Edge-choosability of planar graphs without adjacent triangles or without 7-cycles

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Abstract

A graph $G$ is edge-$L$-colorable, if for a given edge assignment $L = \{L(e) : e \in E(G)\}$, there exists a proper edge-coloring $\phi$ of $G$ such that $\phi(e) \in L(e)$ for all $e \in E(G)$. If $G$ is edge-$L$-colorable for every edge assignment $L$ with $|L(e)| \geq k$ for $e \in E(G)$, then $G$ is said to be edge-$k$-choosable. In this paper, we prove that if $G$ is a planar graph with maximum degree $\Delta(G) \neq 5$ and without adjacent 3-cycles, or with maximum degree $\Delta(G) \neq 5, 6$ and without 7-cycles, then $G$ is edge-$(\Delta(G) + 1)$-choosable.

Keywords: Planar graph; Edge-coloring; Choosability; Triangle; Cycle

1. Introduction

All graphs considered in this paper are simple, finite, and undirected. For a planar graph $G$, we denote its vertex set, edge set, face set, maximum degree, and minimum degree by $V(G), E(G), F(G), \Delta(G), \delta(G)$. A triangle is synonymous with a 3-cycle.

An edge-coloring of a graph $G$ is a mapping $\phi$ from $E(G)$ to the set of colors $\{1, 2, \ldots, k\}$ for some positive integer $k$. An edge-coloring is called proper if adjacent edges receive different colors. The edge chromatic number $\chi'(G)$ is the smallest integer $k$ such that $G$ has a proper edge-coloring into the set $\{1, 2, \ldots, k\}$. We say that $L$ is an edge assignment for the graph $G$ if it assigns a list $L(e)$ of possible colors to each edge $e$ of $G$. If $G$ has a proper edge-coloring $\phi$ such that $\phi(e) \in L(e)$ for any edge $e$ of $G$, then we say that $G$ is edge-$L$-colorable or $\phi$ is an edge-$L$-coloring of $G$. The graph $G$ is edge-$k$-choosable if it is edge-$L$-colorable for every edge assignment $L$ satisfying $|L(e)| \geq k$ for any edge $e \in E(G)$. The edge choice number $\chi'_l(G)$ of $G$ is the smallest $k$ such that $G$ is edge-$k$-choosable.

The following conjecture was formulated independently by Vizing, by Gupta, by Alberson and Collins, and by Bollobás and Harris (see [5] or [8]), and it is well known as the List Coloring Conjecture.

Conjecture 1. If $G$ is a multigraph, then $\chi'_l(G) = \chi'(G)$.
The conjecture has been proved for a few special cases, such as bipartite multigraphs [4], complete graphs of odd order [6], multicircuits [16], graphs with \( \Delta(G) \geq 12 \) which can be embedded in a surface of non-negative characteristic [3], and outerplanar graphs [13]. Vizing (see [10]) proposed a weaker conjecture as follows.

**Conjecture 2.** Every graph \( G \) is edge-(\( \Delta(G) + 1 \))-choosable.

An earlier result of Harris [7] shows that \( \chi'_c(G) \leq 2\Delta(G) - 2 \) if \( G \) is a graph with \( \Delta(G) \geq 3 \). This implies Conjecture 2 for the case \( \Delta(G) = 3 \). In 1999, Juvan, Mohar and Škrekovski [9] settled the case for \( \Delta(G) = 4 \). Conjecture 2 has also been confirmed for other special cases such as complete graphs [6], graphs with girth at least \( 8\Delta(G) (\ln \Delta(G) + 1.1) \) [10], planar graphs with \( \Delta(G) \geq 9 \) [2], and planar graphs with \( \Delta(G) \neq 5 \) and without two 3-cycles sharing a common vertex [14]. Suppose that \( G \) is a planar graph without \( k \)-cycles for some fixed integer \( 3 \leq k \leq 6 \). Then it was shown that Conjecture 2 holds if \( G \) satisfies one of following conditions: (i) either \( k = 3 \) or \( k = 4 \) and \( \Delta(G) \neq 5 \) [17]; (ii) \( k = 4 \) [11]; (iii) \( k = 5 \) [15]; (iv) \( k = 6 \) and \( \Delta(G) \neq 5 \) [12].

In this paper, we will prove the following theorems which show that Conjecture 2 is true in some cases.

**Theorem 1.1.** Every planar graph \( G \) without adjacent triangles is edge-\( k \)-choosable, where \( k = \max(7, \Delta(G) + 1) \).

**Theorem 1.2.** Every planar graph \( G \) without 7-cycles is edge-\( k \)-choosable, where \( k = \max(8, \Delta(G) + 1) \).

Let \( G \) be a connected graph (not necessarily planar). It is well known that \( G \) is edge-(\( \Delta(G) + 1 \))-choosable for \( \Delta(G) \leq 2 \) and in particular \( G \) is edge-2-choosable if \( G \) is an even cycle. From the results of [7,9], \( G \) is edge-(\( \Delta(G) + 1 \))-choosable if \( \Delta(G) = 3 \) or \( \Delta(G) = 4 \). Thus we have the following theorem by Theorems 1.1 and 1.2.

**Theorem 1.3.** If \( G \) is a planar graph with \( \Delta(G) \neq 5 \) and without adjacent triangles, or with \( \Delta(G) \neq 5, 6 \) and without 7-cycles, then \( G \) is edge-(\( \Delta(G) + 1 \))-choosable.

2. Structural lemmas of some planar graphs

Let us introduce some notations and definitions. Let \( G \) be a planar graph. We use \( d_G(v) \) (for short, \( d(v) \)) to denote the degree of a vertex \( v \in V(G) \). A vertex \( v \) is called a \( k \)-vertex or a \( k^+ \)-vertex if \( d(v) = k \) or if \( d(v) \geq k \), respectively. For any face \( f \in F(G) \), the degree of \( f \), denoted by \( d(f) \), is the number of edges incident with it, where each cut edge is counted twice, and we write \( f = u_1u_2\cdots u_nu_1 \) if \( u_1, u_2, \ldots, u_n \) are the boundary vertices of \( f \) in the clockwise order. A \( k \)-face or a \( k^+ \)-face is a face of degree \( k \) or of degree at least \( k \), respectively. A \( k \)-face \( f \) is called simple if the boundary of \( f \) forms a cycle of length \( k \). If \( f \) is not a simple face, then \( f \) must contain two cycles, say \( C_1, C_2 \), such that \( C_1 \) and \( C_2 \) share a common vertex. This implies that \( d(f) \geq 6 \). Thus every face of degree at most 5 is simple. For a vertex \( v \in V(G) \), let \( m_k(v) \) denote the number of \( k \)-faces incident with \( v \) for \( k \geq 3 \), and let \( n_k(v) \) denote the number of \( k \)-vertices adjacent to \( v \). We sometimes use \( (d_1, d_2, \ldots, d_n) \) to represent a cycle (or a face) whose boundary vertices are of degree \( d_1, d_2, \ldots, d_n \) in the clockwise order in the graph \( G \). Let \( \delta(f) \) denote the minimum degree of vertices incident with \( f \).

A subgraph \( H \) of the graph \( G \) with \( \Delta(G) = 6 \) is called a special subgraph of \( G \) if it has the structure in Fig. 1. In Fig. 1, the vertex \( v \), called the center vertex of \( H \), is also called a special vertex of \( G \).

In the proofs of Lemmas 2.1 and 2.2, we use the technique of discharging, which was used to prove Four Color Theorem [1].
Lemma 2.1. Let $G$ be a planar graph without adjacent triangles. Then $G$ contains one of the following configurations

(A1) An edge $uv$ with $d(u) + d(v) \leq \max\{8, \Delta(G) + 2\}$.

(A2) A $(3, \Delta(G), 3, \Delta(G))$-cycle.

(A3) A special vertex $v$ incident with two $(3, 6, 6)$-faces and one $(4, 5, 6)$-face.

Proof. The proof is carried out by contradiction. Let $G$ be a minimal counterexample to the lemma in terms of the number of vertices and edges. Then $G$ is a connected planar graph with $\delta(G) \geq 3$ by lacking of (A1). Euler’s formula $|V(G)| - |E(G)| + |F(G)| = 2$ can be rewritten as $(6|E(G)| - 8|V(G)|) + (2|E(G)| - 8|F(G)|) = -16$. It follows from $\sum_{v \in V(G)} d(v) = \sum_{f \in F(G)} d(f) = 2|E(G)|$ that

$$\sum_{v \in V(G)} (3d(v) - 8) + \sum_{f \in F(G)} (d(f) - 8) = -16.$$

Let $w$ denote the weight function defined on $V(G) \cup F(G)$ by $w(v) = 3d(v) - 8$ if $v \in V(G)$ and $w(f) = d(f) - 8$ if $f \in F(G)$. So we have $\sum_{x \in V(G) \cup F(G)} w(x) = -16$. We are going to redistribute these weights, not changing their sum, so that the new weight $w^*(x)$ becomes non-negative for all $x \in V(G) \cup F(G)$. Thus the following contradiction is produced and henceforth the proof is completed.

$$0 \leq \sum_{x \in V(G) \cup F(G)} w^*(x) = \sum_{x \in V(G) \cup F(G)} w(x) = -16.$$

Our discharging rules are defined as follows.

(R1) From each 3-vertex to its incident 3-face, transfer 1.

(R2) From each 4-vertex to each of its incident face $f$, where $3 \leq d(f) \leq 7$, transfer 1.

(R3) From each 5-vertex to each of its incident face $f$, where $3 \leq d(f) \leq 7$, transfer

$$2, \text{ if } d(f) = 3;$$

$$1, \text{ otherwise.}$$

(R4) From each 6-vertex $v$ with $m_3(v) \leq 2$ to each of its incident face $f$, where $3 \leq d(f) \leq 7$, transfer

$$2, \text{ if } d(f) = 3;$$

$$\frac{w(v) - 2m_3(v)}{6 - m_3(v)}, \text{ otherwise.}$$

(R5) From each 6-vertex $v$ with $m_3(v) = 3$ to each of its incident face $f$, where $3 \leq d(f) \leq 7$, transfer

$$2, \text{ if } d(f) = 3 \text{ and } f \text{ is not a } (5, 5, 6)\text{-face;}$$

$$1, \text{ if } f \text{ is a } (5, 5, 6)\text{-face;}$$

$$3, \text{ if } d(f) = 4 \text{ and } \delta(f) \geq 4 \text{ or } 5 \leq d(f) \leq 7;$$

$$\frac{3}{2}, \text{ if } f \text{ is either a } (6, 3, 6, 4)\text{-face or a } (6, 3, 6, 5)\text{-face;}$$

$$\frac{5}{4}, \text{ otherwise.}$$

(R6) From each 7$^+$-vertex $v$ to each of its incident face $f$, where $3 \leq d(f) \leq 7$, transfer

$$2, \text{ if } d(f) = 3;$$

$$\frac{3}{2}, \text{ otherwise.}$$

Let $\gamma(x \to y)$ denote the amount transferred out of an element $x$ into another element $y$ according to the above rules. Then $G$ has the following properties.

(P1) Since $G$ does not contain adjacent triangles, every $k$-vertex, where $k \geq 3$, is incident with at most $\lfloor \frac{k}{2} \rfloor$ 3-faces.

(P2) Let $uv$ be any edge of $G$. Then $d(u) + d(v) \geq \max\{9, \Delta(G) + 3\}$. This implies that $d(v) = \Delta(G) \geq 6$ if $v$ neighbors a 3-vertex.
Lemma 2.1

An edge $u \rightarrow v$ is a $6$-vertex with $m_3(v) \leq 2$ and $f$, where $4 \leq d(f) \leq 7$, is a face incident with $v$, then $\gamma(v \rightarrow f) = \frac{w(v)-2m_3(v)}{6-m_3(v)} \geq \min\{\frac{10-2\times 1}{6-1}, \frac{10-2\times 2}{6-2}\} = \frac{3}{2}$.

(P4) Let $f$ be a $4$-face with $\delta(f) = 3$ and let $v$ be a $6^+$-vertex incident with $f$. If $d(v) \geq 7$, then $\gamma(v \rightarrow f) = \frac{3}{2}$.

Assume that $d(v) = 6$. If $m_3(v) \leq 2$, then $\gamma(v \rightarrow f) = \frac{3}{2}$ by (P3). Otherwise, we have $m_3(v) = 3$. If $f$ is either a $(6, 3, 6)$-face or $(6, 3, 6, 5)$-face, then $\gamma(v \rightarrow f) = \frac{3}{2}$. Otherwise, $\gamma(v \rightarrow f) = \frac{3}{4}$. Thus in any case, we have $\gamma(v \rightarrow f) \geq \frac{5}{4}$.

(P5) If $v$ is a special vertex of $G$ and the total number of $(6, 3, 6, 4)$-faces or $(6, 3, 6, 5)$-faces incident with $v$ is exactly two, then $v$ is incident with exactly one $(5, 5, 6)$-face.

Next, we show that (P5) is true. Since $v$ is a special vertex of $G$, we have $d(v) = 6$ and $m_3(v) = 3$. The fact that the total number of $(6, 3, 6, 4)$-faces or $(6, 3, 6, 5)$-faces incident with $v$ is exactly two implies that $v$ is incident with two $(3, 6, 6)$-faces and $n_3(v) + n_5(v) = 2$. So $n_3(v) = n_2(v) = n_4(v) + n_5(v) = 2$. Since $G$ is lacking in (A3), the remaining 3-face with which $v$ is incident is a $(5, 5, 6)$-face.

We shall show that $w^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$. Suppose that $v$ is a $k$-vertex of $G$. If $k = 3$, then $v$ is incident with at most one 3-face by (P1). Thus $w^*(v) \geq w(v) - 1 = 0$. If $k = 4$, then $w^*(v) \geq w(v) - 4 \times 1 = 0$. If $k = 3$, then $m_3(v) \leq 2$ by (P1). Thus $w^*(v) = w(v) - m_3(v) \geq 2 - (5 - m_3(v)) \times 1 = 7 - 5 - m_3(v) \geq 0$.

Now suppose that $k \geq 6$. If $m_3(v) \leq 2$, then $w^*(v) = (v) - m_3(v) \times 2 - (6 - m_3(v) \times 6 - m_3(v)) \times 6 = 0$. Otherwise, we have $m_3(v) = 3$. In this case, if $v$ is incident with a $5^+$-face, then $w^*(v) = w(v) - 3 \times 2 - 2 \times 3 = 0$. Next we assume that $k = 6$ and $m_3(v) = m = 3$. It is easy to verify that the total number of $(6, 3, 6, 4)$-faces or $(6, 3, 6, 5)$-faces incident with $v$ is at most two. If the total number of $(6, 3, 6, 4)$-faces or $(6, 3, 6, 5)$-faces incident with $v$ is exactly two, then $v$ is incident with exactly one $(5, 5, 6)$-face by (P5). So $w^*(v) = w(v) - 2 \times 1 - 3 \times 2 = \frac{5}{2} > 0$ by (R5). Otherwise, there is at most one 4-face $f$ incident with $v$ which receives $\frac{3}{2}$ from $v$. Thus $w^*(v) \geq w(v) - 3 \times 2 - 2 \times 2 - 2 \times \frac{5}{2} = 0$.

If $k \geq 7$, then $w^*(v) \geq w(v) - 3 \times 2 - 2 \times 2 - 2 \times \frac{5}{2} = 0$.

Next, we consider the case that $f = v_1v_2v_3v_4v_1$ is a 4-face. If $\delta(f) \geq 4$, then $w^*(f) = w(f) + 4 \times 1 = 0$. Now assume that $\delta(f) = 3$. Without loss of generality, let $d(v_1) = 3$. Then $d(v_2) = d(v_4) = \Delta(G) \geq 6$ and $d(v_3) \geq 4$ by lacking of (A1) and (A2). If $\Delta(G) \geq 7$, then $w^*(f) = w(f) + 2 \times \frac{3}{2} + 1 = 0$. Otherwise, we have $d(v_2) = d(v_4) = 6$ and $d(v_3) \geq 4$. Then $w^*(f) = w(f) + 2 \times 2 = 0$. If $d(v_3) = 5$, then $w^*(f) = w(f) + 1 + 2 \times 2 = 0$. Thus $w^*(f) = w(f) + 1 + 2 \times 2 = 0$. Now suppose that $d(v_4) \geq 5$. If $f$ is a $(5, 5, 6)$-face, then $w^*(f) = w(f) + 1 + 2 \times 2 = 0$. Otherwise, $w^*(f) = w(f) + 1 + 2 \times 2 > 0$.

Finally, let $f$ be a $k$-face of $G$, where $5 \leq d(f) \leq 7$. Then $f$ is incident with at least three $4^+$-vertices by lacking of (A1), even if $f$ is not a simple face. So $w^*(f) \geq w(f) + 3 \times 1 \geq 0$. This completes the proof of Lemma 2.1. ■

Lemma 2.2. Every planar graph $G$ without 7-cycles contains one of the following configurations.

(B1) An edge $uv$ with $d(u) + d(v) \leq \max\{9, \Delta(G) + 2\}$.

(B2) A $(3, \Delta(G), 3, \Delta(G))$-cycle.

Proof. The proof is carried out by contradiction. Let $G$ be a minimal counterexample to the lemma in terms of number of vertices and edges. Then $G$ is a connected planar graph with $\Delta(G) \geq 3$ by lacking of (B1). Moreover, the following configurations are excluded from $G$.

(C1) a simple 7-face;

(C2) a $k$-vertex, where $k \geq 6$, is incident with at least $(k - 1)$ 3-faces.
Let $w$ denote the weight function defined on $V(G) \cup F(G)$ by $w(v) = 3d(v) - 8$ if $v \in V(G)$ and $w(f) = d(f) - 8$ if $f \in F(G)$. Applying Euler’s formula $|V(G)| - |E(G)| + |F(G)| = 2$, we can show that $\sum_{v \in V(G) \cup F(G)} w(v) = -16$. Weights will be transferred according to the following rules.

(R1) From each 3-vertex $v$ to each of its incident 3-face $f$, transfer

1. if $m_3(v) = 1$;
2. $\frac{1}{3}$, otherwise.

(R2) From each 4-vertex $v$ to each of its incident face $f$, where $3 \leq d(f) \leq 6$, transfer 1.

(R3) From each 5-vertex $v$ to each of its incident 3-face $f$, transfer

1. if $m_3(v) = 5$;
2. $\frac{3}{2}$, if $m_3(v) = 4$;
3. if $v$ is incident with at least two $8^+$-faces;
4. $\frac{5}{3}$, otherwise.

(R4) From each 5-vertex $v$ to each of its incident face $f$, where $4 \leq d(f) \leq 6$, transfer 1.

(R5) From each 6-vertex $v$ to each of its incident face $f$, where $3 \leq d(f) \leq 6$, transfer

1. if $d(f) = 3$;
2. if $4 \leq d(f) \leq 6$.

(R6) From each 7$^+$-vertex to each of its incident 3-face $f$, transfer

1. $\frac{7}{3}$, if $\delta(f) = 3$ and the 3-vertex incident with $f$ transfers $\frac{1}{3}$ to $f$;
2. otherwise.

(R7) From each 7$^+$-vertex to each of its incident face $f$, where $4 \leq d(f) \leq 6$, transfer

1. if $d(f) = 4$ and $\delta(f) = 3$;
2. otherwise.

Note that $G$ have the following properties.

(P1) Let $uv$ be any edge of $G$. Then $d(u) + d(v) \geq \max\{10, \Delta(G) + 3\}$. This implies that $d(v) = \Delta(G) \geq 7$ if $v$ neighbors a 3-vertex.

(P2) Let $v$ be a 5-vertex of $G$ with $m_3(v) = 5$. If $u$ is a neighbor of $v$, then $u$ is incident with at least two $8^+$-faces. Thus $\gamma(u \rightarrow f) \geq 2$ for any 3-face $f$ incident with $u$.

(P3) Let $v$ be a 5-vertex of $G$ with $m_3(v) = 4$. Then it must be one of the cases in Fig. 2. Furthermore, $d(v_i) \geq 5$ and $v_i$ is incident with at least two $8^+$-faces in Fig. 2 for $i \in \{1, 2, 4, 5\}$.

(P4) Let $v$ be a $k$-vertex, where $k \geq 7$. If $v$ is incident with three continuous 3-faces, then $v$ is incident with at least one $8^+$-face.

Now we show that (P4) is true. Let $f_1, f_2, \ldots, f_k$ be the faces incident with $v$ in the clockwise order. Without loss of generality, let $d(f_1) = d(f_2) = d(f_3) = 3$. If $d(f_4) \geq 8$, then we are done. Otherwise, we have $d(f_4) \in \{3, 4, 5\}$, since $G$ is free of 7-cycles. We first consider the case that $d(f_4) = 3$. If $d(f_5) \geq 8$, then we are done. Otherwise, we have $d(f_5) = 4$ or $d(f_5) = 5$. This implies that $d(f_6) \geq 8$. Next we consider the case that $d(f_4) = 4$. If $d(f_5) \geq 8$, then we are done. Otherwise, $d(f_5) = 3$. This implies that $d(f_6) \geq 8$. Finally, we have $d(f_6) = 5$. This implies that $d(f_5) \geq 8$.

(P5) Let $v$ be a 7-vertex with $m_3(v) = 4$. Then $v$ is incident with at least one $8^+$-face.

We shall show that (P5) is true. Let $f_1, f_2, \ldots, f_7$ be the faces incident with $v$ in the clockwise order. If $v$ is incident with three continuous 3-faces, then $v$ is incident with at most one $8^+$-face by (P4). Otherwise, it must be one of the cases in Fig. 3. If $d(f_1) = d(f_2) = d(f_3) = d(f_4) = 3$ (see Fig. 3 (a)), then $d(f_5) \geq 8$. Assume that $d(f_1) = d(f_2) = d(f_4) = d(f_6) = 3$ (see Fig. 3 (b)). If $d(f_3) = 4$ or 5, then $d(f_5) \geq 8$. Otherwise, $d(f_5) \geq 8$. 


(P_6) Let \( v \) be a 7-vertex \( v \) with \( n_3(v) = 5 \). Then \( v \) is incident with at least one \( 8^+ \)-face. Furthermore, if \( v \) is incident with exactly one \( 8^+ \)-face, then it must be the case in Fig. 4.

In Fig. 4, if \( d(v_1) = 3 \), then \( v_1 \) is incident with at most one 3-face and \( d(v_2) = 7 \). Thus \( \gamma(v \rightarrow f_i) = 2 \) for \( i \in \{1, 2\} \). If \( d(v_2) = 3 \), then \( d(v_i) \geq 4 \) for \( i \in \{3, 4, 5\} \) by lacking of \((B_2)\). Thus \( \gamma(v \rightarrow f_i) = 2 \) for \( i \in \{3, 4\} \). Otherwise, we have that \( d(v_1) \geq 4 \) and \( d(v_2) \geq 4 \). Thus \( \gamma(v \rightarrow f_i) = 2 \) for \( i \in \{1, 2\} \). In any case, \( v \) is incident with at least two 3-faces which receive 2 from \( v \).

We shall show that \( w^*(x) \geq 0 \) for all \( x \in V(G) \cup F(G) \). Suppose that \( v \) is a \( k \)-vertex. If \( k = 3 \), then \( w^*(v) \geq w(v) - \max\{3 \times \frac{1}{2}, 1\} = 0 \). If \( k = 4 \), then \( w^*(v) \geq w(v) - 4 \times 1 = 0 \). Let \( k = 5 \). If \( n_3(v) = 5 \), then \( w^*(v) = w(v) - 5 \times \frac{2}{3} = 0 \). If \( n_3(v) = 4 \), then \( w^*(v) \geq w(v) - 4 \times \frac{3}{3} - 1 = 0 \). Now suppose that \( n_3(v) \leq 3 \). If \( v \) is incident with at least two \( 8^+ \)-faces, then \( w^*(v) \geq w(v) - 3 \times 2 = 1 > 0 \). Otherwise, \( w^*(v) \geq w(v) - 3 \times \frac{5}{3} - 2 \times 1 = 0 \). If \( k = 6 \), then \( v \) is incident with at most four 3-faces by \((C_2)\). Thus \( w^*(v) \geq w(v) - 4 \times 2 - 2 \times 1 = 0 \). Suppose that \( k = 7 \). If \( v \) is incident with at most three 3-faces, then \( w^*(v) \geq w(v) - 3 \times \frac{7}{3} - 4 \times \frac{3}{2} = 0 \). If \( v \) is incident with four 3-faces, then \( v \) is incident with at least one \( 8^+ \)-face by \((P_5)\). Thus \( w^*(v) \geq w(v) - 4 \times \frac{7}{2} - 2 \times \frac{3}{2} = \frac{2}{3} > 0 \).
Otherwise, $v$ is incident with exactly five 3-faces. In this case, if the other faces incident with $v$ are $8^+$-faces, then $w^*(v) \geq w(v) - 5 \times \frac{7}{3} = \frac{1}{2} > 0$. Otherwise, it must be the case in Fig. 4 by (P$_6$). Thus $v$ is incident with at least two 3-faces which receive 2 from $v$ and then $w^*(v) \geq w(v) - 3 \times \frac{7}{3} - 2 \times 2 - \frac{3}{2} = \frac{1}{2} > 0$. Let $k = 8$. If $v$ is incident with at most four 3-faces, then $w^*(v) \geq w(v) - 4 \times \frac{7}{3} - 4 \times \frac{3}{2} = \frac{2}{3} > 0$. Now suppose that $v$ is incident with five 3-faces. If $v$ is incident with three continuous 3-faces, then $v$ is incident with at least one $8^+$-face by (P$_4$). Thus $w^*(v) \geq w(v) - 5 \times \frac{7}{3} - 2 \times \frac{3}{2} = \frac{4}{3} > 0$. Otherwise, it must be the case in Fig. 5. It is easy to verify that $v$ is incident with at least one $5^+$-face and then $w^*(v) \geq w(v) - 5 \times \frac{7}{3} - 2 \times \frac{3}{2} - 1 = \frac{1}{3} > 0$. If $v$ is incident with six 3-faces, then $v$ is incident with at least one $8^+$-face. Thus $w^*(v) \geq w(v) - 6 \times \frac{7}{3} - \frac{3}{2} = \frac{5}{6} > 0$.

Let $k \geq 9$. If $m_3(v) \leq k - 3$, then $w^*(v) \geq w(v) - (k - 3) \times \frac{7}{3} - 3 \times \frac{3}{2} = \frac{2}{3}k - \frac{11}{2} > 0$. Otherwise, $n_3(v) = k - 2$ and $v$ is incident with at least one $8^+$-face. Thus $w^*(v) \geq w(v) - (k - 2) \times \frac{7}{3} - \frac{3}{2} = \frac{2}{3}k - \frac{29}{6} > 0$.

Let $f$ be any face of $G$. Clearly, $w^*(f) = w(f) \geq 0$ if $d(f) \geq 8$. Now suppose that $f = v_1v_2v_3v_4$ is a 3-face with $d(v_1) \leq d(v_2) \leq d(v_3)$. If $d(v_1) = 3$, then $d(v_2) = d(v_3) = \Delta(G) \geq 7$ by (P$_1$). Thus $w^*(f) \geq w(f) + \min\{1 + 2 \times \frac{7}{3}, \frac{1}{2} + 2 \times \frac{7}{3}\} = 0$. If $d(v_1) = 4$, then $d(v_2) \geq 6$ and $d(v_3) \geq 6$. Thus $w^*(f) \geq w(f) + 1 + 2 \times 2 = 0$. Now suppose that $d(v_1) = 5$. If there is a 5-vertex $v \in \{v_1, v_2, v_3\}$ satisfying $n_3(v) = 5$, then $w^*(f) \geq w(f) + \frac{7}{3} + 2 \times 2 = \frac{2}{3} > 0$. Otherwise, $\gamma(v_i \rightarrow f) \geq \frac{3}{2}$ for $i = 1, 2, 3$. In this case, if $d(v_3) \geq 6$, then $\gamma(v_3 \rightarrow f) = 2$ and $w^*(f) \geq w(f) + 2 \times \frac{3}{2} = 0$. Now suppose that $d(v_1) = d(v_2) = d(v_3) = 5$. Without loss of generality, let $\gamma(v_1 \rightarrow f) \leq \gamma(v_2 \rightarrow f) \leq \gamma(v_3 \rightarrow f)$. If $\gamma(v_1 \rightarrow f) \geq \frac{3}{2}$, then $w^*(f) \geq w(f) + 3 \times \frac{3}{2} > 0$. Otherwise, we have $\gamma(v_1 \rightarrow f) = \frac{3}{2}$. It implies that $n_3(v_1) = 4$. It follows from (P$_3$) that either $v_2$ or $v_3$ is incident with at least two $8^+$-faces and then $w^*(f) \geq w(f) + 2 \times \frac{3}{2} = 0$. Now suppose that $d(v_1) \geq 6$, then $w^*(f) = w(f) + 3 \times 2 = 1 > 0$.

Let $f = v_1v_2v_3v_4v_5$ be a 4-face. If $d(f) = 3$, without loss of generality, let $d(v_1) = 3$. Then $d(v_2) = d(v_3) = \Delta(G) \geq 7$ and $d(v_4) \geq 4$. Thus $w^*(f) \geq w(f) + 2 \times \frac{3}{2} + 1 = 0$. Otherwise, $w^*(f) \geq w(f) + 4 \times 1 = 0$. If $5 \leq d(f) \leq 6$, then $f$ is incident with at least three $4^+$-vertices. Thus $w^*(f) \geq w(f) + 3 \times 1 \geq 0$.

This completes the proof of Lemma 2.2. ■

3. Proof of theorems

In this section, we will prove the Theorems 1.1 and 1.2 by contradiction.

Proof of Theorem 1.1. The proof is carried out by contradiction. Let $G$ be a minimal counterexample to the theorem. Then there is an edge assignment $L$ with $|L(e)| \geq k$ for all $e \in E(G)$, where $k = \max\{7, \Delta(G) + 1\}$, such that $G$ is not edge-$L$-colorable. By Lemma 2.1, we consider three cases as follows.

Case 1. $G$ contains an edge $uv$ with $d(u) + d(v) \leq \max\{8, \Delta(G) + 2\}$. Consider the graph $G' = G - uv$. Then $G'$ has an edge-$L$-coloring $\phi$. Since there exists at most $\max\{6, \Delta(G)\}$ edges adjacent to $uv$ and $|L(uv)| \geq \max\{7, \Delta(G) + 1\}$, we can color $uv$ with some color from $L(uv)$ that was not used by $\phi$ on the edges adjacent to $uv$. It is easy to see that the resulting coloring is an edge-$L$-coloring of $G$. This contradicts the choice of $G$.

Case 2. There is a 4-cycle $C = v_1v_2v_3v_4v_1$ such that $d(v_1) = d(v_2) = 3$, and $d(v_3) = d(v_4) = \Delta(G)$. Let $G'$ be the subgraph of $G$ obtained by deleting the edges on $C$. Then $G'$ has an edge-$L$-coloring $\phi$. Define an edge
assignment $L'$ of $C$ such that $L'(e) = L(e) \setminus \{\phi(e')|e' \in E(G')\}$ is adjacent to $e$ in $G$} for each $e \in E(C)$. It is easy to inspect that $|L'(e)| \geq 2$ for each $e \in E(C)$. Thus $C$ is edge-$L'$-colorable and hence $G$ is edge-$L$-colorable, which is a contradiction.

Case 3. $G$ contains a special vertex $v$ of $C$ which is incident with two $(3, 6, 6)$-faces and one $(4, 5, 6)$-face. Let $H$ be the special subgraph containing $v$ as shown in Fig. 1. Without loss of generality, we assume that $d(v_1) = d(v_3) = 3$, $d(v_2) = d(v_4) = 6$, $d(v_5) = 4$, and $d(v_6) = 5$. Let $G'$ be the subgraph of $G$ obtained by deleting the edges on $H$. Then $G'$ has an edge-$L$-coloring $\phi$. Define an edge assignment $L'$ of $H$ such that $L'(e) = L(e) \setminus \{\phi(e')|e' \in E(G')\}$ is adjacent to $e$ in $G$} for each $e \in E(H)$. It is easy to inspect that $|L'(v_i v_{i+1})| \geq 2$ for $i = 1, 3, 5$, $|L'(v_1 v_2)| \geq 6$ for $i = 1, 3$, $|L'(v_i v_4)| \geq 3$ for $i = 2, 4$, $|L'(v_2 v_5)| \geq 5$, and $|L'(v_4 v_6)| \geq 4$. If $|L'(v_1 v_2)| \geq 3$, then color $v_2, v_4, v_3 v_4, v_5 v_6, v_3 v_5, v_4 v_5, v_1 v_2$, and $v_1 v_2$, successively. Otherwise, since $|L'(v_2 v_5)| \geq 3$, there exists $v \in L'(v_2 v_5) \setminus L'(v_1 v_2)$. We color $v_2, v_4, v_5 v_6, v_3 v_5, v_4 v_5, v_1 v_2$, successively. Thus $H$ is edge-$L'$-colorable, and hence $G$ is edge-$L$-colorable, which is a contradiction. This completes the proof. ■

Proof of Theorem 1.2. The proof is carried out by contradiction. Let $G$ be a minimal counterexample to the theorem. Then there is an edge assignment $L$ with $|L(e)| \geq k$ for all $e \in E(G)$, where $k = \max\{8, \Delta(G) + 1\}$, such that $G$ is not edge-$L$-colorable. By Lemma 2.2, we consider two cases as follows.

Case 1. $G$ contains an edge $uv$ with $d(u) + d(v) \leq \max\{9, \Delta(G) + 2\}$. Consider the graph $G' = G - uv$. Then $G'$ has an edge-$L$-coloring $\phi$. Since there exists at most $\max\{7, \Delta(G)\}$ edges adjacent to $uv$ and $|L(uv)| \geq \max\{8, \Delta(G) + 1\}$, we can color $uv$ with some color from $L(uv)$ that was not used by $\phi$ on the edges adjacent to $uv$, which is a contradiction.

Case 2. There is a 4-cycle $C = v_1 v_2 v_3 v_4 v_1$ such that $d(v_1) = d(v_3) = 3$, and $d(v_2) = d(v_4) = \Delta(G)$. Let $G'$ be the subgraph of $G$ obtained by deleting the edges on $C$. Then $G'$ has an edge-$L$-coloring $\phi$. Define an edge assignment $L'$ of $C$ such that $L'(e) = L(e) \setminus \{\phi(e')|e' \in E(G')\}$ is adjacent to $e$ in $G$} for each $e \in E(C)$. It is easy to inspect that $|L'(e)| \geq 2$ for each $e \in E(C)$. Thus $C$ is edge-$L'$-colorable and hence $G$ is edge-$L$-colorable, which is a contradiction. This completes the proof. ■

References