak{f}$ as $\mathfrak{f} = \sum_{i=0}^{m_1} f_i\, x_1^i$ with $f_i \in \mathbb{R}[x_2,\dots,x_d]$.
    If $m_1 \neq 0$ do
    - F2. Find d affinely independent points $u^2,\dots,u^{d+1} \in \mathbb{R}^{d-1}$ such that $\mathfrak{f}(x_1,u^i) \neq 0$, $2 \leqslant i \leqslant d+1$.
    - F3. For $2 \leqslant i \leqslant d+1$ determine the real roots $\mathcal{Z}_i$ of the univariate polynomials $\mathfrak{f}(x_1,u^i) = 0$. Let $\mathcal{Z}_0 = \{x_1 \in \mathbb{R}\colon \mathfrak{f}(x_1,0,\dots,0) = 0\}$.
    - F4. For each $(z_2,\dots,z_{d+1}) \in \mathcal{Z}_2 \times \dots \times \mathcal{Z}_{d+1}$ do
      - F5. Determine the solution $l = (l_0,l_2,\dots,l_d)^{\mathrm{T}} \in \mathbb{R}^d$ of the linear system $(1, {u^i}^{\mathrm{T}})l = z_i$, $2 \leqslant i \leqslant d+1$.
      - F6. Let $\mathfrak{l} = -l_0 + x_1 - \sum_{i=2}^{d} l_i \cdot x_i$. If $\mathfrak{f}(x_1 - \mathfrak{l}, x_2,\dots,x_d) = 0$ then $\mathfrak{l}$ is a linear factor and the remainder $\widehat{\mathfrak{f}} = \mathfrak{f}/\mathfrak{l}$ is given by the polynomial $\mathfrak{q} = \sum_{i=0}^{m_1} q_i\, x_1^i$ defined in (4.9).
- If for all solutions $l = (l_0,l_2,\dots,l_d)^{\mathrm{T}} \in \mathbb{R}^d$ the corresponding linear polynomial $\mathfrak{l}$ is not a factor of $\mathfrak{f}$ then $\mathfrak{f}$ posseses no linear factors.

Lemma 4.4. *Let* $\mathfrak{f} \in \mathbb{R}[x_1, \ldots, x_d]$, $\mathfrak{f} \neq 0$. *Then Algorithm* 4.1 *finds a linear factor* $\mathfrak{l} \in \mathbb{R}[x_1, \ldots, x_d]$ *of* $\mathfrak{f}$ *and determines the polynomials* $\mathfrak{f}/\mathfrak{l}$, *if* $\mathfrak{f}$ *has a linear factor at all.*

Proof. Let $\mathfrak{l} = l_0 + \sum_{i=1}^{d} l_i x_i$ be a linear factor of $\mathfrak{f}$ and let us assume without loss of generality that $l_1 = -1$. Let $\bar{\mathfrak{l}} = l_0 + l_2 x_2 + \cdots + l_d x_d \in \mathbb{R}[x_2, \ldots, x_d]$ and $l = (l_0, l_2, \ldots, l_d)^{\mathrm{T}}$. By Remark 4.1 we have $\mathfrak{f}(\bar{\mathfrak{l}}, x_2, \ldots, x_d) = 0$ and for the points u^i we get $0 = \mathfrak{f}(\bar{\mathfrak{l}}(u^i), u^i) = \mathfrak{f}((1, {u^i}^{\mathrm{T}})l, u^i)$ and hence $(1, {u^i}^{\mathrm{T}})l \in \mathcal{Z}_i$, $i = 2, \ldots, d+1$. By the choice of the vectors u^i each linear system has a unique solution.

Finally, we remark that the $(d-1)$-linearly independent points u^i can be found quite easily. If the leading coefficient of f with respect to x_1 is constant, then we set $u^i = e^{i-1} \in \mathbb{R}^{d-1}$, $2 \leqslant i \leqslant d$, and $u^{d+1} = 0$, where e^i denotes the ith unit vector. Otherwise we perturb these points a bit. □

Since each polynomial can be written as a unique product of irreducible polynomials we can apply the above algorithm iteratively to the computed remainders in order to determine all linear factors of a given polynomial. Obviously, we can also use Algorithm 4.1 to find a common linear factor of two or more polynomials. Hence we have the following corollary.

Corollary 4.1. *Let* $\mathcal{I}$ *be a set of polynomials* $\mathfrak{f}_i \in \mathbb{R}[x_1, \ldots, x_d]$, $i \in I$. *Then there exists an algorithm which computes all common linear factors of the polynomials* $\mathfrak{f}_i$, $i \in I$, *and the set* $\mathcal{W}^{d-1}(\mathcal{I})$.

For special polynomials we have a simple test to decide whether one polynomial is a factor of another one.

Remark 4.2. Let $\mathfrak{f} = \sum_{i=0}^{m} f_i \cdot x_1^{m-i} \in \mathbb{R}[x_1, \ldots, x_d]$, $\mathfrak{g} = \sum_{i=0}^{n} g_i \cdot x_1^{n-i} \in \mathbb{R}[x_1, \ldots, x_d]$, $m \geqslant n \geqslant 1$, $f_i, g_i \in \mathbb{R}[x_2, \ldots, x_d]$ and $g_0 \in \mathbb{R}\backslash\{0\}$. Then there exists an algorithm which decides whether $\mathfrak{g}$ is a factor of $\mathfrak{f}$ and determines the polynomial $\mathfrak{h} = \mathfrak{f}/\mathfrak{g}$ if $\mathfrak{g}$ is a factor of $\mathfrak{f}$.

Proof. Without loss of generality let $g_0 = 1$. By definition $\mathfrak{g}$ is a factor of $\mathfrak{f}$ if and only if there exists a polynomial $\mathfrak{h} = \sum_{i=0}^{m-n} h_i \cdot x_1^{m-n-i}$, $h_i \in \mathbb{R}[x_2, \ldots, x_d]$, such that $\mathfrak{g} \cdot \mathfrak{h} = \mathfrak{f}$. Hence by comparing the coefficients we get

$$f_k = \sum_{j=\max\{0, n-m+k\}}^{\min\{k,n\}} g_j \cdot h_{k-j}, \quad k = 0, \ldots, m.$$

Since $g_0 = 1$ the first $m - n$ identities determine uniquely the coefficients h_j, $j = 0, \ldots, m-n$, and the remaining identities can be used as a verification whether $\mathfrak{g}$ is really a factor of $\mathfrak{f}$, i.e., whether $\mathfrak{h} = \mathfrak{f}/\mathfrak{g}$. □

Next we describe algorithms how we can find all the isolated affine subspaces for some very special polynomials in two variables.

Lemma 4.5. *Let* $\mathfrak{f} = f_0 \cdot x^2 + f_1 \cdot x + f_2 \in \mathbb{R}[x, y]$ *with* $f_i \in \mathbb{R}[y]$ *and* $\deg(f_i) \leqslant i$. *Then there exists an algorithm which computes* $\mathcal{V}_{\mathrm{aff}}(\mathfrak{f})$.

Proof. Since $\mathcal{V}(\mathfrak{f})$ is a conic section, the set of all isolated affine subspaces of $\mathcal{V}(\mathfrak{f})$ consists either of one or of two lines or of an isolated root or it is empty. Using well-known formula for conic sections one can determine all of them. □

Lemma 4.6. *Let* $\mathfrak{f} \in \mathbb{R}[x, y]$ *with* $\deg(\mathfrak{f}) \leqslant 4$ *and* $\mathfrak{f} \neq 0$. *Then there exists an algorithm which computes a set* $\widetilde{\mathcal{V}}_{\mathrm{aff}}(\mathfrak{f})$ *in the sense of* (4.7).

Proof. Let $\mathfrak{f} = \sum_{i=0}^{4} f_i \cdot x^{4-i}$, $f_i \in \mathbb{R}[y]$ and $\deg(f_i) \leqslant i$. On account of Corollary 4.1 we determine all linear factors of $\mathfrak{f}$ and the set $\mathcal{W}^1(\mathfrak{f})$ containing $\mathcal{V}^1_{\mathrm{aff}}(\mathfrak{f})$. Hence it remains to determine the isolated roots, i.e., $\mathcal{V}^0_{\mathrm{aff}}(\mathfrak{f})$. Since none of the isolated roots lies in a 1-dimensional subspace of $\mathcal{W}^1(\mathfrak{f})$, we can divide $\mathfrak{f}$ by the linear factors corresponding to the elements of $\mathcal{W}^1(\mathfrak{f})$. Therefore, in the following we assume that $\mathfrak{f}$ has no linear factors. If $\mathfrak{f}$ is an univariate polynomial in x, say, having a real root $\overline{x}$, then $-\overline{x} + x$ would be a linear factor of $\mathfrak{f}$. Thus if $\mathfrak{f}$ is an univariate polynomial we set $\widetilde{\mathcal{V}}_{\mathrm{aff}}(\mathfrak{f}) = \mathcal{W}^1(\mathfrak{f})$. Hence let $\deg(\mathfrak{f}, x) \geqslant \deg(\mathfrak{f}, y) \geqslant 1$.

If $(\overline{x}, \overline{y})$ is an isolated root then $\nabla\mathfrak{f}(\overline{x}, \overline{y}) = 0$ and thus we can look for all isolated roots of the set $\{(x, y) \in \mathbb{R}^2\colon\ \mathfrak{f}(x, y) = \mathfrak{f}_x(x, y) = 0\}$, where $\mathfrak{f}_x$ denotes the partial derivative of $\mathfrak{f}$ with respect to x, i.e., $\mathfrak{f}_x = \sum_{i=0}^{3} \widetilde{f}_i \cdot x^{3-i}$, with $\widetilde{f}_i \in \mathbb{R}[y]$, $\deg(\widetilde{f}_i) \leqslant i$. Now let $\mathrm{res} = \mathrm{res}(\mathfrak{f}, \mathfrak{f}_x, x) \in \mathbb{R}[y]$ be the resultant of these two polynomials. If $\mathrm{res} \neq 0$ then we determine $\mathcal{V}(\mathrm{res})$ and afterwards $\mathcal{W}_0 = \{(x, y)\colon\ \mathfrak{f} = \mathfrak{f}_x = 0,\ y \in \mathcal{V}(\mathrm{res})\}$. Observe, that for any fixed $\overline{y} \in \mathcal{V}(\mathrm{res})$ the polynomial $\mathfrak{f}(x, \overline{y})$ cannot vanish, because otherwise it contains a linear factor (see Remark 4.1). In this case we set $\widetilde{\mathcal{V}}_{\mathrm{aff}}(\mathfrak{f}) = \mathcal{W}_0 \cup \mathcal{W}^1(\mathfrak{f})$.

It remains to consider the case $\mathrm{res} = 0$ and therefore we may assume $\deg(\mathfrak{f}, x) \geqslant 2$. By Lemma 4.3 we know that $\mathfrak{f}$ and $\mathfrak{f}_x$ have a common factor $\mathfrak{g}$ which has positive degree in x. Moreover, it is not hard to see that $\mathrm{res}(\mathfrak{f}, \mathfrak{f}_x) = 0$ implies that $\mathfrak{f}$ is divisible by some $\mathfrak{h}^2$, $\mathfrak{h} \in \mathbb{R}[x, y]$ with $\deg(\mathfrak{h}, x) \geqslant 1$. Hence, if $\deg(\mathfrak{f}, x) \leqslant 3$ then $\mathfrak{f}$ can be written as a product of polynomials $\mathfrak{f}_k$ with $\deg(\mathfrak{f}_k, x) = 1$ which shows that $\mathfrak{f}$ has no isolated roots.

So let $\deg(\mathfrak{f}, x) = 4$. Then the common factor $\mathfrak{g}$ of $\mathfrak{f}$, $\mathfrak{f}_x$ has degree 1, 2 or 3 with respect to the variable x. If $\deg(\mathfrak{g}, x) = 1$ then it is a linear factor, because the leading coefficient f_0 is a constant. Also, if $\deg(\mathfrak{g}, x) = 3$ then $\mathfrak{f}/\mathfrak{g} \in \mathbb{R}[x, y]$ is a linear factor of $\mathfrak{f}$. Hence $\deg(\mathfrak{g}, x) = 2$ and it can be written as $\mathfrak{g} = \sum_{i=0}^{2} g_i x^{2-i}$ with $g_i \in \mathbb{R}[y]$, $\deg(g_i) \leqslant i$. Thus $\mathfrak{f}_x/\mathfrak{g}$ is a linear factor and we can determine $\mathfrak{g}$ by the following procedure. For each linear factor $\mathfrak{l}$ of $\mathfrak{f}_x$ let $\mathfrak{g}_{\mathfrak{l}} = \mathfrak{f}_x/\mathfrak{l}$. Then we have to test whether $\mathfrak{g}_{\mathfrak{l}}$ is a factor of $\mathfrak{f}$. This can be done with the algorithm described in Remark 4.2. So we can assume that we have found the factor $\mathfrak{g}$ of $\mathfrak{f}$ with $\deg(\mathfrak{g}) = 2$. Let $\overline{\mathfrak{g}} = \mathfrak{f}/\mathfrak{g}$. Then the problem is reduced to the determination of the isolated roots of the two polynomials $\mathfrak{g}$ and $\overline{\mathfrak{g}}$, which can be solved by Lemma 4.5. Hence we set $\widetilde{\mathcal{V}}_{\mathrm{aff}}(\mathfrak{f}) = \mathcal{V}^0_{\mathrm{aff}}(\mathfrak{g}) \cup \mathcal{V}^0_{\mathrm{aff}}(\overline{\mathfrak{g}}) \cup \mathcal{W}^1(\mathfrak{f})$. $\square$

Lemma 4.7. *Let* $\mathfrak{f}, \mathfrak{g} \in \mathbb{R}[x, y]$ *with* $\mathfrak{f}, \mathfrak{g} \neq 0$ *and* $\deg(\mathfrak{f}) \leqslant 4$, $\deg(\mathfrak{g}) \leqslant 3$. *Then there exists an algorithm which computes a set* $\widetilde{\mathcal{V}}_{\mathrm{aff}}(\mathfrak{f}, \mathfrak{g})$ *in the sense of* (4.7).

Proof. We proceed as in the proof of the last lemma. First we determine all common linear factors of these two polynomials and the set $\mathcal{W}^1(\mathfrak{f}, \mathfrak{g})$ containing $\mathcal{V}^1_{\mathrm{aff}}(\mathfrak{f}, \mathfrak{g})$ (cf. Corollary 4.1). Then we divide the polynomials by these common linear factors and it remains to find the isolated roots $\mathcal{V}^0_{\mathrm{aff}}(\mathfrak{f}, \mathfrak{g})$. To this end we may assume that $\deg(\mathfrak{f}, x), \deg(\mathfrak{f}, y) \geqslant 1$.

Let $\mathrm{res} = \mathrm{res}(\mathfrak{f}, \mathfrak{g}, x) \in \mathbb{R}[y]$. If $\mathrm{res} \neq 0$ then we determine $\mathcal{V}(\mathrm{res})$, $\mathcal{W}_0 = \{(x, y)\colon\ \mathfrak{f} = \mathfrak{f}_x = 0,\ y \in \mathcal{V}(\mathrm{res})\}$ and we set $\widetilde{\mathcal{V}}_{\mathrm{aff}}(\mathfrak{f}) = \mathcal{W}_0 \cup \mathcal{W}^1(\mathfrak{f})$. Therefore we can assume $\mathrm{res} = 0$. Since $\deg(\mathfrak{g}) \leqslant 3$ we know that if the polynomial $\mathfrak{g}$ is not irreducible then it has a linear factor $\mathfrak{l}$, say. Using Algorithm 4.1 we can determine such a linear factor $\mathfrak{l}$ as well as the remainder $\widehat{\mathfrak{g}} = \mathfrak{g}/\mathfrak{l}$. Next we split our system in two systems, namely

$$\mathcal{I}_1: \quad \mathfrak{f} = \widehat{\mathfrak{g}} = 0$$

and

$$\mathcal{I}_2: \quad \mathfrak{f} = \mathfrak{l} = 0.$$

Since $\mathfrak{l}$ is a linear factor and $\mathfrak{g}$ and $\mathfrak{f}$ are assumed to be free of common linear factors the second system can be solved by substituting one variable in $\mathfrak{f}$ via $\mathfrak{l}$. Since $\deg(\widehat{\mathfrak{g}}) < \deg(\mathfrak{g})$ we can apply recursively the previous argumentation to the system $\mathcal{I}_1$. Thus, if $\mathfrak{g}$ is not irreducible we set $\widetilde{\mathcal{V}}_{\mathrm{aff}}(\mathfrak{f}, \mathfrak{g}) = \mathcal{V}^0_{\mathrm{aff}}(\mathcal{I}_1) \cup \mathcal{V}^0_{\mathrm{aff}}(\mathcal{I}_2) \cup \mathcal{W}^1(\mathfrak{f}, \mathfrak{g})$.

If $\mathfrak{g}$ is irreducible, and since res $= 0$, the polynomial $\mathfrak{g}$ has to be a factor of $\mathfrak{f}$. Hence the determination of $\mathcal{V}^0_{\mathrm{aff}}(\mathfrak{f}, \mathfrak{g})$ is reduced to the calculation of the isolated roots of $\mathfrak{g}$. With the algorithm of Lemma 4.6 we can find a set $\widetilde{\mathcal{V}}_{\mathrm{aff}}(\mathfrak{g}) \subset \mathcal{V}(\mathfrak{g})$ containing $\mathcal{V}_{\mathrm{aff}}(\mathfrak{g})$ and we set $\widetilde{\mathcal{V}}_{\mathrm{aff}}(\mathfrak{f}, \mathfrak{g}) = \widetilde{\mathcal{V}}_{\mathrm{aff}}(\mathfrak{g}) \cup \mathcal{W}^1(\mathfrak{f}, \mathfrak{g})$. □

Lemma 4.8. *For $j = 1, \dots, 3$ let $\mathfrak{f}^j = f_0^j \cdot x^2 + f_1^j \cdot x + f_2^j \in \mathbb{R}[x, y, z]$, $f_i^j \in \mathbb{R}[y, z]$, with $\deg(f_i^j) \leqslant i$. Let $f_0^2 = 0$, $f_1^2 \neq 0$, $\mathrm{res}_{1,2} = \mathrm{res}(\mathfrak{f}^1, \mathfrak{f}^2, x)$, $\mathrm{res}_{2,3} = \mathrm{res}(\mathfrak{f}^2, \mathfrak{f}^3, x)$ and let $\mathrm{res}_{1,2} \neq 0$ or $\mathrm{res}_{2,3} \neq 0$. Then for each $L \in \mathcal{V}_{\mathrm{aff}}(\mathfrak{f}^1, \mathfrak{f}^2, \mathfrak{f}^3)$ there exists an $M \in \mathcal{V}_{\mathrm{aff}}(\mathrm{res}_{1,2}, \mathrm{res}_{2,3}) \cup W^1(\mathrm{res}_{1,2}, \mathrm{res}_{2,3})$ such that $L \subset \{(x, y, z) \in \mathbb{R}^3\colon (y, z) \in M\}$.*

Proof. Without loss of generality let $\mathrm{res}_{1,2} \neq 0$ and let $L \in \mathcal{V}_{\mathrm{aff}}(\mathfrak{f}^1, \mathfrak{f}^2, \mathfrak{f}^3)$. By L^π we denote the orthogonal projection of L onto the plane $\{(x, y, z) \in \mathbb{R}^3\colon x = 0\}$. Obviously, L^π is an affine subspace and by the definition of resultants we have $\mathrm{res}_{1,2}(y, z) = \mathrm{res}_{2,3}(y, z) = 0$ for all $(y, z) \in L^\pi$. Since $\mathrm{res}_{1,2} \neq 0$ we have $\dim(L^\pi) \in \{0, 1\}$. If $\dim(L^\pi) = 1$, then L^π itself corresponds to a common linear factor of $\mathrm{res}_{1,2}$ and $\mathrm{res}_{2,3}$ and thus $L^\pi \in W^1(\mathrm{res}_{1,2}, \mathrm{res}_{2,3})$. Therefore let $L^\pi = \{(y^0, z^0)\}$ and so L is either an isolated 1-dimensional subspace or an isolated root of $\mathcal{V}(\mathfrak{f}^1, \mathfrak{f}^2, \mathfrak{f}^3)$. If (y^0, z^0) is contained in an 1-dimensional subspace of $W^1(\mathrm{res}_{1,2}, \mathrm{res}_{2,3})$ the statement is certainly true. Thus we may assume that $\mathfrak{l}(y^0, z^0) \neq 0$ for every common linear factor $\mathfrak{l}$ of $\mathrm{res}_{1,2}$ and $\mathrm{res}_{2,3}$ and we have to show that $u^0 = (y^0, z^0)$ is an isolated root of $\mathcal{V}_{\mathrm{aff}}(\mathrm{res}_{1,2}, \mathrm{res}_{2,3})$. Suppose the contrary and let $u^1 = (y^1, z^1) \in \mathcal{V}(\mathrm{res}_{1,2}, \mathrm{res}_{2,3})$ such that there exists a path $\mathcal{P} \subset \mathcal{V}(\mathrm{res}_{1,2}, \mathrm{res}_{2,3})$, $\mathcal{P} = \{u^t\colon t \in [0, 1]\}$, connecting u^0 and u^1. Then $\mathcal{P} \neq \mathrm{conv}\{u^0, u^1\}$, because otherwise $\mathrm{aff}\{u^0, u^1\} \subset \mathcal{V}(\mathrm{res}_{1,2}, \mathrm{res}_{2,3})$ is a 1-dimensional set containing u^0. Hence we can assume $f_1^2(u^t) \neq 0$, $t \in (0, 1)$, and by Lemma 4.3(ii) there exist $a^t, b^t \in \mathbb{C}$ such that $\mathfrak{f}^1(a^t, u^t) = \mathfrak{f}^2(a^t, u^t) = 0 = f^2(b^t, u^t) = \mathfrak{f}^3(b^t, u^t)$, $t \in (0, 1)$. Since $f_0^2 = 0$ and $f_1^2(u^t) \neq 0$ we get $a^t = b^t \in \mathbb{R}$. However this shows that L is not an isolated affine subspace of $\mathcal{V}(\mathfrak{f}^1, \mathfrak{f}^2, \mathfrak{f}^3)$. □

Of course the last lemma also implies the following corollary.

Corollary 4.2. *For $j = 1, 2$ let $\mathfrak{f}^j = f_0^j \cdot x^2 + f_1^j \cdot x + f_2^j \in \mathbb{R}[x, y, z]$, $f_i^j \in \mathbb{R}[y, z]$, with $\deg(f_i^j) \leqslant i$. Let $f_0^2 = 0$, $f_1^2 \neq 0$ and let $\mathrm{res}_{1,2} = \mathrm{res}(\mathfrak{f}^1, \mathfrak{f}^2, x) \neq 0$. Then for each $L \in \mathcal{V}_{\mathrm{aff}}(\mathfrak{f}^1, \mathfrak{f}^2)$ there exists an $M \in \mathcal{V}_{\mathrm{aff}}(\mathrm{res}_{1,2}) \cup W^1(\mathrm{res}_{1,2})$ such that $L \subset \{(x, y, z) \in \mathbb{R}^3\colon (y, z) \in M\}$.*

Proof. We set $f^3 = f^1$ and apply Lemma 4.8. □

Using Corollary 4.1 and the previous lemmas we can make the resultant approach practicable.

Theorem 4.1. *Let $\mathfrak{p}(x,y,z)$ be a polynomial with total degree at most* 3. *Let* $\mathcal{V}(\nabla\mathfrak{p}) = \{(x,y,z) \in \mathbb{R}^3\colon\ \nabla p = 0\}$. *There exists an algorithm computing a set $\widetilde{\mathcal{V}}_{\mathrm{aff}}(\nabla\mathfrak{p})$ in the sense of* (4.7).

Proof. First we note that after some scaling, subtractions, and renumbering we may assume that $\mathcal{V}(\nabla\mathfrak{p})$ is given by the following system $\mathcal{I}$ of polynomial equations (see (4.5)):

$$\mathcal{I}\colon\quad \begin{aligned} \mathfrak{p}_1 &= \phantom{\kappa\cdot x^2+{}} \mathfrak{l}_1\cdot x + \mathfrak{q}_1 = 0,\\ \mathfrak{p}_2 &= \phantom{\kappa\cdot x^2+{}} \mathfrak{l}_2\cdot x + \mathfrak{q}_2 = 0,\\ \mathfrak{p}_3 &= \kappa\cdot x^2 + \mathfrak{l}_3\cdot x + \mathfrak{q}_3 = 0, \end{aligned} \tag{4.10}$$

where $\kappa \in \{0,1\}$, $\mathfrak{l}_i \in \mathbb{R}[y,z]$ are linear polynomials and $\mathfrak{q}_i \in \mathbb{R}[y,z]$ with $\deg(\mathfrak{q}_i) \leqslant 2$. We may further assume that $\deg(\mathfrak{p}_1) \leqslant \deg(\mathfrak{p}_2) \leqslant \deg(\mathfrak{p}_3)$. Depending on the number of non-trivial polynomials and the number of variables in $\mathcal{I}$ we have to distinguish several cases. Obviously, if all polynomials in (4.10) vanish then we have $\mathcal{V}(\nabla\mathfrak{p}) = \mathbb{R}^3$ and we have to do nothing.

(0) $\mathcal{I}$ consists of one or two or three polynomials in only one variable.
Then we can determine $\mathcal{V}(\nabla\mathfrak{p})$ by any algorithm computing the roots of an univariate polynomial.

(1,2) $\mathcal{I}$ consists of one polynomial in two variables.
Without loss of generality let $\mathcal{V}(\mathcal{I}) = \{(x,y,z) \in \mathbb{R}^3\colon\ \mathfrak{q}_3(y,z) = 0\} = \mathbb{R} \times \mathcal{V}(\mathfrak{q}_3)$, with $\mathcal{V}(q_3) \subset \mathbb{R}^2$. Via the algorithm of Lemma 4.5 we can determine $\mathcal{V}(\mathfrak{q}_3)$ and we set $\widetilde{\mathcal{V}}_{\mathrm{aff}}(\mathcal{I}) = \{\mathbb{R} \times \mathcal{C}\colon\ \mathcal{C} \in \mathcal{V}_{\mathrm{aff}}(\mathfrak{q}_3)\}$.

(2,2) $\mathcal{I}$ consists of two polynomials in two variables.
Without loss of generality let

$$\mathcal{V}(\mathcal{I}) = \bigl\{(x,y,z) \in \mathbb{R}^3\colon\ \mathfrak{q}_2(y,z) = \mathfrak{q}_3(y,z) = 0\bigr\} = \mathbb{R} \times \mathcal{V}(\mathfrak{q}_2,\mathfrak{q}_3).$$

Using the Algorithm of Lemma 4.7 we can determine a set $\widetilde{\mathcal{V}}_{\mathrm{aff}}(\mathfrak{q}_2,\mathfrak{q}_3) \subset \mathbb{R}^2$ (cf. (4.7)) and we set $\widetilde{\mathcal{V}}_{\mathrm{aff}}(\mathcal{I}) = \{\mathbb{R} \times \mathcal{C}\colon\ \mathcal{C} \in \widetilde{\mathcal{V}}_{\mathrm{aff}}(\mathfrak{q}_2,\mathfrak{q}_3)\}$.

(3,2) $\mathcal{I}$ consists of three polynomials in two variables.
Without loss of generality let

$$\mathcal{V}(\mathcal{I}) = \bigl\{(x,y,z) \in \mathbb{R}^3\colon\ \mathfrak{q}_1(y,z) = \mathfrak{q}_2(y,z) = \mathfrak{q}_3(y,z) = 0\bigr\} = \mathbb{R} \times \mathcal{V}(\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3).$$

We may assume that both variables occur in all three polynomials and that the polynomials are linearly independent. Otherwise we can reduce this case to one of the previous ones. First, by Corollary 4.1 we determine all common linear factors and the set $\mathcal{W}^1(\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3)$ containing $\mathcal{V}^1_{\mathrm{aff}}(\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3)$ and hence we may assume that $\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3$ have no common linear factors and both variables occur in the polynomials. Next we compute the resultant $\mathrm{res}_{2,3} = \mathrm{res}(\mathfrak{q}_2,\mathfrak{q}_3,y) \in \mathbb{R}[z]$. If $\mathrm{res}_{2,3} \neq 0$ then we determine $\mathcal{W} = \{(y,z)\colon\ \mathfrak{q}_1 = \mathfrak{q}_2 = \mathfrak{q}_3 = 0,\ z \in \mathcal{V}(\mathrm{res}_{2,3})\}$. Observe that for a fixed $\bar{z}$ not all three polynomials $\mathfrak{q}_i$ can vanish, because otherwise they have a common linear factor. In this case we set $\widetilde{\mathcal{V}}_{\mathrm{aff}}(\mathcal{I}) = \{\mathbb{R} \times \mathcal{C}\colon\ \mathcal{C} \in \mathcal{W}^1(\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3) \cup \mathcal{W}\}$.
It remains to consider the case $\mathrm{res}_{2,3} = 0$. Since $\mathfrak{q}_2$, $\mathfrak{q}_3$ are assumed to be linearly independent and since $\deg(\mathfrak{q}_i) \leqslant 2$ the common factor (cf. Lemma 4.3(iii)) has to be a linear polynomial $\mathfrak{l}_{2,3}$ which can be determined via Algorithm 4.1. Then we consider the two systems

$$\mathcal{I}_1\colon\quad \mathfrak{q}_1 = 0,\quad \mathfrak{l}_{2,3} = 0,$$

$$\mathcal{I}_2\colon\quad \mathfrak{q}_1 = 0,\quad \mathfrak{q}_2/\mathfrak{l}_{2,3} = 0,\quad \mathfrak{q}_3/\mathfrak{l}_{2,3} = 0.$$

Both systems are free of common linear factors and since both systems contain linear polynomials we can easily determine $\mathcal{V}(\mathcal{I}_1)$ and $\mathcal{V}(\mathcal{I}_2)$ by substitution. We set

$$\widetilde{\mathcal{V}}_{\text{aff}}(\mathcal{I}) = \{\mathbb{R} \times \mathcal{C}\colon \mathcal{C} \in \mathcal{W}^1(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3) \cup \mathcal{V}(\mathcal{I}_1) \cup \mathcal{V}(\mathcal{I}_2)\}.$$

- In the remaining cases we first determine the set $\mathcal{W}^2(\mathcal{I})$ and therefore we may always assume that the given partial derivatives have no common linear factors.

(1,3) $\mathcal{I}$ consists of one polynomial in three variables.
(a) Let $\mathcal{V}(\mathcal{I}) = \{(x, y, z) \in \mathbb{R}^3\colon \mathfrak{l}_3 \cdot x + \mathfrak{q}_3 = 0\}$. Then we may assume $\mathfrak{l}_3 \neq 0$ and the dimension of any affine subspace of $\mathcal{V}(\mathcal{I})$ is 2. Therefore we set $\widetilde{\mathcal{V}}(\mathcal{I}) = \mathcal{W}^2(\mathcal{I})$.
(b) Let $\mathcal{V}(\mathcal{I}) = \{(x, y, z) \in \mathbb{R}^3\colon \mathfrak{f} = x^2 + \mathfrak{l}_3 \cdot x + \mathfrak{q}_3 = 0\}$. Let $\widetilde{\mathfrak{q}} = \mathfrak{l}_3 \cdot \mathfrak{l}_3/4 - \mathfrak{q}_3 \in \mathbb{R}[y, z]$. For every $(x^*, y^*, z^*) \in \mathcal{V}(\mathcal{I})$ we have $x^* = -\mathfrak{l}_3(y^*, z^*)/2 \pm \sqrt{\widetilde{\mathfrak{q}}(y^*, z^*)}$ and hence $\widetilde{\mathfrak{q}}(y^*, z^*) \geqslant 0$. If $\widetilde{\mathfrak{q}}(y^*, z^*) > 0$ then we can always find a neighborhood U of (y^*, z^*) such that $\widetilde{\mathfrak{q}}(y, z) \geqslant 0$ for all $(y, z) \in U$ and (x^*, y^*, z^*) belongs to a 2-dimensional connected component of $\mathcal{V}(\mathfrak{f})$. Hence we may set (cf. Lemma 4.5)

$$\widetilde{\mathcal{V}}_{\text{aff}}(\mathcal{I}) = \mathcal{W}^2(\mathcal{I}) \cup \bigcup_{\mathcal{C} \in \mathcal{V}_{\text{aff}}(\widetilde{\mathfrak{q}})} \{(-\mathfrak{l}_3(y, z)/2, y, z)\colon (y, z) \in \mathcal{C}\}.$$

(2,3) $\mathcal{I}$ consists of two polynomials in three variables.
Then we may assume without loss of generality that the variable x occurs in both polynomials.
(a) Let

$$\mathcal{V}(\mathcal{I}) = \{(x, y, z) \in \mathbb{R}^3\colon \mathfrak{p}_1 = \mathfrak{l}_1 \cdot x + \mathfrak{q}_1 = 0,\ \mathfrak{p}_2 = \mathfrak{l}_2 \cdot x + \mathfrak{q}_2 = 0\}$$

with $\mathfrak{l}_1 \neq 0$, $\mathfrak{l}_2 \neq 0$ and we can assume that $\mathfrak{p}_1, \mathfrak{p}_2$ are linearly independent. Let $\text{res}_{1,2} \in \mathbb{R}[y, z]$ be the resultant of these two polynomials with respect to x. Since the polynomials are linearly independent, $\deg(\mathfrak{p}_1), \deg(\mathfrak{p}_2) \leqslant 2$, and since we have assumed that they have no common linear factors the resultant $\text{res}_{1,2}$ cannot vanish. By definition $\text{res}_{1,2} \in \mathbb{R}[y, z]$ is a polynomial of total degree at most 3. By Lemma 4.6 we can find a set $\widetilde{\mathcal{V}}_{\text{aff}}(\text{res}_{1,2})$ containing $\mathcal{V}_{\text{aff}}(\text{res}_{1,2})$ and for each $\mathcal{C} \in \widetilde{\mathcal{V}}_{\text{aff}}(\text{res}_{1,2})$ we consider the system

$$\mathcal{I}_{\mathcal{C}}\colon \quad \mathfrak{p}_1(x, y, z) = 0, \quad \mathfrak{p}_2(x, y, z) = 0, \quad (y, z) \in \mathcal{C}.$$

Since $\mathcal{C}$ is a 0- or 1-dimensional affine subspace, $\mathcal{I}_{\mathcal{C}}$ is a system of at most two polynomials in at most two variables. Hence by the previous cases we can determine a set $\widetilde{\mathcal{V}}_{\text{aff}}(\mathcal{I}_{\mathcal{C}}) \subset \mathcal{V}(\mathcal{I}_{\mathcal{C}})$ containing $\mathcal{V}_{\text{aff}}(\mathcal{I}_{\mathcal{C}})$ and we set

$$\widetilde{\mathcal{V}}_{\text{aff}}(\mathcal{I}) = \mathcal{W}^2(\mathcal{I}) \cup \bigcup_{\mathcal{C} \in \widetilde{\mathcal{V}}_{\text{aff}}(\text{res}_{1,2})} \widetilde{\mathcal{V}}_{\text{aff}}(\mathcal{I}_{\mathcal{C}}).$$

On account of Corollary 4.2 we have $\mathcal{V}_{\text{aff}}(\mathcal{I}) \subset \widetilde{\mathcal{V}}_{\text{aff}}(\mathcal{I})$.
(b) Let

$$\mathcal{V}(\mathcal{I}) = \{(x, y, z) \in \mathbb{R}^3\colon \mathfrak{p}_1 = \mathfrak{l}_1 \cdot x + \mathfrak{q}_1 = 0,\ \mathfrak{p}_2 = x^2 + \mathfrak{l}_2 \cdot x + \mathfrak{q}_2 = 0\}$$

with $\mathfrak{l}_1 \neq 0$. Then we can proceed as in the case above. The only difference is that the total degree of $\text{res}_{1,2}$ is at most 4.

(3,3) $\mathcal{I}$ consists of three polynomials in three variables.
Then we may assume without loss of generality that the variable x occurs in at least two of the polynomials, $\mathfrak{l}_2 \neq 0$ (cf. (4.10)) and that each of the three polynomials contains at least two variables. Now we compute $\text{res}_{1,2} = \text{res}(\mathfrak{p}_1, \mathfrak{p}_2, x)$ and $\text{res}_{2,3} = \text{res}(\mathfrak{p}_2, \mathfrak{p}_3, x)$.
(a) $\text{res}_{1,2} = 0$.
By Lemma 4.3 we know that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ have a common factor with positive degree in x. Thus $\mathfrak{p}_1$, $\mathfrak{p}_2$ are either linearly dependent or they have a common linear factor. If they are linearly dependent we can proceed as in case (2,3)(b). So let $\mathfrak{l}$ be a common linear factor and let $\widetilde{\mathfrak{p}}_1 = \mathfrak{p}_1/\mathfrak{l}$, $\widetilde{\mathfrak{p}}_2 = \mathfrak{p}_2/\mathfrak{l}$. Observe, $\widetilde{\mathfrak{p}}_1, \widetilde{\mathfrak{p}}_2$ are linear polynomials. Next we consider the two systems

$$\mathcal{I}_1\colon \quad \mathfrak{l} = 0, \quad \mathfrak{p}_3 = 0,$$

$$\mathcal{I}_2\colon \quad \widetilde{\mathfrak{p}}_1 = 0, \quad \widetilde{\mathfrak{p}}_2 = 0, \quad \mathfrak{p}_3 = 0.$$

Since $\mathcal{V}_{\text{aff}}(\mathcal{I}) \subset \mathcal{V}_{\text{aff}}(\mathcal{I}_1) \cup \mathcal{V}_{\text{aff}}(\mathcal{I}_2)$ it suffices to consider $\mathcal{V}_{\text{aff}}(\mathcal{I}_i)$. Since both systems contain linear polynomials we can reduce them to systems in at most 2 variables which can be handled by one of the methods described in one of the previous cases.
(b) $\text{res}_{2,3} = 0$. It is not hard to see that also in this case we can split our system in two systems containing linear polynomials and we can proceed as before.
(c) $\text{res}_{1,2} \neq 0$ and $\text{res}_{2,3} \neq 0$. Since $\deg(\text{res}_{1,2}) \leqslant 3$ and $\deg(\text{res}_{2,3}) \leqslant 4$ we can use the algorithm of Lemma 4.7 in order to determine a set $\widetilde{\mathcal{V}}_{\text{aff}}(\text{res}_{1,2}, \text{res}_{2,3})$ containing $\mathcal{V}_{\text{aff}}(\text{res}_{1,2}, \text{res}_{2,3})$. Now, for each $\mathcal{C} \in \widetilde{\mathcal{V}}_{\text{aff}}(\text{res}_{1,2}, \text{res}_{2,3})$ let

$$\mathcal{I}_{\mathcal{C}}\colon \quad \mathfrak{p}_i(x, y, z) = 0, \quad i = 1, \dots, 3,\ (y, z) \in \mathcal{C}.$$

Since $\mathcal{C}$ is a 0- or 1-dimensional affine subspace $\mathcal{I}_{\mathcal{C}}$ is a system of at most three polynomials in at most two variables. Hence by the previous cases we can determine a set $\widetilde{\mathcal{V}}_{\text{aff}}(\mathcal{I}_{\mathcal{C}})$ containing $\mathcal{V}_{\text{aff}}(\mathcal{I}_{\mathcal{C}})$ and we set

$$\widetilde{\mathcal{V}}_{\text{aff}}(\mathcal{I}) = \mathcal{W}^2(\mathcal{I}) \cup \bigcup_{\mathcal{C} \in \widetilde{\mathcal{V}}_{\text{aff}}(\text{res}_{1,2}, \text{res}_{2,3})} \widetilde{\mathcal{V}}(\mathcal{I}_{\mathcal{C}}).$$

On account of Lemma 4.8 we have $\mathcal{V}_{\text{aff}}(\mathcal{I}) \subset \widetilde{\mathcal{V}}_{\text{aff}}(\mathcal{I})$. $\square$

Remark. Up to now we have only discussed polynomials arising in case I of Algorithm 3.2. Case II can be treated completely similar to case I. In the cases III and IV we have seven hyperplanes determining the function $f_{H_{l_1},\dots,H_{l_7}}$, but it is easy to see that Lemma 4.1 keeps true. Hence also in these cases we just have to examine the isolated affine subspaces of polynomials of total degree at most three. Indeed, with some extra effort one can show that in the cases III and IV it suffices to consider matrices $(A_{i_1,\dots,i_7})$ with $\text{rank}(A_{i_1,\dots,i_7}) = 7$. However, we note that in case IV we have to replace the 6th hyperplane H_{l_6} by the hyperplane $\widetilde{H}_{l_6}$ given in Theorem 2.1. The hyperplane $\widetilde{H}_{l_6}$ can easily be constructed, if it exists at all.

Altogether, using the notation of Algorithm 3.2 we have the following result.

Corollary 4.3. *There exists an algorithm which computes a set $\mathcal{V}_{H_{l_1},\dots,H_{l_k}}$ consisting of finitely many affine subspaces $A_1, \dots, A_r$ of $\mathcal{S}_{H_{l_1},\dots,H_{l_k}}$ such that $f_{H_{l_1},\dots,H_{l_k}}$ is constant on any affine subspace A_i and each local minimum of $f_{H_{l_1},\dots,H_{l_k}}$ is contained in one of the spaces A_i.*

Observe, we do not determine the local minima $\mathcal{M}_{H_{l_1},\dots,H_{l_k}}$ of the function $f_{H_{l_1},\dots,H_{l_k}}$, but just a set of affine subspaces containing this set. It remains to check whether such an affine subspace A_i of the corollary contains an admissible lattice. To this end we use the criteria of Minkowski as given in Lemma 2.3, i.e., we are only interested in admissible lattices having a basis $W \in A_i$ such that the conditions of Lemma 2.3 are satisfied.

Lemma 4.9. *There exists an algorithm which finds for any affine subspace $A \in \mathcal{V}_{H_{l_1},\dots,H_{l_k}}$ a basis W of an admissible lattice* (*in the sense of Lemma* 2.3) *or asserts that no such lattice exists.*

Proof. For simplification we assume that $H_{l_i} = H_i$, $1 \leqslant i \leqslant k$. With $r = \dim A$, $r \in \{0, 3\}$, we may write

$$A = \left\{ W \in \mathbb{R}^{3\times 3}\colon\ W = C + \sum_{i=1}^{r} \lambda_i M_i \right\},$$

for suitable matrices $C, M_i \in \mathbb{R}^{3\times 3}$. Now for $r \geqslant 1$ and for $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathbb{R}^r$ let $W(\lambda) = C + \sum_{i=1}^{r} \lambda_i M_i$. If $r = 0$ we set $W(\lambda) = C$. According to Lemma 2.3 and Theorem 2.1 we have to find a $\lambda \in \mathbb{R}^r$ such that

$$\mathcal{U}^j_{W(\lambda)} \subset \operatorname{bd} P, \tag{4.11}$$

with $j = 1$ in case I, $j = 2$ in case II, and $j = 3$ in the cases III and IV. Furthermore, in the cases I and II we have the additional restrictions

$$\begin{aligned} &\text{Case I:} \quad (-1,1,1)_{W(\lambda)},\ (1,-1,1)_{W(\lambda)},\ (1,1,-1)_{W(\lambda)} \notin P, \\ &\text{Case II:} \quad (1,1,1)_{W(\lambda)} \notin P. \end{aligned} \tag{4.12}$$

Let us denote by $u^i_{W(\lambda)}$, $1 \leqslant i \leqslant k$, the vectors of the test set $\mathcal{U}^j_{W(\lambda)}$. This means we have $k = 6$ if $j \leqslant 2$, and $k = 7$ otherwise. Observe, by construction we know that all vectors $u^i_{W(\lambda)}$ lie in supporting hyperplanes of the polytope. Therefore, if $r = 0$ we can verify (4.11) and (4.12) by just checking the facet defining inequalities of the polytope for the corresponding points $u^i_{W(\lambda)}$, etc. Thus in the following we assume $r \geqslant 1$. Then we define a set $\mathcal{A}$ by (cf. notation at the beginning of this section)

$$\mathcal{A} = \left\{ \lambda \in \mathbb{R}^r\colon\ \mathcal{U}^j_{W(\lambda)} \subset \operatorname{bd}(P) \right\} = \left\{ \lambda \in \mathbb{R}^r\colon\ u^i_{W(\lambda)} \in H^+_{i_j} \text{ for all } F_{i_j} \in \mathcal{N}\left(F^i\right),\ 1 \leqslant i \leqslant k \right\}. \tag{4.13}$$

Using standard methods from Linear Programming we can easily decide whether $\mathcal{A} = \emptyset$ or we can find a point $\lambda^* \in \mathcal{A}$. If $\mathcal{A} = \emptyset$ then the affine subspace does not contain an admissible lattice. So let $\lambda^* \in \mathcal{A}$. In the cases III and IV or if $W(\lambda^*)$ also satisfies (4.12) in the cases I and II, we are done and we have found an admissible lattice. So let us assume that we are in case I or II, $\mathcal{A} \neq \emptyset$ and $W(\lambda^*)$ violates (4.12). The most simple way to decide whether there exists a $\lambda \in \mathcal{A}$ satisfying (4.12) is the following. In case I we consider for $i, j, l \in \{1, \dots, n\}$ the sets

$$\begin{aligned} \mathcal{A}_{i,j,l} = \big\{ \lambda \in \mathbb{R}^r\colon\ \lambda \in \mathcal{A} \text{ and } (-1,1,1)_{W(\lambda)} \in H^-_i, \\ (1,-1,1)_{W(\lambda)} \in H^-_j,\ (1,1,-1)_{W(\lambda)} \in H^-_l \big\}. \end{aligned} \tag{4.14}$$

In case II we set for $1 \leqslant i \leqslant n$

$$\mathcal{A}_i = \left\{ \lambda \in \mathbb{R}^r\colon\ \lambda \in \mathcal{A} \text{ and } (1,1,1)_{W(\lambda)} \in H^-_i \right\}. \tag{4.15}$$

Obviously, in the case I (case II) there exists an admissible lattice in the affine subspace A if and only if there exist $i, j, l \in \{1, \dots, n\}$ ($i \in \{1, \dots, n\}$) such that $\mathcal{A}_{i,j,l} \neq \emptyset$ ($\mathcal{A}_i \neq \emptyset$). Again, using tools from Linear Programming we can either find a λ^* in one of sets $\mathcal{A}_{i,j,l}$ ($\mathcal{A}_i$), and thus an admissible lattice $W(\lambda^*)$, or we know that A contains no admissible lattices. □

Remark. Instead of considering the n^3 (n) feasibility problems in (4.14) ((4.15)) one can argue as follows. First let us assume that we are in case I and that $W(\lambda^*)$ violates one of the restrictions in (4.12). Then we claim that we do not have to consider this case further: Of course, if each $W(\lambda)$ violates (4.12) for all $\lambda \in \mathcal{A}$ satisfying (4.11) then this is trivial. Hence suppose that there exists a $\lambda^\circ \in \mathcal{A}$ satisfying (4.12) such that $W(\lambda^\circ)\mathbb{Z}^3$ is a critical lattice of P. Then there exists a $\mu \in \operatorname{conv}\{\lambda^\circ, \lambda^*\} \subset \mathcal{A}$ such that one of the points of (4.12) with respect to $W(\mu)$ lies in the boundary of P and the other points are not contained in the interior of P. By definition, $W(\mu)\mathbb{Z}^3$ is a critical lattice of P, too. Without loss of generality let $(1, 1, -1)_{W(\mu)} \in \operatorname{bd} P$ and $(1, -1, 1)_{W(\mu)}, (-1, 1, 1)_{W(\mu)} \notin \operatorname{int} P$. Furthermore, let H_7 be a supporting hyperplane of a facet of P containing $(1, 1, -1)_{W(\mu)}$. Now let $w^i(\mu)$ be the ith column vector of $W(\mu)$ and let $\overline{W}(\mu)$ be the matrix with columns $w^1(\mu) - w^3(\mu)$, $w^2(\mu)$ and $-w^2(\mu) + w^3(\mu)$. Then $\overline{W}(\mu)$ is just another basis of the lattice $W(\mu)\mathbb{Z}^3$ and we get

$$
\begin{aligned}
&(1,0,0)_{W(\mu)} = (1,1,1)_{\overline{W}(\mu)}, &&(0,1,0)_{W(\mu)} = (0,1,0)_{\overline{W}(\mu)},\\
&(0,0,1)_{W(\mu)} = (0,1,1)_{\overline{W}(\mu)}, &&(1,-1,0)_{W(\mu)} = (1,0,1)_{\overline{W}(\mu)},\\
&(1,0,-1)_{W(\mu)} = (1,0,0)_{\overline{W}(\mu)}, &&(0,-1,1)_{W(\mu)} = (0,0,1)_{\overline{W}(\mu)},\\
&(1,1,-1)_{W(\mu)} = (1,1,0)_{\overline{W}(\mu)}.
\end{aligned}
$$

Thus the test set $\mathcal{U}^1_{W(\mu)}$ plus the additional point $(1, 1, -1)_{W(\mu)}$ is equivalent to the test set $\mathcal{U}^3_{\overline{W}(\mu)}$. Since $\overline{W}(\mu)\mathbb{Z}^3$ is a critical lattice of P it follows from the work of Minkowski (cf. [20, p. 27]) that we shall find a basis of this lattice in case III. In case II we can apply the argumentation and this means that in all cases it is sufficient to determine only one point of the set $\mathcal{A}$.

To sum it up, we finally have the following algorithm for determining a densest lattice packing of a 3-dimensional polytope $P \in \mathcal{K}^3$ (cf. Algorithm 3.2).

Algorithm 4.2 (Densest lattice packing of a 3-polytope).
Input: A polytope $P \in \mathcal{K}^3$ given by the supporting hyperplanes H_i, $1 \leqslant i \leqslant m$, or by its vertices.
Output: A densest packing lattice of P.
- Find the supporting hyperplanes H_i, $1 \leqslant i \leqslant n$, of the facets F_i of the polytope $P_0 = (P - P) \in \mathcal{K}^3_0$ (cf. (2.1)) and compute the lattice description of P_0. With respect to P_0 do
- For each of the cases I–IV of Theorem 2.1 do
  I1. Compute the sets $\mathcal{G}(F_i)$ and $\mathcal{G}$ with Algorithm 3.1.
  - For three facets $F_{l_1}, F_{l_2}, F_{l_3}$ satisfying $F_{l_2} \in \mathcal{G}(F_{l_1})$ and $F_{l_3} \in \mathcal{G}(F_{l_1}) \cap \mathcal{G}(F_{l_2})$ do
    - For every choice of facets F_{l_i}, $4 \leqslant i \leqslant k$, with $(F_{l_1}, F_{l_2}, F_{l_6}), (F_{l_2}, F_{l_3}, F_{l_4}), (F_{l_1}, F_{l_3}, F_{l_5}) \in \mathcal{G}$ do
      R0. If $\widetilde{\mathcal{S}}_{H_{l_1},\dots,H_{l_k}} \neq \emptyset$ (cf. (3.2)) do
        R1. Determine $\mathcal{S}_{H_{l_1},\dots,H_{l_k}}$ (cf. (4.3)).
        If $\operatorname{rank}(A_{i_1,\dots,i_k}) \geqslant 6$ (cf. Lemma 4.1 and the remark after Theorem 4.1) do

R2. Determine a set $\mathcal{V}_{H_{l_1},\ldots,H_{l_k}}$ consisting of finitely many affine subspaces $A_1,\ldots,A_r$ of $\mathcal{S}_{H_{l_1},\ldots,H_{l_k}}$ such that $f_{H_{l_1},\ldots,H_{l_k}}$ is constant on any affine subspace A_i and each local minimum of $f_{H_{l_1},\ldots,H_{l_k}}$ is contained in one of the spaces A_i (cf. Corollary 4.3).

R3. For each affine subspace $A \in \mathcal{V}_{H_{l_1},\ldots,H_{l_k}}$ find a basis W such that $W \cdot \mathbb{Z}^d$ satisfies the criterion (1) of Lemma 2.3 in the first case and criterion (2) in the remaining cases, or asserts that such lattice does not exist (cf. Lemma 4.9).

- Among all calculated admissible lattices find one with minimal determinant. The corresponding lattice is a critical lattice of P_0 and a densest packing lattice of P.

5. Densest lattice packings of the regular and Archimedean polytopes

In this section we present densest packing lattices Λ^* of all regular and Archimedean polytopes. Since a densest packing lattice depends on the representation of the polytope, and in order to make the results more transparent, we shall also give the coordinates of our representations of the polytopes used for the algorithm. For a polytope P let $f=(f_0,f_1,f_2)$ be its f-vector, i.e., f_i is the number of i-faces. Futhermore we shall use the following abbrevations. Let $\tau=(1/2)(1+\sqrt{5})$ and let D_3 be the fcc-lattice with basis $(1,1,0)^{\mathrm{T}}$, $(1,0,1)^{\mathrm{T}}$, $(0,1,1)^{\mathrm{T}}$. Finally, let P_{rtc} be the so called rhombic triacontahedron given by

$$\begin{aligned} P_{\mathrm{rtc}} = \Big\{ x\in\mathbb{R}^3\colon\ |\tau x_i|\leqslant 1,\ \left|\frac{1}{2}x_1+\frac{\tau}{2}x_2+\frac{\tau+1}{2}x_3\right|\leqslant 1,& \\ \left|\frac{\tau}{2}x_1+\frac{\tau+1}{2}x_2+\frac{1}{2}x_3\right|\leqslant 1,\ \left|\frac{\tau+1}{2}x_1+\frac{1}{2}x_2+\frac{\tau}{2}x_3\right|\leqslant 1\Big\}.& \end{aligned} \tag{5.1}$$

For the identification of the polytopes we shall use the Wythoff symbols.

- *Tetrahedron*, $3\,|\,2\,3$, $f=(4,6,4)$ (cf. [16]).

$$P_{\mathrm{t}}=\{x\in\mathbb{R}^3\colon\ x_1+x_2+x_3\leqslant 1,\ -x_1-x_2+x_3\leqslant 1,\ -x_1+x_2-x_3\leqslant 1,\ x_1-x_2-x_3\leqslant 1\},$$

$$\Lambda^*(P_{\mathrm{t}})=2\big((1,-\tfrac{1}{6},-\tfrac{1}{6})^{\mathrm{T}},(-\tfrac{1}{6},1,-\tfrac{1}{6})^{\mathrm{T}},(-\tfrac{1}{6},-\tfrac{1}{6},1)^{\mathrm{T}}\big)\mathbb{Z}^3,$$

$$\delta^*(P_{\mathrm{t}})=\tfrac{18}{49}\approx 0.367346938.$$

- *Cube*, $3\,|\,2\,4$, $f=(8,12,6)$.

$$P_{\mathrm{c}}=\{x\in\mathbb{R}^3\colon\ |x_i|\leqslant 1\},\qquad \Lambda^*(P_{\mathrm{c}})=2\mathbb{Z}^3,\qquad \delta^*(P_{\mathrm{c}})=1.$$

- *Octahedron*, $4\,|\,2\,3$, $f=(6,12,8)$ (cf. [20]).

$$P_{\mathrm{o}}=\{x\in\mathbb{R}^3\colon\ |x_1|+|x_2|+|x_3|\leqslant 1\},$$

$$\Lambda^*(P_{\mathrm{o}})=2\big((\tfrac{1}{3},\tfrac{1}{2},\tfrac{1}{6})^{\mathrm{T}},(-\tfrac{1}{6},-\tfrac{1}{3},\tfrac{1}{2})^{\mathrm{T}},(-\tfrac{1}{2},\tfrac{1}{6},-\tfrac{1}{3})^{\mathrm{T}}\big)\mathbb{Z}^3,$$

$$\delta^*(P_{\mathrm{o}})=\tfrac{18}{19}\approx 0.947368421.$$

- *Dodecahedron*, $3\mid 2\,5$, $f=(20,30,12)$.

$$P_{\mathrm{d}}=\{x\in\mathbb{R}^3\colon\ |\tau x_1|+|x_2|\leqslant 1,\ |\tau x_2|+|x_3|\leqslant 1,\ |\tau x_3|+|x_1|\leqslant 1\},$$
$$\Lambda^*\big((1+\tau)P_{\mathrm{d}}\big)=2D_3,\qquad \delta^*(P_{\mathrm{d}})=\frac{2+\tau}{4}\approx 0.904508497.$$

- *Icosahedron*, $5\mid 2\,3$, $f=(12,30,20)$.

$$\begin{aligned}P_{\mathrm{i}}=\{x\in\mathbb{R}^3\colon\ &|x_1|+|x_2|+|x_3|\leqslant 1,\ |\tau x_1|+|(1/\tau)x_3|\leqslant 1,\\ &|\tau x_2|+|(1/\tau)x_1|\leqslant 1,\ |\tau x_3|+|(1/\tau)x_2|\leqslant 1\},\end{aligned}$$
$$\Lambda^*\big((1+\tau)P_{\mathrm{i}}\big)=2\big(w^1(\overline{x}),w^2(\overline{x}),w^3(\overline{x})\big)\mathbb{Z}^3,\quad\text{where}$$
$$w^1(\overline{x})=\begin{pmatrix}\big(-\frac{33}{8}-\frac{39}{8}\sqrt{5}\big)\overline{x}^2+\big(\frac{39}{4}+\frac{33}{4}\sqrt{5}\big)\overline{x}-\frac{11}{4}-\frac{3}{2}\sqrt{5}\\ \big(-\frac{1}{4}-\frac{1}{4}\sqrt{5}\big)\overline{x}+1+\frac{1}{2}\sqrt{5}\\ \big(\frac{33}{8}+\frac{39}{8}\sqrt{5}\big)\overline{x}^2+\big(-\frac{19}{2}-8\sqrt{5}\big)\overline{x}+\frac{13}{4}+\frac{3}{2}\sqrt{5}\end{pmatrix},$$
$$w^2(\overline{x})=\begin{pmatrix}\big(-\frac{33}{40}\sqrt{5}-\frac{39}{8}\big)\overline{x}^2+\big(\frac{41}{20}\sqrt{5}+\frac{35}{4}\big)\overline{x}-\frac{5}{2}-\frac{23}{20}\sqrt{5}\\ \big(\frac{5}{4}+\frac{1}{4}\sqrt{5}\big)\overline{x}-1-\frac{1}{2}\sqrt{5}\\ \big(-\frac{33}{40}\sqrt{5}-\frac{39}{8}\big)\overline{x}^2+\big(\frac{9}{5}\sqrt{5}+\frac{15}{2}\big)\overline{x}-\frac{3}{20}\sqrt{5}\end{pmatrix}, \tag{5.2}$$
$$w^3(\overline{x})=\Big(\big(\tfrac{1}{2}\sqrt{5}+\tfrac{3}{2}\big)\overline{x}-2-\sqrt{5},\quad \overline{x},\quad 0\Big)^{\mathrm{T}},$$

and $\overline{x}$ is the unique root with $\overline{x}\in(1,2)$ of

$$1086\,x^3+\big(-1063-111\sqrt{5}\big)x^2+\big(15\sqrt{5}+43\big)x+102+44\sqrt{5},$$
$$\delta^*(P_{\mathrm{i}})=\frac{5(1+\tau)}{|\det(w^1(\overline{x}),w^2(\overline{x}),w^3(\overline{x}))|}\approx 0.836357445.$$

- *Cubeoctahedron*, $2\mid 3\,4$, $f=(12,24,14)$ (cf. [16]).

$$P_{\mathrm{co}}=\{x\in\mathbb{R}^3\colon\ x\in P_{\mathrm{c}}\cap 2\cdot P_{\mathrm{o}}\},$$
$$\Lambda^*(P_{\mathrm{co}})=\Lambda^*(P_{\mathrm{t}}),\qquad \delta^*(P_{\mathrm{co}})=\frac{45}{49}\approx 0.918367346.$$

- *Icosidodecahedron*, $2\mid 3\,5$, $f=(30,60,32)$.

$$P_{\mathrm{id}}=\{x\in\mathbb{R}^3\colon\ x\in P_{\mathrm{i}}\cap P_{\mathrm{d}}\},$$
$$\Lambda^*\big((1+\tau)P_{\mathrm{id}}\big)=2D_3,\qquad \delta^*(P_{\mathrm{id}})=\frac{14+17\tau}{48}\approx 0.864720371.$$

- *Rhombic cubeoctahedron*, $3\,4\mid 2$, $f=(24,48,26)$.

$$\begin{aligned}P_{\mathrm{rco}}=\{x\in\mathbb{R}^3\colon\ &|x_1|+|x_2|\leqslant 2,\ |x_1|+|x_3|\leqslant 2,\ |x_2|+|x_3|\leqslant 2,\\ &\text{and } x\in\sqrt{2}\cdot P_{\mathrm{c}}\cap\big(4-\sqrt{2}\big)P_{\mathrm{o}}\},\end{aligned}$$
$$\Lambda^*(P_{\mathrm{rco}})=2D_3,\qquad \delta^*(P_{\mathrm{rco}})=\frac{16\sqrt{2}-20}{3}\approx 0.875805666.$$

- *Rhombic icosidodecahedron*, $3\,5\,|\,2$, $f=(60,120,62)$.

$$P_{\mathrm{rid}}=\{x\in\mathbb{R}^3\colon\ x\in(3\tau+2)\cdot P_{\mathrm{rtc}}\cap(4\tau+1)\cdot P_{\mathrm{i}}\cap\big(3(1+\tau)\big)\cdot P_{\mathrm{d}}\},$$

$$\Lambda^*(P_{\mathrm{rid}})=2\left(\left(\frac{\tau-1}{4\tau+2},\frac{7}{2},\frac{9\tau+4}{4\tau+2}\right)^{\mathrm{T}},\left(\frac{9\tau+4}{4\tau+2},\frac{\tau-1}{4\tau+2},\frac{7}{2}\right)^{\mathrm{T}},\left(\frac{7}{2},\frac{9\tau+4}{4\tau+2},\frac{\tau-1}{4\tau+2}\right)^{\mathrm{T}}\right)\mathbb{Z}^3,$$

$$\delta^*(P_{\mathrm{rid}})=\frac{8\tau+46}{36\tau+15}\approx 0.804708487.$$

- *Truncated cube*, $2\,3\,|\,4$, $f=(24,36,14)$.

$$P_{\mathrm{trc}}=\{x\in\mathbb{R}^3\colon\ x\in P_{\mathrm{c}}\cap\big(1+\sqrt{2}\big)P_{\mathrm{o}}\},$$

$$\Lambda^*(P_{\mathrm{trc}})=2\big((1,-\alpha,0)^{\mathrm{T}},(0,1,-\alpha)^{\mathrm{T}},(-\alpha,0,1)^{\mathrm{T}}\big)\mathbb{Z}^3,\quad \alpha=\frac{(2-\sqrt{2})}{3},$$

$$\delta^*(P_{\mathrm{trc}})=\frac{9}{5+3\sqrt{2}}\approx 0.973747688.$$

- *Truncated octahedron*, $2\,4\,|\,2$, $f=(24,36,14)$.

$$P_{\mathrm{tro}}=\{x\in\mathbb{R}^3\colon\ x\in P_{\mathrm{c}}\cap\tfrac{3}{2}P_{\mathrm{o}}\},$$

$$\Lambda^*(P_{\mathrm{tro}})=2\big((1,0,0)^{\mathrm{T}},(1,1,0)^{\mathrm{T}},\big(\tfrac12,\tfrac12,-\tfrac12\big)^{\mathrm{T}}\big)\mathbb{Z}^3,\qquad \delta^*(P_{\mathrm{tro}})=1.$$

- *Truncated dodecahedron*, $2\,3\,|\,5$, $f=(60,90,32)$.

$$P_{\mathrm{trd}}=\left\{x\in\mathbb{R}^3\colon\ x\in(1+\tau)\cdot P_{\mathrm{d}}\cap\frac{7+12\tau}{3+4\tau}\cdot P_{\mathrm{i}}\right\},$$

$$\Lambda^*(P_{\mathrm{trd}})=2D_3,\qquad \delta^*(P_{\mathrm{trd}})=\frac14\cdot\frac{5\tau+16}{6\tau-3}\approx 0.897787626.$$

- *Truncated icosahedron*, $2\,5\,|\,3$, $f=(60,90,32)$.

$$P_{\mathrm{tri}}=\{x\in\mathbb{R}^3\colon\ x\in(1+\tau)\cdot P_{\mathrm{i}}\cap(4/3+\tau)\cdot P_{\mathrm{d}}\},$$

$$\Lambda^*(P_{\mathrm{tri}})=\Lambda^*\big((1+\tau)P_{\mathrm{i}}\big),$$

$$\delta^*(P_{\mathrm{tri}})=\frac{43\sqrt{5}+125}{108\cdot|\det(w^1(\overline{x}),w^2(\overline{x}),w^3(\overline{x}))|}\approx 0.7849877759\quad(\text{cf. }(5.2)).$$

- *Truncated cubeoctahedron*, $2\,3\,4\,|$, $f=(48,72,26)$.

$$\begin{aligned}P_{\mathrm{trco}}=\{x\in\mathbb{R}^3\colon\ &|x_1|+|x_2|\leqslant 2+3\sqrt{2},\ |x_2|+|x_3|\leqslant 2+3\sqrt{2},\\ &|x_2|+|x_3|\leqslant 2+3\sqrt{2}\text{ and }x\in\big(2\sqrt{2}+1\big)P_{\mathrm{c}}\cap\big(3\sqrt{2}+3\big)P_{\mathrm{o}}\},\end{aligned}$$

$$\begin{aligned}\Lambda^*(P_{\mathrm{trco}})=2\Big(&\big(2\sqrt{2}+1,-2\sqrt{2}-\tfrac12+\alpha,2\sqrt{2}+\tfrac12-\alpha\big)^{\mathrm{T}},\\ &\big(\tfrac14\sqrt{2}-\tfrac34+\tfrac12\alpha,-\tfrac34\sqrt{2}+\tfrac14+\tfrac12\alpha,2\sqrt{2}+1\big)^{\mathrm{T}},\\ &\big(\tfrac74+\tfrac74\sqrt{2}-\tfrac12\alpha,\tfrac12+\alpha,\tfrac54\sqrt{2}+\tfrac34-\tfrac12\alpha\big)^{\mathrm{T}}\Big)\mathbb{Z}^3,\end{aligned}$$

where $\alpha=\frac16\sqrt{33}(\sqrt{2}+1)$,

$$\delta^*(P_{\mathrm{trco}})=\tfrac{99}{992}\sqrt{66}-\tfrac{231}{1984}\sqrt{33}+\tfrac{2835}{992}\sqrt{2}-\tfrac{6615}{1984}\approx 0.849373252.$$

- *Truncated icosidodecahedron*, $2\,3\,5\,|$, $f = (120, 180, 62)$.

$$P_{\mathrm{trid}} = \left\{x \in \mathbb{R}^3\colon\ x \in (5\tau + 4) \cdot P_{\mathrm{rtc}} \cap (6\tau + 3) \cdot P_{\mathrm{i}} \cap \left(5(1+\tau)\right) \cdot P_{\mathrm{d}}\right\},$$
$$\Lambda^*\left(\tfrac{1}{5} P_{\mathrm{trid}}\right) = 2d_3, \qquad \delta^*(P_{\mathrm{trid}}) = \tfrac{2}{5}\tau + \tfrac{9}{50} \approx 0.827213595.$$

- *Truncated tetrahedron*, $2\,3\,|\,3$, $f = (12, 18, 8)$.

$$P_{\mathrm{trt}} = \left\{x \in \mathbb{R}^3\colon\ x \in 5 \cdot P_{\mathrm{t}} \cap -3 \cdot P_{\mathrm{t}}\right\},$$
$$\Lambda^*(P_{\mathrm{trt}}) = 2\left(\left(\tfrac{2}{3}, 2, \tfrac{4}{3}\right)^{\mathrm{T}}, \left(2, -\tfrac{4}{3} - \tfrac{2}{3}\right)^{\mathrm{T}}, \left(-\tfrac{4}{3}, \tfrac{2}{3}, -2\right)^{\mathrm{T}}\right)\mathbb{Z}^3,$$
$$\delta^*(P_{\mathrm{trt}}) = \tfrac{207}{304} \approx 0.680921053.$$

- *Snub cube*, $|\,2\,3\,4$, $f = (24, 60, 38)$.
  Let P_{sc} be the snub cube such that the 6 quadrangle facets lie in the hyperplanes $\{x \in \mathbb{R}^3\colon\ x_i = \pm 1\}$, $1 \leqslant i \leqslant 3$, and let y^* be the unique real solution of $y^3 + y^2 + y = 1$.

$$\Lambda^*(P_{\mathrm{sc}}) = 2 \cdot \left((1,0,0)^{\mathrm{T}}, (0,0,1)^{\mathrm{T}}, \left(\frac{1}{2}, \frac{1}{y^*} - 1, -\frac{1}{2}\right)^{\mathrm{T}}\right),$$
$$\delta^*(P_{\mathrm{sc}}) = \tfrac{1}{2} + \tfrac{1}{6}y^* + \tfrac{2}{3}\left(y^*\right)^2 \approx 0.78769996.$$

- *Snub dodecahedron*, $|\,2\,3\,5$, $f = (60, 150, 92)$.
  Let P_{sd} be the snub dodecahedron such that the 12 pentagonal facets lie in the supporting hyperplanes of the facets of the dodecahedron $(1+\tau)P_{\mathrm{d}}$.

$$\Lambda^*(P_{\mathrm{sd}}) = 2D_3, \qquad \delta^*(P_{\mathrm{sd}}) = \frac{\mathrm{vol}(P_{\mathrm{sd}})}{16} \approx 0.788640117.$$

Theorem 5.1. *The above list contains the densities of a densest lattice packing of all regular and Archimedean polytopes.*

Proof. Algorithm 4.2. □

Remarks.
(a) In order to get exact values for the densities as given in the above list we first use a numerical implementation of Algorithm 4.2, by which we determine an optimal selection $F_{l_1}, \ldots, F_{l_k}$ of facets corresponding to a critical lattice. Then for this special choice of factes we carry out the steps R1–R3 with the symbolic computer algebra system Maple V Release 5.
(b) The last three polytopes P_{trt}, P_{sc}, P_{sd} are not centrally symmetric and thus one has to calculate the difference bodies first. The difference body of P_{sc} is a polytope with 74 facets, whereas $\frac{1}{2}(P_{\mathrm{sd}} - P_{\mathrm{sd}})$ has already 182 facets. We also have computed the densest packing lattice of the dual polytopes of the Archimedean polytopes, the so called Catalan polytopes. Thereby we had to determine packing lattices of polytopes with more than 380 facets. The CPU time for the determination of the densest packing lattices of all regular and Archimedean polytopes is about 5.5 hours on a PC with a 266 Mhz Pentium II processor.

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