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Hyers–Ulam stability of linear differential equations of first order, II

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Abstract

Let X be a complex Banach space and let I be an open interval. For given functions $g : I \to \mathbb{C}$, $h : I \to X$ and $\varphi : I \to [0, \infty)$, we will solve the differential inequality $\|y'(t) + g(t)y(t) + h(t)\| \leq \varphi(t)$ for the class of continuously differentiable functions $y : I \to X$ under some integrability conditions.

Keywords: Hyers–Ulam stability; Hyers–Ulam–Rassias stability; Differential equation

1. Introduction

Let X be a normed space and let I be an open interval. Assume that for any function $f : I \to X$ satisfying the differential inequality

$$\|a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) + h(t)\| \leq \varepsilon$$

for all $t \in I$ and for some $\varepsilon \geq 0$, there exists a solution $f_0 : I \to X$ of the differential equation

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) + h(t) = 0$$

such that $\|f(t) - f_0(t)\| \leq K(\varepsilon)$ for any $t \in I$, where $K(\varepsilon)$ is an expression for ε only. Then, we say that the above differential equation has the Hyers–Ulam stability.

If the above statement is also true when we replace ε and $K(\varepsilon)$ by $\varphi(t)$ and $\Phi(t)$, where $\varphi, \Phi : I \to [0, \infty)$ are functions not depending on f and f_0 explicitly, then we say that the corresponding differential equation has the Hyers–Ulam–Rassias stability (or the generalized Hyers–Ulam stability).

We may apply these terminologies for other differential equations. For more detailed definitions of the Hyers–Ulam stability and the Hyers–Ulam–Rassias stability, refer to [1,2].

Alsina and Ger were the first authors who investigated the Hyers–Ulam stability of differential equations: They proved in [3] that if a differentiable function $f : I \to \mathbb{R}$ is a solution of the differential inequality $|y'(t) - y(t)| \leq \varepsilon$, where I is an open subinterval of $\mathbb{R}$, then there exists a solution $f_0 : I \to \mathbb{R}$ of the differential equation $y'(t) = y(t)$ such that $|f(t) - f_0(t)| \leq 3\varepsilon$ for any $t \in I$.

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This result of Alsina and Ger has been generalized by Takahasi et al.: they proved in [4] that the Hyers–Ulam stability holds for the Banach space valued differential equation $y'(t) = \lambda y(t)$ (see also [5]).

In [6], Miura et al. also proved the Hyers–Ulam stability of linear differential equations of first order, $y'(t) + g(t)y(t) = 0$, where $g(t)$ is a continuous function, while the author [7] proved the Hyers–Ulam stability of differential equations of the form $c(t)y'(t) = y(t)$.

The aim of this work is to improve the results of [6,7] by proving the Hyers–Ulam–Rassias stability of the following nonhomogeneous linear differential equation of first order:

$$y'(t) + g(t)y(t) + h(t) = 0. \tag{1}$$

We assume that X is a complex Banach space and $I = (a, b)$ is an arbitrary interval. We moreover assume that $g : I \to \mathbb{C}$, $h : I \to X$, and $\varphi : I \to [0, \infty)$ are functions such that for arbitrary $c \in I$, $g(t)$ and $\exp\{\int_a^t g(u)\mathrm{d}u\}h(t)$ are integrable on (a, c), and further such that $\varphi(t)\exp\{\Re(\int_a^t g(u)\mathrm{d}u)\}$ is integrable on I, where we use $\Re(\omega)$ to denote the real part of complex numbers ω.

We prove in Theorem 1 that if a continuously differentiable function $y : I \to X$ satisfies the differential inequality

$$\|y'(t) + g(t)y(t) + h(t)\| \le \varphi(t) \tag{2}$$

for all $t \in I$, then there exists a unique solution $y_0(t)$ of the differential equation (1) such that

$$\|y(t) - y_0(t)\| \le \exp\left\{-\Re\left(\int_a^t g(u)\mathrm{d}u\right)\right\}\int_t^b \varphi(v)\exp\left\{\Re\left(\int_a^v g(u)\mathrm{d}u\right)\right\}\mathrm{d}v$$

for all $t \in I$.

2. Main results

In the following theorem, we will prove the Hyers–Ulam–Rassias stability of the nonhomogeneous linear differential equation (1). More precisely, we solve the differential inequality (2) for the class of functions $y : I \to X$.

Theorem 1. *Let X be a complex Banach space and let $I = (a, b)$ be an open interval, where $a, b \in \mathbb{R} \cup \{\pm\infty\}$ are arbitrarily given with $a < b$. Assume that $g : I \to \mathbb{C}$ and $h : I \to X$ are continuous functions such that $g(t)$ and $\exp\{\int_a^t g(u)\mathrm{d}u\}h(t)$ are integrable on (a, c) for each $c \in I$. Moreover, suppose $\varphi : I \to [0, \infty)$ is a function such that $\varphi(t)\exp\{\Re(\int_a^t g(u)\mathrm{d}u)\}$ is integrable on I. If a continuously differentiable function $y : I \to X$ satisfies the differential inequality (2) for all $t \in I$, then there exists a unique $x \in X$ such that*

$$\begin{aligned}\left\|y(t) - \exp\left\{-\int_a^t g(u)\mathrm{d}u\right\}\left(x - \int_a^t \exp\left\{\int_a^v g(u)\mathrm{d}u\right\}h(v)\mathrm{d}v\right)\right\| \\ \le \exp\left\{-\Re\left(\int_a^t g(u)\mathrm{d}u\right)\right\}\int_t^b \varphi(v)\exp\left\{\Re\left(\int_a^v g(u)\mathrm{d}u\right)\right\}\mathrm{d}v\end{aligned} \tag{3}$$

for every $t \in I$.

Proof. For simplicity, we use the following notation:

$$z(t) := \exp\left\{\int_a^t g(u)\mathrm{d}u\right\}y(t) + \int_a^t \exp\left\{\int_a^v g(u)\mathrm{d}u\right\}h(v)\mathrm{d}v$$

for each $t \in I$. By making use of this notation and by (2), we get

$$\begin{aligned}\|z(t) - z(s)\| &= \left\|\exp\left\{\int_a^t g(u)\mathrm{d}u\right\}y(t) - \exp\left\{\int_a^s g(u)\mathrm{d}u\right\}y(s)\right. \\ &\quad \left. + \int_s^t \exp\left\{\int_a^v g(u)\mathrm{d}u\right\}h(v)\mathrm{d}v\right\| \\ &= \left\|\int_s^t \frac{\mathrm{d}}{\mathrm{d}v}\left[\exp\left\{\int_a^v g(u)\mathrm{d}u\right\}y(v)\right]\mathrm{d}v\right.\end{aligned}$$

$$
\begin{aligned}
&\quad + \int_s^t \exp\left\{\int_a^v g(u)\mathrm{d}u\right\} h(v)\mathrm{d}v\Bigg\| \\
&= \left\|\int_s^t \exp\left\{\int_a^v g(u)\mathrm{d}u\right\}\{y'(v)+g(v)y(v)+h(v)\}\mathrm{d}v\right\| \\
&\leq \left|\int_s^t \exp\left\{\Re\left(\int_a^v g(u)\mathrm{d}u\right)\right\}\varphi(v)\mathrm{d}v\right| \qquad (4)
\end{aligned}
$$

for any $s, t \in I$.

Since $\varphi(t)\exp\{\Re(\int_a^t g(u)\mathrm{d}u)\}$ is assumed to be integrable on I, we may select $t_0 \in I$, for any given $\varepsilon > 0$, such that $s, t \geq t_0$ implies $\|z(t) - z(s)\| < \varepsilon$. That is, $\{z(s)\}_{s\in I}$ is a Cauchy net and hence there exists an $x \in X$ such that $z(s)$ converges to x as $s \to b$, since X is complete.

Finally, it follows from (4) and the above argument that for any $t \in I$,

$$
\begin{aligned}
&\left\|y(t) - \exp\left\{-\int_a^t g(u)\mathrm{d}u\right\}\left(x - \int_a^t \exp\left\{\int_a^v g(u)\mathrm{d}u\right\}h(v)\mathrm{d}v\right)\right\| \\
&= \left\|\exp\left\{-\int_a^t g(u)\mathrm{d}u\right\}(z(t)-x)\right\| \\
&\leq \exp\left\{-\Re\left(\int_a^t g(u)\mathrm{d}u\right)\right\}\|z(t)-z(s)\| \\
&\quad + \exp\left\{-\Re\left(\int_a^t g(u)\mathrm{d}u\right)\right\}\|z(s)-x\| \\
&\leq \exp\left\{-\Re\left(\int_a^t g(u)\mathrm{d}u\right)\right\}\left|\int_s^t \varphi(v)\exp\left\{\Re\left(\int_a^v g(u)\mathrm{d}u\right)\right\}\mathrm{d}v\right| \\
&\quad + \exp\left\{-\Re\left(\int_a^t g(u)\mathrm{d}u\right)\right\}\|z(s)-x\| \\
&\quad \to \exp\left\{-\Re\left(\int_a^t g(u)\mathrm{d}u\right)\right\}\int_t^b \varphi(v)\exp\left\{\Re\left(\int_a^v g(u)\mathrm{d}u\right)\right\}\mathrm{d}v
\end{aligned}
$$

as $s \to b$, since $z(s) \to x$ as $s \to b$.

It now remains to prove the uniqueness of x. Assume that $x_1 \in X$ also satisfies the inequality (3) in place of x. Then, we have

$$
\begin{aligned}
&\left\|\exp\left\{-\int_a^t g(u)\mathrm{d}u\right\}(x_1 - x)\right\| \\
&\quad \leq 2\exp\left\{-\Re\left(\int_a^t g(u)\mathrm{d}u\right)\right\}\int_t^b \varphi(v)\exp\left\{\Re\left(\int_a^v g(u)\mathrm{d}u\right)\right\}\mathrm{d}v
\end{aligned}
$$

for any $t \in I$. It follows from the integrability hypotheses that

$$
\|x_1 - x\| \leq 2\int_t^b \varphi(v)\exp\left\{\Re\left(\int_a^v g(u)\mathrm{d}u\right)\right\}\mathrm{d}v \to 0
$$

as $t \to b$. This implies the uniqueness of x. $\square$

Remark 1. We may now remark that

$$
y(t) = \exp\left\{-\int_a^t g(u)\mathrm{d}u\right\}\left(x - \int_a^t \exp\left\{\int_a^v g(u)\mathrm{d}u\right\}h(v)\mathrm{d}v\right)
$$

is the general solution of the differential equation (1), where x is an arbitrary element of X.

Corollary 2. *Let X be a complex Banach space and let $I = (a, b)$ be an open interval, where $a, b \in \mathbb{R} \cup \{\pm\infty\}$ are arbitrarily given with $a < b$. Assume that $g : I \to \mathbb{C}$ and $h : I \to X$ are continuous functions such that $g(t)$ and $\exp\{\int_b^t g(u)\mathrm{d}u\}h(t)$ are integrable on (c, b) for every $c \in I$. Moreover, suppose $\varphi : I \to [0,\infty)$ is a function* *such that* $\varphi(t)\exp\{\Re(\int_b^t g(u)\mathrm{d}u)\}$ *is integrable on* I. *If a continuously differentiable function* $y : I \to X$ *satisfies the differential inequality* (2) *for all* $t \in I$, *then there exists a unique* $x \in X$ *such that*

$$\left\| y(t) - \exp\left\{-\int_b^t g(u)\mathrm{d}u\right\}\left(x - \int_b^t \exp\left\{\int_b^v g(u)\mathrm{d}u\right\} h(v)\mathrm{d}v\right)\right\| \le \exp\left\{-\Re\left(\int_b^t g(u)\mathrm{d}u\right)\right\}\int_a^t \varphi(v)\exp\left\{\Re\left(\int_b^v g(u)\mathrm{d}u\right)\right\}\mathrm{d}v \tag{5}$$

for any $t \in I$.

Proof. Let $J = (-b, -a)$ and define functions $g_1 : J \to \mathbb{C}$, $h_1 : J \to X$, $y_1 : J \to X$, and $\varphi_1 : J \to [0, \infty)$ by $g_1(t) = g(-t)$, $h_1(t) = h(-t)$, $y_1(t) = y(-t)$, and $\varphi_1(t) = \varphi(-t)$, respectively.

Using these definitions, we may transform the inequality (2) into

$$\|y_1'(t) - g_1(t)y_1(t) - h_1(t)\| \le \varphi_1(t)$$

for each $t \in J$.

Since $g(t)$ is integrable on (c, b) with $c \in I$, so is $-g_1(t)$ on $(-b, -c)$ with $-c \in J$. Similarly, other integrability hypotheses imply that for any $-c \in J$,

$$\exp\left\{\int_{-b}^t (-g_1(u))\mathrm{d}u\right\}(-h_1(t)) \quad \text{and} \quad \varphi_1(t)\exp\left\{\Re\left(\int_{-b}^t (-g_1(u))\mathrm{d}u\right)\right\}$$

are integrable on $(-b, -c)$ and on J, respectively. Hence, we can now use Theorem 1 to conclude that there exists a unique $x \in X$ such that

$$\left\| y_1(t) - \exp\left\{\int_{-b}^t g_1(u)\mathrm{d}u\right\}\left(x + \int_{-b}^t \exp\left\{-\int_{-b}^v g_1(u)\mathrm{d}u\right\} h_1(v)\mathrm{d}v\right)\right\| \le \exp\left\{\Re\left(\int_{-b}^t g_1(u)\mathrm{d}u\right)\right\}\int_t^{-a} \varphi_1(v)\exp\left\{-\Re\left(\int_{-b}^v g_1(u)\mathrm{d}u\right)\right\}\mathrm{d}v$$

for any $t \in J$. Indeed, we can verify by some tedious substitutions that this inequality is equivalent to that in (5). □

Remark 2. We remark that for any element x of X,

$$y(t) = \exp\left\{-\int_b^t g(u)\mathrm{d}u\right\}\left(x - \int_b^t \exp\left\{\int_b^v g(u)\mathrm{d}u\right\} h(v)\mathrm{d}v\right)$$

is a solution of the differential equation (1).

Remark 3. If we replace $\mathbb{C}$ by $\mathbb{R}$ in the proofs of both Theorem 1 and Corollary 2, we can see that Theorem 1 and Corollary 2 are also true for a real Banach space X. In both cases, we naturally assume that g is a real-valued continuous function.

3. Examples

In this section, we will introduce some examples for linear differential equations of first order which have the Hyers–Ulam–Rassias stability.

Example 1. If we set $h(t) \equiv 0$ and $\varphi(t) \equiv \varepsilon$ in Theorem 1, we obtain the following result: Let X be a complex Banach space and let $I = (a, b)$ be an open interval, where $a, b \in \mathbb{R} \cup \{\pm\infty\}$. Assume that $g : I \to \mathbb{C}$ is a continuous and integrable function on (a, c) for each $c \in I$ such that $\exp\{\Re(\int_a^t g(u)\mathrm{d}u)\}$ is integrable on I. If a continuously differentiable function $y : I \to X$ satisfies the differential inequality

$$\|y'(t) + g(t)y(t)\| \le \varepsilon$$

for all $t \in I$, then there exists a unique $x \in X$ such that

$$\left\| y(t) - \exp\left\{ -\int_a^t g(u)\mathrm{d}u \right\} x \right\| \le \varepsilon \exp\left\{ -\Re\left(\int_a^t g(u)\mathrm{d}u \right) \right\} \int_t^b \exp\left\{ \Re\left(\int_a^v g(u)\mathrm{d}u \right) \right\} \mathrm{d}v$$

for each $t \in I$ (cf. [6]).

Example 2. Let $g < 0$ and h be fixed real numbers, let $I = (a, \infty)$ be an open interval with $a \in \mathbb{R}$, and let $\varphi : I \to \mathbb{R}$ be a polynomial in t with real coefficients. Assume that a continuously differentiable function $y : I \to \mathbb{R}$ satisfies the differential inequality

$$|y'(t) + gy(t) + h| \le \varphi(t)$$

for all $t \in I$.

We can easily verify that the choices of g, h, φ and I are consistent with the hypotheses of Theorem 1. Hence, according to Theorem 1 and Remark 3, there exists a unique $c_0 \in \mathbb{R}$ such that

$$\left| y(t) - c_0 \mathrm{e}^{-gt} + \frac{h}{g}\left(1 - \mathrm{e}^{-g(t-a)}\right) \right| \le \mathrm{e}^{-gt} \int_t^\infty \varphi(v) \mathrm{e}^{gv} \mathrm{d}v$$

for any $t \in I$. Further, we know that $y_0(t) = c_0 \mathrm{e}^{-gt} - \frac{h}{g}(1 - \mathrm{e}^{-g(t-a)})$ is a (particular) solution of the differential equation $y'(t) + gy(t) + h = 0$.

If we set $\varphi(t) \equiv \varepsilon$ and $I = (a, \infty)$ with $a \ge 0$ in the above statement, then there exists a unique solution $y_0(t)$ of the differential equation $y'(t) + gy(t) + h = 0$ such that

$$|y(t) - y_0(t)| \le -\frac{\varepsilon}{g}$$

for all $t \in I$. (We may compare this result with [3] or [4].)

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